Nonlocality and the Critical Reynolds Numbers of the Minimum State Magnetohydrodynamic Turbulence

Ye Zhou[1] and Sean Oughton[2]
[1]Lawrence Livermore National Laboratory, Livermore, California, 94550, USA
[2]Department of Mathematics, University of Waikato, Hamilton, New Zealand

May 17, 2011

Magnetohydrodynamic (MHD) systems can be strongly nonlinear (turbulent) when their kinetic and magnetic Reynolds numbers are high, as is the case in many astrophysical and space plasma flows. Unfortunately these high Reynolds numbers are typically much greater than those currently attainable in numerical simulations of MHD turbulence. A natural question to ask is how can researchers be sure that their simulations have reproduced all of the most influential physics of the flows and magnetic fields? In this Report, a metric is defined to indicate whether the necessary physics of interest has been captured. It is found that current computing resources will typically not be sufficient to achieve this minimum state metric.

I. INTRODUCTION

Magnetohydrodynamic (MHD) turbulence [1, 2] has been widely employed as a physical model in simulations and modeling of space physics systems and astrophysics systems. As is well known, the number of degrees of freedom in turbulent flows can be estimated using non-dimensional parameters such as the Reynolds number (Re) and magnetic Reynolds number (Rm). These can be interpreted as ratios of the nonlinear terms to the dissipative terms in the governing MHD equations. In space physics and astrophysics, estimates for Re and Rm are often in excess of 10^5, sometimes by many orders of magnitude. Direct numerical simulation of such high Reynolds number systems would require resolutions that are well beyond what can be achieved using current and foreseeable supercomputers. Thus, it is highly desirable to determine whether the computationally feasible simulations—with much lower Re and Rm—still capture the most important physics of the flows of interest, despite the inevitable differences associated with the lower Reynolds numbers. Here we employ the *minimum state* concept [3] along with recent results on the wavenumber locality of nonlinear interactions in MHD turbulence [4] to estimate the minimum Reynolds numbers needed for accurate simulation of the energy-containing range in incompressible MHD turbulence.

The equations of incompressible three-dimensional MHD are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nu\nabla^2\boldsymbol{u} = -\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \nabla p, \qquad (1)$$

$$\frac{\partial \boldsymbol{b}}{\partial t} + \eta\nabla^2\boldsymbol{b} = -\boldsymbol{u}\cdot\nabla\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}, \qquad (2)$$

along with the solenoidality constraints $\nabla\cdot\boldsymbol{u} = 0$ and $\nabla\cdot\boldsymbol{b} = 0$ [e.g., 1]. Here, $\boldsymbol{u}$ is the fluid velocity, $\boldsymbol{b}$ the magnetic field, expressed in Alfvén speed units, and p the total pressure. Equations (1)–(2) are written so that the nonlinear terms are isolated on the RHS, along with the pressure gradient. Note that the nonlinear terms all have the same structure, $\sim \boldsymbol{\alpha}\cdot\nabla\boldsymbol{\beta}$, where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be either $\boldsymbol{u}$ or $\boldsymbol{b}$.

We begin by discussing the basic requirement of a minimum state, namely capturing the key physics of the flow of interest. From an applications perspective the most important group of scales is often the energy-containing range.

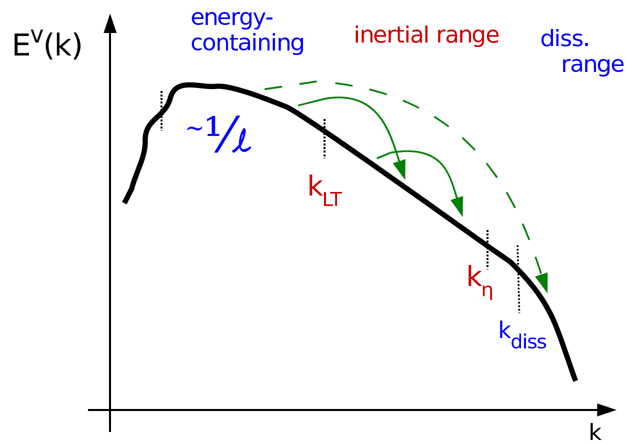

FIG. 1. (Color online) Sketch of a kinetic energy spectrum indicating the energy-containing, inertial, and dissipation ranges, and their wavenumber boundaries. The idea behind the minimum state is that the inertial range should be long enough so that direct interactions between modes in the energy-containing and dissipation ranges are energetically weak, indicated by the dashed (green) arrow. Some "strong" interactions are indicated via the solid (green) arrows.

