Hypercontractive Inequality for Pseudo-Boolean Functions of Bounded Fourier Width

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Abstract

A function $f : \{-1,1\}^n \to \mathbb{R}$ is called pseudo-Boolean. It is well-known that each pseudo-Boolean function f can be written as $f(x) = \sum_{I\in\mathcal{F}} \hat{f}(I)\chi_I(x)$, where $\mathcal{F} \subseteq \{I : I \subseteq [n]\}$, $[n] = \{1,2,\ldots,n\}$, and $\chi_I(x) = \prod_{i\in I} x_i$ and $\hat{f}(I)$ are non-zero reals. The degree of f is $\max\{|I| : I \in \mathcal{F}\}$ and the width of f is the minimum integer ρ such that every $i \in [n]$ appears in at most ρ sets in $\mathcal{F}$. For $i \in [n]$, let $\mathbf{x}_i$ be a random variable taking values 1 or -1 uniformly and independently from all other variables $\mathbf{x}_j$, $j \neq i$. Let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$. The p-norm of f is $||f||_p = (\mathbb{E}[|f(\mathbf{x})|^p])^{1/p}$ for any $p \geq 1$. It is well-known that $||f||_q \geq ||f||_p$ whenever $q > p \geq 1$. However, the higher norm can be bounded by the lower norm times a coefficient not directly depending on f: if f is of degree d and $q > p > 1$ then $||f||_q \leq \left(\frac{q-1}{p-1}\right)^{d/2} ||f||_p$. This inequality is called the Hypercontractive Inequality. We show that one can replace d by ρ in the Hypercontractive Inequality for each $q > p \geq 2$ as follows: $||f||_q \leq ((2r)!\rho^{r-1})^{1/(2r)}||f||_p$, where $r = \lceil q/2 \rceil$. For the case $q = 4$ and $p = 2$, which is important in many applications, we prove a stronger inequality: $||f||_4 \leq (2\rho+1)^{1/4}||f||_2$.

1 Introduction

Fourier analysis of *pseudo-Boolean functions*[1], i.e., functions $f : \{-1,1\}^n \to \mathbb{R}$, has been used in many areas of computer science (cf. [1, 5, 10, 13, 14]), social choice theory (cf. [6, 11, 12]), combinatorics, learning theory, coding theory, and many others (cf. [13, 14]). We will use the following well-known and easy to prove fact [13]: each function $f : \{-1,1\}^n \to \mathbb{R}$ can be uniquely written as

$$f(x) = \sum_{I\in\mathcal{F}} \hat{f}(I)\chi_I(x), \tag{1}$$

[1]Often functions $f : \{0,1\}^n \to \mathbb{R}$ are called pseudo-Boolean [3]. In Fourier Analysis, the Boolean domain is often assumed to be $\{-1,1\}^n$ rather than the more usual $\{0,1\}^n$ and we will follow this assumption in our paper.

where $\mathcal{F} \subseteq \{I : I \subseteq [n]\}$, $[n] = \{1, 2, \ldots, n\}$, and $\chi_I(x) = \prod_{i \in I} x_i$ and $\hat{f}(I)$ are non-zero reals. Formula (1) is the Fourier expansion of f and $\hat{f}(I)$ are the Fourier coefficients of f. The right hand size of (1) is a polynomial and the degree $\max\{|I| : I \in \mathcal{F}\}$ of this polynomial will be called the *degree* of f. For $i \in [n]$, let ρ_i be the number of sets $I \in \mathcal{F}$ such that $i \in I$. Let us call $\rho = \max\{\rho_i : i \in [n]\}$ the *Fourier width* (or, just *width*) of f. The Fourier width was introduced in [9] without giving it a name.

The degree and width can be viewed as dual parameters in the following sense. Consider a bipartite graph G with partite sets V and T, where V is the set of variables in f and T is the set of terms in f in (1), and zt is an edge in G if z is a variable in $t \in T$. Note that the degree of f is the maximum degree of a vertex in T and the width of f is the maximum degree of a vertex in V. These two parameters are quite independent. Indeed, while the function $f(x) = \prod_{i=1}^{n} x_i$ has degree n and width 1, the function $f(x) = \sum_{1 \leq i < j \leq n} x_i x_j$ has degree 2 and width $n-1$.

