COMPACTIFICATION OF THE PRYM MAP FOR NON CYCLIC TRIPLE COVERINGS

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Abstract. According to [LO], the Prym variety of any non-cyclic étale triple cover $f : Y \to X$ of a smooth curve X of genus 2 is a Jacobian variety of dimension 2. This gives a map from the moduli space of such covers to the moduli space of Jacobian varieties of dimension 2. We extend this map to a proper map Pr of a moduli space ${}_{S_3}\widetilde{\mathcal{M}}_2$ of admissible S_3-covers of genus 7 to the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces. The main result is that $Pr : {}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2$ is finite surjective of degree 10.

1. Introduction

Let $f : Y \to X$ denote a non-cyclic cover of degree 3 of a smooth projective curve X of genus 2. The Prym variety $P = P(f)$ of f is by definition the complement of the image of the pullback map of Jacobians $f^* : JY \to JX$ with respect to the canonical polarization of JY. It is easy to see that the canonical polarization of JY restricts to the 3-fold of a principal polarization Ξ on P. This induces a morphism Pr, called Prym map, of the moduli space $\mathcal{R}^{nc}_{2,3}$ of connected étale non-cyclic degree-3 covers of curves of genus 2 into the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces. In [LO] we showed that the image of Pr is contained in the Jacobian locus $\mathcal{J}_2$ and moreover that

$$Pr : \mathcal{R}^{nc}_{2,3} \to \mathcal{J}_2$$

is of degree 10 onto its image. The main aim of this paper is to determine the image of Pr, we will see that Pr is not surjective, and extend this map to a proper surjective map.

For this it turns out to be convenient to shift the point of view slightly. In [LO, Proposition 4.1] we saw that taking the Galois closure gives a bijection between the set of connected non-cyclic étale f covers of above and the set of étale Galois covers $h : Z \to X$, with Galois group the symmetric group S_3 of order 6. Hence, if we denote by ${}_{S_3}\mathcal{M}_2$ the moduli space of étale Galois covers of smooth curves of genus 2 with Galois group S_3 as constructed for example in [ACG, Theorem 17.2.11], we get a morphism which we denote by the same symbol,

$$Pr : {}_{S_3}\mathcal{M}_2 \to \mathcal{J}_2,$$

and also call the Prym map. Then we use the compactification ${}_{S_3}\overline{\mathcal{M}}_2$ of ${}_{S_3}\mathcal{M}_2$ by admissible S_3-covers as constructed in [ACG, Chapter 17] (based on [ACV]) to define the

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extended Prym map. In fact, consider the following subset of ${}_{S_3}\overline{\mathcal{M}}_2$:

$${}_{S_3}\widetilde{\mathcal{M}}_2 := \left\{ [h : Z \to X] \in {}_{S_3}\overline{\mathcal{M}}_2 \;\middle|\; \begin{array}{l} p_a(Z) = 7 \text{ and for any node } z \in Z \\ \text{the stabilizer } \operatorname{Stab}(z) \text{ is of order } 3 \end{array} \right\}.$$

Then ${}_{S_3}\widetilde{\mathcal{M}}_2$ is a non-empty open set of a component of ${}_{S_3}\overline{\mathcal{M}}_2$ containing the smooth S_3-covers ${}_{S_3}\mathcal{M}_2$. For any $[h : Z \to X] \in {}_{S_3}\widetilde{\mathcal{M}}_2$ let Y denote the quotient of Z by a subgroup of order 2 of S_3. We show that the kernel $P = P(f)$ of the map $f : Y \to X$ is a principally polarized abelian surface. Hence we get an extended map $Pr : {}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2$, which is modular and which we denote by the same symbol and also call the Prym map. Clearly P does not depend on the choice of the subgroup of order 2. Our main result is the following theorem.

Theorem *The Prym map $Pr : {}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2$ is a finite surjective morphism of degree* 10.

In fact, we can be more precise. Consider the following stratification of ${}_{S_3}\widetilde{\mathcal{M}}_2$:

$${}_{S_3}\widetilde{\mathcal{M}}_2 = {}_{S_3}\mathcal{M}_2 \sqcup R \sqcup S,$$

where R denotes the set of covers of ${}_{S_3}\widetilde{\mathcal{M}}_2$ with X singular, but irreducible, and S denotes the complement of ${}_{S_3}\mathcal{M}_2 \sqcup R$ in ${}_{S_3}\widetilde{\mathcal{M}}_2$. As for $\mathcal{A}_2$, let $\mathcal{E}_2$ denote the closed subset of $\mathcal{A}_2$ consisting of products of elliptic curves with canonical principal polarisation. For any smooth curve C of genus 2 and any 3 Weierstrass points w_1, w_2, w_3 of C let $\varphi_{2(w_1+w_2+w_3)}$ denote the map $C \to \mathbb{P}^1$ defined by the pencil $(\lambda(2(w_1+w_2+w_3)) + \mu(2(w_4+w_5+w_6))_{(\lambda,\mu)\in\mathbb{P}^1}$, where w_4, w_5, w_6 are the complementary Weierstrass points. The map $\varphi_{2(w_1+w_2+w_3)}$ factorizes via the hyperelliptic cover and a $3:1$ map $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$. With this notation we define the following subsets of $\mathcal{J}_2$,

$$\mathcal{J}_2^u := \{JC \in \mathcal{J}_2 \mid \; \exists\, w_1, w_2, w_3 \text{ in } C \text{ such that } \bar{f} \text{ is simply ramified}\},$$

$$\mathcal{J}_2^r := \{JC \in \mathcal{J}_2 \mid \; \exists\, w_1, w_2, w_3 \text{ in } C \text{ such that } \bar{f} \text{ is not simply ramified}\}.$$

So we have

$$\mathcal{A}_2 = \mathcal{J}_2^u \cup \mathcal{J}_2^r \sqcup \mathcal{E}_2.$$

We show that the Prym map restricts to finite surjective morphisms $Pr : {}_{S_3}\mathcal{M}_2 \to \mathcal{J}_2^u$, $Pr : S \to \mathcal{E}_2$ and to a finite morphism $R \to \mathcal{J}_2^r$. We then prove that the extended Prym map is proper and of degree 10, this implies the theorem.

Recall that an even spin curve of genus 2 is a pair consisting of a smooth curve of genus 2 and an even theta characteristic on it and that every curve of genus 2 admits exactly 10 even theta characteristics. The degree of the Prym map being 10 suggests that the moduli spaces ${}_{S_3}\widetilde{\mathcal{M}}_2$ and the moduli space of even spin curves should be related. An in fact they are. We will work out details in the forthcoming paper [LO1].

The first part of the paper is devoted to prove that the Prym map $Pr : {}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2$ is proper. We apply a method used already in [B] and [F] to show the properness of a Prym map: we consider an open set of ${}_{S_3}\overline{\mathcal{M}}_2$, which is seemingly bigger than ${}_{S_3}\widetilde{\mathcal{M}}_2$, namely the set of S_3-covers satisfying condition $(**)$ (see Section 3). In Section 5 we classify these

S_3-covers and use this to show that this set coincides with ${}_{S_3}\widetilde{\mathcal{M}}_2$. In Sections 6 we deduce from this the properness of the extended Prym map. In Sections 7 and 8 we study the restriction of Pr to R and S. Finally, Section 9 contains the proof of the above mentioned theorem.

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2. Admissible S_3-covers

In this section we recall some notions and results which we need subsequently. Let $\mathcal{X} \to S$ be a family of connected nodal curves of arithmetic genus g and $d \geq 2$ be an integer. A family of *degree d admissible covers* of $\mathcal{Z}$ over S is a finite morphism $\mathcal{Z} \to \mathcal{X}$ such that,

(1) the composition $\mathcal{Z} \to S$ is a family of nodal curves;
(2) every node of a fiber of $\mathcal{Z} \to S$ maps to a node of the corresponding fiber of $\mathcal{X} \to S$;
(3) away from the nodes $\mathcal{Z} \to S$ is étale of constant degree d;
(4) if the node z lies over the node x of the corresponding fibre of $\mathcal{X} \to S$, the two branches near z map to the two branches near x with the same ramification index $r \geq 1$.

If G is a finite group, a G-cover $\mathcal{Z} \to \mathcal{X}$ is called a family of *admissible G-covers* if in addition to (1) and (2) it satisfies

(3′) $\mathcal{Z} \to \mathcal{X}$ is a principal G-bundle away from the nodes;
(4′) if ξ and η are local coordinates of the two branches near z, any element of the stabilizer $\mathrm{Stab}_G(z)$ acts as
$$(\xi, \eta) \mapsto (\zeta\xi, \zeta^{-1}\eta)$$
where ζ is a primitive r-th root of the unity for some positive integer r.

In the case of $S = \operatorname{Spec}\mathbb{C}$ we just speak of an admissible degree d- (respectively G-) cover. Clearly, (3′) and (4′) imply (3) and (4). So an admissible G-cover is an admissible d-cover with $d = |G|$. In the case of an admissible G-cover, the ramification index at any node z over x equals the order of the stabilizer of z and depends only on x. It is called the *index* of the G-cover $\mathcal{Z} \to \mathcal{X}$ at x. Note that, for any admissible G-covering $Z \to X$, the curve Z is stable if and only if and only if X is stable.

In this paper we are interested in the case $G = S_3$, with
$$S_3 := \langle \sigma, \tau \mid \sigma^3 = \tau^2 = \tau\sigma\tau\sigma = 1 \rangle.$$
If $f : Z \to X$ is any S_3-covering of nodal curves, then all curves in the following diagram

(2.1)
$$\begin{tikzcd}
& Z \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "h"] & \\
Y = Z/\langle\tau\rangle \arrow[dr, "f"'] & & D = Z/\langle\sigma\rangle \arrow[dl, "g"] \\
& X = Z/S_3 &
\end{tikzcd}$$

have only ordinary nodes too. Here p and g are of degree 2, q is cyclic of degree 3 and f is non-cyclic of degree 3. There is, of course, an analogous diagram for any family of S_3-coverings.

For any node z of Z such that the generator of $\operatorname{Stab} z := Stab_{S_3}(z)$ does not exchange the 2 branches of Z at z, the subgroup $\operatorname{Stab} z$ is cyclic. This follows from the fact that $\operatorname{Stab} z$ injects into the automorphism group of the tangent space to a branch at z. It is then easy to see that any node z of Z is of one of the following types:

(1) $|\operatorname{Stab} z| = 1$. The orbit $S_3(z)$ consists of 6 nodes and its image $x = h(z)$ is a node of X.
(2) $|\operatorname{Stab} z| = 2$ and the generator of $\operatorname{Stab} z$ does not exchange the 2 branches of Z at z. The orbit $S_3(z)$ consists of 3 nodes and its image $x = h(z)$ is a node of X.
(3) $|\operatorname{Stab} z| = 3$. The generator of σ of $\operatorname{Stab} z$ acts on each branch of Z in z, the orbit $S_3(z)$ consists of 2 nodes and its image $x = h(z)$ is a node of X.
(4) $|\operatorname{Stab} z| = 2$ and the generator of $\operatorname{Stab} z$ does exchanges the 2 branches of Z at z. The orbit $S_3(z)$ consists of 3 nodes and its image $x = h(z)$ is smooth in X.
(5) $\operatorname{Stab} z = S_3$. In this case τ exchanges the 2 branches of Z at z, the orbit $S_3(z)$ consists of z alone and $x = h(z)$ is smooth in X.

We call the nodes *of type* (1), ..., (5) respectively.

Note that if the S_3-covering $Z \to X$ is admissible, then every node of Z is of type (1), (2) or (3). If z is or type (1), then all maps in (2.1) are étale near z and its images. If z is a node of type (2), then q and f are étale near z and $p(z)$ and p and g are ramified at both branches near z and $q(z)$. If z is a node of type (3), then p and g are étale near z and $q(z)$ and q and f are ramified of index 3 at both branches near z and $p(z)$.

In order to describe the norm map of f we need the following description of the divisors. We have

$$\operatorname{Div}(Y) = \bigoplus_{x \in Y_{sm}} \mathbb{Z} \cdot x + \bigoplus_{s \in Y_{sing}} \mathcal{K}^*_{Y,s}/\mathcal{O}^*_{Y,s}$$

and similarly for $\operatorname{Div}(X)$, where $\mathcal{K}_Y$ (resp. $\mathcal{K}_X$) is the ring of rational function of Y (resp. X). Let $n_Y : \tilde{Y} \to Y$ and $n_X : \tilde{X} \to X$ be the normalizations. For a node y of Y denote $n_Y^{-1}(y) = \{y_1, y_2\}$ and ν_i the valuation of y_i for $i = 1, 2$. We obtain an isomorphism

$$\mathcal{K}^*_{Y,y}/\mathcal{O}^*_{Y,y} \simeq \mathbb{C}^* \times \mathbb{Z} \times \mathbb{Z} \quad \text{defined by} \quad \phi \mapsto \left(\frac{\phi(s_1)}{\phi(s_2)}, \nu_1(\phi), \nu_2(\phi) \right)$$

and similarly for $\mathcal{K}^*_{X,f(y)}/\mathcal{O}^*_{X,f(y)}$. With these identifications we have the following lemma.

