

# Combinatorial optimization over two random point sets

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## Abstract

We analyze combinatorial optimization problems over a random pair of points  $(\mathcal{X}, \mathcal{Y})$  in  $\mathbb{R}^d$  of equal cardinal. Typical examples include the matching of minimal length, the traveling salesperson tour constrained to alternate between points of each set, or the connected bipartite  $r$ -regular graph of minimal length. As the cardinal of the sets goes to infinity, we investigate the convergence of such bipartite functionals.

## 1 Introduction

This work pertains to the probabilistic study of Euclidean combinatorial optimization problems. The starting point in this field is the celebrated theorem of Beardwood, Halton and Hammersley [2] about the traveling salesperson problem. It ensures that given a sequence  $(X_i)_{i \geq 1}$  of independent random variables on  $\mathbb{R}^d$ ,  $d \geq 2$  with common law  $\mu$  of bounded support, then almost surely

$$\lim_{n \rightarrow \infty} n^{\frac{1}{d}-1} T(X_1, \dots, X_n) = \beta_d \int f^{1-\frac{1}{d}}.$$

Here  $\beta_d$  is a constant depending only on the dimension,  $f$  is the density of the absolutely continuous part of  $\mu$  and

$$T(X_1, \dots, X_n) = \inf_{\sigma \in \mathcal{S}_n} \sum_{i=1}^{n-1} |X_{\sigma(i+1)} - X_{\sigma(i)}| + |X_{\sigma(1)} - X_{\sigma(n)}|$$

is the length (for the canonical Euclidean distance) of the shortest tour through the points  $X_1, \dots, X_n$ . In the above formula  $\mathcal{S}_n$  stands for the set of permutations of  $\{1, 2, \dots, n\}$ . Very informally, this result supports the following interpretation: when the number of points  $n$  is large, for  $\mu$  almost every  $x$ , if the salesperson is at  $X_i = x$  then the distance to the next point in the optimal tour is comparable to  $\beta(d)(nf(x))^{-1/d}$  if  $f(x) > 0$  and of lower order otherwise. This should be compared to the fact that the distance from  $X_i = x$  to  $\{X_j, j \leq n \text{ and } j \neq i\}$  also stabilizes at the same rate.

Later, Papadimitriou [9] and Steele [14] have initiated a general theory of Euclidean functionals  $F(\{X_1, \dots, X_n\})$  that satisfy almost sure limits of this type. We refer the reader to the monographs of Steele [15] and Yukich [19] for a full treatment of this now mature theory, and present a short outline. It is convenient to consider multisets rather than sets, so throughout the paper  $\{x_1, \dots, x_n\}$  will stand for a multiset (the elements are unordered but may be repeated). The umbrella theorem in [19] puts forward the following three features of a functional  $F$  on finite multisets of  $\mathbb{R}^d$ :

- $F$  is 1-homogeneous if it is translation invariant and dilation covariant:

$$F(a + \lambda \mathcal{X}) = \lambda F(\mathcal{X})$$

for all finite multisets  $\mathcal{X}$ , all  $a \in \mathbb{R}^d$  and  $\lambda \in \mathbb{R}^+$ .

- The key assumption is subadditivity:  $F$  is subadditive if there exists a constant  $C > 0$  such that for all multisets  $\mathcal{X}, \mathcal{Y}$  in the unit cube  $[0, 1]^d$ ,

$$F(\mathcal{X} \cup \mathcal{Y}) \leq F(\mathcal{X}) + F(\mathcal{Y}) + C.$$

As noted by Rhee in [12], this assumption implies that there is another constant  $C'$  such that for all multiset in  $[0, 1]^d$ ,

$$|F(\mathcal{X})| \leq C' (\text{card}(\mathcal{X}))^{1-\frac{1}{d}}. \quad (1)$$

Hence the worst case for  $n$  points is at most in  $n^{1-\frac{1}{d}}$  and the above mentioned theorems show that the average case is of the same order.

- The third important property is smoothness (or regularity). A functional  $F$  on finite multisets  $\mathbb{R}^d$  is smooth if there is a constant  $C''$  such that for all multisets  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$  in  $[0, 1]^d$ , it holds

$$|F(\mathcal{X} \cup \mathcal{Y}) - F(\mathcal{X} \cup \mathcal{Z})| \leq C'' \left( \text{card}(\mathcal{Y})^{1-\frac{1}{d}} + \text{card}(\mathcal{Z})^{1-\frac{1}{d}} \right).$$

These three properties are enough to show upper limits for  $F$ , on the model of the Beardwood, Halton, Hammersley theorem. To have the full limits, the umbrella theorem of [19] also requires to check a few more properties of a so-called boundary functional associated with  $F$ .

Next, let us present a classical optimization problem which does not enter the above picture. Given two multi-subsets of  $\mathbb{R}^d$  with the same cardinality,  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_n\}$ , the cost of the minimal bipartite matching of  $\mathcal{X}$  and  $\mathcal{Y}$  is defined as

$$M_1(\mathcal{X}, \mathcal{Y}) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|,$$

where the minimum runs over all permutations of  $\{1, \dots, n\}$ . It is well-known that  $n^{-1}M_1(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)$  coincides with the power of the  $L_1$ -Wasserstein distance between the empirical distributions

$$W_1\left(\frac{1}{n} \sum_i \delta_{X_i}, \frac{1}{n} \sum_i \delta_{Y_i}\right),$$

hence it is easily seen to tend to 0, for example when  $\mu$  has bounded support. Recall that given two finite measures  $\mu_1, \mu_2$  on  $\mathbb{R}^d$  with the same total mass,

$$W_1(\mu_1, \mu_2) = \inf_{\pi \in \Pi(\mu_1, \mu_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| d\pi(x, y),$$

where  $\Pi(\mu_1, \mu_2)$  is the set of measures on  $(\mathbb{R}^d)^2$  having  $\mu_1$  as first marginal and  $\mu_2$  as second marginal (see e.g. [10, 18] for more background). Note that for all finite multisets  $\mathcal{X}, \mathcal{Y}$  in  $[0, 1]^d$  with  $\text{card}(\mathcal{X}) = \text{card}(\mathcal{Y})$ ,

$$M_1(\mathcal{X}, \mathcal{Y}) \leq \sqrt{d} \text{card}(\mathcal{X}),$$

and equality holds for some well-chosen configurations of any cardinal (all elements in  $\mathcal{X}$  at  $(0, \dots, 0)$  and all elements in  $\mathcal{Y}$  at  $(1, \dots, 1)$ ). Hence, an interesting feature of  $L$  (as well as others bipartite Euclidean optimization functionals) is that the growth bound assumption (1) fails, hence it is not subadditive in the above sense. However Dobrić and Yukich were able to prove the following theorem:

**Theorem 1** ([4]). *Let  $d \geq 3$  be an integer. Assume that  $\mu$  is a probability measure on  $\mathbb{R}^d$  having a bounded support. Consider mutually independent random variables  $(X_i)_{i \geq 1}$  and  $(Y_j)_{j \geq 1}$  having distribution  $\mu$ . Then, almost surely,*

$$\lim_n n^{\frac{1}{d}-1} M_1(\{X_1, \dots, X_n\}, \{Y_1, \dots, Y_n\}) = \beta_1(d) \int_{\mathbb{R}^d} f^{\frac{d-1}{d}},$$

where  $f(x) dx$  is the absolutely continuous part of  $\mu$  and  $\beta_1(d)$  is a constant depending only on the dimension  $d$ .

The proof of Dobrić and Yukich is very specific to the bipartite matching as it uses from the start the Kantorovich-Rubinstein dual representation of the optimal transportation cost. It is not adapted to a general treatment of bipartite functionals. The starting point of our work was recent paper of Boutet de Monvel and Martin [3] which (independently of [4]) establishes the convergence of the bipartite matching for uniform variables on the unit cube, without using the dual formulation of the transportation cost.

Building on their approach we are able to propose a soft approach of bipartite functionals, based on appropriate notions of subadditivity and regularity. These properties allow to establish upper estimates on upper limits. In order to deal with lower limits we adapt to the bipartite setting the ideas of boundary functionals exposed in [19]. We are able to explicitly construct such functionals for a class of optimization problems involving families of graphs with good properties, and to establish full convergence for absolutely continuous laws. Finally we introduce a new notion of inverse subadditivity which allows to deal with singular parts.

This viewpoint sheds a new light on the result of Dobrić and Yukich, that we extend in other respects, by considering power distance costs, and unbounded random variables satisfying certain tail assumptions. Note that in the classical theory of Euclidean functionals, the analogous question for unbounded random variables was answered in Rhee [13] and generalized in [19].

Let us illustrate our results in the case of the bipartite matching with power distance cost : given  $p > 0$  and two multi-subsets of  $\mathbb{R}^d$ ,  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_n\}$ , define

$$M_p(\mathcal{X}, \mathcal{Y}) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^p,$$

where the minimum runs over all permutations of  $\{1, \dots, n\}$ . Note that we have the same result for the bipartite travelling salesperson problem, and that our generic approach puts forward key properties that allow to establish similar facts for other functionals. As mentioned in the title, our results apply to relatively high dimension. More precisely, if the length of edges are counted to a power  $p$ , our study applies to dimensions  $d > 2p$  only.

**Theorem 2.** *Let  $0 < 2p < d$ . Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  with absolutely continuous part  $f(x) dx$ . We assume that for some  $\alpha > \frac{4dp}{d-2p}$ ,*

$$\int |x|^\alpha d\mu(x) < +\infty.$$

*Consider mutually independent random variables  $(X_i)_{i \geq 1}$  and  $(Y_j)_{j \geq 1}$  having distribution  $\mu$ . Then there is a constant  $\beta_p(d)$  depending only on  $(p, d)$  such that the following convergence holds almost surely*

$$\lim_n n^{\frac{p}{d}-1} M_p(\{X_1, \dots, X_n\}, \{Y_1, \dots, Y_n\}) = \beta_p(d) \int_{\mathbb{R}^d} f^{1-\frac{p}{d}},$$

*provided one of the following hypothesis is verified:*

- $d \in \{1, 2\}$
- $d \geq 3$  and  $p \in (0, 1]$ ,
- $d \geq 3$ ,  $p \in (0, (\sqrt{2d-3} + 1)/2)$  and  $\mu$  is absolutely continuous with respect to Lebesgue's measure,
- $\mu$  is the uniform distribution over a bounded set  $\Omega \subset \mathbb{R}^d$  with positive Lebesgue measure.

Assuming only  $\alpha > \frac{2dp}{d-2p}$ , we can establish convergence in probability. A basic concentration inequality implies that if  $\mu$  has bounded support the convergence holds also in  $L^q$  for all  $q \geq 1$ .

We suspect that the above conditions are due to technical difficulties that we have not been able to solve, and that the convergence should hold for  $0 < p < d/2$  (note that  $(\sqrt{2d-3} + 1)/2 \leq d/2$  with equality for  $d = 2$  only). Assuming just  $0 < p < d/2$ , we are only able to prove almost sure upper limit for general distributions in arbitrary dimension:

$$\limsup_n n^{\frac{p}{d}-1} M_p(\{X_1, \dots, X_n\}, \{Y_1, \dots, Y_n\}) \leq \beta_p(d) \int_{\mathbb{R}^d} f^{1-\frac{p}{d}}.$$

The paper is organized as follows: Section 2 presents the key properties for bipartite functionals (homogeneity, subadditivity and regularity) and gathers useful preliminary statements. Section 3 establishes the convergence for uniform samples on the cube. Section 4 proves upper bounds on the upper limits. These two sections essentially rely on classical subadditive methods, nevertheless a careful analysis is

needed to control the differences of cardinalities of the two samples in small domains. In Section 5, we introduce some examples of bipartite functionals. The lower limits are harder to prove and require a new notion of penalized boundary functionals. It is however difficult to build an abstract theory there, so in Section 6, we will first present the proof for bipartite matchings with power distance cost, and put forward a few lemmas which will be useful for other functionals. We then check that for a natural family of Euclidean combinatorial optimization functionals defined in §5.3, the lower limit also holds. This family includes the bipartite traveling salesman tour. Finally, Section 7 mentions possible variants and extensions.

## 2 A general setting

Let  $\mathcal{M}_d$  be the set of all finite multisets contained in  $\mathbb{R}^d$ . We consider a bipartite functional:

$$L : \mathcal{M}_d \times \mathcal{M}_d \rightarrow \mathbb{R}^+.$$

Let  $p > 0$ . We shall say that  $L$  is  $p$ -homogeneous if for all multisets  $\mathcal{X}, \mathcal{Y}$ , all  $a \in \mathbb{R}^d$  and all  $\lambda \geq 0$ ,

$$L(a + \lambda\mathcal{X}, a + \lambda\mathcal{Y}) = \lambda^p L(\mathcal{X}, \mathcal{Y}). \quad (\mathcal{H}_p)$$

Here  $a + \lambda\{x_1, \dots, x_k\}$  is by definition  $\{a + \lambda x_1, \dots, a + \lambda x_k\}$ . For shortness, we call the above property  $(\mathcal{H}_p)$ . Note that a direct consequence is that  $L(\emptyset, \emptyset) = 0$ .

The functional  $L$  satisfies the regularity property  $(\mathcal{R}_p)$  if there exists a number  $C$  such that for all multisets  $\mathcal{X}, \mathcal{Y}, \mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$ , denoting by  $\Delta$  the diameter of their union, the following inequality holds

$$L(\mathcal{X} \cup \mathcal{X}_1, \mathcal{Y} \cup \mathcal{Y}_1) \leq L(\mathcal{X} \cup \mathcal{X}_2, \mathcal{Y} \cup \mathcal{Y}_2) + C\Delta^p (\text{card}(\mathcal{X}_1) + \text{card}(\mathcal{X}_2) + \text{card}(\mathcal{Y}_1) + \text{card}(\mathcal{Y}_2)). \quad (\mathcal{R}_p)$$

The above inequality implies in particular an easy size bound:  $L(\mathcal{X}, \mathcal{Y}) \leq C\Delta^p (\text{card}(\mathcal{X}) + \text{card}(\mathcal{Y}))$  when  $L(\emptyset, \emptyset) = 0$ .

Eventually,  $L$  verifies the subadditivity property  $(\mathcal{S}_p)$  if there exists a number  $C$  such that for every  $k \geq 2$  and all multisets  $(\mathcal{X}_i, \mathcal{Y}_i)_{i=1}^k$ , denoting by  $\Delta$  the diameter of their union, the following inequality holds

$$L\left(\bigcup_{i=1}^k \mathcal{X}_i, \bigcup_{i=1}^k \mathcal{Y}_i\right) \leq \sum_{i=1}^k L(\mathcal{X}_i, \mathcal{Y}_i) + C\Delta^p \sum_{i=1}^k \left(1 + |\text{card}(\mathcal{X}_i) - \text{card}(\mathcal{Y}_i)|\right). \quad (\mathcal{S}_p)$$

*Remark 1.* A less demanding notion of "geometric subadditivity" could be introduced by requiring the above inequality only when the multisets  $\mathcal{X}_i \cup \mathcal{Y}_i$  lie in disjoint parallelepipeds (see [19] where such a notion is used in order to encompass more complicated single sample functionals). It is clear from the proofs that some of our results hold assuming only geometric subadditivity (upper limit for bounded absolutely continuous laws for example). We will not push this idea further in this paper.

We will see later on that suitable extensions of the bipartite matching, of the bipartite traveling salesperson problem, and of the minimal bipartite spanning tree with bounded maximal degree satisfy all these properties. Our main generic result on bipartite functionals is the following.

**Theorem 3.** *Let  $d > 2p > 0$  and let  $L$  be a bipartite functional on  $\mathbb{R}^d$  with the properties  $(\mathcal{H}_p)$ ,  $(\mathcal{R}_p)$  and  $(\mathcal{S}_p)$ . Consider a probability measure  $\mu$  on  $\mathbb{R}^d$  such that there exists  $\alpha > \frac{4dp}{d-2p}$  with*

$$\int |x|^\alpha d\mu(x) < +\infty.$$

*Consider mutually independent random variables  $(X_i)_{i \geq 1}$  and  $(Y_j)_{j \geq 1}$  having distribution  $\mu$ . Let  $f$  be a density function for the absolutely continuous part of  $\mu$ , then, almost surely,*

$$\limsup_{n \rightarrow \infty} \frac{L(\{X_1, \dots, X_n\}, \{Y_1, \dots, Y_n\})}{n^{1-\frac{p}{d}}} \leq \beta_L \int f^{1-\frac{p}{d}},$$

*for some constant  $\beta_L$  depending only on  $L$ . Moreover, if  $\mu$  is the uniform distribution over a bounded set  $\Omega$  with positive Lebesgue measure, then there is equality: almost surely,*

$$\lim_{n \rightarrow \infty} \frac{L(\{X_1, \dots, X_n\}, \{Y_1, \dots, Y_n\})}{n^{1-\frac{p}{d}}} = \beta_L \text{Vol}(\Omega)^{\frac{p}{d}}.$$

Beyond uniform distributions, lower limits are harder to obtain. In Section 6, we will state a matching lower bound for a subclass of bipartite functionals which satisfy the properties  $(\mathcal{H}_p)$ ,  $(\mathcal{R}_p)$  and  $(\mathcal{S}_p)$  (see the forthcoming Theorem 38 and, for the bipartite traveling salesperson tour, Theorem 39).

*Remark 2.* Let  $B(1/2) = \{x \in \mathbb{R}^d : |x| \leq 1/2\}$  be the Euclidean ball of radius  $1/2$  centered at the origin. It is immediate that the functional  $L$  satisfies the regularity property  $(\mathcal{R}_p)$  if it satisfies property  $(\mathcal{H}_p)$  and if for all multisets  $\mathcal{X}, \mathcal{Y}, \mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$  in  $B(1/2)$ ,

$$L(\mathcal{X} \cup \mathcal{X}_1, \mathcal{Y} \cup \mathcal{Y}_1) \leq L(\mathcal{X} \cup \mathcal{X}_2, \mathcal{Y} \cup \mathcal{Y}_2) + C(\text{card}(\mathcal{X}_1) + \text{card}(\mathcal{X}_2) + \text{card}(\mathcal{Y}_1) + \text{card}(\mathcal{Y}_2)). \quad (\mathcal{R})$$

Similarly,  $L$  will enjoy the subadditivity property  $(\mathcal{S}_p)$  if it satisfies property  $(\mathcal{H}_p)$  and if for every  $k \geq 2$  and all multisets  $(\mathcal{X}_i, \mathcal{Y}_i)_{i=1}^k$  in  $B(1/2)$ ,

$$L\left(\bigcup_{i=1}^k \mathcal{X}_i, \bigcup_{i=1}^k \mathcal{Y}_i\right) \leq \sum_{i=1}^k L(\mathcal{X}_i, \mathcal{Y}_i) + C \sum_{i=1}^k \left(1 + |\text{card}(\mathcal{X}_i) - \text{card}(\mathcal{Y}_i)|\right). \quad (\mathcal{S})$$

The set of assumptions  $(\mathcal{H}_p)$ ,  $(\mathcal{R}_p)$ ,  $(\mathcal{S}_p)$  is thus equivalent to the set of assumptions  $(\mathcal{H}_p)$ ,  $(\mathcal{R})$ ,  $(\mathcal{S})$ .

## 2.1 Consequences of regularity

### 2.1.1 Poissonization

For technical reasons, it is convenient to consider the poissonized version of the above problem. Let  $(X_i)_{i \geq 1}, (Y_i)_{i \geq 1}$  be mutually independent variables with distribution  $\mu$ . Considering independent variables  $N_1, N_2$  with Poisson distribution  $\mathcal{P}(n)$ , the random sets  $\{X_1, \dots, X_{N_1}\}$  and  $\{Y_1, \dots, Y_{N_2}\}$  are independent Poisson point processes with intensity measures  $n\mu$ . For shortness, we set

$$L(n\mu) := L(\{X_1, \dots, X_{N_1}\}, \{Y_1, \dots, Y_{N_2}\}).$$

When  $d\mu(x) = f(x) dx$  we write  $L(nf)$  instead of  $L(n\mu)$ . Note that whenever we are dealing with Poisson processes,  $n \in (0, +\infty)$  is not necessarily an integer. More generally  $L(\nu)$  makes sense for any finite measure, as the value of the functional  $L$  for two independent Poisson point processes with intensity  $\nu$ .