For $i \in [n]$, let $\mathbf{x}_i$ be a random variable taking values 1 or -1 uniformly and independently from all other variables $\mathbf{x}_j$, $j \neq i$. Let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$. Then $f(\mathbf{x})$ is a random variable and the *p-norm* of f is $||f||_p = (\mathbb{E}[|f(\mathbf{x})|^p])^{1/p}$ for any $p \geq 1$. It is easy to show that $||f||_2^2 = \sum_{I \in \mathcal{F}} \hat{f}(I)^2$, which is Parseval's Identity for pseudo-Boolean functions. It is well-known and easy to show that $||f||_q \geq ||f||_p$ whenever $q \geq p \geq 1$. However, the higher norm can be bounded by the lower norm times a coefficient not depending on f: if f is of degree d then

$$||f||_q \leq \left(\frac{q-1}{p-1}\right)^{d/2} ||f||_p. \tag{2}$$

The last inequality is called the *Hypercontractive Inequality.* (In fact, the Hypercontractive Inequality is often stated differently, but the Hypercontractive Inequality in the original form and (2) are equivalent.) Since $||f||_2$ is easy to compute, the Hypercontractive Inequality is quite useful for $p = 2$ and is often used for $p = 2$ and $q = 4$; this special case of the Hypercontractive Inequality has been applied in many papers on algorithmics, social choice theory and many other areas, see, e.g., [1, 2, 6, 8, 9, 10, 11, 12] and was given special proofs (cf. [7] and the extended abstract of [12]). We will call this case the *(4,2)-Hypercontractive Inequality.*

The coefficient before $||f||_p$ in (2) is not optimal. For example, it is easy to show that $||f||_4 \leq 3^{1/4}||f||_2$ for $d = 1$ [13]. This is a strong inequality, but it is proved only for a restricted class of pseudo-Boolean functions, linear functions. In fact, one can extend this inequality to a larger class of pseudo-Boolean functions, those of width 1 (i.e., all functions of the form $c_0 + \sum_{j=1}^{m-1} c_j \chi_{K_j}(x)$, where $K_1, \ldots, K_{m-1}$ is a partition of $[n]$ into non-empty subsets). This is a consequences of Theorem 1 which replaces the coefficient $3^{d/2}$ before $||f||_2$ in the (4,2)-Hypercontractive Inequality by $(2\rho + 1 - \frac{2\rho}{m})^{1/4}$, where ρ is the width of f and $m = |\mathcal{F}|$. Clearly, for many functions $2\rho + 1 < 9^d$ (since $\rho \leq 2^{n-1}$ we will always have $2\rho + 1 < 9^d$ whenever $d > 0.32n$) and then Theorem 1 provides an important special case of the Hypercontractive Inequality with a smaller coefficient. In fact, Theorem 1 improves Lemma 7 in [9]. While in Lemma 7 [9], the coefficient before $||f||_2$ is $(2\rho^2)^{1/4}$ $(\rho \geq 2)$, in Theorem 1, we decrease it to $(2\rho + 1 - \frac{2\rho}{m})^{1/4}$. We provide examples showing that this coefficient is tight.

Due to Theorem 1, we know that the width can replace the degree as a parameter in the coefficient before $||f||_2$ in the (4,2)-Hypercontractive Inequality. A natural question is whether the same is true in the general case of the Hypercontractive Inequality for pseudo-Boolean functions. We show that we can replace d by ρ for each $q \geq p \geq 2$ as follows: $||f||_q \leq ((2r)!\rho^{r-1})^{1/(2r)}||f||_p$, where $r = \lceil q/2 \rceil$.

Note that the value of the coefficient before $||f||_p$ in the Hypercontractive Inequality can be important for proving some results. For example, the main result in [9] has three parts, where in order to prove Part 2 the $(4, 2)$-Hypercontractive Inequality is used with the bound $3^{d/2}$, but in order to prove Part 3 a $(4, 2)$-Hypercontractive Inequality using the width is required. Lemma 7 in [9] was sufficient for that purpose, but Theorem 1 in this paper would give a better result.

2 (4,2)-Hypercontractive Inequality

In (1), let $\mathcal{F} = \{I_1, \ldots, I_m\}$, $f_j(x) = \hat{f}(I_j)\chi_{I_j}(x)$ and $w_j = \hat{f}(I_j)$, $j \in [m]$. If $\emptyset \in \mathcal{F}$, we will assume that $I_1 = \emptyset$.