Lemma 2.1. *Let y be a node of Y and $(\gamma, m, n) \in \mathcal{K}^*_{Y,y}/\mathcal{O}^*_{Y,y}$.*
(a) *If y is of type* (1) *or* (2)*, then $f_*(\gamma, m, n) = (\gamma, m, n)$;*
(b) *If y is of type* (3)*, then $f_*(\gamma, m, n) = (\gamma^3, m, n)$.*

Proof. (a) is a consequence of the fact that f is étale near y. (b) follows from diagram (2.1) using the facts that the maps p and g are étale near the corresponding nodes and the analogous statement for the cyclic map q which was shown in [F, p.64]. □

Let JX and JY denote the (generalized) Jacobians of X and Y, i.e. the components of $\operatorname{Pic}(X)$ and $\operatorname{Pic}(Y)$ of (multi-) degree 0 and let $\operatorname{Nm}_f : \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ be the norm map. Since the norm of $\mathcal{K}_Y/\mathcal{K}_X$ maps $\mathcal{O}_Y$ into $\mathcal{O}_X$ we get the diagram of exact sequences

(2.2)
$$\begin{tikzcd}
\mathcal{K}_Y^* \arrow[r] \arrow[d, "N_{\mathcal{K}_Y/\mathcal{K}_X}"] & \mathrm{Div}(Y) \arrow[r] \arrow[d, "f_*"] & \mathrm{Pic}(Y) \arrow[r] \arrow[d, "\mathrm{Nm}_f"] & 0 \\
\mathcal{K}_X^* \arrow[r] & \mathrm{Div}(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & 0
\end{tikzcd}$$

The norm map induces a morphism $\mathrm{Nm}_f : JX \to JY$. We define the Prym variety associated to f as

$$P(f) := (\mathrm{Ker}\,\mathrm{Nm}_f)^0.$$

In general the kernel of the norm can have several conected components. However, consider the following condition

$$(*)\begin{cases} \text{let } h : Z \to X \text{ be an admissible } S_3\text{-cover} \\ \text{such that all nodes of } Z \text{ are of type (3).} \end{cases}$$

Then we have,

Lemma 2.2. *Let $h : Z \to X$ be an S_3-cover satisfying $(*)$. Then* $\mathrm{Ker}\,\mathrm{Nm}_f$ *is an abelian subvariety P of JY.*

Proof. Let n_3 be the number of nodes and s be the number of irreducible components of X. Then we have the following exact diagram.

(2.3)
$$\begin{tikzcd}
0 \arrow[r] & T_3 \arrow[r] \arrow[d, hook] & K \arrow[r] \arrow[d, hook] & R \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & T_Y \arrow[r] \arrow[d, "Nm_f"] & JY \arrow[r, "n_Y^*"] \arrow[d, "\mathrm{Nm}_f"] & J\tilde{Y} \arrow[r] \arrow[d, "\mathrm{Nm}_{\tilde{f}}"] & 0 \\
0 \arrow[r] & T_X \arrow[r] & JX \arrow[r, "n_X^*"] & J\tilde{X} \arrow[r] & 0
\end{tikzcd}$$

with

$$T_Y \simeq T_X \simeq \mathbb{C}^{*n_3-s+1}.$$

The kernel R of $\mathrm{Nm}_{\tilde{f}}$ is an abelian subvariety of $J\tilde{Y}$, since $\tilde{Y}$ and $\tilde{X}$ are disjoint unions of smooth projective curves and on every component $Nm_{\tilde{f}}$ is non-cyclic (and ramified) of degree 3. Moreover, according to Lemma 2.1 the kernel T_3 of Nm_f is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{n_3-s+1}$. This implies that the kernel K of Nm_f is a compact subgroup of JY.

It remains to show that K is connected, i.e. $K = P$. This is certainly the case for any S_3-cover of smooth projective curves. The assertion follows from this by considering a connected family of S_3-covers the general member of which is smooth and which contains the given cover $f : Z \to X$ as a special fiber. We get a family of maps Nm_f in an obvious way. Since for the general member of this family the kernel of Nm_f is connected, it cannot be disconnected for a special member. □

Proposition 2.3. *Let $h : Z \to X$ be an S_3-cover satisfying $(*)$ and let $\rho : P \to \hat{P}$ be the polarization induced by the principal polarization of JY. Then*

$$|\,\mathrm{Ker}\,\rho\,| = 3^{2p_a(X)}.$$

Proof. Consider the isogeny

$$\beta : P \times J\tilde{X} \to J\tilde{Y},\ (L, M) \mapsto n_Y^* L \otimes \tilde{f}^* M.$$

Then $\mathrm{Ker}(\beta)$ is a maximal isotropic subgroup of the kernel of the polarization on $P \times J\tilde{X}$ given by the pullback of the principal polarization on $J\tilde{Y}$. Consider again the commutative diagram (2.3) with $K = P$ according to Lemma 2.2. An element $(L, M) \in P \times J\tilde{X}$ is in the kernel of β if and only if

$$n_Y^* L \otimes \tilde{f}^* M \simeq \mathcal{O}_{\tilde{Y}}.$$

Let $M = n_X^* M'$, for some $M' \in JX$, then $L \otimes f^* M' \in \mathrm{Ker}\, n_Y^* = T_Y$. Since T_3 is finite, we have $T_Y = f^*(T_X)$. This implies that $L = f^* M''$ for some $M'' \in JX$. Since $\mathrm{Nm}_f(L) \simeq \mathcal{O}_X$ we obtain $(M'')^3 \simeq \mathcal{O}_X$, which implies

$$L^3 \simeq \mathcal{O}_Y \quad \text{and} \quad M^3 \simeq \mathrm{Nm}_{\tilde{f}}\, \tilde{f}^* M \simeq \mathrm{Nm}_{\tilde{f}}\, n_Y^* L^{-1} = n_X^* \,\mathrm{Nm}_f\, L^{-1} \simeq \mathcal{O}_{\tilde{X}}.$$

So $\mathrm{Ker}\, \beta \subset P[3] \times J\tilde{X}[3]$ and therefore $\mathrm{Ker}\, \rho \subset P[3]$. Moreover,

$$\begin{aligned} \tilde{f}^* M &= n_Y^* L^{-1} \\ &= n_Y^* f^* (M'')^{-1} \\ &= \tilde{f}^* n_X^* (M'')^{-1}. \end{aligned}$$

Since $\tilde{f}$ is non-cyclic on each irreducible component of $\tilde{Y}$, $\tilde{f}^*$ is injective. Then $M = n_X^* (M'')^{-1}$. This shows that

$$\mathrm{Ker}\, \beta = \{(f^* a, n_X^* a^{-1}) \mid a \in JX[3],\ f^* a \in P\}.$$

Since f^* is injective when X is singular or n_X^* is injective when X is non-singular, we conclude that $\mathrm{Ker}\, \beta \simeq \{a \in JX[3] \mid f^* a \in P\}$. Since $\mathrm{Nm} \circ f^*$ is the multiplication by 3, $f^*(JX[3]) \subset \mathrm{Ker}\, \mathrm{Nm}_f = P$, by the previous lemma. Hence we obtain

$$\mathrm{Ker}\, \beta \simeq JX[3].$$

Let $t = \dim T_Y = \dim T_X = n_3 - s + 1$ and denote $\mu : J\tilde{Y} \to \widehat{J\tilde{Y}}$ the canonical principal polarization. Then the dimension over $\mathbb{F}_3$ of the kernel of the pullback of μ on $P \times J\tilde{X}$ is

$$\begin{aligned} \dim_{\mathbb{F}_3} \mathrm{Ker}(\hat{\beta} \circ \mu \circ \beta) &= 2 \dim_{\mathbb{F}_3} \mathrm{Ker}\, \beta \\ &= 2 \dim_{\mathbb{F}_3} JX[3] \\ &= 2(2p_a(X) - t) \end{aligned}$$

On the other hand we clearly have

$$\dim_{\mathbb{F}_3} \mathrm{Ker}(\hat{\beta} \circ \mu \circ \beta) = \dim_{\mathbb{F}_3} (\mathrm{Ker}\, \rho \times J\tilde{X}[3]).$$

Hence $\dim_{\mathbb{F}_3} \mathrm{Ker}\, \rho = 4p_a(X) - 2t - \dim_{\mathbb{F}_3} J\tilde{X}[3] = 2p_a(X)$. $\square$

Corollary 2.4. *Let $h : Z \to X$ be an S_3-cover satisfying $(*)$. Then the polarization $\rho : P \to \hat{P}$ is three times a principal polarization if and only* $\dim P = p_a(X)$.

Proof. We have $\mathrm{Ker}\, \rho$ is a subset of $P[3]$ of cardinality $3^{2p_a(X)}$ and this is equal to the cardinality of $P[3]$ if and only if $\dim P = p_a(X)$. $\square$

Corollary 2.5. *Let $f : Z \to X$ be an S_3-cover satisfying $(*)$ and let $X_1, \ldots, X_s$ be the irreducible components of X. Then the canonical polarization of P is three times a principal polarization if and only if*

$$\sum_{i=1}^{s} p_g(X_i) = s - n_3 + 1.$$

where as above n_3 denotes the number of nodes of X. In this case $\dim P = 2$.

Proof. Since s is also the number of components of Y and n_3 the number of nodes of Y, we have

$$p_a(Y) = \sum_{i=1}^{s} p_g(Y_i) + n_3 - s + 1$$

and

$$p_a(X) = \sum_{i=1}^{s} p_g(X_i) + n_3 - s + 1.$$

Hence $\dim P = p_a(Y) - p_a(X) = p_a(X)$ if and only if

$$\sum_{i=1}^{s} p_g(Y_i) = 2\sum_{i=1}^{s} p_g(X_i) + n_3 - s + 1. \tag{2.4}$$

The covering $Y \to X$ is doubly ramified exactly over each branch of the nodes of X. So if $2r_i$ denotes the order of the ramification divisor of the normalization $\tilde{Y}_i \to \tilde{X}_i$ of $Y_i \to X_i$ we have

$$\sum_{i=1}^{s} r_i = 2n_3.$$

On the other hand, by the Hurwitz formula, $p_g(Y_i) = 3p_g(X_i) - 2 + r_i$. So (2.4) gives

$$3\sum_{i=1}^{s} p_g(X_i) - 2s + 2n_3 = 2\sum_{i=1}^{s} p_g(X_i) + n_3 - s + 1.$$

This implies the first assertion. Finally we have

$$\dim P = p_a(X) = \sum_{i=1}^{s} p_g(X_i) + n_3 - s + 1 = s - n_3 + 1 + n_3 - s + 1 = 2.$$

□

3. The condition $(**)$

As in [B, Section 5] we shall need to study the Prym variety P of π under a more general assumption than the hypothesis $(*)$ of the previous section. Let Z be a connected curve with only ordinary nodes and an S_3-action and consider the diagram (2.1). Throughout we assume condition $(4')$ of the definition of an S_3-cover, i.e. that at each node of type (3), if one local parameter is multiplied under σ by ζ, then the other is multiplied by ζ^2

for some third root of unity ζ (see [F, Remark 2.1]). Moreover, we assume that the action satisfies the following condition

$$(**) \begin{cases} P \text{ is an abelian variety;} \\ \sigma \text{ and } \tau \text{ are not the identity on any component of } Z; \\ p_a(Z) = 6p_a(X) - 5. \end{cases}$$

The number

$$n_i := |\{\text{nodes } z \text{ of } Z \text{ of type (i)}\}| \cdot \frac{|\operatorname{Stab} z|}{6}$$

obviously is a non-negative integer. For $i = 1, 2, 3$ it coincides with the number of nodes of X of type (i). The number

$$c_i := |\{\text{components } Z_i \text{ of } Z \text{ with } |\operatorname{Stab} Z_i| = i\}| \cdot \frac{i}{6} \quad \text{for } i = 1, 2, 3, 6$$

is the number of components of X. Finally define

$$r_i := |\{\text{smooth points } z \text{ of } Z \text{ with } |\operatorname{Stab} z| = i\}| \quad \text{for } i = 2, 3.$$

Proposition 3.1. *The assumptions* $(**)$ *are equivalent to*

$$\begin{cases} r_2 = r_3 = n_4 = n_5 = 0, \\ 2n_1 + n_2 = 2c_1 + c_2. \end{cases}$$

In particular $h : Z \to X$ *is an admissible* S_3*-cover.*

Proof. Let $\tilde{Z}$ and $\tilde{X}$ denote the normalizations of Z and X. The induced covering $\tilde{h} : \tilde{Z} \to \tilde{X}$ is ramified exactly at points of $\tilde{Z}$ lying over non-singular points fixed by σ or an element of order 2 and at nodes of types (2), (3) and (5). Hence, by the Hurwitz formula:

$$p_a(\tilde{Z}) = 6p_a(\tilde{X}) - 5 + \frac{r_2}{2} + r_3 + 3n_2 + 4n_3 + 2n_5.$$

So

$$\begin{aligned} p_a(Z) &= p_a(\tilde{Z}) + 6n_1 + 3n_2 + 2n_3 + 3n_4 + n_5 \\ &= 6p_a(\tilde{X}) - 5 + \frac{r_2}{2} + r_3 + 6n_1 + 6n_2 + 6n_3 + 3n_4 + 3n_5. \end{aligned}$$

The nodes of X come from nodes of Z of types (1), (2) and (3), hence:

$$p_a(X) = p_a(\tilde{X}) + n_1 + n_2 + n_3.$$

Therefore

$$p_a(Z) = 6p_a(X) - 5 + \frac{r_2}{2} + r_3 + 3n_4 + 3n_5.$$

Hence the condition $p_a(Z) = 6p_a(X) - 5$ is equivalent to $r_2 = r_3 = n_4 = n_5 = 0$.