Assume for a moment that the measure  $\mu$  has a bounded support, of diameter  $\Delta$ . The regularity property ensures that

$$|L(\{X_1, \dots, X_n\}, \{Y_1, \dots, Y_n\}) - L(\{X_1, \dots, X_{N_1}\}, \{Y_1, \dots, Y_{N_2}\})| \leq C\Delta^p (|N_1 - n| + |N_2 - n|).$$

Note that  $\mathbb{E}|N_i - n| \leq (\mathbb{E}(N_i - n)^2)^{1/2} = \text{Var}(N_i) = \sqrt{n}$ . Hence the difference between  $\mathbb{E}L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)$  and  $\mathbb{E}L(n\mu)$  is at most a constant times  $\sqrt{n} = o(n^{1-p/d})$  when  $d > 2p$ . Hence in this case, the original quantity and the poissonized version are the same in average at the relevant scale  $n^{1-p/d}$ . The boundedness assumption can actually be relaxed. To show this, we need a lemma.

**Lemma 4.** *Let  $\alpha > 0$ ,  $n > 0$  and let  $\mu$  be a probability measure on  $\mathbb{R}^d$  such that for all  $t > 0$ ,  $\mu(\{x; |x| \geq t\}) \leq ct^{-\alpha}$ . Let  $\mathcal{X}, \mathcal{Y}$  be two independent Poisson point processes of intensity  $n\mu$  and  $T_n = \max\{|Z| : Z \in \mathcal{X} \cup \mathcal{Y}\}$ . Then, for all  $0 < \gamma < \alpha$  there exists a constant  $K = K(c, \alpha, \gamma)$  such that for all  $n \geq 1$ ,*

$$\mathbb{E}[T_n^\gamma]^{1/\gamma} \leq Kn^{1/\alpha}.$$

*Moreover the same conclusion holds if  $\mathcal{X} = \{X_1, \dots, X_n\}$ ,  $\mathcal{Y} = \{Y_1, \dots, Y_n\}$  are two mutually independent sequences of  $n$  variables with distribution  $\mu$ .*

*Proof.* For  $t \geq 0$ , let  $A_t = \{x \in \mathbb{R}^d : |x| \geq t\}$  and  $g(t) = \int_{A_t} d\mu$ . By assumption,  $\mu(A_t) \leq ct^{-\alpha}$ . We start with the Poisson case. Since  $\mathcal{X}, \mathcal{Y}$  are independent, we have  $\mathbb{P}(T_n < t) = \mathbb{P}(\mathcal{X} \cap A_t = \emptyset)^2 = e^{-2n\mu(A_t)}$ .

Therefore, using  $1 - e^{-u} \leq \min(1, u)$ ,

$$\begin{aligned}
\mathbb{E}[T_n^\gamma] &= \gamma \int_0^\infty t^{\gamma-1} \mathbb{P}(T_n \geq t) dt \\
&= \gamma \int_0^\infty t^{\gamma-1} (1 - e^{-2n\mu(A_t)}) dt \\
&\leq \gamma \int_0^{n^{1/\alpha}} t^{\gamma-1} dt + \int_{n^{1/\alpha}}^\infty 2nct^{\gamma-\alpha-1} dt \\
&= n^{\gamma/\alpha} + \frac{2c}{\alpha - \gamma} n^{\gamma/\alpha},
\end{aligned}$$

For the second case, since  $\mathbb{P}(T_n \geq t) = 1 - (1 - \mu(A_t))^{2n} \leq \min(1, 2n\mu(A_t))$  the same conclusion holds.  $\square$

**Proposition 5.** *Let  $d > 2p > 0$ . Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  such that  $\int |x|^\alpha d\mu(x) < +\infty$  for some  $\alpha > \frac{2dp}{d-2p}$ . Let  $(X_i)_{i \geq 1}, (Y_i)_{i \geq 1}$  be mutually independent variables with distribution  $\mu$ . If  $L$  satisfies the regularity property  $(\mathcal{R}_p)$  then*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n) - \mathbb{E}L(n\mu)}{n^{1-\frac{p}{d}}} = 0.$$

*Remark 3.* We have not yet proved the finiteness of the above integrals. This will be done later. The proof below show that the expectations are finite at the same time. So the above statement is established with the convention  $\infty - \infty = 0$ .

*Proof.* Let  $N_1$  and  $N_2$  be Poisson random variables with mean value  $n$ . Let  $T = \max\{|Z| : Z \in \{X_1, \dots, X_{N_1}\} \cup \{Y_1, \dots, Y_{N_2}\}\}$  and  $S = \max\{|Z| : Z \in \{X_1, \dots, X_n\} \cup \{Y_1, \dots, Y_n\}\}$ , with the convention that the maximum over an empty set is 0. The regularity property ensures that

$$|L(\{X_1, \dots, X_n\}, \{Y_1, \dots, Y_n\}) - L(\{X_1, \dots, X_{N_1}\}, \{Y_1, \dots, Y_{N_2}\})| \leq C(T+S)^p (|N_1 - n| + |N_2 - n|).$$

Taking expectation gives, using Cauchy-Schwarz inequality and the bound  $(a+b)^q \leq \max(1, 2^{q-1})(a^q + b^q)$  valid for  $a, b, q > 0$

$$\begin{aligned}
&|\mathbb{E}L(\{X_1, \dots, X_n\}, \{Y_1, \dots, Y_n\}) - \mathbb{E}L(\{X_1, \dots, X_{N_1}\}, \{Y_1, \dots, Y_{N_2}\})| \\
&\leq c_p \left( \mathbb{E}[T^{2p}] + \mathbb{E}[S^{2p}] \right)^{\frac{1}{2}} \left( \mathbb{E}[|N_1 - n|^2] + \mathbb{E}[|N_2 - n|^2] \right)^{\frac{1}{2}} \\
&= c_p \sqrt{2n} \left( \mathbb{E}[T^{2p}] + \mathbb{E}[S^{2p}] \right)^{\frac{1}{2}}
\end{aligned}$$

Since  $\alpha > 2p$ , by Lemma 4, for some  $c > 0$  and all  $n \geq 1$ ,  $\mathbb{E}[T^{2p}] \leq cn^{2p/\alpha}$  and  $\mathbb{E}[S^{2p}] \leq cn^{2p/\alpha}$ . Hence the above difference of expectations is at most a constant times  $n^{\frac{p}{\alpha} + \frac{1}{2}}$ , which is negligible with respect to  $n^{1-\frac{p}{d}}$  since  $\alpha$  is assumed to be large enough.  $\square$

### 2.1.2 Approximations

**Proposition 6.** *Assume that a bipartite functional  $L$  satisfies the regularity property  $(\mathcal{R}_p)$ . Let  $m, n > 0$  and  $\mu$  be a probability measure with support included in a set  $Q$ . Then*

$$\mathbb{E}L(n\mu) \leq \mathbb{E}L(m\mu) + C \text{diam}(Q)^p |m - n|.$$

*Proof.* Assume  $n < m$  (the other case is treated in the same way). Let  $(X_i)_{i \geq 1}, (Y_i)_{i \geq 1}, N_1, N_2, K_1, K_2$  be mutually independent random variables, such that for all  $i \geq 1$ ,  $X_i$  and  $Y_i$  have law  $\mu$ , and for  $j \in \{1, 2\}$ , the law of  $N_j$  is  $\mathcal{P}(n)$  and the law of  $K_j$  is  $\mathcal{P}(m-n)$ . Then  $M_i = N_i + K_i$  is  $\mathcal{P}(m)$ -distributed. Then  $\{X_1, \dots, X_{N_1}\}$  and  $\{Y_1, \dots, Y_{N_2}\}$  are independent Poisson point processes of intensity  $n\mu$ , while  $\{X_1, \dots, X_{M_1}\}$  and  $\{Y_1, \dots, Y_{M_2}\}$  are independent Poisson point processes of intensity  $m\mu$ . By the regularity property,

$$L(\{X_1, \dots, X_{N_1}\}, \{Y_1, \dots, Y_{N_2}\}) \leq L(\{X_1, \dots, X_{N_1+K_1}\}, \{Y_1, \dots, Y_{N_2+K_2}\}) + C \text{diam}(Q)^p (K_1 + K_2).$$

Taking expectations gives the claim.  $\square$

Applying the above inequality for  $m = 0$  gives a weak size bound on  $\mathbb{E}L(\nu)$ .

**Corollary 7.** *Assume that  $L$  satisfies  $(\mathcal{R}_p)$  and  $L(\emptyset, \emptyset) = 0$  (a consequence of e.g.  $(\mathcal{H}_p)$ ), then if  $\nu$  is a finite measure with support included in a set  $Q$ ,*

$$\mathbb{E}L(\nu) \leq C \text{diam}(Q)^p \nu(Q).$$

Recall the total variation distance of two probability measures on  $\mathbb{R}^d$  is defined as

$$d_{\text{TV}}(\mu, \mu') = \sup\{|\mu(A) - \mu'(A)| : A \text{ Borel set of } \mathbb{R}^d\}.$$

**Proposition 8.** *Assume that  $L$  satisfies  $(\mathcal{R}_p)$ . Let  $\mu, \mu'$  be two probability measures on  $\mathbb{R}^d$  with bounded supports. Set  $\Delta$  be the diameter of the union of their supports. Then*

$$\mathbb{E}L(n\mu) \leq \mathbb{E}L(n\mu') + 4C\Delta^p n d_{\text{TV}}(\mu, \mu').$$

*Proof.* The difference of expectations is estimated thanks to a proper coupling argument. Let  $\pi$  be a probability measure on  $\mathbb{R}^d \times \mathbb{R}^d$  having  $\mu$  as its first marginal and  $\mu'$  as its second marginal. We consider mutually independent random variables  $N_1, N_2, (X_i, X'_i)_{i \geq 1}, (Y_i, Y'_i)_{i \geq 1}$  such that  $N_1, N_2$  are  $\mathcal{P}(n)$  distributed and for all  $i \geq 1$ ,  $(X_i, X'_i)$  and  $(Y_i, Y'_i)$  are distributed according to  $\pi$ . Then the random multisets

$$\mathcal{X} = \{X_1, \dots, X_{N_1}\} \quad \text{and} \quad \mathcal{Y} = \{Y_1, \dots, Y_{N_2}\}$$

are independent Poisson point processes with intensity measure  $n\mu$ . Similarly  $\mathcal{X}' = \{X'_1, \dots, X'_{N_1}\}$  and  $\mathcal{Y}' = \{Y'_1, \dots, Y'_{N_2}\}$  are independent Poisson point processes with intensity measure  $n\mu'$ .

The regularity property ensures that

$$\begin{aligned} & L(\{X_1, \dots, X_{N_1}\}, \{Y_1, \dots, Y_{N_2}\}) \\ & \leq L(\{X'_1, \dots, X'_{N_1}\}, \{Y'_1, \dots, Y'_{N_2}\}) + 2C\Delta^p \left( \sum_{i=1}^{N_1} \mathbb{1}_{X_i \neq X'_i} + \sum_{j=1}^{N_2} \mathbb{1}_{Y_j \neq Y'_j} \right). \end{aligned}$$

Taking expectations yields

$$\begin{aligned} \mathbb{E}L(n\mu) & \leq \mathbb{E}L(n\mu') + 2C\Delta^p \mathbb{E} \left( \sum_{i=1}^{N_1} \mathbb{P}(X_i \neq X'_i) + \sum_{j=1}^{N_2} \mathbb{P}(Y_j \neq Y'_j) \right) \\ & = \mathbb{E}L(n\mu') + 4C\Delta^p n \pi(\{(x, y) \in (\mathbb{R}^d)^2; x \neq y\}). \end{aligned}$$

Optimizing the later term on the coupling  $\pi$  yields the claimed inequality involving the total variation distance.  $\square$

**Corollary 9.** *Assume that the functional  $L$  satisfies the regularity property  $(\mathcal{R}_p)$ . Let  $m > 0$ ,  $Q \subset \mathbb{R}^d$  be measurable with positive Lebesgue measure and let  $f$  be a nonnegative locally integrable function on  $\mathbb{R}^d$ . Let  $\alpha = \int_Q f / \text{vol}(Q)$  be the average value of  $f$  on  $Q$ . It holds*

$$\mathbb{E}L(m f \mathbb{1}_Q) \leq \mathbb{E}L(m\alpha \mathbb{1}_Q) + 2Cm \text{diam}(Q)^p \int_Q |f(x) - \alpha| dx.$$

*Proof.* We simply apply the total variation bound of the previous lemma with  $n = m \int_Q f = m\alpha \text{vol}(Q)$ ,  $d\mu(x) = f(x) \mathbb{1}_Q(x) dx / \int_Q f$  and  $d\mu'(x) = \mathbb{1}_Q(x) dx / \text{vol}(Q)$ . Note that

$$2d_{\text{TV}}(\mu, \mu') = \int \left| \frac{f(x) \mathbb{1}_Q(x)}{\int_Q f} - \frac{\mathbb{1}_Q(x)}{\text{vol}(Q)} \right| dx = \frac{\int_Q |f(x) - \alpha| dx}{\int_Q f}.$$

$\square$

### 2.1.3 Average is enough

It is known since the works of Rhee and Talagrand that concentration inequalities often allow to deduce almost sure convergence from convergence in average. This is the case in our general setting.

**Proposition 10.** *Let  $L$  be a bipartite functional on multisets of  $\mathbb{R}^d$ , satisfying the regularity property  $(\mathcal{R}_p)$ . Assume  $d > 2p > 0$ . Let  $\mu$  be a probability measure  $\mu$  on  $\mathbb{R}^d$  with  $\int |x|^\alpha d\mu(x) < +\infty$ . Consider independent variables  $(X_i)_{i \geq 1}$  and  $(Y_i)_{i \geq 1}$  with distribution  $\mu$ .*

*If  $\alpha > 2dp/(d - 2p)$  then the following convergence holds in probability:*

$$\lim_{n \rightarrow \infty} \frac{L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n) - \mathbb{E}L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)}{n^{1-\frac{p}{d}}} = 0.$$

*Moreover if  $\alpha > 4dp/(d - 2p)$ , the convergence happens almost surely, and if  $\mu$  has bounded support, then it also holds in  $L^q$  for any  $q \geq 1$ .*

*Proof.* This is a simple consequence of Azuma's concentration inequality. It is convenient to  $Z(n) = (X_1, \dots, X_n, Y_1, \dots, Y_n)$ . Assume first that the support of  $\mu$  is bounded and let  $\Delta$  denote its diameter. By the regularity property, modifying one point changes the value of the functional by at most a constant:

$$|L(Z_1, \dots, Z_{2n}) - L(Z_1, \dots, Z_{i-1}, Z'_i, Z_{i+1}, \dots, Z_{2n})| \leq 2C\Delta^p.$$

By conditional integration, we deduce that the following martingale difference:

$$d_i := \mathbb{E}(L(Z(n)) | Z_1, \dots, Z_i) - \mathbb{E}(L(Z(n)) | Z_1, \dots, Z_{i-1})$$

is also bounded  $|d_i| \leq 2C\Delta^p$  almost surely. Recall that Azuma's inequality states that

$$\mathbb{P}\left(\left|\sum_{i=1}^k d_i\right| > t\right) \leq 2e^{-\frac{t^2}{2\sum_i \|d_i\|_\infty^2}}.$$

Therefore, we obtain that

$$\mathbb{P}\left(\left|L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n) - \mathbb{E}L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)\right| > t\right) \leq 2e^{-\frac{t^2}{16nC^2\Delta^{2p}}}, \quad (2)$$

and there is a number  $C'$  (depending on  $\Delta$  only) such that

$$\mathbb{P}\left(\left|\frac{L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n) - \mathbb{E}L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)}{n^{1-\frac{p}{d}}}\right| > t\right) \leq 2e^{-C't^2n^{1-\frac{2p}{d}}}.$$

When  $d > 2p$ , we may conclude by the Borel-Cantelli lemma.

If  $\mu$  is not assumed to be of bounded support, let  $S := \max\{|Z_i|; i \leq 2n\}$ . A conditioning argument allows to use the above method. Let  $s > 0$  and  $B(s) = \{x; |x| \leq s\}$ . Given  $\{S \leq s\}$ , the variables  $\{X_1, \dots, X_n\}$  and  $\{Y_1, \dots, Y_n\}$  are mutually independent sequences with distribution  $\mu|_{B(s)}/\mu(B(s))$ . Hence, applying (2) for  $\mu|_{B(s)}/\mu(B(s))$  instead of  $\mu$  and  $2s$  instead of  $\Delta$ , for any  $t > 0$ ,

$$\mathbb{P}\left(\left|\frac{L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)}{n^{1-\frac{p}{d}}} - \frac{\mathbb{E}L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)}{n^{1-\frac{p}{d}}}\right| > t \mid S \leq s\right) \leq 2 \exp\left(-\frac{n^{1-\frac{2p}{d}}t^2}{c_p s^{2p}}\right).$$

Hence for  $\delta > 0$  to be chosen later,

$$\begin{aligned} u_n &:= \mathbb{P}\left(\left|\frac{L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)}{n^{1-\frac{p}{d}}} - \frac{\mathbb{E}L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)}{n^{1-\frac{p}{d}}}\right| > t\right) \\ &\leq \mathbb{P}(S > n^{\frac{1}{\delta}}) + 2 \exp\left(-\frac{n^{1-\frac{2p}{d}-\frac{2p}{\delta}}t^2}{c_p}\right). \end{aligned}$$

Since  $\mathbb{P}(S > u) = 1 - (1 - \mu(B(s)))^{2n} \leq 2n\mu(B(s)) \leq 2n(\int |x|^\alpha d\mu(x))/u^\alpha$ , we get that for some constant  $c$  and any  $\delta > 0$ ,

$$u_n \leq cn^{1-\frac{\alpha}{\delta}} + 2 \exp\left(-\frac{n^{1-\frac{2p}{d}-\frac{2p}{\delta}}t^2}{c_p}\right).$$

Since  $\alpha > 2dp/(d-2p)$  we may choose  $\gamma \in [2dp/(d-2p), \alpha]$ , which ensures that the latter quantities tend to zero as  $n$  increases. This shows the convergence in probability to 0 of

$$\frac{L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)}{n^{1-\frac{\alpha}{2}}} - \frac{\mathbb{E}L(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n)}{n^{1-\frac{\alpha}{2}}}.$$

If  $\alpha > 4dp/(d-2p)$  we may choose we may choose  $\delta \in [2dp/(d-2p), \alpha/2]$ , which ensures that  $\sum_n u_n < +\infty$ . The Borel-Cantelli lemma yields the almost sure convergence to 0.  $\square$

## 2.2 Consequences of subadditivity

We start with a very general statement, which is however not very precise when the measures do not have disjoint supports.

**Proposition 11.** *Let  $L$  satisfy  $(\mathcal{S}_p)$ . Let  $\mu_1, \mu_2$  be finite measures on  $\mathbb{R}^d$  with supports included in a set  $Q$ . Then*

$$\mathbb{E}L(\mu_1 + \mu_2) \leq \mathbb{E}L(\mu_1) + \mathbb{E}L(\mu_2) + 2C \text{diam}(Q)^p \left(1 + \sqrt{\mu_1(Q)} + \sqrt{\mu_2(Q)}\right).$$

*Proof.* Consider four independent Poisson point processes  $\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$  such that for  $i \in \{1, 2\}$ , the intensity of  $\mathcal{X}_i$  and of  $\mathcal{Y}_i$  is  $\mu_i$ . It is classical [8] that the random multiset  $\mathcal{X}_1 \cup \mathcal{X}_2$  is a Poisson point process with intensity  $\mu_1 + \mu_2$ . Also,  $\mathcal{Y}_1 \cup \mathcal{Y}_2$  is an independent copy of the latter process. Applying the subadditivity property,

$$\begin{aligned} L(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) \\ \leq L(\mathcal{X}_1, \mathcal{Y}_1) + L(\mathcal{X}_2, \mathcal{Y}_2) + C \text{diam}(Q)^p (1 + |\text{card}(\mathcal{X}_1) - \text{card}(\mathcal{Y}_1)| + 1 + |\text{card}(\mathcal{X}_2) - \text{card}(\mathcal{Y}_2)|). \end{aligned}$$

Since  $\text{card}(\mathcal{X}_i)$  and  $\text{card}(\mathcal{Y}_i)$  are independent with Poisson law of parameter  $\mu_i(Q)$  (the total mass of  $\mu_i$ ),

$$\mathbb{E}|\text{card}(\mathcal{X}_i) - \text{card}(\mathcal{Y}_i)| \leq \left(\mathbb{E}(\text{card}(\mathcal{X}_i) - \text{card}(\mathcal{Y}_i))^2\right)^{\frac{1}{2}} = \sqrt{2\text{var}(\text{card}(\mathcal{X}_i))} = \sqrt{2\mu_i(Q)}.$$

Hence, taking expectations in the former estimate leads to the claimed inequality.  $\square$

Partition techniques are essential in the probabilistic theory of Euclidean functionals. The next statement allows to apply them to bipartite functionals. In what follows, given a multiset  $\mathcal{X}$  and a set  $P$ , we set  $\mathcal{X}(P) := \text{card}(\mathcal{X} \cap P)$ . If  $\mu$  is a measure and  $f$  a nonnegative function, we write  $f \cdot \mu$  for the measure having density  $f$  with respect to  $\mu$ .