The integrity of the evolution of modes in this range can be protected by demanding that the (direct) interactions between them and modes in the dissipation range are weak [3]. In such situations the energy-containing and dissipation range scales will be separated by an inertial range, through which the energy originally resident at energy-containing scales cascades to smaller scales [5]. Moreover, the modes in the energy-containing range will then interact dominantly with themselves and modes in the inertial range. It seems likely that there will be critical values of Re and Rm below which this requirement cannot be satisfied. These Reynolds numbers define a *minimum state* flow. See Figure 1.

To quantify these ideas we will extend a criterion developed for Navier–Stokes (NS) turbulence [3] to the MHD case. Specifically, a minimum state flow is defined as one for which the (normalized) energy flux at the high-k end of the inertial range is half that at the low-k end [3], where $k = |\boldsymbol{k}|$ is the

Fourier wavenumber. The remainder of the paper provides the necessary definitions and details required to estimate the Reynolds numbers for a minimum state.

II. INERTIAL RANGE BOUNDARIES

To calculate the minimum state we require estimates of the wavenumbers which bound the kinetic and magnetic inertial ranges. In particular, their scaling with Reynolds number is needed. Let ℓ denote the outer scale, or correlation length, of the velocity field, and let ℓ_B be the equivalent quantity for $\boldsymbol{b}$. Further, let $\tilde{u}$ and $\tilde{b}$ be characteristic rms values for the velocity and magnetic fields. Standard definitions of the (outer scale) kinetic and magnetic Reynolds numbers are $\mathrm{Re} = \tilde{u}\ell/\nu$ and $\mathrm{Rm} = \tilde{u}\ell/\eta$. We will also make use of an alternative magnetic Reynolds number, $\mathrm{Rm}^* = \tilde{B}\ell_B/\eta$, which is based entirely on typical magnetic quantities [6]. The Kolmogorov dissipation scale for the kinetic energy is defined in the usual way as $\ell_{\mathrm{diss}} = (\nu^3/\epsilon)^{1/4}$, where ϵ is the kinetic energy dissipation rate.

For NS turbulence, Dimotakis [7] suggested defining the inertial range as the set of scales which lie below the Liepmann–Taylor scale [8],

$$\lambda_{\mathrm{LT}} = \frac{2\pi}{k_{\mathrm{LT}}} \approx 5\,\mathrm{Re}^{-1/2}\ell, \tag{3}$$

and above the inner viscous scale [e.g., 9],

$$\lambda_\nu = \frac{2\pi}{k_\nu} \approx 50\ell_{\mathrm{diss}} \approx 50\,\mathrm{Re}^{-3/4}\ell. \tag{4}$$

Operationally, the latter is defined as the scale where the spectrum departs from the $\approx -5/3$ powerlaw [9–11]. The energy-containing range is thus treated as having $k \lesssim k_{\mathrm{LT}}$, and the inertial range as $k \in [k_{\mathrm{LT}}, k_\nu]$. We assume that the same boundaries hold for MHD turbulence (but see the final section) and define the magnetic versions analogously:

$$\zeta_{\mathrm{LT}} = \frac{2\pi}{k^B_{\mathrm{LT}}} \approx 5\,\mathrm{Rm}^{*-1/2}\ell_B, \tag{5}$$

$$\zeta_\eta = \frac{2\pi}{k_\eta} \approx 50\,\mathrm{Rm}^{*-3/4}\ell_B. \tag{6}$$

III. ENERGY FLUXES AND LOCALITY

For turbulent systems, the flux of energy in Fourier space is a central concept, and numerous investigations of it have been performed for both NS [e.g., 12–14] and MHD [e.g., 4, 15–21] systems. Each of the nonlinear terms in Eqs. (1)–(2) is associated with such a flux, which we denote herein as $\Pi_{\alpha\beta}$. An important feature of the flux functions is their scaling with wavenumber, which provides information on the extent to which the contributing interactions are local in spectral space. The different scaling properties of these fluxes will be important in determining the minimum state Reynolds numbers.