In order to express the condition that P is an abelian variety, let $\tilde{Y}$ denote the normalization of Y. From the diagram

$$\begin{tikzcd} 0 \arrow[r] & \tilde{T} \arrow[r] \arrow[d] & JY \arrow[r] \arrow[d, "N_f"] & J\tilde{Y} \arrow[r] \arrow[d, "N_{\tilde{f}}"] & 0 \\ 0 \arrow[r] & T \arrow[r] & JX \arrow[r] & J\tilde{X} \arrow[r] & 0 \end{tikzcd}$$

and the surjectivity of the norm map it follows that P is an abelian variety if and only if $\dim \tilde{T} = \dim T$. Now we have

$$\dim \tilde{T} = 3n_1 + 2n_2 + n_3 - 3c_1 - 2c_2 - c_3 - c_6 + 1.$$

For the summand $2n_1$ (and similarly $2c_1$) note that for a transitive action of S_3 on a set of 3 elements, τ fixes one element and exchanges the 2 other elements (see Lemma 4.5). Similarly we have

$$\dim T = n_1 + n_2 + n_3 - c_1 - c_2 - c_3 - c_6 + 1.$$

So $\dim \tilde{T} = \dim T$ if and only if $2n_1 + n_2 = 2c_1 + c_2$. □

Apart from the curves of Theorem 2.3 and its corollaries there are some other curves with S_3-action which lead to principally polarized Prym varieties. These are the analogues of those occurring in [B, Section 5] and [F, Section 2]. However, in the case of interest for us, i.e. X of genus 2, they do not occur (see Corollary 5.7 below).

4. Some auxiliary results

Let X be a stable curve of arithmetic genus 2. In the next section we determine the S_3-covers $h : Y \to X$ satisfying condition $(**)$. For this we need some lemmata which we collect in this section.

Proposition 4.1. *Every smooth non-hyperelliptic curve Z of genus 3 with S_3-action has quotient $Z/S_3 \simeq \mathbb{P}^1$. The automorphism σ has 2 and τ has 8 fixed points on Z.*

Proof. Every non-hyperelliptic curve Z of genus 3 with S_3-action has an equation (see [V])

$$z_0^3 z_2 + z_1^3 z_2 + z_0^2 z_1^2 + a z_0 z_1 z_2^2 + b z_2^4 = 0.$$

The group S_3 is generated by

$$\sigma : \begin{cases} z_0 & \mapsto & \zeta z_0 \\ z_1 & \mapsto & \zeta^2 z_1 \\ z_2 & \mapsto & z_2 \end{cases} \qquad \text{and} \qquad \tau : \begin{cases} z_0 & \mapsto & z_1 \\ z_2 & \mapsto & z_2 \end{cases}$$

where ζ denotes a primitive third root of unity. The quotient $D = Z/\langle \sigma \rangle$ is an elliptic curve, since σ has exactly 2 fixed points, namely $(1:0:0)$ and $(0:1:0)$. An equation of D is

$$x_0^2 x_1 x_2 + x_0 x_1^2 x_2 + x_0^2 x_1^2 + a x_0 x_1 x_2^2 + b x_2^4 = 0$$

and the map $Z \to D$ is given by $x_0 = z_0^3,\ x_1 = z_1^3,\ x_2 = z_1 z_2 z_3$. The involution τ induces an involution $\overline{\tau}$ on D, which is given by $x_0 \mapsto x_1,\ x_2 \mapsto x_2$. Since $\overline{\tau}$ admits exactly 4 fixed points with multiplicities, the quotient $Z/S_3 = D/\langle \overline{\tau} \rangle$ is of genus 0. □

Proposition 4.2. *Every smooth hyperelliptic curve Z of genus 3 with S_3-action has an elliptic curve as quotient Z/S_3. There is a one-dimensional family of such curves Z.*

Proof. Every hyperelliptic curve Z of genus 3 with S_3-action has an affine equation (see [I])

$$y^2 = x(x^3 - 1)(x^3 - a^3)$$

with $a^3 \neq 0, 1$. The group S_3 is generated by

$$\sigma : \begin{cases} x & \mapsto & \zeta x \\ y & \mapsto & \zeta^2 y \end{cases} \quad \text{and} \quad \tau : \begin{cases} x & \mapsto & a x^{-1} \\ y & \mapsto & -a^2 x^4 y \end{cases}$$

The fixed points of σ are a subset of the set of branch points $B = \{0, \infty, 1, \zeta, \zeta^2, a, a\zeta, a\zeta^2\}$. Hence the quotient map $Z \to E = Z/\langle\sigma\rangle$ is ramified exactly at the 2 points over 0 and ∞, which implies that E is an elliptic curve. Since τ permutes 0 and ∞ as well as the 2 orbits $\{1, \zeta, \zeta^2\}$ and $\{a, a\zeta, a\zeta^2\}$ and $\langle\sigma\rangle$ is a normal subgroup of S_3, τ induces a fixed point free automorphism $\overline{\tau}$ of E. Hence $Z/S_3 = E/\langle\tau\rangle$ is an elliptic curve. $\square$

Proposition 4.3. *There is no smooth genus* 2 *curve with automorphism σ of order* 3 *such that the quotient $Z/\langle\sigma\rangle$ is an elliptic curve.*

Proof. According to [Bo] every smooth curve of genus 2 admitting an automorphism of order 3 has an affine equation

$$y^2 = (x^3 - a^3)(x^3 - a^{-3})$$

with $a \neq 0$ and not a 6th root of unity and σ is given by

$$x \mapsto \zeta x \quad \text{and} \quad y \mapsto y.$$

with a primitive third root of unity ζ. Hence the quotient $Z/\langle\sigma\rangle$ satisfies the equation $y^2 = (v - a^2)(v - a^{-1})$ which implies that it is rational. $\square$

Lemma 4.4. *Let $h : Z \to X$ be an S_3-cover of connected nodal curves, with*

$$X = X^1 \cup X^2.$$

If $x \in X^1 \cap X^2$, then $Z^i := h^{-1}(X^i)$ is smooth at each point of $h^{-1}(x)$, for $i = 1, 2$.

Proof. Suppose z is a singular point of Z^i, with $h(z) = x$. Since Z is connected, there exists $z' \in Z^1 \cap Z^2$ with $h(z') = x$, which is necessarily a smooth point of Z^i. This contradicts the fact that z and z' are in the same S_3-orbit. $\square$

Lemma 4.5. *Let S denote a set consisting of* 3 *points Z_1, Z_2, Z_3, with a nontrivial S_3-action. Then the points may be labelled in such a way that τ fixes Z_1 and exchanges Z_2 and Z_3 and either*

(1) $\sigma(Z_i) = Z_{i+1}$ *for* $i = 1, 2, 3$ *(where $Z_4 = Z_1$) or*
(2) $\sigma(Z_i) = Z_i$ *for* $i = 1, 2, 3$.

The proof is straightforward and will be omitted.

Corollary 4.6. *Let $Z = Z_1 \cup Z_2 \cup Z_3$ be a nodal curve such that $Z_i \cap Z_{i+1}$ $(Z_4 = Y_1)$ consists of one point n_i for $i = 1, 2, 3$. Then any S_3-cover $h : Z \to X$ does not satisfy condition* $(**)$.

Proof. Consider the induced S_3-action on the dual graph of Z and let the notation be as in the previous lemma.

First assume case (1), i.e. σ permutes the Z_i and $\tau(Z_1) = Z_1$. Then

$$\tau(n_2) = \tau(Z_2 \cap Z_3) = Z_3 \cap Z_2 = n_2.$$

Since $\tau(Z_2) = Z_3$, the action on n_2 is of type (4), that is, $n_4 \neq 0$, contradicting Proposition 3.1.

Assume case (2), so $\sigma(Z_i) = Z_i$ for $i = 1, 2, 3$. In this case the node $n_2 = \tau(n_2)$ is of type (5), contradicting Proposition 3.1. $\square$

Lemma 4.7. *Let Γ be a connected graph with a transitive S_3-action consisting of 6 vertices and 6 nodes. Then the vertices Z_i and the nodes n_i can be labeled in such a way that*

$$\sigma = (Z_1, Z_3, Z_5)(Z_2, Z_4, Z_6) \quad \textit{and} \quad \tau = (Z_1, Z_2)(Z_3, Z_6)(Z_4, Z_5)$$

on the vertices, and $n_i = Z_i \cap Z_{i+1}$ where $Z_7 = Z_1$, with

$$\sigma = (n_1, n_3, n_5)(n_2, n_4, n_6) \quad \textit{and} \quad \tau = (n_1)(n_4)(n_2, n_6)(n_3, n_5).$$

Here the notation is the cycle-notation of permutations.

In particular, up to isomorphisms, there is only one transitive S_3-action on Γ.

The proof is straightforward and will be omitted.

Corollary 4.8. *Let Z be a connected nodal curve with an S_3-action consisting of $s = 6$ irreducible components and $\delta \leq 6$ nodes such that the quotient $X = Z/S_3$ is irreducible. Then the image of any node of Z is a smooth point of X. In particular, $h : Z \to X$ is not an admissible S_3-covering.*

Proof. The S_3-action induces a transitive action on the dual graph Γ of Z. Since there is no connected graph with 6 vertices and at most 5 edges, we necessarily have $\delta = 6$. Let the notation be as in the previous lemma with components Z_i and nodes n_i. The quotient $D := Z/\langle\sigma\rangle$ consists of 2 components $D_1 = q(Z_1) = q(Z_3) = q(Z_5)$ and $D_2 = q(Z_2) = q(Z_4) = q(Z_6)$ intersecting transversally in 2 points $q(n_1)$ and $q(n_2)$. The involution $\bar{\tau}$, induced by τ in D, interchanges D_1 and D_2 so the quotient $X = D/\bar{\tau}$ is smooth. $\square$

Proposition 4.9. *Let $h : Z \to X$ be an S_3-cover of connected nodal curves, with X stable of arithmetic genus 2 consisting of 2 components X^1 and X^2 intersecting in one point.*

If the cover satifies condition $(**)$, then $Z^j = h^{-1}(X^j)$ consists of at most 3 irreducible components for $j = 1$ and 2.

Proof. Suppose Z^1 has 6 irreducible components. If the curves X^1 and X^2 are smooth, then the 6 components are disjoint according to Lemma 4.4. In order to have Z connected, Z^2 has to be irreducible and smooth. Hence $Z^2 \to X^2$ is an étale map, so $g(Z^2) = 1$. This is a contradiction, since there is no elliptic curve with a non-trivial S_3-action.

The components X^1 and X^2 cannot have more than 1 node, since X is stable of arithmetic genus 2. If one of the components of X, say X^1, has an ordinary double point x_1, then Z^1 has 6 ordinary double points, because the maps $Z^1_i \to X^1$ are birational. But then the cover $Z^1 \to X^1$ satisfies the conditions of Corollary 4.8. So the image of the node is a smooth point of X contradicting the admissiblity of the covering h. $\square$

5. S_3-covers satisfying $(**)$

In this section we determine the S_3-covers $Z \to X$ satisfying condition $(**)$ with $p_a(X) = 2$. There are 6 types of non-smooth stable curves of arithmetic genus 2 which will be considered separately. In the first 2 propositions we assume that X is irreducible. So let $h : Z \to X$ be an S_3-cover satisfying $(**)$ and denote

$$Z = \cup_{i=1}^s Z_i$$

with irreducible components Z_i. Since S_3 acts transitively on the set of components, the number s can take the values $1, 2, 3$ or 6. Moreover, we have the following formula

$$7 = p_a(Z) = \sum_{i=1}^{s} g_i - s + \delta + 1 = s(g_1 - 1) + \delta + 1, \tag{5.1}$$

where g_i denotes the geometric genus of Z_i and δ the number of nodes of Z. Note that $g_i = g_1 \geq 1$ for all i.

Suppose first that X is of geometric genus 1 with one node x. If r denotes its ramification index, we have as usual

$$r\delta = 6.$$

Proposition 5.1. *Let $h : Z = \bigcup_{i=1}^{s} Z_i \to X$ be an S_3-cover of an irreducible curve X of geometric genus 1 with one ordinary double point. Then only in the following cases the cover satisfies condition $(**)$:*

(a) $s = 1,\ r = 3,\ \delta = 2$. *$Z$ is an irreducible curve of geometric genus 5 with 2 nodes.*

(b) $s = 2,\ r = 3,\ \delta = 2$. *The normalization of Z consists of 2 copies of a hyperelliptic curve of genus 3, admitting an automorphism σ of order 3 with 2 fixed points which are glued together transversally at opposite fixed points of σ.*

In both cases there is a 2-dimensional family of such coverings. If $\tilde{f} : \tilde{Y} \to \tilde{X}$ denotes the normalization of f, in both cases the Prym variety P is an extension of the Prym variety of $\tilde{f}$ by the group $\mathbb{Z}/3\mathbb{Z}$.