**Proposition 12.** *Assume that the functional  $L$  satisfies  $(\mathcal{S}_p)$ . Consider a finite partition  $Q = \cup_{P \in \mathcal{P}} P$  of a subset of  $\mathbb{R}^d$  and let  $\nu$  be a measure on  $\mathbb{R}^d$  with  $\nu(Q) < +\infty$ . Then*

$$\mathbb{E}L(\mathbb{1}_Q \cdot \nu) \leq \sum_{P \in \mathcal{P}} \mathbb{E}L(\mathbb{1}_P \cdot \nu) + 3C \text{diam}(Q)^p \sum_{P \in \mathcal{P}} \sqrt{\nu(P)}.$$

*Proof.* Consider  $\mathcal{X}, \mathcal{Y}$  two independent Poisson point processes with intensity  $\nu$ . Note that  $\mathcal{X} \cap P$  is a Poisson point process with intensity  $\mathbb{1}_P \cdot \nu$ , hence  $\mathcal{X}(P)$  is a Poisson variable with parameter  $\nu(P)$ . We could apply the subadditivity property to  $(\mathcal{X} \cap P)_{P \in \mathcal{P}}, (\mathcal{Y} \cap P)_{P \in \mathcal{P}}$ , which yields

$$L(\mathcal{X} \cap Q, \mathcal{Y} \cap Q) \leq \sum_{P \in \mathcal{P}} L(\mathcal{X} \cap P, \mathcal{Y} \cap P) + C \text{diam}(Q)^p \sum_{P \in \mathcal{P}} (1 + |\mathcal{X}(P) - \mathcal{Y}(P)|).$$

Nevertheless, doing this gives a contribution at least  $C \text{diam}(Q)^p$  to cells which do not intersect the multisets  $\mathcal{X}, \mathcal{Y}$ . To avoid this rough estimate, we consider the cells which meet at least one of the multisets:

$$\tilde{\mathcal{P}} := \{P \in \mathcal{P}; \mathcal{X}(P) + \mathcal{Y}(P) \neq 0\}.$$

We get that

$$\begin{aligned}
L(\mathcal{X} \cap Q, \mathcal{Y} \cap Q) &\leq \sum_{P \in \tilde{\mathcal{P}}} L(\mathcal{X} \cap P, \mathcal{Y} \cap P) + C \text{diam}(Q)^p \sum_{p \in \tilde{\mathcal{P}}} (1 + |\mathcal{X}(P) - \mathcal{Y}(P)|) \\
&\leq \sum_{P \in \mathcal{P}} L(\mathcal{X} \cap P, \mathcal{Y} \cap P) + C \text{diam}(Q)^p \sum_{p \in \mathcal{P}} \mathbb{1}_{\mathcal{X}(P) + \mathcal{Y}(P) \neq 0} (1 + |\mathcal{X}(P) - \mathcal{Y}(P)|) \\
&\leq \sum_{P \in \mathcal{P}} L(\mathcal{X} \cap P, \mathcal{Y} \cap P) + C \text{diam}(Q)^p \sum_{p \in \mathcal{P}} (\mathbb{1}_{\mathcal{X}(P) + \mathcal{Y}(P) \neq 0} + |\mathcal{X}(P) - \mathcal{Y}(P)|).
\end{aligned}$$

Since  $\mathcal{X}(P)$  and  $\mathcal{Y}(P)$  are independent Poisson variables with parameter  $\nu(P)$ ,

$$\mathbb{P}(\mathcal{X}(P) + \mathcal{Y}(P) \neq 0) = 1 - e^{-2\nu(P)} \text{ and } \mathbb{E}|\mathcal{X}(P) - \mathcal{Y}(P)| \leq \sqrt{2\nu(P)}.$$

Hence, taking expectation and using the bound  $1 - e^{-t} \leq \min(1, t) \leq \sqrt{t}$ ,

$$\mathbb{E}L(\mathbb{1}_Q \cdot \nu) \leq \sum_{P \in \mathcal{P}} \mathbb{E}L(\mathbb{1}_P \cdot \nu) + 2\sqrt{2}C \text{diam}(Q)^p \sum_{p \in \mathcal{P}} \sqrt{\nu(P)}.$$

□

The next statement deals with iterated partitions, which are very useful in the study of combinatorial optimisation problems, see e.g. [15, 19]. If  $\mathcal{P}$  is a partition, we set  $\text{diam}(\mathcal{P}) = \max_{P \in \mathcal{P}} \text{diam}(P)$  (the maximal diameter of its cells).

**Corollary 13.** *Assume that the functional  $L$  satisfies  $(\mathcal{S}_p)$ . Let  $Q \subset \mathbb{R}^d$  and  $\mathcal{Q}_1, \dots, \mathcal{Q}_k$  be a sequence of finer and finer finite partitions of  $Q$ . Let  $\nu$  be a measure on  $\mathbb{R}^d$  with  $\nu(Q) < +\infty$ . Then*

$$\mathbb{E}L(\mathbb{1}_Q \cdot \nu) \leq \sum_{q \in \mathcal{Q}_k} \mathbb{E}L(\mathbb{1}_q \cdot \nu) + 3C \sum_{i=1}^k \text{diam}(\mathcal{Q}_{i-1})^p \sum_{q \in \mathcal{Q}_i} \sqrt{\nu(q)},$$

where by convention  $\mathcal{Q}_0 = \{Q\}$  is the trivial partition.

*Proof.* We start with applying Proposition 12 to the partition  $\mathcal{Q}_1$  of  $Q$ :

$$\mathbb{E}L(\mathbb{1}_Q \cdot \nu) \leq \sum_{q \in \mathcal{Q}_1} \mathbb{E}L(\mathbb{1}_q \cdot \nu) + 3C \text{diam}(\mathcal{Q}_0)^p \sum_{q \in \mathcal{Q}_1} \sqrt{\nu(q)}.$$

Next for each  $q \in \mathcal{Q}_1$  we apply the proposition again for the partition of  $q$  induced by  $\mathcal{Q}_2$  and iterate the process  $k - 2$  times. □

### 3 Uniform cube samples

We introduce a specific notation for  $n \in (0, +\infty)$ ,

$$\bar{L}(n) := \mathbb{E}L(n \mathbb{1}_{[0,1]^d}).$$

We point out the following easy consequence of the homogeneity properties of Poisson point processes.

**Lemma 14.** *If  $L$  satisfies the homogeneity property  $(\mathcal{H}_p)$  then for all  $a \in \mathbb{R}^d$ ,  $\rho > 0$  and  $n > 0$*

$$\mathbb{E}L(n \mathbb{1}_{a+[0,\rho]^d}) = \rho^p \bar{L}(n\rho^d).$$

The following theorem is obtained by adapting to our abstract setting the line of reasoning of Boutet de Monvel and Martin in the paper [3] which was devoted to the bipartite matching:

**Theorem 15.** *Let  $d > 2p$  be an integer. Let  $L$  be a bipartite functional on  $\mathbb{R}^d$  satisfying the properties  $(\mathcal{H}_p)$ ,  $(\mathcal{R}_p)$  and  $(\mathcal{S}_p)$ . Then there exists  $\beta_L \geq 0$  such that*

$$\lim_{n \rightarrow \infty} \frac{\bar{L}(n)}{n^{1-\frac{p}{d}}} = \beta_L.$$

*Proof.* Let  $m \geq 1$  be an integer. Let  $K \in \mathbb{N}$  such that  $2^K \leq m < 2^{K+1}$ . Set  $Q_0 = [0, a]^d$  where  $a := 2^{K+1}/m > 1$ . Let  $\mathcal{Q}_0 = \{Q_0\}$ . We consider a sequence of finer and finer partitions  $\mathcal{Q}_j$ ,  $j \geq 1$  where  $\mathcal{Q}_j$  is a partition of  $Q_0$  into  $2^{jd}$  cubes of size  $a2^{-j}$  (throughout the paper, this means that the interior of the cells are open cubes of such size, while their closure is a closed cube of the same size. We do not describe precisely how the points in the boundaries of the cubes are partitioned, since it is not relevant for the argument). One often says that  $\mathcal{Q}_j$ ,  $j \geq 1$  is a sequence of dyadic partitions of  $Q_0$ .

A direct application of Corollary 13 for the partitions  $\mathcal{Q}_1, \dots, \mathcal{Q}_{K+1}$  and the measure  $n \mathbb{1}_{[0,1]^d}(x) dx$  gives

$$\bar{L}(n) = \mathbb{E}L(n \mathbb{1}_{[0,1]^d}) = \sum_{q \in \mathcal{Q}_{K+1}} \mathbb{E}L(n \mathbb{1}_{q \cap [0,1]^d}) + 3C \sum_{j=1}^{K+1} \text{diam}(\mathcal{Q}_{j-1})^p \sum_{q \in \mathcal{Q}_j} \sqrt{n \text{Vol}(q \cap [0,1]^d)}.$$

Note that  $\mathcal{Q}_{K+1}$  is a partition into cubes of size  $1/m$ , so that its intersection with  $[0,1]^d$  induces an (essential) partition of the unit cube into  $m^d$  cubes of side-length  $1/m$ . Hence, in the first sum, there are  $m^d$  terms which are equal, thanks to translation invariance and Lemma 14 to  $\mathbb{E}L(n \mathbb{1}_{[0, m^{-1}]^d}) = m^{-p} \bar{L}(nm^{-d})$ . The remaining terms of the first sum vanish. In order to deal with the second sum of the above estimate, we simply use the fact that  $\mathcal{Q}_j$  contains  $2^{jd}$  cubical cells of size  $a2^{-j} = 2^{K+1-j}/m \leq 2^{1-j}$ . Hence their individual volumes are at most  $2^{d(1-j)}$ . These observations allow to rewrite the above estimate as

$$\begin{aligned} \bar{L}(n) &\leq m^{d-p} \bar{L}(nm^{-d}) + 3C \sum_{j=1}^{K+1} \text{diam}([0, 2^{2-j}]^d)^p 2^{jd} \sqrt{n 2^{d(1-j)}} \\ &= m^{d-p} \bar{L}(nm^{-d}) + 3C \sqrt{n} \text{diam}([0, 1]^d)^p \sum_{j=1}^{K+1} 2^{p(2-j) + \frac{d}{2}(j+1)}. \end{aligned}$$

Hence, there is a number  $D$  depending only on  $p, d$  and  $C$  such that

$$\bar{L}(n) \leq m^{d-p} \bar{L}(nm^{-d}) + D \sqrt{n} 2^{K(\frac{d}{2}-p)} \leq m^{d-p} \bar{L}(nm^{-d}) + D \sqrt{n} m^{\frac{d}{2}-p}.$$

Let  $t > 0$ . Setting,  $n = m^d t^d$  and  $f(u) = \bar{L}(u^d)/u^{d-p}$ , the latter inequality reads as

$$f(mt) \leq f(t) + Dt^{p-\frac{d}{2}},$$

and is valid for all  $t > 0$  and  $m \in \mathbb{N}^*$ . Since  $f$  is continuous (Proposition 6 shows that  $u \mapsto M(u)$  is Lipschitz) and  $\lim_{t \rightarrow +\infty} t^{p-\frac{d}{2}} = 0$ , it follows that  $\lim_{t \rightarrow +\infty} f(t)$  exists (we refer to [3] for details).  $\square$

*Remark 4.* The above constant  $\beta_L$  is positive as soon as  $L$  satisfies the following natural condition: for all  $x_1, \dots, x_n, y_1, \dots, y_n$  in  $\mathbb{R}^d$ ,  $L(\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}) \geq c \sum_i \text{dist}(x_i, \{y_1, \dots, y_n\})^p$ . To see this, one combines Proposition 5 and the lower estimate given in [16].

## 4 Upper bounds, upper limits

### 4.1 A general upper bound

**Lemma 16.** *Let  $d > 2p$  and let  $L$  be a bipartite functional satisfying  $(\mathcal{S}_p)$ ,  $(\mathcal{R}_p)$  and  $L(\emptyset, \emptyset) = 0$ . Then there exists a constant  $D$  such that, for all finite measures  $\nu$ ,*

$$\mathbb{E}L(\nu) \leq D \text{diam}(Q)^p \min(\nu(Q), \nu(Q)^{1-\frac{2}{d}}),$$

where  $Q$  contains the support of  $\nu$ .

*Proof.* Thanks to corollary 7, it is enough to deal with the case  $\nu(Q) \geq 2^d$  (or any other positive number). First note that we may assume that  $Q$  is a cube (given a set of diameter  $\Delta$ , one can find a cube containing it, with diameter no more than  $c$  times  $\Delta$  where  $c$  only depends on the norm). We consider a sequence

of dyadic partitions of  $Q$ ,  $(\mathcal{P}_\ell)_{\ell \geq 0}$ , where for  $\ell \in \mathbb{N}$ ,  $\mathcal{P}_\ell$  divides  $Q$  into  $2^{\ell d}$  cubes of side-length  $2^{-\ell}$  times the one of  $Q$ . Let  $k \in \mathbb{N}^*$  to be chosen later. By Corollary 13, we have the following estimate

$$\mathbb{E}L(\nu) \leq \sum_{P \in \mathcal{P}_k} \mathbb{E}L(\mathbb{1}_P \cdot \nu) + 3C \sum_{\ell=1}^k (2^{-\ell+1} \text{diam}(Q))^p \sum_{P \in \mathcal{P}_\ell} \sqrt{\nu(P)}. \quad (3)$$

Thanks to Corollary 7, the first term of the right-hand side of (3) is at most

$$\sum_{P \in \mathcal{P}_k} C (2^{-k} \text{diam}(Q))^p \nu(P) = C 2^{-kp} (\text{diam}(Q))^p \nu(Q).$$

By the Cauchy-Schwarz inequality

$$\sum_{P \in \mathcal{P}_\ell} \sqrt{\nu(P)} \leq (2^{\ell d})^{\frac{1}{2}} \left( \sum_{P \in \mathcal{P}_\ell} \nu(P) \right)^{\frac{1}{2}} = 2^{\frac{\ell d}{2}} \sqrt{\nu(Q)}.$$

Hence the second term of the right-hand side of (3) is at most

$$3C (2 \text{diam}(Q))^p \sum_{\ell=1}^k 2^{\ell(\frac{d}{2}-p)} \sqrt{\nu(Q)} \leq C' 2^{k(\frac{d}{2}-p)} (\text{diam}(Q))^p \sqrt{\nu(Q)}.$$

This leads to

$$\mathbb{E}L(\nu) \leq (\text{diam}(Q))^p \left( C 2^{-kp} \nu(Q) + C' 2^{k(\frac{d}{2}-p)} \sqrt{\nu(Q)} \right).$$

Choosing  $k = \lfloor \frac{1}{d} \log_2 \nu(Q) \rfloor \geq 1$  completes the proof.  $\square$

## 4.2 The upper limit for densities

**Theorem 17.** *Let  $d > 2p$ . Let  $L$  be a bipartite functional on  $\mathbb{R}^d$  satisfying the properties  $(\mathcal{H}_p)$ ,  $(\mathcal{R}_p)$ ,  $(\mathcal{S}_p)$ . Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$  be an integrable function with bounded support. Then*

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}L(nf)}{n^{1-\frac{p}{d}}} \leq \beta_L \int_{\mathbb{R}^d} f^{1-\frac{p}{d}},$$

where  $\beta_L$  is the constant appearing in Theorem 15.

*Proof.* By a scaling argument, we may assume that the support of  $f$  is included in  $[0, 1]^d$  and  $\int f = 1$  (the case  $\int f = 0$  is trivial). We consider a sequence of dyadic partitions  $(\mathcal{P}_\ell)_{\ell \in \mathbb{N}}$  of  $[0, 1]^d$ : for  $\ell \in \mathbb{N}$ ,  $\mathcal{P}_\ell$  divides  $[0, 1]^d$  into  $2^{\ell d}$  cubes of side-length  $2^{-\ell}$ . Let  $k \in \mathbb{N}^*$  to be chosen later. Corollary 13 gives

$$\mathbb{E}L(nf) \leq \sum_{P \in \mathcal{P}_k} \mathbb{E}L(nf \mathbb{1}_P) + 3C \sum_{\ell=1}^k (2^{-\ell+1} \text{diam}([0, 1]^d))^p \sum_{P \in \mathcal{P}_\ell} \sqrt{n \int_P f}. \quad (4)$$

By the Cauchy-Schwarz inequality

$$\sum_{P \in \mathcal{P}_\ell} \sqrt{\int_P f} \leq (2^{\ell d})^{\frac{1}{2}} \left( \sum_{P \in \mathcal{P}_\ell} \int_P f \right)^{\frac{1}{2}} = 2^{\frac{\ell d}{2}} \left( \int f \right)^{\frac{1}{2}} = 2^{\frac{\ell d}{2}}.$$

Hence the second term of the right-hand side of (4) is at most

$$3C (2 \text{diam}([0, 1]^d))^p \sqrt{n} \sum_{\ell=1}^k 2^{\ell(\frac{d}{2}-p)} \leq c_d n^{\frac{1}{2}} 2^{k(\frac{d}{2}-p)}.$$

Let  $\alpha_P$  be the average of  $f$  on  $P$ , then applying Corollary 9 to the first terms of (4) leads to

$$\mathbb{E}L(nf) \leq \sum_{P \in \mathcal{P}_k} \left( \mathbb{E}L(n \alpha_P \mathbb{1}_P) + 2C n \text{diam}(P)^p \int_P |f - \alpha_P| \right) + c_d n^{\frac{1}{2}} 2^{k(\frac{d}{2}-p)}.$$

Each  $P$  in the sum is a square of side length  $2^{-k}$ , hence using homogeneity (see Lemma 14)

$$\mathbb{E}L(nf) \leq \sum_{P \in \mathcal{P}_k} \left( 2^{-kp} M(n\alpha_P 2^{-kd}) + n c'_d 2^{-kp} \int_P |f - \alpha_P| \right) + c_d n^{\frac{1}{2}} 2^{k(\frac{d}{2}-p)}. \quad (5)$$

Let us recast this inequality with more convenient notation. We set  $g(t) = \bar{L}(t)/t^{1-p/d}$  and we define the piecewise constant function

$$f_k = \sum_{P \in \mathcal{P}_k} \alpha_P \mathbb{1}_P = \sum_{P \in \mathcal{P}_k} \frac{\int_P f(x) dx}{\text{Vol}(P)} \mathbb{1}_P.$$