Using direct numerical simulation databases, Domaradzki et al. [4] calculated normalized versions of the four energy flux functions, which they denoted as $\Pi_{\alpha\beta}(k|k_c)$. These represent the flux of energy to wavevectors with magnitudes greater than k_c, due to wavevector triads which have at least one member with a magnitude less than k (and normalized by the total flux through k_c for the particular $\boldsymbol{\alpha}\cdot\nabla\boldsymbol{\beta}$ term). Plotting these as a function of k/k_c reveals approximate powerlaw scaling for three of the four normalized fluxes. (See Figure 2 in [4].) Here we express their results in terms of the *scale disparity parameter* [13, 14, 22],

$$s = \frac{\max(k,p,q)}{\min(k,p,q)}, \tag{7}$$

where k, p, and q are the magnitudes of wavevectors making up an interacting triad $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$. This re-expression is convenient since $k/k_c \approx 1/s$ and thus, $\Pi_{\alpha\beta}(k|k_c) \approx \Pi_{\alpha\beta}(1/s)$. The scale disparity parameter is a measure of the elongation of the triads and has been used to characterize the degree of locality of interactions [e.g., 23].

The scalings observed by Domaradzki et al. [4] are

$$\Pi_{uu}(s) \sim \Pi_{ub}(s) \propto s^{-2/3}, \tag{8}$$

and

$$\Pi_{bb}(s) \propto s^{-1/3}. \tag{9}$$

The flux Π_{bb} is associated with removing the kinetic energy from the velocity field. It is the least local of these three flux functions. These numerical results are consistent with theoretical predictions [21]. Note that for NS turbulence, theory and simulations [13, 14, 24–30] suggest that $\Pi(s) \sim s^{-4/3}$, a scaling which is considerably more local than the MHD results.

The remaining flux function, Π_{bu}, is associated with the process of energy transfer to the magnetic field. The same study [4] found that it was non-universal and that it did not follow an s^{-M} scaling law. It does, however, decrease faster than Π_{uu} and Π_{ub}. While the reason for the different behavior of Π_{bu} is at present not clear, it is fortunate that the falloff is so steep since this suggests extremely local interactions for the term. Thus its detailed form will not affect the analysis here.

IV. MINIMUM STATE

For large enough Reynolds numbers, the energy-containing range of a turbulent flow will have very weak *direct* interactions with the dissipation range. As noted above, the minimum state is the lowest Reynolds number flow of this kind [3]. In flows that have a shorter inertial range, there will be significant direct interactions between the energy-containing and dissipation ranges, and the integrity of the energy-containing range modes will not be maintained. To ensure that strong direct couplings between these two ranges are absent we need to quantify what 'strong' means in this context, and then determine the length of the inertial range in a minimum state.

Here we define the direct interactions between the energy-containing and dissipation ranges as weak if the energy flux at the high-k end of the inertial range (e.g., at k_ν) is at most half that at the low-k end (e.g., at k_{LT}).