Proof. Let f be an S_3-covering satisfying $(**)$. Then $s \neq 3$ according to Corollary 4.6 and $s \neq 6$ by Corollary 4.8. So suppose $s \leq 2$. Then $c_1 = c_2 = 0$ and hence $n_1 = n_2 = 0$. This implies $r = 3$ and $\delta = 2$. If $s = 1$, we are in case (1) by the Hurwitz formula and if $s = 2$ we are in case (b) again using the Hurwitz formula.

Concerning the existence statement, there is a 2-dimensional family of curves X of genus 1 with one node. Hence it suffices to show that, for a fixed such curve X, there exist only finitely many admissible S_3-covers in the cases (a) and (b).

As for case (a), according to [KK, 3,1,(1)], every elliptic curve E admits finitely many S_3-covers of genus 5. The ramification divisor of the normalization $\tilde{h} : \tilde{Z} \to \tilde{X}$ of h is of degree 8. Since 8 is not divisible by 3, $\tilde{h}$ is doubly ramified at 2 pairs of points. Gluing them pairwise together we get case (a).

In case (b), there are finitely many cyclic coverings Z^1 of the normalization of X of degree and genus 3, ramified exactly over the 2 preimages of the node. For each such cover Z^1 take a copy Z^2 of Z^1 and glue both copies transversally at opposite ramification points. This gives the desired cover. The last assertion is obvious. □

Now assume that X be an irreducible rational curve with 2 nodes x_1 and x_2. If δ_i denotes the number of nodes of Z over x_i and r_i their ramification index for $i = 1$ and 2, we clearly have $\delta_1 + \delta_2 = \delta$ and $r_i\delta_i = 6$. We assume $\delta_1 \geq \delta_2$.

Proposition 5.2. *Let $h : Z = \cup_{i=1}^{s} Z_i \to X$ be an S_3-cover of a rational curve X with 2 ordinary double points x_1 and x_2. Then only in the following case the cover satisfies condition $(**)$:*

$s = 2,\ r_1 = r_2 = 3,\ \delta_1 = \delta_2 = 2$ and the normalization of Z consists of 2 curves of genus 2, admitting an automorphism of order 3 with 4 fixed points, which are glued together pairwise transversally at fixed points of σ.

There is a 1-dimensional family of such curves. If $\tilde{Y}$ denotes the normalization of Y, the Prym variety P is an extension of the Prym variety of $J\tilde{Y}$ by the group $(\mathbb{Z}/3\mathbb{Z})^2$.

Proof. Let h be an S_3-covering satisfying $(**)$. Suppose that $s \leq 2$. Then $c_1 = c_2 = 0$ implying $n_1 = n_2 = 0$. Hence $r_1 = r_2 = 3$ and $\delta_1 = \delta_2 = 2$. If $s = 1$, then (5.1) gives $g_1 = 3$. So the normalization of Z is a curve of genus 3 with an S_3-action such that σ admits 8 fixed points. This contradicts Proposition 4.1. If $s = 2$, then (5.1) gives $g_1 = 2$ and we are in the case of the proposition.

If $s = 3$, then $c_1 = 0, c_2 = 1$ and hence $n_1 = 0, n_2 = 1$. So, up to labeling, we have $r_1 = 2, \delta_1 = 3$ and $r_2 = 3, \delta_2 = 2$. On the other hand, (5.1) says $9 = 3g_1 + \delta_1 + \delta_2$ which gives a contradiction.

Finally, if $s = 6$, then each of the components of the normalization of Z is isomorphic to $\mathbb{P}^1$ and the ramification indices of the nodes are $r_1 = r_2 = 1$. So $c_1 = 1,\ n_1 = 2$ and $c_2 = n_2 = 0$, which contradicts $(**)$.

Concerning the existence statement, there is a one-dimensional family of smooth curves of genus 2 admitting an automorphism of order 3 (see the proof of proposition 4.3). Each of the curves is a degree-3 cover of $\mathbb{P}^1$ doubly ramified in exactly 4 points. Take 2 copies of these curves and glue them transversally pairwise at 2 of their ramification points. The last assertion is obvious. □

The remaining 4 types of non-smooth stable curves X of genus 2 have 2 irreducible components which we denote by X^1 and X^2. Let $h : Z \to X$ be an S_3-cover satisfying $(**)$. For $j = 1, 2$ we denote

$$Z^j := h^{-1}(X^j) = \cup_{i=1}^{s_j} Z_i^j$$

with irreducible components Z_i^j. The analogue of (5.1) in this case is

$$7 = p_a(Z) = \sum_{j=1}^{2} s_j(g_1^j - 1) + \delta + 1. \tag{5.2}$$

Proposition 5.3. *Let $h : Z = \cup_{j=1}^2 \cup_{i=1}^{s_j} Z_i^j \to X = X^1 \cup X^2$ be an S_3-cover with elliptic curves X^i and $X^1 \cap X^2 = \{x\}$. Then only in the following case the cover satisfies condition $(**)$:*

$s_1 = s_2 = 1,\ r = 3,\ \delta = 2$. Z^1 and Z^2 are smooth hyperelliptic curves of genus 3 intersecting transversally in 2 points z_1 and z_2. For $j = 1, 2$ the map $Z^j \to X^j$ is an S_3-covering ramified exactly at z_1 and z_2.

There is a 2-dimensional family of such coverings. If P^i denotes the Prym variety of the covering $Y^j \to X^j$ for $j = 1$ and 2, the Prym variety P is isomorphic to $P^1 \times P^2$ as principally polarized abelian varieties.

Proof. Let g_j denote the geometric genus of Z_i^j for $j = 1$ and 2. By Proposition 4.9 we may assume

$$3 \geq s_1 \geq s_2 \geq 1.$$

Suppose $s_1 \leq 2$. Then $c_1 = c_2 = 0$ and hence $n_1 = n_2 = 0$. This implies $r = 3$ and thus $\delta = 2$. If $s_1 = s_2 = 1$, Hurwitz formula implies $g_j = 3$. By Propositions 4.1 and 4.2, Z_j is hyperelliptic for $j = 1, 2$. This gives the case of the proposition.

Suppose $s_1 = 2$. The case $s_2 = 2$ cannot exist, since there is no connected graph with 4 vertices and 2 edges. If $s_2 = 1$, then each component Z_i^1 is smooth and maps 3:1 to X^1 with exactly one doubly ramified point. Hence it is of genus 2, which is a contradiction, since by Proposition 4.3 there is no curve of genus 2 with an automorphism of order 3 with quotient an elliptic curve.

If $s_1 = 3, s_2 = 1$ or 2, then $c_1 = 0, c_2 = 1$ implying $n_1 = 0, n_2 = 1$. This gives $r = 2$ and $\delta = 3$. But then $Z_j^1 \to X^1$ would be a 2:1 cover ramified exactly in one point, a contradiction. Finally suppose $s_1 = s_2 = 3$. So $c_1 = 0, c_2 = 2$ and hence either $n_1 = 0, n_2 = 2$ or $n_1 = 1, n_2 = 0$. Both cases cannot occur, since X has only one node.

As for the existence statement, according to Proposition 4.2, there is a one-dimensional family of hyperelliptic curves Z of genus 3 with S_3-action and quotient Z/S_3 an elliptic curve. Take 2 of them and intersect them transversally at the two ramification points of σ. The last assertion is obvious. □

Proposition 5.4. *Let $h : Z = \cup_{j=1}^2 \cup_{i=1}^{s_j} Z_i^j \to X = X^1 \cup X^2$ be an S_3-cover with a nodal rational curve X^1 and an elliptic curve X^2 such that $X^1 \cap X^2 = \{x\}$. Then only in the following case the cover satisfies condition $(**)$:*

$s_1 = 2, s_2 = 1, r = r_1 = 3, \delta = \delta_1 = 2$. The components Z_1^1 and Z_2^1 are copies of the elliptic curve admitting an automorphism σ of order 3 with 3 fixed points, 2 of which are glued together transversally at opposite fixed points of σ. The third fixed point in Z_i^1 for $i = 1$ and 2 is glued transversally to Z^2. The curve Z^2 is hyperelliptic of genus 3 with an S_3-action, doubly ramified in the 2 intersection points with Z^1.

There is a 1-dimensional family of such coverings. If $\tilde{Y}^1$ denotes the normalization of Y^1 and P^2 the Prym variety of the covering $Y^2 \to Z^2$, the Prym variety P is an extension of $J\tilde{Y}^1 \times P^2$ by the group $\mathbb{Z}/3\mathbb{Z}$.

Proof. Let x_1 denote the node of X^1, δ (respectively δ_1) the number of nodes over x (respectively x_1) and r (respectively r_1) their ramification index. As above, let g_j again denote the geometric genus of Z_i^j for $j = 1, 2$.

According to Proposition 4.9, we may assume that $s_j = 1, 2$ or 3 for $j = 1, 2$. In any case, the components have non-trivial stabilizer, i.e. $c_1 = 0$.

If $s_j \leq 2$ for $j = 1, 2$, then $c_2 = 0$ implying $n_1 = n_2 = 0$. This gives $r = r_1 = 3$ and $\delta = \delta_1 = 2$. If $s_1 = s_2 = 1$, then applying the Hurwitz formula gives $g_1 = 1$ and $g_2 = 3$. But there is no elliptic curve with a non-trivial S_3-action.

If $s_1 = 1, s_2 = 2$, then (5.2) implies $g_1 + 2g_2 = 5$. By the Hurwitz formula we get $g_1 = 1$, a contradiction as in the previous case.

If $s_1 = 2, s_2 = 1$, then (5.2) gives $2g_1 + g_2 = 5$. Since the map $Z^2 \to X^2$ is ramified, we have $g_2 \geq 2$. Moreover, by Proposition 4.3 $g_2 > 2$ and hence $g_2 = 3$. On the other hand, by the Hurwitz formula both components of Z^1 are of genus 1 and we are in the case of the proposition.

If $s_1 = 2,\ s_2 = 3$, or $s_2 = 3,\ s_1 = 2$, the 2 components must intersect the 3 components equally often which is impossible.

In the case $s_1 = s_2 = 2$ we have $c_2 = 0$ and $n_1 = n_2 = 0$, which implies $r_1 = r = 3$. Hence the 3:1 map $Z_i^2 \to X_2$ is doubly ramified in 1 point and the Hurwitz formula yields $g_2 = 2$, but by Proposition 4.3 there is no such covering.

Finally, suppose $s_1 = s_2 = 3$ then $c_2 = 2$, which implies that either $n_1 = 1, n_2 = 0$ or $n_1 = 0, n_2 = 2$. The first case contradicts the fact that Z^1 and Z^2 consist of 3 components. If $n_1 = 0, n_2 = 1$, then $\delta = \delta_1 = 3$. So for example $Z_1^2 \to X^2$ would be a double cover ramified at 1 point only, a contradiction.

As for the existence statement, we only note that, according to Proposition 4.2, there is a one-dimensional family hyperelliptic curves of genus 3 with S_3-action and only finitely many possibilities for the curve Z^1. So, gluing Z_1 and Z_2 as in the proposition gives a 1-dimensional family of S_3-covers satisfying $(**)$. The last assertion is obvious. □

Proposition 5.5. *Let $h : Z = \cup_{j=1}^2 \cup_{i=1}^{s_j} Z_i^j \to X = X^1 \cup X^2$ be an S_3-cover with X^j rational with 1 node for $j = 1, 2$ and $X^1 \cap X^2 = \{x\}$. Then only in the following case the cover satisfies condition $(**)$:*

$s_1 = s_2 = 2$ and all ramification indices are 3. For $i, j = 1, 2$, the normalization of Z_i^j is the unique elliptic curve with an automorphism of order 3. The normalization of $Z_i^j \to X^j$ has 3 ramification points. Z_i^j intersects Z_{3-i}^j in 2 of them and Z_i^{3-j} in the last one.

There are only finitely many coverings of this type. If $\tilde{Y}^j$ denotes the normalization of Y^j for $j = 1$ and 2, the Prym variety P is an extension of the abelian surface $\tilde{Y}^1 \times \tilde{Y}^2$ by the group $(\mathbb{Z}/3\mathbb{Z})^2$.

Proof. For $i = 1, 2$, let x_i denote the node of X^i, δ_i (respectively δ) the number of nodes over x_i (respectively x) and r_i (respectively r) their ramification index. Let g_j again denote the geometric genus of Z_i^j for $j = 1, 2$.

According to Proposition 4.9 we may assume $3 \geq s_1 \geq s_2 \geq 1$. In any case, the components have non-trivial stabilizer, i.e. $c_1 = 0$.

If $s_1 \leq 2$ and $s_2 = 1$, then $c_2 = 0$ and hence $n_1 = n_2 = 0$. So all ramification indices are 3. In particular $\delta = \delta_2 = 2$. Hence $Z^2 \to \mathbb{P}^1$ is a 6-fold cover doubly ramified at 6 points which gives $g_2 = 1$. But there is no elliptic curve with non-trivial S_3-action.