It is plain that  $\int f_k = \int f < +\infty$ . Moreover, by Lebesgue's theorem,  $\lim_{k \rightarrow \infty} f_k = f$  holds for almost every point  $x$ . Inequality (5) amounts to

$$\begin{aligned} \frac{\mathbb{E}L(nf)}{n^{1-\frac{p}{d}}} &\leq \sum_{P \in \mathcal{P}_k} \left( g(n\alpha_P 2^{-kd}) \alpha_P^{1-\frac{p}{d}} 2^{-kd} + n^{\frac{p}{d}} c'_d 2^{-kp} \int_P |f - f_k| \right) + c_d n^{\frac{p}{d}-\frac{1}{2}} 2^{k(\frac{d}{2}-p)} \\ &= \sum_{P \in \mathcal{P}_k} \left( \int_P g(n f_k 2^{-kd}) f_k^{1-\frac{p}{d}} + n^{\frac{p}{d}} c'_d 2^{-kp} \int_P |f - f_k| \right) + c_d n^{\frac{p}{d}-\frac{1}{2}} 2^{k(\frac{d}{2}-p)} \\ &= \int g(n 2^{-kd} f_k) f_k^{1-\frac{p}{d}} + c'_d n^{\frac{p}{d}} 2^{-kp} \int |f - f_k| + c_d (n^{\frac{1}{d}} 2^{-k})^{p-\frac{d}{2}}. \end{aligned}$$

If there exists  $k_0$  such that  $f = f_{k_0}$  then we easily get the claim by setting  $k = k_0$  and letting  $n$  go to infinity (since  $g$  is bounded and converges to  $\beta_L$  at infinity, see Lemma 16 and Theorem 15). On the other hand, if  $f_k$  never coincides almost surely with  $f$ , we use a sequence of numbers  $k(n) \in \mathbb{N}$  such that

$$\lim_n k(n) = +\infty, \quad \lim_n n^{\frac{1}{d}} 2^{-k(n)} = +\infty \quad \text{and} \quad \lim_n n^{\frac{1}{d}} 2^{-k(n)} \left( \int |f - f_{k(n)}| \right)^{\frac{1}{p}} = 0. \quad (6)$$

Assuming its existence, the claim follows easily: applying the inequality for  $k = k(n)$  and taking upper limits gives

$$\limsup_n \frac{\mathbb{E}L(nf)}{n^{1-\frac{p}{d}}} \leq \limsup_n \int g(n 2^{-k(n)d} f_{k(n)}) f_{k(n)}^{1-\frac{p}{d}}.$$

Since  $\lim f_{k(n)} = f$  a.e., it is easy to see that the limit of the latter integral is  $\beta_L \int f^{1-\frac{p}{d}}$ : first the integrand converges almost everywhere to  $\beta_L f^{1-\frac{p}{d}}$  (if  $f(x) = 0$  this follows from the boundedness of  $g$ ; if  $f(x) \neq 0$  then the argument of  $g$  is going to infinity). Secondly, the sequence of integrands is supported on the unit cube and is uniformly integrable since

$$\int \left( g(n 2^{-k(n)d} f_{k(n)}) f_{k(n)}^{1-\frac{p}{d}} \right)^{\frac{d}{d-p}} \leq (\sup g)^{\frac{d}{d-p}} \int f_{k(n)} = (\sup g)^{\frac{d}{d-p}} \int f < +\infty.$$

It remains to establish the existence of a sequence of integers  $(k(n))_n$  satisfying (6). Note that since  $f_k \geq 0$ ,  $\int f_k = \int f = 1$  and a.e.  $\lim f_k = f$ , it follows from Scheffé's lemma that  $\lim_k \int |f - f_k| = 0$ . Hence  $\varphi(k) = (\sup_{j \geq k} \int |f - f_j|)^{-d/p}$  is non-decreasing with an infinite limit. We derive the existence of a sequence with the following stronger properties

$$\lim_n k(n) = +\infty, \quad \lim_n \frac{n}{(2^d)^{k(n)}} = +\infty \quad \text{and} \quad \lim_n \frac{n}{(2^d)^{k(n)} \varphi(k(n))} = 0 \quad (7)$$

as follows. Set  $\gamma = 2^d$ . Since  $\gamma^k \sqrt{\varphi(k-1)}$  is increasing with infinite limit

$$[\gamma \sqrt{\varphi(0)}, +\infty) = \cup_{k \geq 1} [\gamma^k \sqrt{\varphi(k-1)}, \gamma^{k+1} \sqrt{\varphi(k)}).$$

For  $n \geq \gamma \sqrt{\varphi(0)}$ , we define  $k(n)$  as the integer such that

$$\gamma^{k(n)} \sqrt{\varphi(k(n)-1)} \leq n < \gamma^{k(n)+1} \sqrt{\varphi(k(n))}.$$

This defines a non-decreasing sequence. It is clear from the above strict inequality that  $\lim_n k(n) = +\infty$ . Hence  $n \gamma^{-k(n)} \geq \sqrt{\varphi(k(n)-1)}$  tends to infinity at infinity. Eventually  $n / (\gamma^{k(n)} \varphi(k(n))) \leq \gamma / \sqrt{\varphi(k(n))}$  tends to zero as required. The proof is therefore complete.  $\square$

### 4.3 Purely singular measures

**Lemma 18.** *Let  $d > 2p$ . Let  $L$  be a bipartite functional on  $\mathbb{R}^d$  with properties  $(\mathcal{R}_p)$  and  $(\mathcal{S}_p)$ . Let  $\mu$  be a finite singular measure on  $\mathbb{R}^d$  having a bounded support. Then*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}L(n\mu)}{n^{1-\frac{d}{2}}} = 0.$$

*Proof.* Let  $Q$  be a cube which contains the support of  $\mu$ . We consider a sequence of dyadic partitions of  $Q$ ,  $(\mathcal{P}_\ell)_{\ell \in \mathbb{N}}$ . For  $\ell \in \mathbb{N}$ ,  $\mathcal{P}_\ell$  divides  $Q$  into  $2^{\ell d}$  cubes of side length  $2^{-\ell}$  times the one of  $Q$ . As in the proof of Lemma 16, a direct application of Corollary 13 gives for  $k \in \mathbb{N}^*$ :

$$\mathbb{E}L(n\mu) \leq \sum_{P \in \mathcal{P}_k} \mathbb{E}L(n\mathbb{1}_P \cdot \mu) + 3C \sum_{\ell=1}^k (2^{-\ell+1} \text{diam}(Q))^p \sum_{P \in \mathcal{P}_\ell} \sqrt{n\mu(P)}. \quad (8)$$

The terms of the first sum are estimated again thanks to the easy bound of Corollary 7: since each  $P$  in  $\mathcal{P}_k$  is a cube of side length  $2^{-k}$  times the one of  $Q$ , it holds

$$\sum_{P \in \mathcal{P}_k} \mathbb{E}L(n\mathbb{1}_P \cdot \mu) \leq \sum_{P \in \mathcal{P}_k} C(2^{-k} \text{diam}(Q))^p n\mu(P) = c_{p,Q} 2^{-kp} n|\mu|.$$

Here  $|\mu|$  is the total mass of  $\mu$ . We rewrite the second term in (8) in terms of the function

$$g_\ell = \sum_{P \in \mathcal{P}_\ell} \frac{\mu(P)}{\lambda(P)} \mathbb{1}_P,$$

where  $\lambda$  stands for Lebesgue's measure. Since  $\lambda(P) = 2^{-\ell d} \lambda(Q)$ , we get that

$$\begin{aligned} \mathbb{E}L(n\mu) &\leq c_{p,Q} 2^{-kp} n|\mu| + 3C (2 \text{diam}(Q))^p \sqrt{n} \sum_{\ell=1}^k 2^{-\ell p} \sum_{P \in \mathcal{P}_\ell} 2^{\frac{\ell d}{2}} \lambda(Q)^{-\frac{1}{2}} \lambda(P) \sqrt{\frac{\mu(P)}{\lambda(P)}} \\ &= c_{p,Q} 2^{-kp} n|\mu| + 3C (2 \text{diam}(Q))^p \lambda(Q)^{-\frac{1}{2}} \sqrt{n} \sum_{\ell=1}^k 2^{\ell(\frac{d}{2}-p)} \int \sqrt{g_\ell}. \end{aligned}$$

By the differentiability theorem, for Lebesgue-almost every  $x$ ,  $g_\ell(x)$  tends to zero when  $\ell$  tends to infinity (since  $\mu$  is singular with respect to Lebesgue's measure). Moreover,  $g_\ell$  is supported on the unit cube and  $\int (\sqrt{g_\ell})^2 = \int g_\ell = |\mu| < +\infty$ . Hence the sequence of functions  $\sqrt{g_\ell}$  is uniformly integrable and we can conclude that  $\lim_{\ell \rightarrow \infty} \int \sqrt{g_\ell} = 0$ . By Cesaro's theorem, the sequence

$$\varepsilon_k = \frac{\sum_{\ell=1}^k 2^{\ell(\frac{d}{2}-p)} \int \sqrt{g_\ell}}{\sum_{\ell=1}^k 2^{\ell(\frac{d}{2}-p)}}$$

also converges to zero, using here that  $d > 2p$ . By an obvious upper bound of the latter denominator, we obtain that there exists a number  $c$  which does not depend on  $(k, n)$  (but depends on  $C, p, d, Q, |\mu|$ ) such that for all  $k \geq 1$

$$\mathbb{E}L(n\mu) \leq c \left( n 2^{-kp} + \sqrt{n} 2^{k(\frac{d}{2}-p)} \varepsilon_k \right),$$

where  $\varepsilon_k \geq 0$  and  $\lim_k \varepsilon_k = 0$ . We may also assume that  $(\varepsilon_k)$  is non-increasing (the inequality remains valid if one replaces  $\varepsilon_k$  by  $\sup_{j \geq k} \varepsilon_j$ ). It remains to choose  $k$  in terms of  $n$  in a proper way. Define

$$\varphi(n) = \sqrt{\varepsilon_{\lfloor \frac{1}{d} \log_2 n \rfloor}} \frac{-1}{2^{-p}}.$$

Obviously  $\lim_n \varphi(n) = +\infty$ . For  $n$  large enough, define  $k(n) \geq 1$  as the unique integer such that

$$2^{k(n)} \leq n^{\frac{1}{d}} \varphi(n) < 2^{k(n)+1}.$$

Setting  $k = k(n)$ , our estimate on the cost of the optimal matching yields

$$\frac{\mathbb{E}L(n\mu)}{n^{1-\frac{p}{d}}} \leq c(d) \left( \frac{2}{\varphi(n)^p} + \varepsilon_{k(n)} \varphi(n)^{\frac{d}{2}-p} \right).$$

It is easy to check that the right hand side tends to zero as  $n$  tends to infinity. Indeed,  $\lim_n \varphi(n) = +\infty$ , hence for  $n$  large enough

$$k(n) \geq \left\lfloor \log_2 \left( n^{\frac{1}{d}} \varphi(n) / 2 \right) \right\rfloor \geq \left\lfloor \frac{1}{d} \log_2 n \right\rfloor.$$

Since the sequence  $(\varepsilon_k)$  is non-increasing, it follows that

$$\varepsilon_{k(n)} \varphi(n)^{\frac{d}{2}-p} \leq \varepsilon_{\lfloor \frac{1}{d} \log_2 n \rfloor} \varphi(n)^{\frac{d}{2}-p} = \sqrt{\varepsilon_{\lfloor \frac{1}{d} \log_2 n \rfloor}}$$

tends to zero when  $n \rightarrow \infty$ . The proof is therefore complete.  $\square$

#### 4.4 General upper limits

The first statement of Theorem 3 is a consequence of Propositions 5, 10, and the following result.

**Theorem 19.** *Let  $d > 2p > 0$ . Let  $L$  be a bipartite functional on  $\mathbb{R}^d$  with the properties  $(\mathcal{H}_p)$ ,  $(\mathcal{R}_p)$  and  $(\mathcal{S}_p)$ . Consider a finite measure  $\mu$  on  $\mathbb{R}^d$  such that there exists  $\alpha > \frac{2dp}{d-2p}$  with*

$$\int |x|^\alpha d\mu(x) < +\infty.$$

Let  $f$  be a density function for the absolutely continuous part of  $\mu$ , then

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}L(n\mu)}{n^{1-\frac{p}{d}}} \leq \beta_L \int f^{1-\frac{p}{d}}. \quad (9)$$

*Remark 5.* Observe that the hypotheses ensure the finiteness of  $\int f^{1-\frac{p}{d}}$ . Indeed Hölder's inequality gives

$$\int_{\mathbb{R}^d} f^{1-\frac{p}{d}} \leq \left( \int_{\mathbb{R}^d} (1 + |x|^\alpha) f(x) dx \right)^{1-\frac{p}{d}} \left( \int_{\mathbb{R}^d} (1 + |x|^\alpha)^{1-\frac{d}{p}} \right)^{\frac{p}{d}}$$

where the latter integral converges since  $\alpha > \frac{2dp}{d-2p} > \frac{dp}{d-p}$ .

*Proof.* Assume first that  $\mu$  has a bounded support. Write  $\mu = \mu_{ac} + \mu_s$  where  $\mu_s$  is the singular part and  $d\mu_{ac}(x) = f(x) dx$ . Applying Proposition 11 to  $\mu_{ac}$  and  $\mu_s$ , dividing by  $n^{1-p/d}$ , passing to the limit and using Theorem 17 and Lemma 18 gives

$$\limsup_n \frac{\mathbb{E}L(n\mu)}{n^{1-\frac{p}{d}}} \leq \limsup_n \frac{\mathbb{E}L(n\mu_{ac})}{n^{1-\frac{p}{d}}} + \limsup_n \frac{\mathbb{E}L(n\mu_s)}{n^{1-\frac{p}{d}}} \leq \beta_L \int f^{1-\frac{p}{d}}.$$

Hence the theorem is established for measures with bounded supports.

Now, let us consider the general case. Let  $B(t) = \{x \in \mathbb{R}^d : |x| \leq t\}$ . Let  $A_0 = B(2)$  and for integer  $\ell \geq 1$ ,  $A_\ell = B(2^{\ell+1}) \setminus B(2^\ell)$ . Now, let  $\mathcal{X} = \{X_1, \dots, X_{N_1}\}$ ,  $\mathcal{Y} = \{Y_1, \dots, Y_{N_2}\}$  be two independent Poisson process of intensity  $n\mu$ , and  $T = \max\{|Z| : Z \in \mathcal{X} \cup \mathcal{Y}\}$ . Applying the subadditivity property like in the proof of Proposition 12, we obtain

$$L(\mathcal{X}, \mathcal{Y}) \leq \sum_{\ell \geq 0} L(\mathcal{X} \cap A_\ell, \mathcal{Y} \cap A_\ell) + CT^p \sum_{\ell \geq 0} \mathbb{1}_{\mathcal{X}(A_\ell) + \mathcal{Y}(A_\ell) \neq 0} (1 + |\mathcal{X}(A_\ell) - \mathcal{Y}(A_\ell)|). \quad (10)$$

Note that the above sums have only finitely many non-zero terms, since  $\mu$  is finite. We first deal with the first sum in the above inequality. By Fubini's Theorem,

$$\mathbb{E} \sum_{\ell \geq 0} \frac{L(\mathcal{X} \cap A_\ell, \mathcal{Y} \cap A_\ell)}{n^{1-\frac{p}{d}}} = \sum_{\ell \geq 0} \mathbb{E} \frac{L(\mathcal{X} \cap A_\ell, \mathcal{Y} \cap A_\ell)}{n^{1-\frac{p}{d}}}.$$

Applying (9) to the compactly supported measure  $\mu|_{A_\ell}$  for every integer  $\ell$  gives

$$\limsup_n \mathbb{E} \frac{L(\mathcal{X} \cap A_\ell, \mathcal{Y} \cap A_\ell)}{n^{1-\frac{p}{d}}} \leq \beta_L \int_{A_\ell} f^{1-\frac{p}{d}}. \quad (11)$$

By Lemma 16, for some constant  $c_d$ ,

$$\mathbb{E} \frac{L(\mathcal{X} \cap A_\ell, \mathcal{Y} \cap A_\ell)}{n^{1-\frac{p}{d}}} \leq c_d 2^{\ell p} \mu(A_\ell)^{1-\frac{p}{d}}.$$

From Markov inequality, with  $m_\alpha = \int |x|^\alpha d\mu(x)$ ,

$$\mu(A_\ell) \leq \mu(\mathbb{R}^d \setminus B(2^\ell)) \leq 2^{-\ell\alpha} m_\alpha.$$

Thus, since  $\alpha > 2pd/(d-2p) > dp/(d-p)$ , the series  $\sum_\ell 2^{\ell p} \mu(A_\ell)^{1-\frac{p}{d}}$  is convergent. We may then apply the dominated convergence theorem, we get from (11) that

$$\limsup_n \mathbb{E} \sum_{\ell \geq 0} \frac{L(\mathcal{X} \cap A_\ell, \mathcal{Y} \cap A_\ell)}{n^{1-\frac{p}{d}}} \leq \beta_L \int f^{1-\frac{p}{d}}.$$

For the expectation of the second term on the right hand side of (10), we use Cauchy-Schwartz inequality,

$$\begin{aligned} & \mathbb{E} \left[ T^p \sum_{\ell \geq 0} \mathbb{1}_{\mathcal{X}(A_\ell) + \mathcal{Y}(A_\ell) \neq 0} (1 + |\mathcal{X}(A_\ell) - \mathcal{Y}(A_\ell)|) \right] \\ & \leq \sum_{\ell \geq 0} \sqrt{\mathbb{E}[T^{2p}]} \sqrt{\mathbb{E}(\mathbb{1}_{\mathcal{X}(A_\ell) + \mathcal{Y}(A_\ell) \neq 0} (1 + |\mathcal{X}(A_\ell) - \mathcal{Y}(A_\ell)|)^2)} \\ & \leq \sqrt{2} \sqrt{\mathbb{E}[T^{2p}]} \sum_{\ell \geq 0} \sqrt{\mathbb{P}(\mathcal{X}(A_\ell) + \mathcal{Y}(A_\ell) \neq 0) + \mathbb{E}[|\mathcal{X}(A_\ell) - \mathcal{Y}(A_\ell)|^2]} \\ & = \sqrt{2} \sqrt{\mathbb{E}[T^{2p}]} \sum_{\ell \geq 0} \sqrt{1 - e^{-2n\mu(A_\ell)} + 2n\mu(A_\ell)} \\ & \leq 2\sqrt{\mathbb{E}[T^{2p}]} \sqrt{n} \sum_{\ell \geq 0} \sqrt{\mu(A_\ell)}, \end{aligned}$$

where we have used  $1 - e^{-u} \leq u$ . As above, Markov inequality leads to

$$\sum_{\ell \geq 0} \sqrt{\mu(A_\ell)} \leq \sqrt{m_\alpha} \sum_{\ell \geq 0} 2^{-\ell \frac{\alpha}{2}} < +\infty.$$

Eventually we apply Lemma 4 with  $\gamma := 2p < 2pd/(d-2) < \alpha$  to upper bound  $\mathbb{E}[T^{2p}]$ . We get that for some constant  $c > 0$  and all  $n > 0$ ,

$$n^{-1+\frac{p}{d}} \mathbb{E} \left[ T^p \sum_{\ell \geq 0} \mathbb{1}_{\mathcal{X}(A_\ell) + \mathcal{Y}(A_\ell) \neq 0} (1 + |\mathcal{X}(A_\ell) - \mathcal{Y}(A_\ell)|) \right] \leq cn^{-\frac{1}{2} + \frac{p}{d} + \frac{p}{\alpha}}.$$

Since  $\alpha > 2dp/(d-2p)$ , the later and former terms tend to zero as  $n$  tends to infinity. The upper bound (9) is proved.  $\square$

## 5 Examples of bipartite functionals

The minimal bipartite matching is an instance of a bipartite Euclidean functional  $M_1(\mathcal{X}, \mathcal{Y})$  over the multisets  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_n\}$ . We may mention at least two other interesting examples: the bipartite traveling salesperson problem over  $\mathcal{X}$  and  $\mathcal{Y}$  is the shortest cycle on the multiset  $\mathcal{X} \cup \mathcal{Y}$  such that the image of  $\mathcal{X}$  is  $\mathcal{Y}$ . Similarly, the bipartite minimal spanning tree is the minimal edge-length spanning tree on  $\mathcal{X} \cup \mathcal{Y}$  with no edge between two elements of  $\mathcal{X}$  or two elements of  $\mathcal{Y}$ .