Theorem 1. *Let $f(x)$ be a pseudo-Boolean function of width $\rho \geq 0$. Then $||f||_4 \leq (2\rho + 1 - \frac{2\rho}{m})^{1/4}||f||_2$.*

Proof. If $\rho = 0$ then $f(x) = c$, where c is a constant and hence $||f||_4 = ||f||_2 = c$. Thus, assume that $\rho \geq 1$. Let S be the set of quadruples $(p_1, p_2, p_3, p_4) \in [m]^4$ such that $\sum_{j=1}^4 |\{i\} \cap I_{p_j}|$ is even for each $i \in [n]$, $S' = \{(p_1, p_2, p_3, p_4) \in S : p_1 = p_2\}$ and $S'' = S \setminus S'$. Note that if a product $f_p(\mathbf{x})f_q(\mathbf{x})f_s(\mathbf{x})f_t(\mathbf{x})$ contains a variable $\mathbf{x}_i$ in only one or three of the factors, then $\mathbb{E}[f_p(\mathbf{x})f_q(\mathbf{x})f_s(\mathbf{x})f_t(\mathbf{x})] = \mathbb{E}[P] \cdot \mathbb{E}(\mathbf{x}_i) = 0$, where P is a polynomial in random variables $\mathbf{x}_l$, $l \in [n] \setminus \{i\}$. Thus,

$$\mathbb{E}[f(\mathbf{x})^4] = \sum_{(p,q,s,t)\in S} \mathbb{E}[f_p(\mathbf{x})f_q(\mathbf{x})f_s(\mathbf{x})f_t(\mathbf{x})].$$

Observe that if $(p, q, s, t) \in S'$ then $p = q$ and $s = t$ and, thus, $\sum_{(p,q,s,t)\in S'} \mathbb{E}[f_p(\mathbf{x})f_q(\mathbf{x})f_s(\mathbf{x})f_t(\mathbf{x})] = \sum_{p=1}^m \sum_{s=1}^m w_p^2 w_s^2$. For a pair $(p, q) \in [m]^2$, let $N(p, q) = |\{(s, t) \in [m]^2 : (p, q, s, t) \in S''\}|$. Let a quadruple $(p, q, s, t) \in S''$. Since $p \neq q$, there must be an i which belongs to just one of the two sets I_p and I_q. Since $(p, q, s, t) \in S''$, i must also belong to just one of the two sets I_s and I_t (two choices). Assume that $i \in I_s$. Then by the definition of ρ, s can be chosen from a subset of $[m]$ of cardinality at most ρ. Once s is chosen, there is a unique choice for t. Therefore, $N(p, q) \leq 2\rho$.

Note that $(p, q, s, t) \in S''$ if and only if $(s, t, p, q) \in S''$ which implies that there are at most $N(p, q)$ tuples in S'' of the form (s, t, p, q). We also have

$$\mathbb{E}[f_p(\mathbf{x})f_q(\mathbf{x})f_s(\mathbf{x})f_t(\mathbf{x})] \leq w_p w_q w_s w_t \leq (w_p^2 w_q^2 + w_s^2 w_t^2)/2.$$

Thus,

$$\sum_{(p,q,s,t)\in S''} \mathbb{E}[f_p(\mathbf{x})f_q(\mathbf{x})f_s(\mathbf{x})f_t(\mathbf{x})] \le \sum_{1\le p\ne q\le m} 2N(p,q)\frac{w_p^2 w_q^2}{2} \le 2\rho \sum_{1\le p\ne q\le m} w_p^2 w_q^2.$$

Hence,

$$\mathbb{E}[f(\mathbf{x})^4] \le \sum_{p=1}^m \sum_{s=1}^m w_p^2 w_s^2 + 2\rho \sum_{1\le p\ne q\le m} w_p^2 w_q^2 = (2\rho+1)\sum_{p=1}^m \sum_{s=1}^m w_p^2 w_s^2 - 2\rho \sum_{p=1}^m w_p^4.$$

We have

$$\frac{\sum_{p=1}^m w_p^4}{\sum_{p=1}^m \sum_{s=1}^m w_p^2 w_s^2} \ge \frac{\sum_{p=1}^m w_p^4}{\sum_{p=1}^m \sum_{s=1}^m [w_p^4/2 + w_s^4/2]} = \frac{\sum_{p=1}^m w_p^4}{m\sum_{p=1}^m w_p^4} = \frac{1}{m}.$$

Thus, $\mathbb{E}[f(\mathbf{x})^4] \le (2\rho + 1 - \frac{2\rho}{m})\left[\sum_{i=1}^m w_i^2\right]^2 = (2\rho + 1 - \frac{2\rho}{m})\mathbb{E}[f(\mathbf{x})^2]^2$. The last equality follows from Parseval's Identity. $\square$

The following two examples show the sharpness of this theorem.