If $s_1 = 3 \geq s_2 \geq 1$, then in any case $\delta = 2$ or $\delta_1 = 2$. This is absurde, since Z_1 has 3 components and over both nodes x and x_1 there are at least 3 nodes.

We are left with the case $s_1 = s_2 = 2$. In this situation $c_2 = n_1 = n_2 = 0$ and $r_1 = r_2 = r = 3$. Hence $Z_i^j \to X^j$ is a 3:1 map doubly ramified at 3 points, which gives $g_j = 1$ for $j = 1, 2$. So the normalization of Z_i^j is the unique elliptic curve with an automorphism of order 3. The curves Z^1 and Z^2 must be connected, since otherwise Z would not be connected. This implies that all Z_i^j are smooth and the components Z_1^j and Z_2^j intersect transversally in 2 points (so $p_a(Z^j) = 3$) and Z^1, Z^2 intersect transversally in 2 points as well.

The existence as well as the last statement are obvious. □

Proposition 5.6. *Suppose $X = X^1 \cup X^2$ with $X^j \simeq \mathbb{P}^1$ for $j = 1$ and 2 such that $X^1 \cap X^2 = \{x_1, x_2, x_3\}$. There is no S_3-cover $h : Y \to X$ satisfying condition $(**)$:*

Proof. Suppose that $h : Z = \cup_{j=1}^2 \cup_{i=1}^{s_j} Z_i^j \to X$ is an S_3-cover satisfying $(**)$ with X as in the proposition. For $i = 1,2,3$, let δ_i denote the number of nodes over x_i and r_i their ramification index. Let g_j again denote the geometric genus of Z_i^j for $j = 1,2$.

According to Proposition 4.9 we may assume $3 \geq s_1 \geq s_2$. By Lemma 4.4, Z^j is a disjoint union of its components Z_i^j for $j = 1,2$, which are moreover smooth, since otherwise X^j would have a node.

Suppose first $2 \geq s_1 \geq s_2 = 1$. Then $c_1 = c_2 = 0$ and thus $n_1 = n_2 = 0$. So $\delta_i = 2$ for all i and the Hurwitz formula gives $g_2 = 1$, contradicting the fact that an elliptic curve does not admit a non-trivial S_3-action.

If $s_1 = 3 \geq s_2 \geq 1$, then every component Z_i^1 intersects Z^2 in exactly 3 points z_j^i, where $f(z_j^i) = x_j$ and the double cover $Z_j^1 \to X^1 \simeq \mathbb{P}^1$ is ramified exactly in these 3 points. However, there is no connected double cover of $\mathbb{P}^1$ ramified in exactly 3 points, a contradiction.

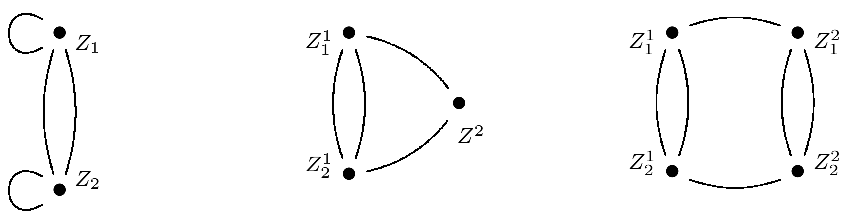
We are left with the case $s_1 = s_2 = 2$. Since σ has to map any of the components Z_i^j into itself, the corresponding quotient map $Z_j^i \to X^j \simeq \mathbb{P}^1$ is doubly ramified in 3 points. So the Hurwitz formula implies that Z_i^j is an elliptic curve for $i, j = 1,2$. So for $j = 1$ and 2 we obtain an S_3-cover $Z_1^j \sqcup Z_2^j \to X^j$ with disjoint elliptic curves Z_1^j and Z_2^j. Then τ is an isomorphism of Z_1^j with Z_2^j. For a general point $z \in Z$ we clearly have $\sigma\tau(z) \neq \tau\sigma^2(z)$. So Z is not an S_3-cover. This completes the proof of the proposition. $\square$

Note that all coverings of Propositions 5.1, $\cdots$, 5.6 satisfy condition $(*)$. So we get as an immediate consequence the first assertion of the following Corollary.

Corollary 5.7. *Any covering $h : Z \to X$ satisfying condition $(**)$ satisfies condition $(*)$. In particular, the Prym variety P of $f : Y \to X$ is a principally polarized abelian variety of dimension 2. For any stable curve X there are only finitely many covers h satisfying condition $(**)$.*

Proof. The last assertion follows from the proof of the propositions. $\square$

The following pictures are the dual graphs of the curves Z in Propositions 5.2, 5.4 and 5.5 from left to right.



6. Properness of the Prym map

Let ${}_{S_3}\mathcal{M}_2$ denote the moduli space of étale Galois covers of smooth curves of genus 2 with Galois group S_3 [ACG, Theorem 17.2.11]. According to [LO], ${}_{S_3}\mathcal{M}_2$ is an irreducible algebraic variety of dimension 3. A (closed) point in it corresponds to a smooth curve Z of genus 7 with an S_3-action and quotient $X = Z/S_3$ of genus 2. The quotient $Y = Z/\langle\tau\rangle$ is of genus 4, non-cyclic of degree 3 over X and the Prym variety $P = P(Y/X)$ is a principally polarized abelian surface. This gives a map

$$Pr : {}_{S_3}\mathcal{M}_2 \to \mathcal{A}_2$$

into the moduli space of principally polarized abelian surfaces, which we call the *Prym map.* In this section we show that one can extend the map to a proper map onto $\mathcal{A}_2$.

Let ${}_{S_3}\overline{\mathcal{M}}_2$ denote the compactification of ${}_{S_3}\mathcal{M}_2$ by admissible S_3-covers of stable curves of genus 2 according to [ACG, Chapter 17]. To be more precise, we denote by ${}_{S_3}\overline{\mathcal{M}}_2$ only the irreducible component containing ${}_{S_3}\mathcal{M}_2$ of the moduli space as defined in [ACG]. Finally, let ${}_{S_3}\widetilde{\mathcal{M}}_2 \subset {}_{S_3}\overline{\mathcal{M}}_2$ be the subset of points corresponding to covers satisfying condition $(**)$. The main result of this section is the following theorem.

Theorem 6.1. *The set* ${}_{S_3}\widetilde{\mathcal{M}}_2$ *is open in* ${}_{S_3}\overline{\mathcal{M}}_2$ *and the Prym map* Pr *extends to a proper morphism*

$$Pr : {}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2,$$

which is modular and which we also call the Prym map.

We mimic the proofs of [B, 6.1], [DS, I,1] and [F, §1] using the results of [ACG, Chapter 17]. Instead of level-n structures one could also use the theory of algebraic stacks to prove the theorem.

Proof. Fix an integer $n \geq 3$ and let ${}_{S_3}\mathcal{M}_2^{(n)}$ denote the moduli space of étale Galois covers of smooth curves of genus 2 with level-n structure and Galois group S_3. As ${}_{S_3}\mathcal{M}_2$, also ${}_{S_3}\mathcal{M}_2^{(n)}$ is an irreducible 3-dimensional variety. With the notation of [ACG, Chapter 17] we have ${}_{S_3}\mathcal{M}_2^{(n)} = M_2[\psi]$, where $\psi : \pi_1(\Sigma) \to G$ is a suitable level structure with Σ a fixed curve of genus 2 and G a subgroup of $H^1(\Sigma, \mathbb{Z}/n\mathbb{Z})$ with quotient S_3. Let ${}_{S_3}\overline{\mathcal{M}}_2^{(n)}$ denote the (irreducible) compactification by stable curves which is constructed in [ACG, Ch. 17, Theorem 4.8]. According to this theorem, there exists a universal family $\mathcal{X} \to {}_{S_3}\overline{\mathcal{M}}_2^{(n)}$ of genus-2 curves with the corresponding structure and we have ${}_{S_3}\overline{\mathcal{M}}_2^{(n)}/Sp(4,\mathbb{Z}/n\mathbb{Z}) \simeq {}_{S_3}\overline{\mathcal{M}}_2$. Let

$$\mathfrak{h} : \mathcal{Z} \to \mathcal{X}$$

denote the corresponding family of Galois covers. It is a family of admissible S_3-covers of stable curves of genus 7 mapping to stable curves of genus 2. Define the family of stable curves $\mathcal{Y} \to {}_{S_3}\overline{\mathcal{M}}_2^{(n)}$ by

$$\mathcal{Y} := \mathcal{Z}/ < \tau > .$$

The covering $\mathfrak{h}$ factorizes via a family non-cyclic 3-fold covers of stable curves

$$\mathfrak{f} : \mathcal{Y} \to \mathcal{X}.$$

According to [BLR], the relative Jacobians $\mathrm{Pic}^0(\mathcal{Y}/{}_{S_3}\overline{\mathcal{M}}_2^{(n)})$ and $\mathrm{Pic}^0(\mathcal{X}/{}_{S_3}\overline{\mathcal{M}}_2^{(n)})$ are families of semi-abelian varieties and the norm defines a morphism

$$\mathrm{Nm} : \mathrm{Pic}^0(\mathcal{Y}/{}_{S_3}\overline{\mathcal{M}}_2^{(n)}) \to \mathrm{Pic}^0(\mathcal{X}/{}_{S_3}\overline{\mathcal{M}}_2^{(n)}).$$

Define the family of Prym varieties $\mathcal{P} \to {}_{S_3}\mathcal{M}_2^{(n)}$ of $\mathfrak{f}$ by

$$\mathcal{P} := \mathrm{Ker}(\mathrm{Nm} : \mathrm{Pic}^0(\mathcal{Y}/{}_{S_3}\overline{\mathcal{M}}_2^{(n)}) \to \mathrm{Pic}^0(\mathcal{X}/{}_{S_3}\overline{\mathcal{M}}_2^{(n)})).$$

Let ${}_{S_3}\widetilde{\mathcal{M}}_2^{(n)}$ denote the set consisting of points $s \in {}_{S_3}\overline{\mathcal{M}}_2^{(n)}$ such that the fibre $\mathcal{P}_s$ is an abelian variety. Hence ${}_{S_3}\widetilde{\mathcal{M}}_2^{(n)}$ coincides with the subset over which the map $\mathcal{P} \to {}_{S_3}\overline{\mathcal{M}}_2^{(n)}$ is proper. This implies also that ${}_{S_3}\widetilde{\mathcal{M}}_2^{(n)}$ is an open set in ${}_{S_3}\overline{\mathcal{M}}_2^{(n)}$. By abuse of notation we denote by $\mathcal{P}$ also the restriction of $\mathcal{P}$ to ${}_{S_3}\widetilde{\mathcal{M}}_2^{(n)}$. According to Proposition 3.1 the set ${}_{S_3}\widetilde{\mathcal{M}}_2^{(n)}$ coincides with the set of points s such that the cover $\mathfrak{h}_s : \mathcal{Z}_s \to \mathcal{X}_s$ satisfies condition $(**)$ and hence, by Corollary 5.7, satisfies condition $(*)$. By Corollary 2.5, for every $s \in {}_{S_3}\widetilde{\mathcal{M}}_2^{(n)}$ the abelian variety $\mathcal{P}_s$ admits a principal polarization Ξ_s. These principal polarizations glue together in the usual way to give a family of principal polarizations, i.e. an isomorphism $\phi_\Xi : \mathcal{P} \to \widehat{\mathcal{P}}$ over ${}_{S_3}\widetilde{\mathcal{M}}_2^{(n)}$. Since all families occuring are flat, we get a flat family of principally polarized abelian varieties of dimension 2 over ${}_{S_3}\widetilde{\mathcal{M}}_2^{(n)}$. Hence we get a morphism

$$p : {}_{S_3}\widetilde{\mathcal{M}}_2^{(n)} \to \mathcal{A}_2$$

into the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces. The proof of [B, Proposition 6.3] also applies in our case and shows that the map p is proper.

Now ${}_{S_3}\widetilde{\mathcal{M}}_2^{(n)} \subset {}_{S_3}\overline{\mathcal{M}}_2^{(n)}$ is stable under the action of the group $Sp(4, \mathbb{Z}/n\mathbb{Z})$. Hence the quotient

$${}_{S_3}\widetilde{\mathcal{M}}_2 := {}_{S_3}\widetilde{\mathcal{M}}_2^{(n)}/Sp(4, \mathbb{Z}/n\mathbb{Z})$$

is a non-empty open set in ${}_{S_3}\overline{\mathcal{M}}_2$. Moreover, since p commutes with this action, we obtain an induced map

$$Pr : {}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2.$$

This is the (extended) Prym map of the theorem. The induced map Pr is proper, since p is. Finally, its restriction to the open set of smooth covers in ${}_{S_3}\widetilde{\mathcal{M}}_2$ clearly is the Prym map of [LO]. □

As an immediate consequence of Theorem 6.1 we obtain

Corollary 6.2. *The extended Prym map* $Pr : {}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2$ *is surjective. In other words, every principally polarized abelian surface occurs as the Prym variety of a non-cyclic degree-3 admissible cover* $f : Y \to X$ *of a stable curve* X *of genus* 2.