## 5.1 Minimal bipartite matching

Fix  $p > 0$ . Given two multi-subsets of  $\mathbb{R}^d$  with the same cardinality,  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_n\}$ , the  $p$ -cost of the minimal bipartite matching of  $\mathcal{X}$  and  $\mathcal{Y}$  is defined as

$$M_p(\mathcal{X}, \mathcal{Y}) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^p,$$

where the minimum runs over all permutations of  $\{1, \dots, n\}$ . It is useful to extend the definition to sets of different cardinalities, by matching as many points as possible: if  $\mathcal{X} = \{X_1, \dots, X_m\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_n\}$  and  $m \leq n$  then

$$M_p(\mathcal{X}, \mathcal{Y}) = \min_{\sigma} \sum_{i=1}^m |X_i - Y_{\sigma(i)}|^p,$$

where the minimum runs over all injective maps from  $\{1, \dots, m\}$  to  $\{1, \dots, n\}$ . When  $n \leq m$  the symmetric definition is chosen  $M_p(\mathcal{X}, \mathcal{Y}) := M_p(\mathcal{Y}, \mathcal{X})$ .

The bipartite functional  $M_p$  is obviously homogeneous of degree  $p$ , i.e. it satisfies  $(\mathcal{H}_p)$ . The next lemma asserts that it also verifies the subadditivity property  $(\mathcal{S}_p)$ . In the case  $p = 1$ , this is the starting point of the paper [3].

**Lemma 20.** *For any  $p > 0$ , the functional  $M_p$  satisfies property  $(\mathcal{S}_p)$  with constant  $C = 1/2$ . More precisely, if  $\mathcal{X}_1, \dots, \mathcal{X}_k$  and  $\mathcal{Y}_1, \dots, \mathcal{Y}_k$  are multisets in a bounded subset  $Q \subset \mathbb{R}^d$ , then*

$$M_p\left(\bigcup_{i=1}^k \mathcal{X}_i, \bigcup_{i=1}^k \mathcal{Y}_i\right) \leq \sum_{i=1}^k M_p(\mathcal{X}_i, \mathcal{Y}_i) + \frac{\text{diam}(Q)^p}{2} \sum_{i=1}^k |\text{card}(\mathcal{X}_i) - \text{card}(\mathcal{Y}_i)|.$$

*Proof.* It is enough to upper estimate of the cost of a particular matching of  $\bigcup_{i=1}^k \mathcal{X}_i$  and  $\bigcup_{i=1}^k \mathcal{Y}_i$ . We build a matching of these multisets as follows. For each  $i$  we choose the optimal matching of  $\mathcal{X}_i$  and  $\mathcal{Y}_i$ . The overall cost is  $\sum_i M_p(\mathcal{X}_i, \mathcal{Y}_i)$ , but we have left  $\sum_i |\text{card}(\mathcal{X}_i) - \text{card}(\mathcal{Y}_i)|$  points unmatched (the number of excess points). Among these points, the less numerous species (there are two species: points from  $\mathcal{X}_i$ 's, and points from  $\mathcal{Y}_i$ 's) has cardinality at most  $\frac{1}{2} \sum_i |\text{card}(\mathcal{X}_i) - \text{card}(\mathcal{Y}_i)|$ . To complete the definition of the matching, we have to match all the points of this species in the minority. We do this in an arbitrary manner and simply upper bound the distance between matched points by the diameter of  $Q$ .  $\square$

The regularity property is established next.

**Lemma 21.** *For any  $p > 0$ , the functional  $M_p$  satisfies property  $(\mathcal{R}_p)$  with constant  $C = 1$ .*

*Proof.* Let  $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, \mathcal{Y}_1, \mathcal{Y}_2$  be finite multisets contained in  $Q = B(1/2)$ . Denote by  $x, x_1, x_2, y, y_1, y_2$  the cardinalities of the multisets and  $a \wedge b$  for  $\min(a, b)$ . We start with an optimal matching for  $M_p(\mathcal{X} \cap \mathcal{X}_2, \mathcal{Y} \cap \mathcal{Y}_2)$ . It comprises  $(x + x_2) \wedge (y + y_2)$  edges. We remove the ones which have a vertex in  $\mathcal{X}_2$  or in  $\mathcal{Y}_2$ . There are at most  $x_2 + y_2$  of them, so we are left with at least  $((x + x_2) \wedge (y + y_2) - x_2 - y_2)_+$  edges connecting points of  $\mathcal{X}$  to points of  $\mathcal{Y}$ . We want to use this partial matching in order to build a (suboptimal) matching of  $\mathcal{X} \cap \mathcal{X}_1$  and  $\mathcal{Y} \cap \mathcal{Y}_1$ . This requires to have globally  $(x + x_1) \wedge (y + y_1)$  edges. Hence we need to add at most

$$(x + x_1) \wedge (y + y_1) - ((x + x_2) \wedge (y + y_2) - x_2 - y_2)_+$$

new edges. We do this in an arbitrary way, and simply upper bound their length by the diameter of  $Q$ . To prove the claim it is therefore sufficient to prove the following inequalities for non-negative numbers:

$$(x + x_1) \wedge (y + y_1) - ((x + x_2) \wedge (y + y_2) - x_2 - y_2)_+ \leq x_1 + x_2 + y_1 + y_2. \quad (12)$$

This is obviously equivalent to

$$\begin{aligned} x + x_1 &\leq x_1 + x_2 + y_1 + y_2 + ((x + x_2) \wedge (y + y_2) - x_2 - y_2)_+ \\ \text{or } y + y_1 &\leq x_1 + x_2 + y_1 + y_2 + ((x + x_2) \wedge (y + y_2) - x_2 - y_2)_+. \end{aligned}$$

After simplification, and noting that  $y_1 \geq 0$  appears only on the right-hand side of the first inequation (and the same for  $x_1$  in the second one), it is enough to show that

$$x \wedge y \leq x_2 + y_2 + ((x + x_2) \wedge (y + y_2) - x_2 - y_2)_+.$$

This is obvious, as by definition of the positive part,  $x \wedge y \leq x_2 + y_2 + ((x \wedge y) - x_2 - y_2)_+$ .  $\square$

## 5.2 Bipartite traveling salesperson tour

Fix  $p > 0$ . Given two multi-subsets of  $\mathbb{R}^d$  with the same cardinality,  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_n\}$ , the  $p$ -cost of the minimal bipartite traveling salesperson tour of  $(\mathcal{X}, \mathcal{Y})$  is defined as

$$T_p(\mathcal{X}, \mathcal{Y}) = \min_{(\sigma, \sigma') \in S_n^2} \sum_{i=1}^n |X_{\sigma(i)} - Y_{\sigma'(i)}|^p + \sum_{i=1}^{n-1} |Y_{\sigma'(i)} - X_{\sigma(i+1)}|^p + |Y_{\sigma'(n)} - X_{\sigma(1)}|^p,$$

where the minimum runs over all pairs of permutations of  $\{1, \dots, n\}$ . We extend the definition to sets of different cardinalities, by completing the longest possible bipartite tour : if  $\mathcal{X} = \{X_1, \dots, X_m\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_n\}$  and  $m \leq n$  then

$$T_p(\mathcal{X}, \mathcal{Y}) = \min_{(\sigma, \sigma')} \sum_{i=1}^m |X_{\sigma(i)} - Y_{\sigma'(i)}|^p + \sum_{i=1}^{m-1} |Y_{\sigma'(i)} - X_{\sigma(i+1)}|^p + |Y_{\sigma'(m)} - X_{\sigma(1)}|^p$$

where the minimum runs over all pairs  $(\sigma, \sigma')$ , with  $\sigma \in S_m$  and  $\sigma'$  is an injective maps from  $\{1, \dots, m\}$  to  $\{1, \dots, n\}$ . When  $n \leq m$  the symmetric definition is chosen  $T_p(\mathcal{X}, \mathcal{Y}) := T_p(\mathcal{Y}, \mathcal{X})$ . This traveling salesperson functional is an instance of a larger class of functionals that we now describe.

## 5.3 Euclidean combinatorial optimization over bipartite graphs

For integers  $m, n$ , we define  $[n] = \{1, \dots, n\}$  and  $[n]_m = \{m+1, \dots, m+n\}$ . Let  $\mathcal{B}_n$  be the set of bipartite graphs with common vertex set  $([n], [n]_n)$  : if  $G \in \mathcal{B}_n$ , the edge set of  $G$  is contained in the set of pairs  $\{i, n+j\}$ , with  $i, j \in [n]$ .

We should introduce some graph definitions. If  $G_1 \in \mathcal{B}_n$  and  $G_2 \in \mathcal{B}_m$  we define  $G_1 + G_2$  as the graph in  $\mathcal{B}_{n+m}$  obtained by the following rule : if  $\{i, n+j\}$  is an edge of  $G_1$  then  $\{i, n+m+j\}$  is an edge of  $G_1 + G_2$ , and if  $\{i, m+j\}$  is an edge of  $G_2$  then  $\{n+i, 2n+m+j\}$  is an edge of  $G_1 + G_2$ . Finally, if  $G \in \mathcal{B}_{n+m}$ , the restriction  $G'$  of  $G$  to  $\mathcal{B}_n$  is the element of  $\mathcal{B}_n$  defined by the following construction rule: if  $\{i, n+m+j\}$  is an edge of  $G$  and  $(i, j) \in [n]^2$  then add  $\{i, n+j\}$  as an edge of  $G'$ .

We consider a collection of subsets  $\mathcal{G}_n \subset \mathcal{B}_n$  with the following properties, there exist constants  $\kappa_0, \kappa \geq 1$  such that for all integers  $n, m$ ,

- (A1) (*not empty*) If  $n \geq \kappa_0$ ,  $\mathcal{G}_n$  is not empty.
- (A2) (*isomorphism*) If  $G \in \mathcal{G}_n$  and  $G' \in \mathcal{B}_n$  is isomorphic to  $G$  then  $G' \in \mathcal{G}_n$ .
- (A3) (*bounded degree*) If  $G \in \mathcal{G}_n$ , the degree of any vertex is at most  $\kappa$ .
- (A4) (*merging*) If  $G \in \mathcal{G}_n$  and  $G' \in \mathcal{G}_m$ , there exists  $G'' \in \mathcal{G}_{n+m}$  such that  $G + G'$  and  $G''$  have all but at most  $\kappa$  edges in common. For  $1 \leq m < \kappa_0$ , it also holds if  $G'$  is the empty graph of  $\mathcal{B}_m$ .
- (A5) (*restriction*) Let  $G \in \mathcal{G}_n$  and  $\kappa_0 + 1 \leq n$  and  $G'$  be the restriction of  $G$  to  $\mathcal{B}_{n-1}$ . Then there exists  $G'' \in \mathcal{G}_{n-1}$  such that  $G'$  and  $G''$  have all but at most  $\kappa$  edges in common.

If  $|\mathcal{X}| = |\mathcal{Y}| = n$ , we define

$$L(\mathcal{X}, \mathcal{Y}) = \min_{G \in \mathcal{G}_n} \sum_{(i,j) \in [n]^2: \{i, n+j\} \in G} |X_i - Y_j|^p.$$

With the convention that the minimum over an empty set is 0. Note that the isomorphism property implies that  $L(\mathcal{X}, \mathcal{Y}) = L(\mathcal{Y}, \mathcal{X})$ . If  $m = |\mathcal{X}| \leq |\mathcal{Y}| = n$ , we define

$$L(\mathcal{X}, \mathcal{Y}) = \min_{(G, \sigma)} \sum_{(i,j) \in [m]^2: \{i, m+j\} \in G} |X_i - Y_{\sigma(j)}|^p, \quad (13)$$

where the minimum runs over all pairs  $(G, \sigma)$ ,  $G \in \mathcal{G}_m$  and  $\sigma$  is an injective maps from  $\{1, \dots, m\}$  to  $\{1, \dots, n\}$ . When  $n \leq m$  the symmetric definition is chosen  $L(\mathcal{X}, \mathcal{Y}) := L(\mathcal{Y}, \mathcal{X})$ .

The case of bipartite matchings is recovered by choosing  $\mathcal{G}_n$  as the set of graphs in  $\mathcal{B}_n$  where all vertices have degree 1. We then have  $\kappa_0 = 1$  and  $\mathcal{G}_n$  satisfies the merging property with  $\kappa = 0$ . It also satisfies the restriction property with  $\kappa = 1$ . The case of the traveling salesperson tour is obtained by choosing  $\mathcal{G}_n$  as the set of connected graphs in  $\mathcal{B}_n$  where all vertices have degree 2, this set is non-empty for  $n \geq \kappa_0 = 2$ . Also this set  $\mathcal{G}_n$  satisfies the merging property with  $\kappa = 4$  (as can be checked by edge switching). The restriction property follows by merging strings into a cycle.

For the minimal bipartite spanning tree, we choose  $\mathcal{G}_n$  as the set of connected trees of  $[2n]$  in  $\mathcal{B}_n$ . It satisfies the restriction property and the merging property with  $\kappa = 1$ . For this choice, however, the maximal degree is not bounded uniformly in  $n$ . We could impose artificially this condition by defining  $\mathcal{G}_n$  as the set of connected graphs in  $\mathcal{B}_n$  with maximal degree bounded by  $\kappa \geq 2$ . We would then get the minimal bipartite spanning tree with maximal degree bounded by  $\kappa$ . It is not hard to verify that the corresponding functional satisfies all the above properties.

Another interesting example is the following. Fix an integer  $r \geq 2$ . Recall that a graph is  $r$ -regular if the degree of all its vertices is equal to  $r$ . We may define  $\mathcal{G}_n$  as the set of  $r$ -regular connected graphs in  $\mathcal{B}_n$ . This set is not empty for  $n \geq \kappa_0 = r$ . It satisfies the first part of the merging property (A4) with  $\kappa = 4$ . Indeed, consider two  $r$ -regular graphs  $G, G'$ , and take any edge  $e = \{x, y\} \in G$  and  $e' = \{x', y'\} \in G'$ . The merging property holds with  $G''$ , the graph obtained from  $G + G'$  by switching  $(e, e')$  in  $(\{x, y\}, \{x', y'\})$ . Up to increasing the value of  $\kappa$ , the second part of the merging property is also satisfied. Indeed, if  $n$  is large enough, it is possible to find  $rm < r\kappa_0 = r^2$  edges  $e_1, \dots, e_{rm}$  in  $G$  with no-adjacent vertices. Now, in  $G''$ , we add  $m$  points from each species, and replace the edge  $e_{ri+q} = \{x, n+y\}$ ,  $1 \leq i \leq m$ ,  $0 \leq q < r$ , by two edges : one between  $x$  and the  $i$ -th point of the second species, and one between  $y$  and the  $i$ -th point of the first species.  $G''$  is then a connected  $r$ -regular graph in  $\mathcal{B}_{n+m}$  with all but at most  $2r^2$  edges in common with  $G$ . Hence, by taking  $\kappa$  large enough, the second part of the merging property holds.

Checking the restriction property (A5) for  $r$ -regular graphs requires a little more care. Let  $r = \kappa_0 + 1 \leq n$  and consider the restriction  $G_1$  of  $G \in \mathcal{B}_n$  to  $\mathcal{B}_{n-1}$ . Our goal is to show that by modifying a small number of edges of  $G_1$ , one can obtain a connected  $r$ -regular bipartite graph on  $\mathcal{B}_{n-1}$ . We first explain how to turn  $G_1$  into a possibly non-connected  $r$ -regular graph. Let us observe that  $G_1$  was obtained from  $G$  by deleting one vertex of each species and the edges to which these points belong. Hence  $G_1$  has vertices of degree  $r$ , and vertices of degree  $r-1$  ( $r$  blue and  $r$  red vertices if the removed points did not share an edge, only  $r-1$  points of each species if the removed points shared an edge). In any case  $G_1$  has at most  $2r$  connected components and  $r$  vertices of each color with one edge missing. The simplest way to turn  $G_1$  into a  $r$  regular graph is to connect each blue vertex missing an edge with a red vertex missing an edge. However this is not always possible as these vertices may already be neighbours in  $G_1$  and we do not allow multiple edges. However given a red vertex  $v_R$  and a blue vertex  $v_B$  of degree  $r-1$  and provided  $n-1 > 2r^2$  there exists a vertex  $v$  in  $G_1$  which is at graph distance at least 3 from  $v_B$  and  $v_R$ . Then open up an edge to which  $v$  belongs and connect its end-points to  $v_R$  and  $v_B$  while respecting the bipartite structure. In the new graph  $v_B$  and  $v_R$  have degree  $r$ . Repeating this operation no more than  $r$  times turns  $G_1$  into a  $r$  regular graphs with at most as many connected components (and the initial and the final graph differ by at most  $3r$  edges). Next we apply the merge operation at most  $2r-1$  times in order to glue together the connected components (this leads to modifying at most  $4(2r-1)$  edges. As a conclusion, provided we choose  $\kappa_0 > 2r^2$ , the restriction property holds for  $\kappa = 11r$ .

We now come back to the general case. From the definition, it is clear that  $L$  satisfies the property  $(\mathcal{H}_p)$ . We are going to check that it also satisfies properties  $(\mathcal{S}_p)$  and  $(\mathcal{R}_p)$ .

**Lemma 22.** *Assume (A1-A4). For any  $p > 0$ , the functional  $L$  satisfies property  $(\mathcal{S}_p)$  with constant  $C = (3 + \kappa_0)\kappa/2$ .*

*Proof.* The proof of is an extension of the proof of Lemma 20. We can assume without loss of generality  $k \geq 2$ . Let  $\mathcal{X}_1, \dots, \mathcal{X}_k$  and  $\mathcal{Y}_1, \dots, \mathcal{Y}_k$  be multisets in  $Q = B(1/2)$ . For ease of notation, let  $x_i = |\mathcal{X}_i|$ ,

$y_i = |\mathcal{Y}_i|$  and  $n = \sum_{i=1}^k x_i \wedge \sum_{i=1}^k y_i$ . If  $n < \kappa_0$ , then from the bounded degree property (A3),

$$L\left(\bigcup_{i=1}^k \mathcal{X}_i, \bigcup_{i=1}^k \mathcal{Y}_i\right) \leq n\kappa \leq \kappa\kappa_0.$$

If  $n \geq \kappa_0$ , it is enough to upper bound the cost for an element  $G$  in  $\mathcal{G}_n$ . For each  $1 \leq i \leq k$ , if  $n_i = x_i \wedge y_i \geq \kappa_0$ , we consider the element  $G_i$  in  $\mathcal{G}_{n_i}$  which reaches the minimum cost of  $L(\mathcal{X}_i, \mathcal{Y}_i)$ . From the merging property (A4), there exists  $G'$  in  $\mathcal{G}_{\sum_i \mathbf{1}_{n_i \geq \kappa_0} n_i}$  whose total cost is at most

$$L' := \sum_i L(\mathcal{X}_i, \mathcal{Y}_i) + \kappa k.$$

It remains at most  $\sum_i \kappa_0 + |x_i - y_i|$  vertices that have been left aside. The less numerous species has cardinal  $m_0 \leq m = (\sum_i \kappa_0 + |x_i - y_i|)/2$ . If  $m_0 \geq \kappa_0$ , from the non-empty property (A1), there exists a graph  $G'' \in \mathcal{G}_{m_0}$  that minimizes the cost of the vertices that have been left aside. From the merging and bounded degree properties, we get

$$L\left(\bigcup_{i=1}^k \mathcal{X}_i, \bigcup_{i=1}^k \mathcal{Y}_i\right) \leq L' + \kappa + \kappa m \leq \sum_i L(\mathcal{X}_i, \mathcal{Y}_i) + \frac{\kappa}{2} \sum_i (3 + \kappa_0 + |x_i - y_i|).$$

If  $m_0 < \kappa_0$ , we apply to  $G'$  the merging property with the empty graph : there exists an element  $G$  in  $\mathcal{G}_n$  whose total cost is at most

$$L\left(\bigcup_{i=1}^k \mathcal{X}_i, \bigcup_{i=1}^k \mathcal{Y}_i\right) \leq L' + \kappa \leq \sum_i L(\mathcal{X}_i, \mathcal{Y}_i) + (k+1)\kappa.$$

We have proved that property  $(\mathcal{S}_p)$  is satisfied for  $C = (3 + \kappa_0)\kappa/2$ .  $\square$

**Lemma 23.** *Assume (A1-A5). For any  $p > 0$ , the functional  $L$  satisfies property  $(\mathcal{R}_p)$  with constant  $C = C(\kappa, \kappa_0)$ .*

*Proof.* Let  $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, \mathcal{Y}_1, \mathcal{Y}_2$  be finite multisets contained in  $B(1/2) = Q$ . Denote by  $x, x_1, x_2, y, y_1, y_2$  the cardinalities of the multisets. As a first step, let us prove that

$$L(\mathcal{X} \cup \mathcal{X}_1, \mathcal{Y} \cup \mathcal{Y}_1) \leq L(\mathcal{X}, \mathcal{Y}) + C(x_1 + y_1). \quad (14)$$

By induction, it is enough to deal with the cases  $(x_1, y_1) = (1, 0)$  and  $(x_1, y_1) = (0, 1)$ . Because of our symmetry assumption, our task is to prove that

$$L(\mathcal{X} \cup \{a\}, \mathcal{Y}) \leq L(\mathcal{X}, \mathcal{Y}) + C. \quad (15)$$

If  $\text{card}(\mathcal{Y}) \leq \text{card}(\mathcal{X})$ , then the latter is obvious: choose an optimal graph for  $L(\mathcal{X}, \mathcal{Y})$  and use it to upper estimate  $L(\mathcal{X} \cup \{a\}, \mathcal{Y})$ . Assume on the contrary that  $\text{card}(\mathcal{Y}) \geq \text{card}(\mathcal{X}) + 1$ . Then there exists  $\mathcal{Y}' \subset \mathcal{Y}$  with  $\text{card}(\mathcal{Y}') = \text{card}(\mathcal{X})$  and  $L(\mathcal{X}, \mathcal{Y}')$ . Let  $b \in \mathcal{Y} \setminus \mathcal{Y}'$ . In order to establish (15), it is enough to show that

$$L(\mathcal{X} \cup \{a\}, \mathcal{Y}' \cup \{b\}) \leq L(\mathcal{X}, \mathcal{Y}') + C,$$

but this is just an instance of the subadditivity property. Hence (14) is established.