Let $f(x) = 1 + \sum_{i=1}^n x_i$. By Parseval's Indentity, $\mathbb{E}[f(\mathbf{x})^2] = n+1$. It is easy to check that $\mathbb{E}[f(\mathbf{x})^4] = (n+1) + \binom{4}{2}\binom{n+1}{2} = 3n^2 + 4n + 1$. Clearly, $\rho = 1$ and $m = n+1$ and, thus, $2\rho + 1 - \frac{2\rho}{m} = 3 - \frac{2}{n+1}$. Also, $\mathbb{E}[f(\mathbf{x})^4]/\mathbb{E}[f(\mathbf{x})^2]^2 = \frac{3n^2+4n+1}{(n+1)^2} = 3 - \frac{2}{n+1}$.

Let $f(x) = \sum_{I\subseteq[n]} \chi_I(x)$. Clearly, $\mathbb{E}[f(\mathbf{x})^2] = m = 2^n$. To compute $\mathbb{E}[f(\mathbf{x})^4]$ observe that when p, q and s are arbitrarily fixed we have $\mathbb{E}[f_p(\mathbf{x})f_q(\mathbf{x})f_s(\mathbf{x})f_t(\mathbf{x})] \ne 0$ for a unique (one in 2^n) choice of t. Hence, $\mathbb{E}[f(\mathbf{x})^4] = m^4/2^n = 2^{3n}$. Thus, $\mathbb{E}[f(\mathbf{x})^4]/\mathbb{E}[f(\mathbf{x})^2]^2 = 2^n$. Observe that $\rho = 2^{n-1}$ and $2\rho + 1 - \frac{2\rho}{m} = 2^n$ as well.

3 Hypercontractive Inequality

A *multiset* may contain multiple appearances of the same element. For multisets we will use the same notation as for sets, but we will stress it when we deal with multisets. We do not attempt to optimize $g(r)$ in the following theorem.

Theorem 2. *Let $f(x)$ be a pseudo-Boolean function of width $\rho \ge 1$. Then for each positive integer r we have $||f||_{2r} \le [g(r)\rho^{r-1}]^{\frac{1}{2r}} \cdot ||f||_2$, where $g(r) = (2r)!$.*

Proof. Observe that $\mathbb{E}[f(\mathbf{x})^{2r}] = \sum \binom{2r}{\alpha_1 \ldots \alpha_m}\mathbb{E}[f_1^{\alpha_1}(\mathbf{x}) \cdots f_m^{\alpha_m}(\mathbf{x})]$, where the sum is taken over all partitions $\alpha_1 + \cdots + \alpha_m = 2r$ of $2r$ into m non-negatives summands. Consider a non-zero term $\mathbb{E}[f_1^{\alpha_1}(\mathbf{x}) \cdots f_m^{\alpha_m}(\mathbf{x})]$. Note that each variable $\mathbf{x}_i$ appears in an even number of the factors in $f_1^{\alpha_1}(\mathbf{x}) \cdots f_m^{\alpha_m}(\mathbf{x})$. We denote the set of all such m-tuples $\alpha = (\alpha_1, \ldots, \alpha_m)$ by $\mathcal{E}$. Then

$$\mathbb{E}[f(\mathbf{x})^{2r}] = \sum_{\alpha\in\mathcal{E}} \binom{2r}{\alpha} \prod_{i=1}^m w_i^{\alpha_i}. \tag{3}$$

It is useful for us to view $f_1^{\alpha_1}(\mathbf{x})\cdots f_m^{\alpha_m}(\mathbf{x})$, $\alpha\in\mathcal{E}$, as a product of $2r$ factors $f_i(\mathbf{x})$, i.e.,

$$\mathbb{E}[f_1^{\alpha_1}(\mathbf{x})\cdots f_m^{\alpha_m}(\mathbf{x})] = \mathbb{E}[f_{t_1}(\mathbf{x})\cdots f_{t_{2r}}(\mathbf{x})].$$

Let I be a subset of the multiset $\{t_1,\ldots,t_{2r}\}$ (I is a multiset). We call I is *nontrivial* if it contains at least two elements (not necessarily distinct). A subset J of I is called *minimally even* if J is nontrivial, $\mathbb{E}[\prod_{i\in J} f_i(\mathbf{x})]\neq 0$ but $\mathbb{E}[\prod_{i\in K} f_i(\mathbf{x})] = 0$ for each nontrivial subset K of the multiset J. If $I_1=\emptyset$ (that is $\emptyset\in\mathcal{F}$) and 1 is an element of I without repetition (i.e., only one copy of 1 is in I), then $\{1\}$ is also called a *minimally even* subset. (Thus, if I contains two or more elements 1 then $\{1,1\}$ is minimally even, but $\{1\}$ is not; however, if I contains just one element 1, then $\{1\}$ is minimally even.)