7. Stratification of ${}_{S_3}\widetilde{\mathcal{M}}_2$

Consider the following stratification of the moduli space ${}_{S_3}\widetilde{\mathcal{M}}_2$:

$${}_{S_3}\widetilde{\mathcal{M}}_2 = {}_{S_3}\mathcal{M}_2 \ \sqcup \ R \ \sqcup \ S$$

with boundary components R and S where

$$R = \cup_{i=1}^2 R_i \quad \text{and} \quad S = \cup_{i=0}^2 S_i.$$

Here R_2, respectively S_2, denotes the 2-dimensional subspace of ${}_{S_3}\widetilde{\mathcal{M}}_2$ of Proposition 5.1, respectively 5.3, R_1 respectively S_1, denotes the 1-dimensional subspace of ${}_{S_3}\widetilde{\mathcal{M}}_2$ of Proposition 5.2 respectively 5.4 and finally S_0 denotes the 0-dimensional subspace of ${}_{S_3}\widetilde{\mathcal{M}}_2$ of Proposition 5.5. Note that R_1 is in the closure of R_2 and S_{i-1} is in the closure of S_i for $i = 1$ and 2. Note moreover that S is closed in ${}_{S_3}\widetilde{\mathcal{M}}_2$ whereas we have for the closure of R,

$$\overline{R} = R \cup S_1 \cup S_0.$$

In this section we will study the images of R_i and S_i under the extended Prym map Pr. First we determine the image $Pr({}_{S_3}\mathcal{M}_2)$ of the open set of smooth covers. We will use the following well known fact: Every principally polarized abelian surface is either the Jacobian of a smooth curve of genus 2 or a canonically polarized product of 2 elliptic curves. We denote by $\mathcal{J}_2 \subset \mathcal{A}_2$ the (open) subset of Jacobians of smooth curves and by $\mathcal{E}_2$ its complement in $\mathcal{A}_2$.

Given a smooth genus-2 curve Σ we denote by $\varphi : \Sigma \to \mathbb{P}^1$ the corresponding hyperelliptic cover. For any 3 Weierstrass points w_1, w_2, w_3 of Σ let $\varphi_{2(w_1+w_2+w_3)}$ denote the map $\Sigma \to \mathbb{P}^1$ defined by the pencil $(\lambda(2(w_1 + w_2 + w_3)) + \mu(2(w_4 + w_5 + w_6))_{(\lambda,\mu)\in\mathbb{P}^1}$ where w_4, w_5, w_6 are the complementary Weierstrass points. According to the construction in [LO], we have the following

Proposition 7.1. *$Pr({}_{S_3}\mathcal{M}_2) = \{J\Sigma \in \mathcal{J}_2 \mid\ \exists\, w_1, w_2, w_3$ Weierstrass points of Σ such that $\varphi_{2(w_1+w_2+w_3)} = \bar{f} \circ \varphi$, and $\bar{f}$ is simply ramified $\}$*

Proof. The inclusion "$\supset$" has been shown in [LO]. Conversely, let $J\Sigma$ be the Prym variety of the cover $f : Y \to X$ associated to an element of ${}_{S_3}\mathcal{M}_2$. According to [LO, Proof of Theorem 5.1], the curve Σ fits into the following commutative diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "f"'] & \mathbb{P}^1 \arrow[d, "\bar{f}"] & \Sigma \arrow[l, "\varphi"'] \arrow[dl, "\psi"] \\
X \arrow[r] & \mathbb{P}^1 &
\end{tikzcd}$$

where the horizontal maps are the hyperelliptic covers, and ψ is given by a pencil $g^1_6 \subset |3K_\Sigma|$ which, up to enumeration, is generated by the 2 divisors $2w_1 + 2w_2 + 2w_3$ and $2w_4 + 2w_5 + 2w_6$. Here the w_i denote the Weierstrass points of Σ.

Suppose now that $\bar{f}$ is not simply ramified, then the branch locus of ψ consists of at most 5 points. Two of these branch points are images of w_1, w_2, w_3 and w_4, w_5, w_6 and the other are the branch locus of $\bar{f}$. Hence there exists a point $q \in \mathbb{P}^1$ outside of branch locus of ψ such that $f^{-1}(q)$ contains the image of a Weierstrass point of Y. Since $\bar{f}$ is étale on the fiber of q, all the 3 points in $\bar{f}^{-1}(q)$ are images of Weierstrass points of Y, contradicting the number of Weierstrass points of Y. □

Proposition 7.2. *The extended Prym map restricts to a surjective morphism (denoted by the same letter)*

$$Pr : \sqcup_{i=0}^2 S_i \to \mathcal{E}_2.$$

Proof. For $i = 2$ we saw that $Pr(S_2) \subset \mathcal{E}_2$ already in Proposition 5.3. This implies the analogous statement for $i = 1$ and 2, since any specialization of a principally polarized product of elliptic curves is itself such a product. Moreover, $\sqcup_{i=0}^2 S_i$ is closed in ${}_{S_3}\widetilde{\mathcal{M}}_2$. So $Pr : \sqcup_{i=0}^2 S_i \to \mathcal{E}_2$ is a proper morphism by Theorem 6.1. Since it is clearly dominant, the surjectivity follows from this. □

Finally, we determine the image of R under the Prym map. Consider first R_1.

Proposition 7.3. $\Pr(R_1) \subset \mathcal{J}_2$.

Proof. Let $f : Y \to X$ be the S_3-cover given by an element of R_1. So Y is an irreducible curve of geometric genus 2 with two nodes and X is a rational irreducible curve with 2 nodes. Let $\tilde{Y}$ and $\tilde{X}$ be their normalizations. Then $Y = \tilde{Y}/(\tilde{y}_1 \sim \tilde{y}_2, \tilde{y}_3 \sim \tilde{y}_4)$, where $\tilde{y}_i$ for $i = 1, \dots, 4$, are doubly ramified points of the triple covering $\tilde{f} : \tilde{Y} \to \tilde{X}$ and $X = \tilde{X}/(\tilde{x}_1 \sim \tilde{x}_2, \tilde{x}_3 \sim \tilde{x}_4)$, with $\tilde{x}_i = \tilde{f}(\tilde{y}_i)$ for $i = 1, \dots, 4$. According to Proposition 5.2, the associated Prym variety P is an extension

$$0 \to (\mathbb{Z}/3\mathbb{Z})^2 \to P \xrightarrow{\pi} J\tilde{Y} \to 0.$$

Suppose that $P \simeq E_1 \times E_2$ as principally polarized abelian surfaces, with E_1, E_2 elliptic curves and $E_1 \times E_2$ with canonical (split) polarization. The pullback $\pi^*(\Theta_{\tilde{Y}})$ of the canonical principal polarization of $J\tilde{Y}$ defines a covering of degree 9, $\pi^*(\Theta_{\tilde{Y}}) \to \Theta_{\tilde{Y}}$, so it contains an irreducible component of genus at least 2. On the other hand, according to the assumption and the construction in Proposition 5.2, $\pi^*(\Theta_{\tilde{Y}})$ defines the 3-fold of the canonical principal polarization on $E_1 \times E_2$. Hence the linear system of this polarization contains an irreducible curve of genus at least 2. This contradicts the Künneth formula, according to which all (reduced) irreducible components of this linear system are elliptic curves. □

As an immediate consequence of Proposition 7.3, we get that the Prym variety of a general element if R_2 is the Jacobian of a genus 2 curve. We will see in the next section that this is the case for every element in R_2.

According to Proposition 7.1, the set

$$\mathcal{J}_2^u := Pr({}_{S_3}\mathcal{M}_2) \subset \mathcal{J}_2$$

is the set of Jacobians which admit Weierstrass points w_1, w_2, w_3 such that the map $\varphi_{2(w_1+w_2+w_3)}$ factors via the hyperelliptic cover and a simply ramified map $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$. It is easy to see that $\mathcal{J}_2^u$ is open in $\mathcal{J}_2$ and thus in $\mathcal{A}_2$. Denote by

$$\mathcal{J}_2^r := \{J\Sigma \in \mathcal{J}_2 \mid \exists\, w_1, w_2, w_3 \text{Weierstrass points in } \Sigma \text{ with} \bar{f} \text{ not simply ramified}\}.$$

Note that $\mathcal{J}_2^u \cap \mathcal{J}_2^r \neq \emptyset$. So the covering

$$\mathcal{A}_2 = \mathcal{J}_2^u \cup \mathcal{J}_2^r \sqcup \mathcal{E}_2.$$

is not a stratification of $\mathcal{A}_2$. The following theorem summarizes the situation

Theorem 7.4. *The stratification* ${}_{S_3}\widetilde{\mathcal{M}}_2 = {}_{S_3}\mathcal{M}_2 \sqcup R \sqcup S$ *and the covering* $\mathcal{A}_2 = \mathcal{J}_2^u \cup \mathcal{J}_2^r \sqcup \mathcal{E}_2$ *are compatible under the Prym map* Pr*. To be more precise, we have*

(1) $Pr({}_{S_3}\mathcal{M}_2) = \mathcal{J}_2^u$,
(2) $Pr(R) \subset \mathcal{J}_2^r$ *and*
(3) $Pr(S) = \mathcal{E}_2$.

Proof. Part (1) has been shown in Proposition 7.1 and part (3) in Proposition 7.2. Moreover, we know from [LO, Theorem 5.1] that $Pr^{-1}(\mathcal{J}_2^u) \supset {}_{S_3}\mathcal{M}_2$. Hence, concerning (2), it suffices to show that $Pr(R) \subset \mathcal{J}_2$. We showed this already for R_1 in Proposition 7.3. It remains to show that $Pr(R_2) \subset \mathcal{J}_2$ and this is the contents of the next section (see Proposition 8.6). □

Remark 7.5. One can show that $Pr(R) = \mathcal{J}_2$, but we do not need this fact (see [LO1]).

8. The image of R_2 under the Prym map

Suppose that we are given an S_3-cover in R_2, which gives the cover $f : Y \to X$. So X is an irreducible curve of geometric genus 1 with one node and normalization $\widetilde{X}$, i.e.

$$X = \tilde{X}/\tilde{x}_1 \sim \tilde{x}_2,$$

with points $\tilde{x}_1 \neq \tilde{x}_2$ of $\tilde{X}$. The curve Y is an irreducible curve of geometric genus 3 with one node y and normalization $\tilde{Y}$, i.e.

$$Y = \tilde{Y}/\tilde{y}_1 \sim \tilde{y}_2,$$

with points $\tilde{y}_1 \neq \tilde{y}_2$ of $\tilde{Y}$ such that $\tilde{f} : \tilde{Y} \to \tilde{X}$ is doubly ramified exactly at $\tilde{y}_1$ and $\tilde{y}_2$. As a degeneration of hyperelliptic curves, Y is hyperelliptic and hence $\tilde{Y}$ is hyperelliptic. As an elliptic curve, $\tilde{X}$ admits a one-dimensional family of double covers of $\mathbb{P}^1$. There is exactly one such double cover such that the square in the following diagram commutes.

(8.1)
$$\begin{tikzcd}
\tilde{Y} \arrow[rr, "h_{\tilde{Y}}"] \arrow[dr, "n_{\tilde{Y}}"] \arrow[dd, "\tilde{f}"'] & & \mathbb{P}^1 \arrow[dd, "\bar{f}"] \\
& Y \arrow[ur, "h_Y"'] \arrow[dd, "f"] & \\
\tilde{X} \arrow[rr, "h_{\tilde{X}}"] \arrow[dr, "n_Y"'] & & \mathbb{P}^1 \\
& X \arrow[ur, "h_X"'] &
\end{tikzcd}$$

We denote the map $\tilde{X} \to \mathbb{P}^1$ by $h_{\tilde{X}}$ and call it (by abuse of notation) the hyperelliptic cover of the elliptic curve $\tilde{X}$. The hyperelliptic involution of $\tilde{Y}$ exchanges the points $\tilde{y}_1$ and $\tilde{y}_2$ and the corresponding involution on $\tilde{X}$ exchanges the points $\tilde{x}_1$ and $\tilde{x}_2$. Hence $\bar{f}$ is doubly ramified at $h_{\tilde{X}}(\tilde{y}_1) = h_{\tilde{X}}(\tilde{y}_2)$ and simply ramified over 2 other points.