In order to prove the regularity property, it remains to show that

$$L(\mathcal{X}, \mathcal{Y}) \leq L(\mathcal{X} \cup \mathcal{X}_2, \mathcal{Y} \cup \mathcal{Y}_2) + C(x_2 + y_2). \quad (16)$$

Again, using induction and symmetry, it is sufficient to establish

$$L(\mathcal{X}, \mathcal{Y}) \leq L(\mathcal{X} \cup \{a\}, \mathcal{Y}) + C. \quad (17)$$

If  $\text{card}(\mathcal{X}) \wedge \text{card}(cY) < \kappa_0$ , then by the bounded degree property  $L(\mathcal{X}, \mathcal{Y}) \leq \kappa\kappa_0 \text{diam}(Q)^p$  and we are done. Assume next that  $\text{card}(\mathcal{X}), \text{card}(\mathcal{Y}) \geq \kappa_0$ . Let us consider an optimal graph for  $L(\mathcal{X} \cup \{a\}, \mathcal{Y})$ .

If  $a$  is not a vertex of this graph (which forces  $\text{card}(\mathcal{X}) \geq \text{card}(\mathcal{Y})$ ) then one can use the same graph to upper estimate  $L(\mathcal{X}, \mathcal{Y})$  and obtain (17). Assume on the contrary that  $a$  is a vertex of this optimal graph. Let us distinguish two cases: if  $\text{card}(\mathcal{X}) \geq \text{card}(\mathcal{Y})$ , then in the optimal graph for  $L(\mathcal{X} \cup \{a\}, \mathcal{Y})$ , at least a point  $b \in \mathcal{X}$  is not used. Consider the isomorphic graph obtained by replacing  $a$  by  $b$  while the other points remain fixed (this leads to the deformation of the edges out of  $a$ . There are at most  $\kappa$  of them by the bounded degree assumption). This graph can be used to upper estimate  $L(\mathcal{X}, \mathcal{Y})$ , and gives

$$L(\mathcal{X}, \mathcal{Y}) \leq L(\mathcal{X} \cup \{a\}, \mathcal{Y}) + \kappa \text{diam}(Q)^p.$$

The second case is when  $a$  is used but  $\text{card}(\mathcal{X}) + 1 \leq \text{card}(\mathcal{Y})$ . Actually, the optimal graph for  $L(\mathcal{X} \cup \{a\}, \mathcal{Y})$  uses all the points of  $\mathcal{X} \cup \{a\}$  and of a subset of same cardinality  $\mathcal{Y}' \subset \mathcal{Y}$ . Choose an element  $b$  in  $\mathcal{Y}'$ . Then  $\mathcal{Y}'' = \mathcal{Y}' \setminus \{b\}$  has the same cardinality as  $\mathcal{X}$ . Obviously  $L(\mathcal{X} \cup \{a\}, \mathcal{Y}) = L(\mathcal{X} \cup \{a\}, \mathcal{Y}'' \cup \{b\})$ . Consider the corresponding optimal bipartite graph. By the restriction property, if we erase  $a$  and  $b$  and their edges, we obtain a bipartite graph on  $(\mathcal{X}, \mathcal{Y}'')$  which differs from an admissible graph of our optimization problem by at most  $\kappa$  edges. Using this new graphs yields

$$L(\mathcal{X}, \mathcal{Y}) \leq \kappa \text{diam}(Q)^p + L(\mathcal{X} \cup \{a\}, \mathcal{Y}'' \cup \{b\}) = \kappa \text{diam}(Q)^p + L(\mathcal{X} \cup \{a\}, \mathcal{Y}).$$

This concludes the proof.  $\square$

## 6 Lower bounds, lower limits

### 6.1 Uniform distribution on a set

In order to motivate the sequel, we start with the simple case where  $f$  is an indicator function. The lower bound is then a direct consequence of Theorem 15 and Theorem 17.

**Theorem 24.** *Let  $d > 2p > 0$ . Let  $L$  be a bipartite functional on  $\mathbb{R}^d$  satisfying the properties  $(\mathcal{H}_p)$ ,  $(\mathcal{R}_p)$ ,  $(\mathcal{S}_p)$ . Let  $\Omega \subset \mathbb{R}^d$  be a bounded set with positive Lebesgue measure. Then*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}L(n\mathbb{1}_\Omega)}{n^{1-\frac{p}{d}}} = \beta_L \text{Vol}(\Omega).$$

*Proof.* Theorem 17 gives directly  $\limsup \mathbb{E}L(n\mathbb{1}_\Omega)/n^{1-\frac{p}{d}} \leq \beta_L \text{Vol}(\Omega)$ . By translation and dilation invariance, we may assume without loss of generality that  $\Omega \subset [0, 1]^d$ . Let  $\Omega_c := [0, 1]^d \setminus \Omega$ . Applying Proposition 12 for the partition  $[0, 1]^d = \Omega \cup \Omega_c$ , gives after division by  $n^{1-p/d}$

$$\frac{\mathbb{E}L(n\mathbb{1}_{[0,1]^d})}{n^{1-\frac{p}{d}}} - \frac{\mathbb{E}L(n\mathbb{1}_{\Omega_c})}{n^{1-\frac{p}{d}}} \leq \frac{\mathbb{E}L(n\mathbb{1}_\Omega)}{n^{1-\frac{p}{d}}} + 3C \text{diam}([0, 1]^d) n^{\frac{p}{d}-\frac{1}{2}} \left( \text{Vol}(\Omega)^{\frac{1}{2}} + \text{Vol}(\Omega_c)^{\frac{1}{2}} \right).$$

Since  $d > 2p$ , letting  $n$  go to infinity gives

$$\begin{aligned} \liminf_n \frac{\mathbb{E}L(n\mathbb{1}_\Omega)}{n^{1-\frac{p}{d}}} &\geq \lim_n \frac{\mathbb{E}L(n\mathbb{1}_{[0,1]^d})}{n^{1-\frac{p}{d}}} - \limsup_n \frac{\mathbb{E}L(n\mathbb{1}_{\Omega_c})}{n^{1-\frac{p}{d}}} \\ &\geq \beta_L - \beta_L \text{Vol}(\Omega_c) = \beta_L \text{Vol}(\Omega), \end{aligned}$$

where we have used Theorem 15 for the limit and Theorem 17 for the upper limit.  $\square$

The argument of the previous proof relies on the fact that the quantity  $\lim n^{1-p/d} \mathbb{E}L(n\mathbb{1}_\Omega) = \beta_L \text{Vol}(\Omega)$  is in a sense additive in  $\Omega$ . This line of reasoning does not pass to functions since  $f \mapsto \int f^{1-p/d}$  is additive only for functions with disjoint supports. The lower limit result requires more work for general densities.

### 6.2 Lower limits for matchings

In order to establish a tight estimate on the lower limit, it is natural to try and reverse the partition inequality given in Proposition 12. This is usually more difficult and there does not exist a general method to perform this lower bound. We shall first restrict our attention to the case of the matching functional  $M_p$  with  $p > 0$ , we define in this subsection

$$L = M_p.$$

### 6.2.1 Boundary functional

Given a matching on the unit cube, one needs to infer from it matchings on the subcubes of a dyadic partition and to control the corresponding costs. The main difficulty comes from the points of a subcube that are matched to points of another subcube. In other words some links of the optimal matching cross the boundaries of the cells. As in the book by Yukich [19], a modified notion of the cost of a matching is used in order to control the effects of the boundary of the cells of a partition. Our argument is however more involved, since the good bound (1) used by Yukich is not available for the bipartite matching.

We define

$$q = 2^{p-1} \wedge 1.$$

Let  $S \subset \mathbb{R}^d$  and  $\varepsilon \geq 0$ . Given multisets  $\mathcal{X} = \{X_1, \dots, X_m\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_n\}$  included in  $S$  we define the penalized boundary-matching cost as follows

$$\begin{aligned} L_{\partial S, \varepsilon}(X_1, \dots, X_m; Y_1, \dots, Y_n) & \\ = \min_{A, B, \sigma} & \left\{ \sum_{i \in A} |X_i - Y_{\sigma(i)}|^p + \sum_{i \in A^c} q \left( d(X_i, \partial S)^p + \varepsilon^p \right) + \sum_{j \in B^c} q \left( d(Y_j, \partial S)^p + \varepsilon^p \right) \right\}, \end{aligned} \quad (18)$$

where the minimum runs over all choices of subsets  $A \subset \{1, \dots, m\}$ ,  $B \subset \{1, \dots, n\}$  with the same cardinality and all bijective maps  $\sigma : A \rightarrow B$ . When  $\varepsilon = 0$  we simply write  $L_{\partial S}$ . Given an optimal choice of  $A, B, \sigma$ , we say that the points  $(X_i)_{i \notin A}$  are matched to the boundary and we denote by  $\mathcal{X}_{\partial S, \varepsilon}$  their number, that is the cardinality of  $A^c = \{1, \dots, m\} \setminus A$  (to be very precise,  $\mathcal{X}_{\partial S, \varepsilon}$  is the maximum over all minimizing triplets  $(A, B, \sigma)$  of the number of points in  $\mathcal{X}$  that are matched to the boundary.) The number  $\mathcal{Y}_{\partial S, \varepsilon}$  is defined in the same way. Notice that in our definition, and contrary to the definition of optimal matching, all points are matched even if  $m \neq n$ . If  $\mathcal{X}$  and  $\mathcal{Y}$  are independent Poisson point processes with intensity  $\nu$  supported in  $S$  and with finite total mass, we write  $L_{\partial S, \varepsilon}(\nu)$  for the random variable  $L_{\partial S, \varepsilon}(\mathcal{X}, \mathcal{Y})$  and  $\mathcal{X}_{\partial S, \varepsilon}(\nu)$  for  $\mathcal{X}_{\partial S, \varepsilon}$ .

The main interest of the notion of boundary matching is that it allows to bound from below the matching cost on a large set in terms of contributions on cells of a partition. The following Lemma establishes a superadditive property of  $L_{\partial S}$  and it can be viewed as a counterpart to the upper bound provided by Proposition 12.

**Lemma 25.** *Assume  $L = M_p$ . Let  $\nu$  be a finite measure on  $\mathbb{R}^d$  and consider a partition  $Q = \cup_{P \in \mathcal{P}} P$  of a subset of  $\mathbb{R}^d$ . Then*

$$\text{diam}(Q)^p \sqrt{2\nu(\mathbb{R}^d)} + \mathbb{E}L(\nu) \geq \mathbb{E}L_{\partial Q}(\mathbf{1}_Q \cdot \nu) \geq \sum_{P \in \mathcal{P}} \mathbb{E}L_{\partial P}(\mathbf{1}_P \cdot \nu).$$

*Proof.* Let  $\mathcal{X} = \{X_1, \dots, X_m\}, \mathcal{Y} = \{Y_1, \dots, Y_n\}$  be multisets included in  $Q$  and  $\mathcal{X}' = \{X_{m+1}, \dots, X_{m+m'}\}, \mathcal{Y}' = \{Y_{n+1}, \dots, Y_{n+n'}\}$  be multisets included in  $Q^c$ . By considering an optimal matching of  $\mathcal{X} \cup \mathcal{X}'$  and  $\mathcal{Y} \cup \mathcal{Y}'$ , we have the lower bound

$$\text{diam}(Q)^p |m + m' - n - n'| + L(\mathcal{X} \cup \mathcal{X}', \mathcal{Y} \cup \mathcal{Y}') \geq L_{\partial Q}(\mathcal{X}, \mathcal{Y}).$$

Indeed, if  $1 \leq i \leq m$  and a pair  $(X_i, Y_{n+j})$ , is matched then  $|X_i - Y_{n+j}| \geq d(X_i, \partial Q)$  and similarly for a pair  $(X_{m+i}, Y_j)$ , with  $1 \leq j \leq n$ ,  $|X_{m+i} - Y_j| \geq d(Y_j, \partial Q)$ . The term  $\text{diam}(Q)^p |m + m' - n - n'|$  takes care of the points of  $\mathcal{X} \cup \mathcal{Y}$  that are not matched in the optimal matching of  $\mathcal{X} \cup \mathcal{X}'$  and  $\mathcal{Y} \cup \mathcal{Y}'$ . We apply the above inequality to  $\mathcal{X}, \mathcal{Y}$  independent Poisson processes of intensity  $\mathbf{1}_Q \cdot \nu$ , and  $\mathcal{X}', \mathcal{Y}'$ , two independent Poisson processes of intensity  $\mathbf{1}_{Q^c} \cdot \nu$ , independent of  $(\mathcal{X}, \mathcal{Y})$ . Then  $\mathcal{X} \cup \mathcal{X}', \mathcal{Y} \cup \mathcal{Y}'$  are independent Poisson processes of intensity  $\nu$ . Taking expectation and bounding the average of the difference of cardinalities in the usual way, we obtain the first inequality.

Now, the second inequality will follow from the superadditive property of the boundary functional:

$$L_{\partial Q}(\mathcal{X}, \mathcal{Y}) \geq \sum_{P \in \mathcal{P}} L_{\partial P}(\mathcal{X} \cap P, \mathcal{Y} \cap P). \quad (19)$$

This is proved as follows. Let  $(A, B, \sigma)$  be an optimal triplet for  $L_{\partial Q}(\mathcal{X}, \mathcal{Y})$ :

$$L_{\partial Q}(\mathcal{X}, \mathcal{Y}) = \sum_{i \in A} |X_i - Y_{\sigma(i)}|^p + \sum_{i \in A^c} qd(X_i, \partial Q)^p + \sum_{j \in B^c} qd(Y_j, \partial Q)^p.$$

If  $x \in Q$ , we denote by  $P(x)$  the unique  $P \in \mathcal{P}$  that contains  $x$ . If  $P(X_i) = P(Y_{\sigma(i)})$  we leave the term  $|X_i - Y_{\sigma(i)}|$  unchanged. On the other hand if  $P(X_i) \neq P(Y_{\sigma(i)})$ , from Hölder's inequality,

$$|X_i - Y_{\sigma(i)}|^p \geq q d(X_i, \partial P(X_i))^p + q d(Y_{\sigma(i)}, \partial P(Y_{\sigma(i)}))^p.$$

Eventually, we apply the inequality

$$d(x, \partial Q) \geq d(x, \partial P(x))$$

in order to take care of the points in  $A^c \cup B^c$ . Combining these inequalities and grouping the terms according to the cell  $P \in \mathcal{P}$  containing the points, we obtain that

$$\begin{aligned} L_{\partial Q}(\mathcal{X}, \mathcal{Y}) &\geq \sum_{P \in \mathcal{P}} \left( \sum_{i \in A; X_i \in P, Y_{\sigma(i)} \in P} |X_i - Y_{\sigma(i)}|^p + \sum_{i \in A; X_i \in P, Y_{\sigma(i)} \notin P} qd(X_i, \partial P)^p \right. \\ &\quad \left. + \sum_{i \in A^c; X_i \in P} qd(X_i, \partial P)^p + \sum_{j \in B; Y_j \in P, j \notin \sigma(\{i; X_i \in P\})} qd(Y_j, \partial P)^p + \sum_{j \in B^c; Y_j \in P} qd(Y_j, \partial P)^p \right) \\ &\geq \sum_{P \in \mathcal{P}} L_{\partial P}(\mathcal{X} \cap P, \mathcal{Y} \cap P), \end{aligned}$$

and we have obtained the inequality (19).  $\square$

The next lemma will be used to reduce to uniform distributions on squares.

**Lemma 26.** *Assume  $L = M_p$ . Let  $\mu, \mu'$  be two probability measures on  $\mathbb{R}^d$  with supports in  $Q$  and  $n > 0$ . Then*

$$\mathbb{E}L_{\partial Q}(n\mu) \leq \mathbb{E}L_{\partial Q}(n\mu') + 4n \text{diam}(Q)^p d_{\text{TV}}(\mu, \mu').$$

Consequently, if  $f$  is a nonnegative locally integrable function on  $\mathbb{R}^d$ , setting  $\alpha = \int_Q f / \text{vol}(Q)$ , it holds

$$\mathbb{E}L_{\partial Q}(nf \mathbb{1}_Q) \leq \mathbb{E}L_{\partial Q}(n\alpha \mathbb{1}_Q) + 2n \text{diam}(Q)^p \int_Q |f(x) - \alpha| dx.$$

*Proof.* The functional  $L_{\partial Q}$  satisfies a slight modification of property  $(\mathcal{R}_p)$ : for all multisets  $\mathcal{X}, \mathcal{Y}, \mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$  in  $Q$ , it holds

$$L_{\partial Q}(\mathcal{X} \cup \mathcal{X}_1, \mathcal{Y} \cup \mathcal{Y}_1) \leq L_{\partial Q}(\mathcal{X} \cup \mathcal{X}_2, \mathcal{Y} \cup \mathcal{Y}_2) + \text{diam}(Q)^p (\text{card}(\mathcal{X}_1) + \text{card}(\mathcal{X}_2) + \text{card}(\mathcal{Y}_1) + \text{card}(\mathcal{Y}_2)).$$

Indeed, we start from an optimal boundary matching of  $L_{\partial Q}(\mathcal{X} \cup \mathcal{X}_2, \mathcal{Y} \cup \mathcal{Y}_2)$ , we match to the boundary the points of  $(\mathcal{X}, \mathcal{Y})$  that are matched to a point in  $(\mathcal{X}_2, \mathcal{Y}_2)$ . There are at most  $\text{card}(\mathcal{X}_2) + \text{card}(\mathcal{Y}_2)$  such points. Finally we match all points of  $(\mathcal{X}_1, \mathcal{Y}_1)$  to the boundary and we obtain a suboptimal boundary matching of  $L_{\partial Q}(\mathcal{X} \cup \mathcal{X}_1, \mathcal{Y} \cup \mathcal{Y}_1)$ . This establishes the above inequality. The statements follow then from the proofs of Proposition 8 and Corollary 9.  $\square$

We will need an asymptotic for the boundary matching for the uniform distribution on the unit cube. This requires to compare the boundary matching and the usual matching costs for the uniform measure on cubes. The crucial point is to estimate the number of points that are matched to the boundary in order to reallocate them.