Let μ_1 be an element in the multiset $T_1:=\{t_1,\ldots,t_{2r}\}$ such that $w_{\mu_1}^2=\max\{w_{t_i}^2:\ t_i\in T_1\}$, and let M_1 be a minimally even subset of T_1 containing μ_1. For $j\geq 2$, let μ_j be an element in the multiset $T_j:=\{t_1,\ldots,t_{2r}\}\setminus(\cup_{i=1}^{j-1}M_i)$ such that $w_{\mu_j}^2=\max\{w_{t_i}^2:\ t_i\in T_j\}$, and let M_j be a minimally even subset of T_j containing μ_j. Let s be the largest j for which μ_j is defined above. Observe that $s\leq r$ as at most one of the minimally even sets $M_1,M_2,\ldots,M_s$ has size one. If $s<r$, for every $j\in\{s+1,s+2,\ldots,r\}$ let μ_j be an element in the multiset T_1 such that $w_{\mu_j}^2=\max\{w_q^2:\ q\in T_1\setminus\{\mu_1,\ldots,\mu_{j-1}\}\}$.

Let $\alpha\in\mathcal{E}$. For every $i\in[m]$, let $\beta_i=\beta_i(\alpha)$ be the number of copies of i in the multiset $\{\mu_1,\ldots,\mu_r\}$. Let $\mathcal{E}':=\{\beta(\alpha):\ \alpha\in\mathcal{E}\}$. The $2r$ terms in $\prod_{t\in T_1} w_t=\prod_{i=1}^m w_i^{\alpha_i}$ can be split into r pairs such that each pair contains exactly one element with its index in the multiset $\{\mu_1,\ldots,\mu_r\}$ and, furthermore, in each pair, the element with its index in the multiset has at least as high an absolute value as the other element. Therefore the following holds.

$$\prod_{i=1}^m w_i^{\alpha_i}\leq\prod_{i=1}^m w_i^{2\beta_i(\alpha)}. \tag{4}$$

For an m-tuple $\beta\in\mathcal{E}'$, let $N(\beta)$ be the number of m-tuples $\alpha\in\mathcal{E}$ such that $\beta=\beta(\alpha)$. We will now give an upper bound on $N(\beta)$, by showing how to construct all possible α with $\beta(\alpha)=\beta$. Let $M=\{\mu_1,\ldots,\mu_r\}$ be the multiset containing β_i copies of i. We first partition M into any number of non-empty subsets. This can be done in at most $r!$ ways, since we can place μ_1 in the "first" subset, μ_2 in the same subset or in the "second" subset, etc. Each of the subsets will be a subset of a minimal even multiset. Thus, while any multiset, M_i', is not a minimally even subset, there is an $\mathbf{x}_j$ of odd total degree in $\prod_{t\in M_i'} f_t(\mathbf{x})$. Thus, to construct a minimally even subset from M_i', we have to add to M_i' an element q such that $f_q(\mathbf{x})$ contains $\mathbf{x}_j$, which restricts q to at most ρ choices. Continuing in this manner, observe that we have at most ρ choices for the r extra elements we need to add. As the very last element we add has to be unique we note that we construct at most $r!\rho^{r-1}$ partitions of T_1 into minimally even subsets in this way. For each such partition, we have $\alpha=(\alpha_1,\ldots,\alpha_m)$, where α_i is the number of occurrences of i in T_1. Note that every α for which

$\beta(\alpha) = \beta$ can be constructed this way, which implies that

$$N(\beta) \le \rho^{r-1} r!. \tag{5}$$

Let $\alpha \in \mathcal{E}$ and $\beta(\alpha) = (\beta_1, \ldots, \beta_m)$. By the construction of $\beta(\alpha)$, each non-zero β_i appears in the multiset $\{\beta_1, \ldots, \beta_m\}$ at least as many times as in $\{\alpha_1, \ldots, \alpha_m\}$. This implies that

$$\binom{2r}{\alpha} / \binom{r}{\beta(\alpha)} \le (2r)!/r!. \tag{6}$$

By Parseval's Identity,

$$\mathbb{E}[f(\mathbf{x})^2]^r = \left(\sum_{i=1}^{m} w_i^2\right)^r = \sum \binom{r}{b_1 \ldots b_m} w_1^{2b_1} \cdots w_m^{2b_m}, \tag{7}$$

where the last sum is taken over all partitions $b_1 + \cdots + b_m = r$ of r into m non-negatives integral summands.