According to [B, Section 2], there is a canonical theta divisor in $\mathrm{Pic}^3(Y)$, namely

$$\Theta_Y := \{L \in \mathrm{Pic}^3(Y) \mid h^0(Y, L) \geq 1\}.$$

The following proposition describes Θ_Y set-theoretically. For this we need some notation. As $n_Y : \tilde{Y} \to Y$ denotes the normalization of Y, we have a surjective morphism

$$\mathrm{Pic}^3(Y) \stackrel{n_Y^*}{\to} \mathrm{Pic}^3(\tilde{Y}).$$

For any $M \in \mathrm{Pic}^3(\tilde{Y})$ we denote by

$$F(M) := (n_Y^*)^{-1}(M),$$

the set of line bundles L in $\mathrm{Pic}^3(Y)$ mapping to M. If we fix an isomorphism

$$\varphi : M_{\tilde{y}_1} \stackrel{\simeq}{\to} M_{\tilde{y}_2},$$

then L is determined by a constant $c \in \mathbb{C}^*$ such that $M_{\tilde{y}_1}$ is glued with $M_{\tilde{y}_1}$ by $c\varphi$. We denote by

$$\Theta(M) := F(M) \cap \Theta_Y,$$

the fibre over M of the restricted map $n_Y^*|_{\Theta_Y} : \Theta_Y \to \mathrm{Pic}^3(\tilde{Y})$. In the sequel we denote

$$Y^0 := Y \setminus \{y\},$$

the smooth locus of Y. The *Abel map* in degree 3 of Y is defined as the morphism

$$\alpha_Y^3 : (Y^0)^3 \to \mathrm{Pic}^3(Y), \qquad (y_1, y_2, y_3) \mapsto \mathcal{O}_Y(y_1 + y_2 + y_3).$$

Let $H_{\tilde{Y}}$ denote the hyperelliptic line bundle of $\tilde{Y}$. Since the line bundles of degree 3 on $\tilde{Y}$ with $h^0 = 2$ are exactly the line bundles $H_{\tilde{Y}}(y)$ for some point $y \in \tilde{Y}$ and y is a base point of the corresponding linear system, we have for $M \in \mathrm{Pic}^3(\tilde{Y})$,

$$h^0(M) = 2 \quad \Leftrightarrow \quad M \simeq H_{\tilde{Y}}(y) \text{ for some } y \in \tilde{Y},$$

$$h^0(M) = 1 \quad \Leftrightarrow \quad M \simeq \mathcal{O}_{\tilde{Y}}(y_1 + y_2 + y_3) \text{ with } M(-y_i) \not\simeq H_{\tilde{Y}} \text{ for } i = 1, 2, 3.$$

The following proposition is easy to prove, but since it is a special case of 2 more general lemmas of [C], we refer to this paper instead.

Proposition 8.1. *For any $M \in \mathrm{Pic}^3(\tilde{Y})$ exactly one of the following cases occurs,*

(1) *If $M \simeq H_{\tilde{Y}}(y)$ for some $y \in \tilde{Y}$, $y \neq \tilde{y}_1, \tilde{y}_2$, then*
$$\Theta(M) = F(M) \text{ and there is a unique } L_M \in \Theta(M) \text{ with } L_M \in \mathrm{Im}(\alpha_Y^3).$$
Moreover, $h^0(L_M) = 2$ and $h^0(L) = 1$ for all $L \in \Theta(M)$, $L \not\simeq L_M$.

(2) *If $M \simeq H_{\tilde{Y}}(\tilde{y}_i)$ for $i = 1$ or 2, then*
$$\Theta(M) = F(M) = F(M) \setminus \mathrm{Im}(\alpha_Y^3).$$
Moreover, $h^0(L) = 1$ for all $L \in \Theta(M)$.

(3) *If $M \simeq \mathcal{O}_{\tilde{Y}}(y_1 + y_2 + y_3)$ with $M(-y_i) \not\simeq H_{\tilde{Y}}$, then $\Theta(M)$ contains exactly one line bundle denoted L_M and*
$$\Theta(M) = L_M \in \mathrm{Im}(\alpha_Y^3).$$
Moreover, $h^0(L_M) = 1$.

(4) *If $M \simeq \mathcal{O}(y_1 + y_2 + \tilde{y}_i)$ for $i = 1$ or 2 with $M(-\tilde{y}_i) \not\simeq H_{\tilde{Y}}$ and $y_j \neq \tilde{y}_{2-j}$ for $j = 1$ and 2, then*
$$\Theta(M) = \emptyset.$$

Proof. We have $h^0(M) \geq 1$, since $\tilde{Y}$ is of genus 3, and $h^0(M) \leq 2$ by Clifford's theorem. Hence exactly one of the 4 cases occurs. Assertion (1) is a special case of [C, Lemma 2.2.4(2)], (2) is a special case of [C, Lemma 2.2.4(3)], (3) a special cases of [C, Lemma 2.2.3(2a)] and finally (4) a special case of [C, Lemma 2.2.3(1)]. □

Remark 8.2. If $M = \mathcal{O}_{\tilde{Y}}(y_1 + y_2 + y_3)$ is as in (1) or (3) in the proposition, then we can choose y_1, y_2, y_3 in such a way that they correspond to points of Y^0, which we denote by the same letters. Then the uniquely determined line bundle L_M of Proposition 8.1 is just $\mathcal{O}_Y(y_1 + y_2 + y_3)$. We have for any $\mathcal{O}_{\tilde{Y}}(y_1 + y_2 + y_3), \mathcal{O}_{\tilde{Y}}(y_1' + y_2' + y_3')$ of type (1) or (3),

$$\mathcal{O}_Y(y_1 + y_2 + y_3) \simeq \mathcal{O}_Y(y_1' + y_2' + y_3') \Leftrightarrow y_1 + y_2 + y_3 \sim_{\tilde{Y}} y_1' + y_2' + y_3', \tag{8.2}$$

where $\sim_{\tilde{Y}}$ denotes linear equivalence on $\tilde{Y}$.

For the rest of this section we fix the following notation. Let $q_1, \dots, q_8$ denote the Weierstrass points of $\tilde{Y}$ and $p_1, \dots, p_4$ the ramification points of $h_{\tilde{X}}$. From the commutativity of the square in the diagram (8.1) we conclude that $f(q_i)$ is one of the points p_j for each $i = 1, \dots, 8$ and there are 2 points p_j such that $\tilde{f}^{-1}(p_j)$ consists of 3 of the points q_i and for the remaining 2, $\tilde{f}^{-1}(p_j)$ contains of 1 of the points q_j. Without loss of generality, we may assume that

$$\tilde{f}(q_1) = \tilde{f}(q_2) = \tilde{f}(q_3) = p_1, \quad \tilde{f}(q_4) = \tilde{f}(q_5) = \tilde{f}(q_6) = p_2$$

and

$$\tilde{f}(q_7) = p_3, \quad \tilde{f}(q_8) = p_4.$$

If H_Y and H_X denote the hyperelliptic line bundles of Y and X, i.e. the line bundles defining the hyperelliptic covers h_Y and h_X, we have

$$f^*(H_X) \simeq H_Y^3.$$

In order to describe the restriction of the theta divisor of JY to the Prym variety P we will work in $\mathrm{Pic}^2(Y)$. For this we consider the following translate of Θ:

$$\Theta_{q_1} := \Theta - q_1 \subset \mathrm{Pic}^2(Y).$$

A priori $\mathrm{Pic}^2(X), \mathrm{Pic}^2(Y), \mathrm{Pic}^2(\tilde{X})$, and $\mathrm{Pic}^2(\tilde{Y})$ are not algebraic groups, but they have a canonical point, namely the hyperelliptic line bundle. Using this, we may consider them as semiabelian varieties and have the following commutative diagram with exact rows,

(8.3)
$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & \tilde{P} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r, "i_Y"] \arrow[d, "Nm_{\mathbb{C}^*}"] & \mathrm{Pic}^2(Y) \arrow[r, "n_Y^*"] \arrow[d, "Nm_f"] & \mathrm{Pic}^2(\tilde{Y}) \arrow[r] \arrow[d, "Nm_{\tilde{f}}"] & 0 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r, "i_X"] & \mathrm{Pic}^2(X) \arrow[r, "n_X^*"] & \mathrm{Pic}^2(\tilde{X}) \arrow[r] & 0
\end{tikzcd}$$

where $Nm_{\mathbb{C}^*}$ is the third power map. The maps i_Y and i_X are defined as follows: Recall that H_Y is defined in terms of $H_{\tilde{Y}}$ by gluing the fibres at $\tilde{y}_1$ and $\tilde{y}_2$. The constant $c \in \mathbb{C}^*$ corresponding to H_Y depends on the gluing. We glue the fibres of $H_{\tilde{Y}}$ in such a way that the constant corresponding to H_Y is 1. In other words we have $H_Y = i_Y(1)$ and similarly for H_X. So $P = \mathrm{Nm}_f^{-1}(H_X)$ and $\tilde{P} = \mathrm{Nm}_{\tilde{f}}^{-1}(H_{\tilde{X}})$.

For the other fibres we proceed similarly: Given $M \in \mathrm{Pic}^2(\tilde{Y})$, we choose a line bundle $L_M \in n_Y^*(M)$ and glue the fibres $M_{\tilde{y}_1}$ and $M_{\tilde{y}_2}$ in such a way that the constant corresponding to L_M is 1. Then we have

$$n_Y^{*\,-1}(M) = \{L_M \otimes i_Y(c) \otimes H_Y^{-1} \mid c \in \mathbb{C}^*\}.$$

If M is of type (1) or (3) of Proposition 8.1, we choose L_M as described there.

Proposition 8.3. *A line bundle $L \in \mathrm{Pic}^2(Y)$ is in the intersection $P \cap \Theta_{q_1}$ if and only if*
(a) *$L \simeq \mathcal{O}_Y(y_1 + y_2 + q_i - q_1))$ with $f(\iota_Y y_2) = f(y_2)$ for $i = 1, 2, 3$ or*
(b) *$L \simeq \mathcal{O}_Y(q_i - q_1) \otimes i_Y(\zeta^j)$ where ζ is a primitive third root of the unity and $i = 1, 2$ or 3, $j = 0, 1$ or 2.*

Proof. The elements in Θ_Y have been described in Proposition 8.1. Let $L \in \Theta_{q_1}$ and set $M = n_Y^*(L(q_1))$. Suppose M is of type (1) of Proposition 8.1, i.e. $M \simeq H_{\tilde{Y}}(y)$ for some $y \in \tilde{Y} \setminus \{\tilde{y}_1, \tilde{y}_2\}$. Then

$$L = \mathcal{O}_Y(y - q_1) \otimes i_Y(c) \quad \text{for some} \quad c \in \mathbb{C}^*.$$

So $\mathrm{Nm}_f(L) = \mathcal{O}_X(f(y) - p_1) \otimes i_X(c^3)$ and this is equal to H_X if and only if $i_X(c^3) = H_X$ and $f(y) = p_1$, that is, $c^3 = 1$ and $y \in f^{-1}(p_1) = \{q_1, q_2, q_3\}$. This gives the line bundles in (b).

Suppose $M \simeq H_{\tilde{Y}}(\tilde{y}_i)$ for $i = 1$ or 2. If $\mathrm{Nm}_f(L) = H_X$, by the commutativity of diagram (8.3), $\mathrm{Nm}_{\tilde{f}}(M(-q_1)) = H_{\tilde{X}}(\tilde{x}_i - p_1) = H_{\tilde{X}}$, which implies $\tilde{x}_i = p_1$, a contradiction.

Suppose now that M is of type (3), i.e. $M \simeq \mathcal{O}_{\tilde{Y}}(y_1 + y_2 + y_3)$ with $M(-y_i) \not\simeq H_{\tilde{Y}}$ for all i. Then $\Theta(M)$ consists only of the line bundle $L_M \in \mathrm{Im}(\alpha_Y^3)$. So

$$\mathrm{Nm}_f(L_M(-q_1)) = \mathrm{Nm}_f(\mathcal{O}(y_1 + y_2 + y_3 - q_1)) = \mathcal{O}_X(f(y_1) + f(y_2) + f(y_3) - p_1).$$

and $\mathrm{Nm}_f(L_M(-q_1)) = H_X$ if and only if

$$\mathcal{O}_X(f(y_1) + f(y_2) + f(y_3)) \simeq H_X \otimes \mathcal{O}_Y(p_1).$$

Since p_1 is a base point of the linear system $|H_X \otimes \mathcal{O}_Y(p_1)|$ we conclude that, after possibly renumerating, we get $y_3 \in f^{-1}(p_1)$ and $f(y_1) + f(y_2) \sim H_X$. This gives the line bundles in (a). □

For $i = 1, 2, 3$, we consider the following sets

$$\begin{aligned} \widetilde{\Xi}_i \;&:=\; \{\mathcal{O}_Y(y + z + q_i - q_1) \in \mathrm{Pic}^2(Y) \mid y, z \neq \tilde{y}_1, \tilde{y}_2,\ f(\iota_Y z) = f(y)\} \\ &\cup\;\; \{\mathcal{O}_Y(q_i - q_1) \otimes i_Y(\zeta^j) \mid j = 0, 1, 2\}, \end{aligned}$$

As a consequence of Proposition 8.3 we obtain,

$$P \cap \Theta_{q_1} = \widetilde{\Xi}_1 \cup \widetilde{\Xi}_2 \cup \widetilde{\Xi}_3.$$

For $i = 1, 2, 3$, define the scheme Ξ_i as the closure of $\widetilde{\Xi}_i \setminus \{\mathcal{O}_Y(q_i - q_1) \otimes i_Y(\zeta^j) \mid j = 0, 1, 2\}$ with reduced subscheme structure.

Lemma 8.4. *The scheme Ξ_i is a complete curve for $i = 1, 2, 3$ and we have the following equality of sets*

$$P \cap \Theta_{q_1} = \Xi_1 \cup \Xi_2 \cup \Xi_3.$$

Proof. Note first that $\Xi_i = \Xi_1 + q_i - q_1$ for $i = 1$ and 2. So for the proof we have only to show the assertion for $i = 1$ and for this it suffices to show that $P \not\in \Theta_{q_1}$, since then $P \cap \Theta_{q_1}$ is a divisor in a surface. The proof of this is very similar to the smooth case as given in [LO, Lemma 4.11] and we omit it.