**Lemma 27.** *Assume  $L = M_p$ ,  $p > 0$ ,  $n > 2$  and  $\varepsilon \in (0, 1]$ . Let  $Q \subset \mathbb{R}^d$  be a cube of side length 1.*

- If  $d = 1$ , then  $\mathbb{E}\mathcal{X}_{\partial Q}(n\mathbb{1}_Q) \leq c(1)\sqrt{n}$ .
- If  $d = 2$  then  $\mathbb{E}\mathcal{X}_{\partial Q}(n\mathbb{1}_Q) \leq c(2)\sqrt{n} \ln n$ .

- If  $d \geq 3$ , then

$$\mathbb{E}\mathcal{X}_{\partial Q}(n\mathbb{1}_Q) \leq c(d)n^{1-\frac{1}{d}} \quad \text{and} \quad \mathbb{E}\mathcal{X}_{\partial Q,\varepsilon}(n\mathbb{1}_Q) \leq c(d)\varepsilon^{1-\frac{d}{2}}\sqrt{n}.$$

By symmetry, the same bounds are valid for  $\mathbb{E}\mathcal{Y}_{\partial Q}(n\mathbb{1}_Q)$  and  $\mathbb{E}\mathcal{Y}_{\partial Q,\varepsilon}(n\mathbb{1}_Q)$ .

*Remark 6.* Recall that  $\mathcal{X}_{\partial Q,\varepsilon}(n\mathbb{1}_Q)$  stands for the number of points of the first species in  $Q$  that are matched to the boundary in the case of independent Poisson point processes of uniform intensity  $n$  and of a penalty  $\varepsilon$ . Also note that the second bound in the lemma tells us that the number of such points is of order  $n^{1-\frac{1}{d}}$ . Hence reallocating these points to match them within the same cell of the partition would create a difference in the cost of order  $n^{1-\frac{1}{d}}$ . This would definitely not allow a sharp estimate of the lower limit of the cost of the matching in the scale of  $n^{1-\frac{p}{d}}$  for  $p \geq 1$ . This observation motivated the study of the  $\varepsilon > 0$  case where the penalty prevents too many points to be matched to the boundary. Note that  $\varepsilon$  should be at least of order  $n^{-1/d}$  to have a better bound. This is consistent with the heuristic saying that in an average optimal matching, points are roughly matched to points at distance  $n^{-1/d}$ , hence a bigger penalty has a real effect.

*Proof.* By translation invariance, we may assume that  $Q = [-1/2, 1/2]^d$  is a unit cube with center at the origin and  $\mathcal{X} \cap Q$  is a Poisson process with intensity  $n\mathbb{1}_Q$ . The proof relies on a nice dyadic subdivision of  $Q$  introduced in Redmond and Yukich [11].

We first consider non-penalized boundary matchings (formally  $\varepsilon = 0$ ). Let  $\ell \geq 1$  be the unique integer such that

$$2^{(\ell-1)d} \leq n < 2^{\ell d}. \quad (20)$$

Let  $U_0 = \frac{1}{3}Q$  and more generally for any integer  $0 \leq j \leq \ell$ , let

$$U_j = \left(1 - \frac{2^{-j+1}}{3}\right) \cdot Q = \left[-\frac{1}{2} + \frac{2^{-j}}{3}, \frac{1}{2} - \frac{2^{-j}}{3}\right]^d.$$

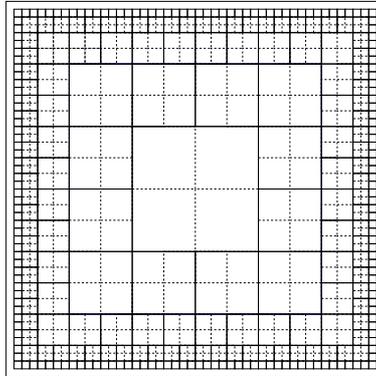


Figure 1: The partition  $\mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4$  in dimension 2.

For each  $1 \leq j \leq \ell$ , the set  $U_j \setminus U_{j-1}$  is a cubical shell of width  $2^{-j}/3$ . It can be (essentially) partitioned by cubes of side length  $2^{-j}/3$ . Indeed the cube  $U_j$  is of side length  $1 - 2^{-j+1}/3$ , so it can be partitioned in

$$\left(\frac{1 - 2^{-j+1}/3}{2^{-j}/3}\right)^d = (3 \cdot 2^j - 2)^d$$

cubes of side length  $2^{-j}/3$ . Notice that the union of the subcubes that are not touching the boundary of  $U_j$  coincides with  $U_{j-1}$ , which is itself partitioned into

$$\left(\frac{1 - 2^{-j+2}/3}{2^{-j}/3}\right)^d = (3 \cdot 2^j - 4)^d$$

cubes of side length  $2^{-j}/3$ . Hence  $U_j \setminus U_{j-1}$  is partitioned by cubes of side length  $2^{-j}/3$  and their number is

$$(3 \cdot 2^j - 2)^d - (3 \cdot 2^j - 4)^d \leq 2d(3 \cdot 2^j - 2)^{d-1} \leq 2d3^{d-1}2^{j(d-1)},$$

using that when  $a \geq b \geq 0$ ,  $a^d - b^d \leq (a-b)da^{d-1}$ . We further partition each subcube into  $2^{md}$  subcubes of side-length  $2^{-m}2^{-j}/3$ , where  $m$  is an integer large enough to guarantee  $2^m > \sqrt{d}$ . We have thus built a partition  $\mathcal{R}_j$  of  $U_j \setminus U_{j-1}$  into at most  $c_1 2^{j(d-1)}$  subcubes with  $c_1 = 2d3^{d-1}2^{md}$ . We also partition  $U_0$  into  $2^{md}$  subcubes, and obtain similarly a partition  $\mathcal{R}_0$  (since Figure 1).

The important property of this partition is that for each  $R \in \mathcal{R}_j$ ,  $0 \leq j \leq \ell$ ,

$$\text{diam}(R) < \text{dist}(R, \partial Q) = \inf_{x \in R} \text{dist}(x, \partial Q).$$

Therefore, if  $\mathcal{X}_{\partial Q} \cap R$  is not empty then  $\mathcal{Y}_{\partial Q} \cap R$  is empty (otherwise, since  $1 \leq 2q$ , we could match two points in these sets and obtain a new matching with a lower cost). It implies

$$\mathcal{X}_{\partial Q}(R) \leq |\mathcal{X}(R) - \mathcal{Y}(R)|.$$

We deduce that

$$\mathcal{X}_{\partial Q}(Q) = \mathcal{X}_{\partial Q}(Q \setminus U_\ell) + \sum_{j=0}^{\ell} \sum_{R \in \mathcal{R}_j} \mathcal{X}_{\partial Q}(R) \leq \mathcal{X}(Q \setminus U_\ell) + \sum_{j=0}^{\ell} \sum_{R \in \mathcal{R}_j} |\mathcal{X}(R) - \mathcal{Y}(R)|.$$

The moat  $Q \setminus U_\ell$  has width  $(2/3)2^{-\ell}$ . Thus taking expectation, for some positive constants  $c_2, c_3$  depending only on the dimension,

$$\begin{aligned} \mathbb{E} \mathcal{X}_{\partial Q}(Q) &\leq \frac{4d}{3} n 2^{-\ell} + \sum_{j=0}^{\ell} c_2 2^{j(d-1)} \sqrt{n 2^{-jd}}. \\ &\leq c_3 \left( n 2^{-\ell} + \sqrt{n} \sum_{j=0}^{\ell} 2^{j(\frac{d}{2}-1)} \right). \end{aligned} \quad (21)$$

For  $d = 1$  the above geometric series is convergent, hence we get  $\mathbb{E} \mathcal{X}_{\partial Q}(Q) \leq c_3(n 2^{-\ell} + 4\sqrt{n})$  which is at most  $c(1)\sqrt{n}$  since  $\ell$  is given by (20). For  $d = 2$ , we get  $\mathbb{E} \mathcal{X}_{\partial Q}(Q) \leq c_3(n 2^{-\ell} + \ell\sqrt{n}) \leq c(2)\sqrt{n} \ln n$ , by our choice of  $\ell$ . If  $d \geq 3$ , the geometric sums appearing in 21 are divergent and can be estimated by their largest term. Hence

$$\mathbb{E} \mathcal{X}_{\partial Q}(Q) \leq c(d) \left( n 2^{-\ell} + \sqrt{n} 2^{\ell(\frac{d}{2}-1)} \right).$$

The claim follows from (20) again.

We now assume that  $0 < \varepsilon \leq 1$  and  $d \geq 3$ . We redefine  $\ell$  as the unique integer such that

$$\frac{2^{-\ell+1}}{3} \leq \varepsilon < \frac{2^{-\ell+2}}{3}.$$

For  $0 \leq j \leq \ell$ , we consider the same partition  $\mathcal{R}_j$  as above. The moat  $Q \setminus U_\ell$  is partitioned into subcubes of side-length  $2^{-\ell-1}/3$ . Each of these subcubes is further partitioned into  $2^{md}$  subcubes. We then obtain a partition  $\mathcal{R}_{\ell+1}$  of  $Q \setminus U_\ell$  into at most  $c_1 2^{(\ell+1)(d-1)}$  subcubes. Also, due to our choice of  $\ell$ , each  $R \in \mathcal{R}_j$ ,  $0 \leq j \leq \ell + 1$ , enjoys the property

$$\text{diam}(R) < \max(\text{dist}(R, \partial Q), \varepsilon).$$

Hence, in an optimal matching it is not possible to have at the same time points from  $\mathcal{X} \cap R$  and from  $\mathcal{Y} \cap R$  that are matched to the boundary. Indeed, it is more economic to match together two points in  $R$  than to match both of them to the boundary since

$$2q(\text{dist}(R, \partial Q)^p + \varepsilon^p) \geq \max(\text{dist}(R, \partial Q)^p, \varepsilon^p) > \text{diam}(R)^p.$$

As above, we get

$$\mathcal{X}_{\partial Q, \varepsilon}(Q) = \sum_{j=0}^{\ell+1} \sum_{R \in \mathcal{R}_j} \mathcal{X}_{\partial Q, \varepsilon}(R) \leq \sum_{j=0}^{\ell+1} \sum_{R \in \mathcal{R}_j} |\mathcal{X}(R) - \mathcal{Y}(R)|.$$

Taking expectation, for some positive constants  $c_4$ ,

$$\mathbb{E}\mathcal{X}_{\partial Q, \varepsilon}(Q) \leq \sum_{j=0}^{\ell+1} 2c_2 2^{j(d-1)} \sqrt{n 2^{-jd}} = c_4 \sqrt{n} 2^{\ell(\frac{d}{2}-1)}.$$

With our choice of  $\ell$ , we obtain the required statement.  $\square$

The next Lemma contains the main new technical argument for the lower bound. Let  $Q = [0, 1]^d$  and denote

$$\bar{L}_{\partial Q}(n) = \mathbb{E}L_{\partial Q}(n\mathbb{1}_Q).$$

**Lemma 28.** *Assume  $L = M_p$ . Assume that either  $d \in \{1, 2\}$  and  $p < d/2$ , or  $d \geq 3$  and  $0 < p < (\sqrt{2d-3} + 1)/2$ , then*

$$\liminf_{n \rightarrow \infty} \frac{\bar{L}_{\partial Q}(n)}{n^{1-\frac{p}{d}}} \geq \beta_L,$$

where  $\beta_L$  is the constant from Theorem 15.

*Proof. Case :  $0 < p < 1$ .* Let  $\varepsilon \in [0, 1]$  and  $\mathcal{X}, \mathcal{Y}$  be independent Poisson point processes with intensity  $n\mathbb{1}_Q$ . We choose an optimal triplet  $(A, B, \sigma)$  in the definition of the boundary matching  $L_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y})$ ,

$$\begin{aligned} L_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) &= \sum_{i \in A} |X_i - Y_{\sigma(i)}|^p + q \sum_{i \notin A} (d(X_i, \partial Q)^p + \varepsilon^p) + q \sum_{j \notin \sigma(A)} (d(Y_j, \partial Q)^p + \varepsilon^p) \\ &\geq \sum_{i \in A} |X_i - Y_{\sigma(i)}|^p \\ &\geq L(\mathcal{X} \setminus \mathcal{X}_{\partial Q, \varepsilon}, \mathcal{Y} \setminus \mathcal{Y}_{\partial Q, \varepsilon}), \end{aligned}$$

where  $\mathcal{X}_{\partial Q, \varepsilon}$  is the set of points of  $\mathcal{X}$  that are matched to the boundary. Hence, applying the regularity property  $(\mathcal{R}_p)$  of Lemma 21, we get

$$L_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) \geq L(\mathcal{X} \setminus \mathcal{X}_{\partial Q, \varepsilon}, \mathcal{Y} \setminus \mathcal{Y}_{\partial Q, \varepsilon}) \geq L(\mathcal{X}, \mathcal{Y}) - d^{\frac{p}{2}} (\mathcal{X}_{\partial Q, \varepsilon}(Q) + \mathcal{Y}_{\partial Q, \varepsilon}(Q)). \quad (22)$$

For  $\varepsilon = 0$ , taking expectations, we get that

$$\mathbb{E}L_{\partial Q}(n\mathbb{1}_Q) \geq \mathbb{E}L(n\mathbb{1}_Q) - d^{\frac{p}{2}} (\mathbb{E}\mathcal{X}_{\partial Q}(n\mathbb{1}_Q) + \mathbb{E}\mathcal{Y}_{\partial Q}(n\mathbb{1}_Q)).$$

Dividing by  $n^{1-\frac{p}{d}}$  and passing to inferior limits in  $n$  yields the claim. Indeed, Theorem 15 ensures that  $\lim_n n^{\frac{p}{d}-1} \mathbb{E}L(n\mathbb{1}_Q) = \beta_L$  and the upper estimates of Lemma 27 give that

$$\lim_n n^{\frac{p}{d}-1} \mathbb{E}\mathcal{X}_{\partial Q}(n\mathbb{1}_Q) = 0$$

and the same for  $\mathbb{E}\mathcal{Y}_{\partial Q}(n\mathbb{1}_Q)$ . Note that in dimension  $d \geq 3$  we have used the additional assumption  $p < 1$ .

*Case :  $d \geq 3$  and  $1 \leq p < (\sqrt{2d-3} + 1)/2$ .* The above argument does not work since our estimate on the number of points matched to the boundary is not negligible with respect to  $n^{1-p/d}$ . We circumvent this difficulty by using the penalized boundary matching  $L_{\partial Q, \varepsilon}$  for some well chosen  $\varepsilon > 0$ . Note that in this case  $q = 1$ . Again, let  $\mathcal{X}, \mathcal{Y}$  be independent Poisson point processes with intensity  $n\mathbb{1}_Q$  and  $\varepsilon > 0$ . Choosing an optimal triplet  $(A, B, \sigma)$  in the definition of the boundary matching  $L_{\partial Q}(\mathcal{X}, \mathcal{Y})$ ,

$$\begin{aligned} L_{\partial Q}(\mathcal{X}, \mathcal{Y}) &= \sum_{i \in A} |X_i - Y_{\sigma(i)}|^p + \sum_{i \notin A} d(X_i, \partial Q)^p + \sum_{j \notin \sigma(A)} d(Y_j, \partial Q)^p \\ &= \sum_{i \in A} |X_i - Y_{\sigma(i)}|^p + \sum_{i \notin A} (d(X_i, \partial Q)^p + \varepsilon^p) + \sum_{j \notin \sigma(A)} (d(Y_j, \partial Q)^p + \varepsilon^p) - (\mathcal{X}_{\partial Q}(Q) + \mathcal{Y}_{\partial Q}(Q))\varepsilon^p \\ &\geq L_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) - (\mathcal{X}_{\partial Q}(Q) + \mathcal{Y}_{\partial Q}(Q))\varepsilon^p. \end{aligned}$$

Taking expectations and using Lemma 27 gives

$$\mathbb{E}L_{\partial Q}(n\mathbb{1}_Q) \geq \mathbb{E}L_{\partial Q,\varepsilon}(n\mathbb{1}_Q) - 2\varepsilon^p \mathbb{E}\mathcal{X}_{\partial Q}(n\mathbb{1}_Q) \geq \mathbb{E}L_{\partial Q,\varepsilon}(n\mathbb{1}_Q) - 2c(d)\varepsilon^p n^{1-\frac{1}{d}}.$$

Hence, if  $\varepsilon_n := \left(\delta n^{\frac{1-p}{d}}\right)^{\frac{1}{p}}$ , with  $0 < \delta < 1$ ,

$$\liminf_{n \rightarrow \infty} \frac{\bar{L}_{\partial Q}(n)}{n^{1-\frac{p}{d}}} \geq \liminf_{n \rightarrow \infty} \frac{\mathbb{E}L_{\partial Q,\varepsilon_n}(n\mathbb{1}_Q)}{n^{1-\frac{p}{d}}} - 2c(d)\delta. \quad (23)$$

We now use (22) and take expectations. We get from Lemma 27 that

$$\mathbb{E}L_{\partial Q,\varepsilon}(n\mathbb{1}_Q) \geq \mathbb{E}L(n\mathbb{1}_Q) - 2d^{\frac{p}{2}} c(d)\sqrt{n}\varepsilon^{1-\frac{d}{2}}.$$

Applied to  $\varepsilon_n$ , dividing by  $n^{1-\frac{p}{d}}$  and taking the limit we obtain thanks to Theorem 15 that

$$\liminf_n \frac{\mathbb{E}L_{\partial Q,\varepsilon_n}(n\mathbb{1}_Q)}{n^{1-\frac{p}{d}}} \geq \beta_L$$

is valid for any  $\delta > 0$  (here we have used our hypothesis on the range of  $p$  in the equivalent form  $d > 2(p^2 - p + 1)$ ). Combining this inequality with (23) and letting  $\delta$  tend to zero gives the claim of the lemma.  $\square$

### 6.2.2 General absolutely continuous measures

We are ready to state and prove

**Theorem 29.** *Assume  $L = M_p$ . Assume that  $d \in \{1, 2\}$  and  $p < d/2$ , or that  $d \geq 3$  and  $0 < p < (\sqrt{2d-3} + 1)/2$ . Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$  be an integrable function with  $\int |x|^\alpha f(x) dx < +\infty$  for some  $\alpha > \frac{2dp}{d-2p}$ . Then*

$$\lim_n \frac{\mathbb{E}L(nf)}{n^{1-\frac{p}{d}}} = \beta_L \int_{\mathbb{R}^d} f^{1-\frac{p}{d}}.$$

*Proof.* From Theorem 19, it is sufficient to check that

$$\liminf_n \frac{\mathbb{E}L(nf)}{n^{1-\frac{p}{d}}} \geq \beta_L \int f^{1-\frac{p}{d}}. \quad (24)$$

Assume first that the support of  $f$  is bounded. By a scaling argument, we may assume that the support of  $f$  is included in  $Q = [0, 1]^d$ . The proof is now similar to the one of Theorem 17. For  $\ell \in \mathbb{N}$ , we consider the partition  $\mathcal{P}_\ell$  of  $[0, 1]^d$  into  $2^{\ell d}$  cubes of side-length  $2^{-\ell}$ . Let  $k \in \mathbb{N}^*$  to be chosen later. For  $P \in \mathcal{P}_k$ ,  $\alpha_P$  denotes the average of  $f$  over  $P$ . Applying Lemma 25, Lemma 26 and homogeneity, we obtain

$$\begin{aligned} 2d^{\frac{p}{2}} \sqrt{n \int f} + \mathbb{E}L(nf) &\geq \mathbb{E}L_{\partial Q}(nf) \geq \sum_{P \in \mathcal{P}_k} \mathbb{E}L_{\partial P}(nf\mathbb{1}_P) \\ &\geq \sum_{P \in \mathcal{P}_k} \left( \mathbb{E}L_{\partial P}(n\alpha_P\mathbb{1}_P) - 2nd^{\frac{p}{2}} 2^{-kp} \int_P |f - \alpha_P| \right) \\ &= \sum_{P \in \mathcal{P}_k} \left( 2^{-kp} \mathbb{E}L_{\partial Q}(n\alpha_P 2^{-kd} \mathbb{1}_Q) - 2nd^{\frac{p}{2}} 2^{-kp} \int_P |f - \alpha_P| \right). \end{aligned}$$

Setting as before  $f_k = \sum_{P \in \mathcal{P}_k} \alpha_P \mathbb{1}_P$  and  $h(t) = \bar{L}_{\partial Q}(t)/t^{\frac{d-1}{d}}$  where  $\bar{L}_{\partial Q}(t) = \mathbb{E}L_{\partial Q}(t\mathbb{1}_Q)$ , the previous inequality reads as

$$2n^{\frac{p}{2}-\frac{1}{2}} d^{\frac{p}{2}} \sqrt{\int f} + \frac{\mathbb{E}L(nf)}{n^{1-\frac{p}{d}}} \geq \mathbb{E}L_{\partial Q}(nf) \geq \int h(n2^{-kd} f_k) f_k^{1-\frac{p}{d}} - 2d^{\frac{p}{2}} n^{\frac{p}{2}} 2^{-kp} \int |f - f_k|.$$

As in the proof of Theorem 17 we may choose  $k = k(n)$  depending on  $n$  in such a way that  $\lim_n k(n) = +\infty$ ,  $\lim_n n^{1/d} 2^{-k(n)} = +\infty$  and  $\lim_n n^{\frac{1}{d}} 2^{-k(n)} \left( \int |f - f_{k(n)}| \right)^{\frac{1}{p}} = 0$ . For such a choice, since  $\liminf_{t \rightarrow +\infty} h(t) \geq \beta_L$  by Lemma 28 and a.e.  $\lim_k f_k = f$ , Fatou's lemma ensures that

$$\liminf_n \int h(n 2^{-k(n)d} f_{k(n)}) f_{k(n)}^{1-\frac{p}{d}} \geq \liminf_n \int_{\{f>0\}} h(n 2^{-k(n)d} f_{k(n)}) f_{k(n)}^{1-\frac{p}{d}} \geq \beta_L \int f^{1-\frac{p}{d}}.$$

Statement (24) easily follows.