Now by (3), (4), (5), (6) and (7), we have

$$\begin{array}{rcl} \mathbb{E}[f(\mathbf{x})^{2r}] & = & \sum_{\alpha \in \mathcal{E}} \binom{2r}{\alpha} \prod_{i=1}^m w_i^{\alpha_i} \\ & \le & \sum_{\alpha \in \mathcal{E}} \binom{2r}{\alpha} (\binom{r}{\beta(\alpha)} / \binom{r}{\beta(\alpha)}) \prod_{i=1}^m w_i^{2\beta_i(\alpha)} \\ & \le & \sum_{\beta \in \mathcal{E}'} N(\beta)((2r)!/r!) \binom{r}{\beta} \prod_{i=1}^m w_i^{2\beta_i} \\ & \le & (2r)! \rho^{r-1} \sum_{\beta \in \mathcal{E}'} \binom{r}{\beta} \prod_{i=1}^m w_i^{2\beta_i} \\ & \le & (2r)! \rho^{r-1} \mathbb{E}[f(\mathbf{x})^2]^r. \end{array}$$

□

We can get a better bound on $N(\beta)$ in the proof of this theorem as follows. Note that the number of partitions of a set of cardinality r into non-empty subsets is called the rth Bell number, B_r, and there is an upper bound on B_r: $B_r < \left(\frac{0.792r}{\ln(r+1)}\right)^r$ [4]. This upper bound is better than the crude one, $B_r \le r!$, that we used in the proof of this theorem, but our bound allowed us to obtain a simple expression for $g(r)$. Moreover, we believe that the following, much stronger, inequality holds.

Conjecture 1. *There exists a constant c such that for every pseudo-Boolean function $f(x)$ of width $\rho \ge 1$ we have $||f||_{2r} \le c\sqrt{r\rho}||f||_2$ for each positive integer r.*

If Conjecture 1 holds then it would be best possible, in a sense, due to the following example. Let $f(x) = \sum_{i=1}^n x_i$. By Parseval's Indentity, $\mathbb{E}[f(\mathbf{x})^2] = n$. We will now give a bound for $\mathbb{E}[f(\mathbf{x})^{2r}]$. Define $(a_1, a_2, \ldots, a_{2r})$ to be a *good* vector if all a_i belong to $[n] = \{1, 2, \ldots, n\}$ and any number from $[n]$ appears in the vector zero times or exactly twice. The number of good vectors is equal to $\binom{n}{r}\frac{(2r!)}{2^r}$, which implies that $\mathbb{E}[f(\mathbf{x})^{2r}] \ge \binom{n}{r}\frac{(2r!)}{2^r} = \frac{n!}{(n-r)!} \times \frac{(2r)!}{2^r r!}$.

Note that $\frac{(2r)!}{2^r r!} = (2r-1)!! > (r/e)^r$ and, when n tends to infinity, $\frac{n!}{(n-r)!}$ tends to $n^r = \mathbb{E}[f(\mathbf{x})^2]^r$. Therefore, the bound in Conjecture 1 (for $\rho = 1$) cannot be less than $c\sqrt{r}$ for some constant c.

Theorem 2 can be easily extended as follows.

Corollary 1. *Let $f(x)$ be a pseudo-Boolean function of width $\rho \geq 1$. Then for each $q > p \geq 2$ we have $||f||_q \leq ((2r)!\rho^{r-1})^{1/(2r)}||f||_p$, where $r = \lceil q/2 \rceil$.*

Proof. Let $r = \lceil q/2 \rceil$. Using Theorem 2 and the fact that $||f||_s \geq ||f||_t$ for each $s > t > 1$, we obtain

$$||f||_q \leq ||f||_{2r} \leq ((2r)!\rho^{r-1})^{1/(2r)}||f||_2 \leq ((2r)!\rho^{r-1})^{1/(2r)}||f||_p.$$

□

4 Further Research

It would be interesting to verify Conjecture 1 and decrease the coefficient before $||f||_p$ in Corollary 1.

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