For the last assertion, certainly the right hand side is contained in the left hand side and the difference consists of at most the finitely many points $\{\mathcal{O}_Y(q_i - q_1) \otimes i_Y(\zeta^j) \mid j = 0, 1, 2\}$. But $P \cap \Theta_{q_1}$ is a divisor in P, so does not contain isolated points. So these points are contained in the right hand side. Together with the first assertion this completes the proof of the lemma. □

Proposition 8.5. *The principal polarization Ξ of the Prym variety P is given by each of the algebraically equivalent divisors $\Xi_1 \equiv \Xi_2 \equiv \Xi_3$.*

Proof. The proof is the same as for [LO, Theorem 4.13] and will be omitted. □

Now we are in a position to prove the main result of this section, which completes the proof of Theorem 7.4.

Proposition 8.6. *The Prym variety of any element of* R_2 *is the Jacobian of a smooth curve of genus* 2.

Proof. Let $f : Y \to X$ be the cover associated to an element of R_2 and let $\Xi := \Xi_1$ denote the curve which, according to Proposition 8.5, defines a principal polarization of the corresponding Prym variety P. Then Ξ is either a smooth genus 2 curve or the union of two elliptic curves intersecting transversally in one point.

Suppose Ξ is reducible. Then also the (non-complete) curve

$$\Xi^0 := \Xi \setminus \{i_Y(\zeta^j) \mid j = 0, 1, 2\}$$

is reducible. The curve Ξ^0 is isomorphic to its image in $\mathrm{Pic}^2(\tilde{Y})$, denoted also by Ξ^0. Since $H_{\tilde{Y}} \notin \Xi^0$, we can and will consider Ξ^0 as a subset in the symmetric product $\tilde{Y}^{(2)}$.

Away from the points $\tilde{y}_1$ and $\tilde{y}_2$ the map $\tilde{f} : \tilde{Y} \to \tilde{X}$ is étale. So for every point $y \in \tilde{Y}$, $y \neq \tilde{y}_1, \tilde{y}_2$ the fibre $\tilde{f}^{-1}\tilde{f}(y)$ consists of exactly 3 points. Let us denote these points by $\tilde{f}^{-1}\tilde{f}(y) = \{y, y', y''\}$; we denote $\tilde{Y}^0 := \tilde{Y} \setminus \{\tilde{y}_1, \tilde{y}_2\}$ and define the curve

$$D := \{(y, \iota_{\tilde{Y}} y'), (y, \iota_{\tilde{Y}} y'') \in \tilde{Y} \times \tilde{Y} \mid y \in \tilde{Y} \setminus \{\tilde{y}_1, \tilde{y}_2\}\}$$

with reduced scheme structure and denote by $\bar{D}$ its completion. The restriction of the canonical map $\tilde{Y} \times \tilde{Y} \to \tilde{Y}^{(2)}$ to D defines a $2:1$ map $D \to \Xi^0$, which extends to the closure $\bar{D} \to \bar{\Xi}^0 = \Xi$.

On the other hand, the projection to the first component gives a $2:1$ map $\bar{D} \to \tilde{Y}$, which is ramified exactly at the points $(\tilde{y}_1, \iota_{\tilde{Y}} \tilde{y}_1) = (\tilde{y}_1, \tilde{y}_2)$ and $(\tilde{y}_2, \iota_{\tilde{Y}} \tilde{y}_2) = (\tilde{y}_2, \tilde{y}_1)$. By the Hurwitz formula we obtain $g(\bar{D}) = 6$. Now, since we are assuming Ξ^0 reducible, it follows that D, and then $\bar{D}$, is also reducible, so $\bar{D} = D_1 \cup D_2$ with $D_1 \cap D_2 = \{(\tilde{y}_1, \tilde{y}_2), (\tilde{y}_2, \tilde{y}_1)\}$ and each component is birational to $\tilde{Y}$. Thus D_i has genus $\geq g(\tilde{Y}) = 3$ for $i = 1, 2$, which implies that the arithmetic genus of $\bar{D}$ is at least 7. This gives a contradiction. Consequently, Ξ is an irreducible genus 2 curve. □

Remark 8.7. It is not difficult to see that, for a given map $\varphi_{2(w_1+w_2+w_3)}$, the corresponding $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ is ramified in the image of the points $\tilde{y}_1$ and $\tilde{y}_2$. This shows directly that $Pr(R_2) \subset \mathcal{J}_2^r$ which we saw already in the proof of Theorem 7.4.

Remark 8.8. In [LO, Proposition 4.18] we saw how to find the curve Σ with $P \simeq J\Sigma$ explicitely in term of the Weierstrass points of the curve Y. The same construction also works for covers of R_2 and R_1. Let $f : Y \to X$ be a cover associated to an element of R_2 and let $\tilde{f} : \tilde{Y} \to \tilde{X}$ be its normalization. Moreover, let the notation be as in this section above. In particular $\tilde{f}(q_1) = \tilde{f}(q_2) = \tilde{f}(q_3) = p_1$ and $\tilde{f}(q_4) = \tilde{f}(q_5) = \tilde{f}(q_6) = p_2$. Considering the canonical map $\pi : \tilde{Y}^{(2)} \to \mathrm{Pic}^2(\tilde{Y})$, we can identify $\mathrm{Pic}^2(\tilde{Y}) \setminus H_{\tilde{Y}}$ with $\tilde{Y}^{(2)} \setminus \pi^{-1}(H_{\tilde{Y}})$. Using this, the Weierstrass points of Σ are given by $[q_1+q_2], [q_1+q_3], [q_2+q_3], [q_4+q_5], [q_4+q_6]$ and $[q_5+q_6]$. An analogous construction works for the elements of R_1.

9. The Prym map is finite

We saw in [LO, Theorem 5.1] that the Prym map $Pr : {}_{S_3}\mathcal{M}_2 \to \mathcal{A}_2$ is finite of degree 10 onto its image. Since ${}_{S_3}\mathcal{M}_2$ is open dense in ${}_{S_3}\widetilde{\mathcal{M}}_2$, the (extended) Prym map $Pr : {}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2$ is also of degree 10. In this section we show that it does not have positive dimensional fibres in the boundary. We keep the notations of the previous section.

Theorem 9.1. *The Prym map $Pr : {}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2$ is finite.*

Proof. According to Theorem 7.4, the Prym map is compatible with the coverings as given there. Hence, since $Pr : {}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2$ is proper according to Theorem 6.1, it suffices to consider the 3 pieces separately and show that they have finite fibres. As mentioned above, this has been proved for $Pr : {}_{S_3}\mathcal{M}_2 \to \mathcal{J}_2^u$ already in [LO][Theorem 5.1].

The proof for $Pr : R \to \mathcal{J}_2^r$ is similar, but for sake of completeness we sketch it here. Consider first the restriction of the Prym map to R_2. Let $f : Y \to X$ be a cover given by an element of R_2 and $Pr(f) = J\Sigma$ its image in $\mathcal{J}_2^r$. Let $\tilde{Y}$ (of genus 3) and $\tilde{X}$ (elliptic curve) denote the normalizations. The corresponding map $\tilde{f} : \tilde{Y} \to \tilde{X}$ fits into a commutative diagram

(9.1)
$$\begin{tikzcd}
\tilde{Y} \arrow[r] \arrow[d, "\tilde{f}"'] & \mathbb{P}^1 \arrow[d, "\bar{f}"'] & \Sigma \arrow[l, "\varphi"'] \arrow[dl, "\psi"] \\
\tilde{X} \arrow[r, "h_{\tilde{X}}"] & \mathbb{P}^1 &
\end{tikzcd}$$

where δ is the unique 2:1 map compatible with the hyperelliptic map of $\tilde{Y}$. The map $\bar{f}$ is doubly ramified at one point and simply ramified at 2 others. If $w_1, \dots, w_6$ denote the Weierstrass points of the curve Σ, the 6:1 map $\psi = \varphi \circ \bar{f}$ (where φ the hyperelliptic cover) is given by a pencil $g_6^1 \subset |3K_\Sigma|$, the ramification divisor of which consists of the 6 Weierstrass points of Σ, the 4 preimages of 2 ramification points over $h_{\tilde{X}}(p_3)$ and $h_{\tilde{X}}(p_4)$ and the 2 preimages of the doubly ramified point over $h_{\tilde{X}}(\tilde{x}_1) = h_{\tilde{X}}(\tilde{x}_2)$. So one of the fibres of ψ is $2w_i + 2w_j + 2w_k$ for some $1 \leq i < j < k \leq 6$ and we denote the map ψ by $\psi_{2(w_i+w_j+w_k)}$. Note that $\psi_{2(w_i+w_j+w_k)} = \psi_{2(w_l+w_m+w_n)}$ if $\{i,j,k,l,m,n\} = \{1, \dots, 6\}$. In particular, ψ has 5 distinct points in his branch locus.

Conversely, the choice of 3 Weierstrass points w_i, w_j, w_k such that $\bar{f}$ is doubly ramified at one point gives 5 marked points on $\mathbb{P}^1$. One can recover an element of $\mathrm{Pr}^{-1}(J\Sigma)$ as follows. Let $\bar{y} \in \mathbb{P}^1$ the doubly ramified point of $\bar{f}$. Consider the uniquely determined elliptic curve $\tilde{X}$ given as a double cover $h_{\tilde{X}} : \tilde{X} \to \mathbb{P}^1$, branched over 4 of the marked points, all but $\bar{x} := \bar{f}(\bar{y})$. Define $\tilde{f} : \tilde{Y} \to \tilde{X}$ as the normalization of the pullback of $\bar{f}$ by $h_{\tilde{X}}$. One verifies that $\tilde{Y}$ is a hyperelliptic curve (of genus 3) and $\tilde{f}$ is non-cyclic and doubly ramified at the 2 points $\tilde{y}_1, \tilde{y}_2$ over $\bar{y} = \bar{f}^{-1}(\bar{x})$. Let

$$Y = \tilde{Y}/\tilde{y}_1 \sim \tilde{y}_2 \quad \text{and} \quad X = \tilde{X}/\tilde{x}_1 \sim \tilde{x}_2,$$

where $h_{\tilde{X}}^{-1}(\bar{x}) = \{\tilde{x}_1, \tilde{x}_2\}$. Then the corresponding cover $f : Y \to X$ defines an element of R_2 which maps to $J\Sigma$ under the Prym map. Since this construction depends only on the choice of the 3 Weierstrass points w_i, w_j, w_k, this implies that the map $Pr|_{R_2} : R_2 \to \mathcal{J}_2^r$ has finite fibres.

For the restriction of Pr to R_1 the argument is similar and will be omitted (here the map $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ has 2 doubly ramified points which lead to 2 singular points of Y). Note that R_1 is connected of dimension 1, so it suffices to show that that $Pr|_{R_1}$ is not constant. This completes the proof of the fact that the map $Pr : R \to \mathcal{J}_2^r$ has finite fibres.

It remains to show that the map $Pr : S \to \mathcal{E}_2$ is finite. First let $f : Y \to X$ be a cover associated to an element of S_2. Then $Y = Y^1 \cup Y^2$ (respectively $X = X^1 \cup X^2$) with smooth curves Y^i of genus 2 (respectively X^i of genus 1) intersecting transversally in 1 point and $f = f_1 \cup f_2$ with $f_i : Y^i \to X^i$ doubly ramified at one point y_i for $i = 1$ and 2. According to Proposition 5.3 we have

$$P(f) = P_1 \times P_2,$$

where $P_i = \mathrm{Ker}(\mathrm{Nm}_{f_i} : JY_i \to JX_i)$ for $i = 1, 2$.

It is enough to show that the Prym variety P_i associated to the non-cyclic 3:1 cover $Y^i \to X^i$ defines a 1-dimensional family, when X^i varies. Let Z^i denote the Galois closure of Y^i/X^i. The Galois group of Z^i/X^i certainly is S_3. Denoting by "$\sim$" isogeny and by $P(\cdot)$ the corresponding Prym varieties, we have

$$JZ^i \sim P(Z^i/Y^i) \times P(Y^i/X^i) \times JX^i.$$

On the other hand, according to [RR],

$$JZ^i \sim P(Y^i/X^i)^2 \times JX^i.$$

This implies that it is enough to show that $P(Z^i/Y^i)$ defines a 1-dimensional family when X^i varies.

Now $Z^i \to Y^i$ is an étale double cover of a hyperelliptic curve and for such a double cover it is very well known that the Prym variety is a product of 2 Jacobians, one of which may be 0 (see [Mu]). In our case, $P(Z^i/Y^i)$ is isomorphic to an elliptic curve D_i which is an étale double cover of X_i. This proves the theorem for the elements in the image of S_2. The same argument works for the Prym varieties in the image of S_1, since they are of the form $P = J\tilde{Y}^1 \times P_2$, where $P_2 = P(Y^2/X^2)$ (see Proposition 5.4). We complete the proof by observing that S_0 is 0-dimensional. $\square$

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