From numerical simulations, the peak of a normalized flux function, $\Pi(s_p)$, can be found, along with the value of s at which it occurs, s_p. Let s_h be the scale disparity parameter where the normalized flux reduces to half of its peak value, i.e., $\Pi(s_h) = \frac{1}{2}\Pi(s_p)$. The s^{-M} scaling properties of Eqs. (8)–(9) lead to

$$\frac{s_h}{s_p} = 2^{(1/M)}. \tag{10}$$

This ratio can be used to determine the values of k_ν and k_η associated with the minimum state. The underlying idea is that Eq. (10) gives the length of the (minimum state) inertial range, in units of the Liepmann–Taylor wavenumber. Hence, we define $k_h = (s_h/s_P)k_{\mathrm{LT}}$ and equate it to the high wavenumber end of the inertial range. For the momentum equation, the least local nonlinear term is Π_{bb}, yielding $s_h/s_p = 8$ and an inertial range wavenumber interval of $[k_{\mathrm{LT}}, 8k_{\mathrm{LT}}]$. For the induction equation we obtain $s_h/s_p = \sqrt{8} \approx 3$ and an inertial range of $[k^B_{\mathrm{LT}}, 3k^B_{\mathrm{LT}}]$.

We are now in position to calculate the critical Reynolds numbers for a minimum state. For the momentum equation we use Eqs. (3) and (4) in $8k_{\mathrm{LT}} = k_\nu$, obtaining

$$\mathrm{Re}_{\mathrm{MS}} \approx 4.1 \times 10^7. \tag{11}$$

Proceeding similarly for the induction equation, $3k^B_{\mathrm{LT}} = k_\eta$ yields

$$\mathrm{Rm}^*_{\mathrm{MS}} \approx 8.1 \times 10^5. \tag{12}$$

This is some 50 times smaller than $\mathrm{Re}_{\mathrm{MS}}$ as a consequence of the more local nature of the nonlinear interactions in the induction equation.

V. COMPARISON WITH FLUID TURBULENCE

It is informative to compare the above results with those for fluid turbulence. We recall that NS turbulence is more local than the MHD case, with $\Pi(s) \sim s^{-4/3}$ [12, 13]. Using $2^{3/4} \approx 2$ in Eq. (10) gives a minimum state Reynolds number of 1.6×10^5 for NS flow [3], which is significantly smaller than the MHD values derived above.

Our minimum state Reynolds numbers are in reasonable accord with results from a perturbative field-theoretic approach for the NS case [31, 32]. In those studies, the nonlocal components of shell-to-shell energy transfers were used to estimate that a turbulent energy flux occurs when $k_{\max}/k_{\min} \approx 2^{16} \approx 10^4$. This value is compatible with the inertial range lengths we found above for the MHD energy fluxes. It would be interesting to perform the MHD version of the study in Ref. [31], to see how the results compare with the ones presented herein.

Another issue is our assumption that the inertial range boundaries carry over essentially unchanged from the NS case to MHD. However, these bounding wavenumbers may scale differently in the two cases. For example, the energy spectrum has often been observed to have a bottleneck feature near the dissipation scale [e.g., 11, 33, 34], but this appears to be more pronounced in NS turbulence than in MHD turbulence [e.g., 35, 36]. Thus, our estimates for k_ν and k_η could be argued to be too small, leading to estimates for the critical Reynolds numbers which are too large. Note that for the NS case, results based on field-theoretic approaches [31] indicate that the spectral bottleneck may not occur for inertial ranges longer than about four decades.

In order to improve understanding of the apparently different features of the NS and MHD bottleneck phenomena it would of course be helpful to have results from MHD studies with (very) large Reynolds numbers. A key element still missing is a major MHD laboratory experiment or observational measurement, analogous to the hydrodynamic wind tunnel results discussed in Ref. [11], for example. Recent MHD computational results [36] indicate that erroneous bottleneck effects can occur when the Reynolds numbers are not large enough. Similar studies have been performed for fluid turbulence [31, 37]. The wavenumber scalings for the energy fluxes contain information about any bottlenecks present in the energy spectra, although this may be hard to extract explicitly. Thus, in that sense the above determinations of the minimum state Reynolds numbers already take account of the bottleneck effects. Further consideration of this interesting issue is beyond the scope of this Brief Report.

VI. SUMMARY

We have extended the concept of a minimum state flow to the case of MHD turbulence, which is a widely used model in space physics and astrophysics applications. By insisting on the integrity of the energy-containing range dynamics, we have determined minimum Reynolds numbers for MHD simulations and experiments below which this condition is unlikely to be satisfied. These ‘critical’ values of $\mathrm{Re}_{\mathrm{MS}} \approx 4.1 \times 10^7$ and $\mathrm{Rm}^*_{\mathrm{MS}} \approx 8.1 \times 10^5$ are rather large, as a consequence of the more nonlocal nature of the the nonlinear terms in the MHD equations (compared to the NS nonlinearity). As far as direct numerical simulations of a minimum state flow are concerned, they are probably not feasible with current computing resources. However, they may become feasible within a few years.

Note that the *numerical* accuracy of a simulation for given Reynolds numbers is a distinct issue, relative to the above discussed ‘physical integrity’ of a simulation. A recent exploration of the accuracy requirements for 2D MHD turbulence [38] concluded that sufficient accuracy is obtained if simulations retain wavenumbers a factor of three greater than the (Kolmogorov) dissipation wavenumber. If a smaller wavenumber range was retained then the accuracy of fourth-order (and higher-order) quantities like the kurtosis was seriously compromised.

In closing we briefly mention some possible extensions and complications associated with the isotropic MHD model employed above. As is well known, the presence of an energetic large-scale (e.g., mean) magnetic field ($\boldsymbol{B}_0$) induces anisotropy in $\boldsymbol{u}$ and $\boldsymbol{b}$ [39–42]. This anisotropy could result in somewhat different critical Reynolds numbers, although the qualitative results presented herein would likely still hold.

Finally, we emphasize that in actual space physics and astrophysics systems the nature of the dissipation mechanisms may be quite different from the uniform viscous and resistive dissipation of Eqs. (1)–(2). In particular, the dissipation scales are not expected to be universal. Plasma effects, such as damping by waves at ion and/or electron gyroradii or inertial lengths may be important [43–46].

Acknowledgments.—This work was performed under the auspices of the Lawrence Livermore National Security, LLC under contract No. DE-AC52-07NA27344 and with support from the University of Waikato Strategic Research Investment Fund. The first author (Y.Z.) is extremely grateful to Professor Sir David Wallace, Director of the Isaac Newton Institute for Mathematical Sciences, University of Cambridge and the organizers of the Partial Differential Equations in Kinetic Theories Programmes, for their kind invitation.

---

[1] D. Biskamp, *Magnetohydrodynamic Turbulence* (CUP, Cambridge, 2003).

[2] Y. Zhou, W. H. Matthaeus, and P. Dmitruk, Rev. Mod. Phys., **76**, 1015 (2004).

[3] Y. Zhou, Phys. Plasmas, **14**, 082701 (2007).

[4] J. A. Domaradzki, B. Teaca, and D. Carati, Phys. Fluids, **22**, 051702 (2010).

[5] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR, **30**, 301 (1941), [Reprinted in Proc. R. Soc. London, Ser. A **434**, 9–13 (1991)].

[6] The distinction between Rm and Rm* is often ignored. In situations where there is equipartition between the kinetic and magnetic energies this can be justified since it is then likely that $\ell \approx \ell_B$ and $\tilde{u} \approx \tilde{B}$. For a detailed discussion, see Zhou and Matthaeus [47].

[7] P. E. Dimotakis, J. Fluid Mech., **409**, 69 (2000).

[8] There are other definitions of the low-k boundary of the inertial range. For example, Pope [9] defines it as one sixth of the longitudinal integral scale, which obviously does not scale with the Reynolds number.

[9] S. B. Pope, *Turbulent Flows* (CUP, Cambridge, UK, 2000).

[10] D. R. Chapman, AIAA J., **17**, 1293 (1979).

[11] S. G. Saddoughi and S. V. Veeravalli, J. Fluid Mech., **268**, 333 (1994).

[12] R. H. Kraichnan, J. Fluid Mech., **47**, 525 (1971).

[13] Y. Zhou, Phys. Fluids A, **5**, 1092 (1993).

[14] Y. Zhou, Phys. Fluids A, **5**, 2511 (1993).

[15] A. Alexakis, P. D. Mininni, and A. Pouquet, Phys. Rev. E, **72**, 046301 (2005).

[16] O. Debliquy, M. K. Verma, and D. Carati, Phys. Plasmas, **12**, 042309 (2005).

[17] M. K. Verma, A. Ayyer, and A. V. Chandra, Phys. Plasmas, **12**, 082307 (2005).

[18] D. Carati, O. Debliquy, B. Knaepen, B. Amd Teaca, and M. Verma, J. Turb., **7**, N51 (2006), doi: 10.1080/14685240600774017.

[19] T. A. Yousef, F. Rincon, and A. A. Schekochihin, J. Fluid Mech., **575**, 111 (2007).

[20] B. Teaca, M. K. Verma, B. Knaepen, and D. Carati, Phys. Rev. E, **79**, 046312 (2009).

[21] H. Aluie and G. L. Eyink, Phys. Rev. Lett., **104**, 081101 (2010).

[22] Y. Zhou, Phys. Rep., **488**, 1 (2010).

[23] M. Lesieur, *Turbulence in Fluids*, 2nd ed. (Kluwer, Dordrecht, The Netherlands, 1990).

[24] Y. Zhou and C. G. Speziale, Applied Mechanics Reviews, **51**, 267 (1998).

[25] T. Gotoh and T. Watanabe, J. Turb., **6**, N33 (2005).

[26] G. L. Eyink, Physica D, **207**, 91 (2005).

[27] J. A. Domaradzki and D. Carati, Phys. Fluids, **19**, 085112 (2007).

[28] J. A. Domaradzki, B. Teaca, and D. Carati, Phys. Fluids, **21**, 025106 (2009).

[29] G. L. Eyink and H. Aluie, Phys. Fluids, **21**, 115107 (2009).

[30] H. Aluie and G. L. Eyink, Phys. Fluids, **21**, 115108 (2009).

[31] M. K. Verma and D. Donzis, J. Phys. A, **40**, 4401 (2007).

[32] M. K. Verma, A. Ayyer, O. Debliquy, S. Kumar, and A. V. Chandra, Pramana — J. Phys., **65**, 297 (2005).

[33] G. Falkovich, Phys. Fluids, **6**, 1411 (1994).

[34] Y. Kaneda, T. Ishihara, M. Yokokawa, K. Itakura, and A. Uno, Phys. Fluids, **15**, L21 (2003).

[35] N. E. L. Haugen, A. Brandenburg, and W. Dobler, Phys. Rev. E, **70**, 016308 (2004).

[36] A. Beresnyak, Phys. Rev. Lett., **106**, 075001 (2011), doi: 10.1103/PhysRevLett.106.075001.

[37] P. K. Yeung and Y. Zhou, Phys. Rev. E, **56**, 1746 (1997).

[38] M. Wan, S. Oughton, S. Servidio, and W. H. Matthaeus, Phys. Plasmas, **17**, 082308 (2010).

[39] H. R. Strauss, Phys. Fluids, **19**, 134 (1976).

[40] D. C. Montgomery, Phys. Scr., **T2/1**, 83 (1982).

[41] J. V. Shebalin, W. H. Matthaeus, and D. Montgomery, J. Plasma Phys., **29**, 525 (1983).

[42] J. C. Higdon, Astrophys. J., **285**, 109 (1984).

[43] R. J. Leamon, C. W. Smith, N. F. Ness, W. H. Matthaeus, and H. K. Wong, J. Geophys. Res., **103**, 4775 (1998).

[44] H. Li, S. P. Gary, and O. Stawicki, Geophys. Res. Lett., **28**, 1347 (2001).

[45] C. W. Smith, K. Hamilton, B. J. Vasquez, and R. J. Leamon, Astrophys. J., **645**, L85 (2006).

[46] W. H. Matthaeus, J. M. Weygand, P. Chuychai, S. Dasso, C. W. Smith, and M. G. Kivelson, Astrophys. J., **678**, L141 (2008).

[47] Y. Zhou and W. H. Matthaeus, Phys. Plasmas, **12**, 056503 (2005).