Now, let us address the general case where the support is not bounded. Let  $\ell \geq 1$  and  $Q = [-\ell, \ell]^d$ . By Lemma 25,

$$2 \text{diam}(Q)^p \sqrt{n \int f} + \mathbb{E}L(nf) \geq \mathbb{E}L_{\partial Q}(nf \mathbf{1}_Q).$$

Also, the above argument has shown that

$$\liminf_n \frac{\mathbb{E}L_{\partial Q}(nf \mathbf{1}_Q)}{n^{1-\frac{p}{d}}} \geq \beta_L \int_Q f^{1-\frac{p}{d}}.$$

We deduce that for any  $Q = [-\ell, \ell]^d$ ,

$$\liminf_n \frac{\mathbb{E}L(nf)}{n^{1-\frac{p}{d}}} \geq \beta_L \int_Q f^{1-\frac{p}{d}}.$$

Taking  $\ell$  arbitrary large we obtain the claimed lower bound.  $\square$

### 6.2.3 Dealing with the singular component

In this section we explain how to extend Theorem 29 from measures with densities to general measures. Given a measure  $\mu$ , we consider its decomposition  $\mu = \mu_{ac} + \mu_s$  into an absolutely continuous part and a singular part.

Our starting point is the following lemma, which can be viewed as an inverse subadditivity property.

**Lemma 30.** *Let  $p \in (0, 1]$  and  $L = M_p$ . Let  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$  be four finite multisets included in a bounded set  $Q$ . Then*

$$L(\mathcal{X}_1, \mathcal{Y}_1) \leq L(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) + L(\mathcal{X}_2, \mathcal{Y}_2) + \text{diam}(Q)^p \left( |\mathcal{X}_1(Q) - \mathcal{Y}_1(Q)| + 2|\mathcal{X}_2(Q) - \mathcal{Y}_2(Q)| \right).$$

*Proof.* Let us start with an optimal matching achieving  $L(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2)$  and an optimal matching achieving  $L(\mathcal{X}_2, \mathcal{Y}_2)$ . Let us view them as bipartite graphs  $G_{1,2}$  and  $G_2$  on the vertex sets  $(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2)$  and  $(\mathcal{X}_2, \mathcal{Y}_2)$  respectively (note that if a point appears more than once, we consider its instances as different graph vertices). Our goal is to build a possibly suboptimal matching of  $\mathcal{X}_1$  and  $\mathcal{Y}_1$ . Assume without loss of generality that  $\mathcal{X}_1(Q) \leq \mathcal{Y}_1(Q)$ . Hence we need to build an injection from  $\sigma : \mathcal{X}_1 \rightarrow \mathcal{Y}_1$  and to upper bound its cost  $\sum_{x \in \mathcal{X}_1} |x - \sigma(x)|^p$ .

To do this, let us consider the graph  $G$  obtained as the union of  $G_{1,2}$  and  $G_2$  (allowing multiple edges when two points are neighbours in both graphs). It is clear that in  $G$  the points from  $\mathcal{X}_1$  and  $\mathcal{Y}_1$  have degree at most one, while the points from  $\mathcal{X}_2$  and  $\mathcal{Y}_2$  have degree at most 2. For each  $x \in \mathcal{X}_1$ , let us consider its connected component  $C(x)$  in  $G$ . Because of the above degree considerations (and since no point is connected to itself in a bipartite graph) it is obvious that  $C(x)$  is a path.

It could be that  $C(x) = \{x\}$ , in the case when  $x$  is a leftover point in the matching corresponding to  $G_{1,2}$ . This means that  $x$  is a point in excess and there are at most  $|\mathcal{X}_1(Q) + \mathcal{X}_2(Q) - (\mathcal{Y}_1(Q) + \mathcal{Y}_2(Q))|$  of them.

Consider now the remaining case, when  $C(x)$  is a non trivial path. Its first edge belongs to  $G_{1,2}$ . If there is a second edge, it has to be from  $G_2$  (since  $G_{1,2}$  as degree at most one). Repeating the argument, we see that the edges of the path are alternately from  $G_{1,2}$  and from  $G_2$ . Note also that the successive vertices are alternately from  $\mathcal{X}_1 \cup \mathcal{X}_2$  and from  $\mathcal{Y}_1 \cup \mathcal{Y}_2$  (see Figure 4). There are three possibilities:

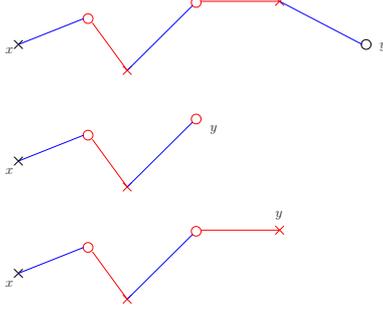


Figure 2: The three possibilities for the path  $C(x)$ . In blue,  $G_{1,2}$ , in red  $G_2$ , the points in  $\mathcal{X}_1 \cup \mathcal{X}_2$  are represented by a cross, points in  $\mathcal{Y}_1 \cup \mathcal{Y}_2$  by a circle.

- The other end of the path is a point  $y \in \mathcal{Y}_1$ . In this case we are done, we have associated a point  $y \in \mathcal{Y}_1$  to our point  $x \in \mathcal{X}_1$ . By the triangle inequality and since  $(a + b)^p \leq a^p + b^p$  due to the assumption  $p \leq 1$ ,  $|x - y|^p$  is upper bounded by the sum of the  $p$ -th powers of the length of the edges in  $C(x)$ .
- The other end of the path is a point  $y \in \mathcal{Y}_2$ . The last edge is from  $G_{1,2}$ . So necessarily,  $y$  has no neighbour in  $G_2$ . This means that it is not matched. There are at most  $|\mathcal{X}_2(Q) - \mathcal{Y}_2(Q)|$  such points in the matching  $G_2$ .
- The other end of the path is a point  $x' \in \mathcal{X}_2$ . The last edge is from  $G_2$ . So necessarily,  $x'$  has no neighbour in  $G_{1,2}$ . This means that it is not matched in  $G_{1,2}$ . As already mentioned there are at most  $|\mathcal{X}_1(Q) + \mathcal{X}_2(Q) - (\mathcal{Y}_1(Q) + \mathcal{Y}_2(Q))|$  such points.

Eventually we have found a way to match the points from  $\mathcal{X}_1$ , apart maybe  $|\mathcal{X}_2(Q) - \mathcal{Y}_2(Q)| + |\mathcal{X}_1(Q) + \mathcal{X}_2(Q) - (\mathcal{Y}_1(Q) + \mathcal{Y}_2(Q))|$  of them. We match the latter points arbitrarily to (unused) points in  $\mathcal{Y}_1$  and upper bound the distances between matched points by  $\text{diam}(Q)$ .  $\square$

As a direct consequence, we obtain:

**Lemma 31.** *Let  $\mu_1$  and  $\mu_2$  be two finite measures supported in a bounded set  $Q$ . Let  $p \in (0, 1]$  and  $L = M_p$  be the bipartite matching functional. Then*

$$\mathbb{E}L(\mu_1) \leq \mathbb{E}L(\mu_1 + \mu_2) + \mathbb{E}L(\mu_2) + 3 \text{diam}(Q)^p (\sqrt{\mu_1(Q)} + \sqrt{\mu_2(Q)}).$$

*Proof.* Let  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$  be four independent Poisson point processes. Assume that for  $i \in \{1, 2\}$ ,  $\mathcal{X}_i$  and  $\mathcal{Y}_i$  have intensity measure  $\mu_i$ . Consequently  $\mathcal{X}_1 \cup \mathcal{X}_2$  and  $\mathcal{Y}_1 \cup \mathcal{Y}_2$  are independent Poisson point processes with intensity  $\mu_1 + \mu_2$ . Applying the preceding lemma 30 and taking expectations yields

$$\mathbb{E}L(\mu_1) \leq \mathbb{E}L(\mu_1 + \mu_2) + \mathbb{E}L(\mu_2) + 2 \text{diam}(Q)^p (\mathbb{E}|\mathcal{X}_1(Q) - \mathcal{Y}_1(Q)| + \mathbb{E}|\mathcal{X}_2(Q) - \mathcal{Y}_2(Q)|).$$

As usual, we conclude using that

$$\mathbb{E}|\mathcal{X}_i(Q) - \mathcal{Y}_i(Q)| \leq \sqrt{\mathbb{E}((\mathcal{X}_i(Q) - \mathcal{Y}_i(Q))^2)} = \sqrt{2 \text{var}(\mathcal{X}_i(Q))} = \sqrt{2\mu_i(Q)}.$$

$\square$

**Theorem 32.** *Assume that  $d \in \{1, 2\}$  and  $p \in (0, d/2)$ , or that  $d \geq 3$  and  $p \in (0, 1]$ . Let  $L = M_p$  be the bipartite matching functional. Let  $\mu$  be a finite measure on  $\mathbb{R}^d$  with bounded support. Let  $f$  be the density of the absolutely continuous part of  $\mu$ . Assume that there exists  $\alpha > \frac{2dp}{d-2p}$  such that  $\int |x|^\alpha d\mu(x) < +\infty$ . Then*

$$\lim_n \frac{\mathbb{E}L(n\mu)}{n^{1-\frac{p}{d}}} = \beta_L \int_{\mathbb{R}^d} f^{1-\frac{p}{d}}.$$

*Proof.* Note that in any case,  $p \leq 1$  is assumed. Let us write  $\mu = \mu_{ac} + \mu_s$  where  $d\mu_{ac}(x) = f(x)dx$  is the absolutely continuous part and  $\mu_s$  is the singular part of  $\mu$ . The upper limit has been established in Theorem 19. It remains to establish the lower limit.

The argument is very simple if  $\mu$  has a bounded support: apply the previous lemma with  $\mu_1 = n\mu_{ac}$  and  $\mu_2 = n\mu_s$ . When  $n$  tends to infinity, observing that  $\sqrt{n}$  is negligible with respect to  $n^{1-\frac{p}{d}}$ , we obtain that

$$\liminf_n \frac{\mathbb{E}L(n\mu_{ac})}{n^{1-\frac{p}{d}}} \leq \liminf_n \frac{\mathbb{E}L(n\mu)}{n^{1-\frac{p}{d}}} + \limsup_n \frac{\mathbb{E}L(n\mu_s)}{n^{1-\frac{p}{d}}}.$$

Observe that the latter upper limit is equal to zero thanks to Theorem 19 applied to a purely singular measures. Eventually  $\liminf_n \frac{\mathbb{E}L(n\mu_{ac})}{n^{1-\frac{p}{d}}} = \beta_L \int_{\mathbb{R}^d} f^{1-\frac{p}{d}}$  by Theorem 29 about absolutely continuous measures.

Let us consider the general case of unbounded support. Let  $Q = [-\ell, \ell]^d$  where  $\ell > 0$  is arbitrary. Let  $\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$  be four independent Poisson point processes, such that  $\mathcal{X}_1$  and  $\mathcal{Y}_1$  have intensity measure  $n\mathbb{1}_Q \cdot \mu_{ac}$ , and  $\mathcal{X}_2$  and  $\mathcal{Y}_2$  have intensity measure  $n(\mu_s + \mathbb{1}_{Q^c} \cdot \mu_{ac})$ . It follows that  $\mathcal{X}_1 \cup \mathcal{X}_2$  and  $\mathcal{Y}_1 \cup \mathcal{Y}_2$  are independent Poisson point processes with intensity  $n\mu$ . Set  $T := \max\{|z|; z \in \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2\}$ . Applying Lemma 30 gives

$$L(\mathcal{X}_1, \mathcal{Y}_1) \leq L(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) + L(\mathcal{X}_2, \mathcal{Y}_2) + c_p T^p (|\text{card}(\mathcal{X}_1) - \text{card}(\mathcal{Y}_1)| - |\text{card}(\mathcal{X}_2) - \text{card}(\mathcal{Y}_2)|).$$

Taking expectations, applying the Cauchy-Schwarz inequality twice and Lemma 4 (note that  $\alpha > 2p$ ) gives

$$\begin{aligned} \mathbb{E}L(nf\mathbb{1}_Q) &\leq \mathbb{E}L(n\mu) + \mathbb{E}L(n(\mu_s + \mathbb{1}_{Q^c}\mu_{ac})) + c_p \sqrt{E[T^{2p}]} \left( \sqrt{2n\mu_{ac}(Q)} + \sqrt{2n(\mu_s(\mathbb{R}^d) + \mu_{ac}(Q^c))} \right) \\ &\leq \mathbb{E}L(n\mu) + \mathbb{E}L(n(\mu_s + \mathbb{1}_{Q^c}\mu_{ac})) + c'_p n^{\frac{p}{\alpha} + \frac{1}{2}}. \end{aligned}$$

Since  $\alpha > \frac{2dp}{d-2p}$  we obtain

$$\liminf_n \frac{\mathbb{E}L(n\mu)}{n^{1-\frac{p}{d}}} \geq \liminf_n \frac{\mathbb{E}L(nf\mathbb{1}_Q)}{n^{1-\frac{p}{d}}} - \limsup_n \frac{\mathbb{E}L(n(\mu_s + \mathbb{1}_{Q^c}\mu_{ac}))}{n^{1-\frac{p}{d}}} \geq \beta_L \left( \int_Q f^{1-\frac{p}{d}} - \int_{Q^c} f^{1-\frac{p}{d}} \right),$$

where we have used Theorem 29 for the lower limit for bounded absolutely continuous measures and Theorem 19 for the upper limit. Recall that  $Q = [-\ell, \ell]^d$ . It remains to let  $\ell$  tend to infinity.  $\square$

Actually, using classical duality techniques (which are specific to the bipartite matching) we can derive the following improvement of Lemma 31, which can be seen as an average monotonicity property:

**Lemma 33.** *Let  $p \in (0, 1]$  and  $L = M_p$ . Let  $\mu_1$  and  $\mu_2$  be two finite measures supported on a bounded subset  $Q \subset \mathbb{R}^d$ . Then*

$$\mathbb{E}L(\mu_1) \leq \mathbb{E}L(\mu_1 + \mu_2) + 3\text{diam}(Q)^p (\sqrt{\mu_1(Q)} + \sqrt{\mu_2(Q)}).$$

*Proof.* Since  $p \in (0, 1]$ , the unit cost  $c(x, y) := |x - y|^p$  is a distance on  $\mathbb{R}^d$ . The Kantorovich-Rubinstein dual representation of the minimal matching cost (or optimal transportation cost) is particularly simple in this case (see e.g. [10, 18, 16]): for  $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}$  two multisets in  $Q$ ,

$$L(\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}) = \sup_{f \in \text{Lip}_{1,0}} \sum_i f(x_i) - f(y_i),$$

where  $\text{Lip}_{1,0}$  denotes the set of function  $f : Q \rightarrow \mathbb{R}$  which are 1-Lipschitzian for the distance  $c(x, y)$  (hence they are  $p$ -Hölderian for the Euclidean distance) and vanish at a prescribed point  $x_0 \in Q$ . Observe that any function in  $\text{Lip}_{1,0}$  is bounded by  $\text{diam}(Q)^p$  pointwise.

Let  $\mathcal{X} = \{X_1, \dots, X_{N_1}\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_{N_2}\}$  be independent Poisson point processes with intensity  $\mu$  of finite mass and supported on a set  $Q$  of diameter  $D < +\infty$ . By definition, on the event  $\{N_1 \leq N_2\}$ ,

$$\begin{aligned} L(\mathcal{X}, \mathcal{Y}) &= \inf_{A \subset \{1, \dots, N_2\}; \text{card}(A)=N_1} L(\{X_i, 1 \leq i \leq N_1\}, \{Y_j, j \in A\}) \\ &= \inf_{A \subset \{1, \dots, N_2\}; \text{card}(A)=N_1} \sup_{f \in \text{Lip}_{1,0}} \left( \sum_{i \leq N_1} f(X_i) - \sum_{j \in A} f(Y_j) \right) \\ &\geq \sup_{f \in \text{Lip}_{1,0}} \left( \sum_{i \leq N_1} f(X_i) - \sum_{j \leq N_2} f(Y_j) \right) - D^p |N_1 - N_2| \end{aligned}$$

where we have used Kantorovich-Rubinstein duality to express the optimal matching of two samples of the same size and used that every  $f \in \text{Lip}_{1,0}$  satisfies  $|f| \leq D^p$  pointwise on  $Q$ . A similar lower bound is valid when  $N_1 \geq N_2$ . Hence, taking expectation and bounding  $E|N_1 - N_2|$  in terms of the variance of the number of points in one process, one gets

$$\mathbb{E}L(\mu) \geq \mathbb{E} \sup_{f \in \text{Lip}_{1,0}} \left( \sum_{i \leq N_1} f(X_i) - \sum_{j \leq N_2} f(Y_j) \right) - D^p \sqrt{2|\mu|}. \quad (25)$$

A similar argument also gives the following upper bound

$$\mathbb{E}L(\mu) \leq \mathbb{E} \sup_{f \in \text{Lip}_{1,0}} \left( \sum_{i \leq N_1} f(X_i) - \sum_{j \leq N_2} f(Y_j) \right) + D^p \sqrt{2|\mu|}. \quad (26)$$

Let  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$  be four independent Poisson point processes. Assume that for  $i \in \{1, 2\}$ ,  $\mathcal{X}_i$  and  $\mathcal{Y}_i$  have intensity  $\mu_i$ . As already mentioned,  $\mathcal{X}_1 \cup \mathcal{X}_2$  and  $\mathcal{Y}_1 \cup \mathcal{Y}_2$  are independent with common intensity  $\mu_1 + \mu_2$ . Given a compact set  $Q$  containing the supports of both measures, and  $x_0 \in Q$  we define the set  $\text{Lip}_{1,0}$ . Using (25),

$$\begin{aligned} \mathbb{E}L(\mu_1 + \mu_2) &= \mathbb{E}L(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) \\ &\geq \mathbb{E} \sup_{f \in \text{Lip}_{1,0}} \left( \sum_{x_1 \in \mathcal{X}_1} f(x_1) - \sum_{y_1 \in \mathcal{Y}_1} f(y_1) + \sum_{x_2 \in \mathcal{X}_2} f(x_2) - \sum_{y_2 \in \mathcal{Y}_2} f(y_2) \right) - D^p \sqrt{2|\mu_1 + \mu_2|} \end{aligned}$$

Now we use the easy inequality  $\mathbb{E} \sup \geq \sup \mathbb{E}$  when  $\mathbb{E}$  is the conditional expectation given  $\mathcal{X}_1, \mathcal{Y}_1$ . Since  $(\mathcal{X}_2, \mathcal{Y}_2)$  are independent from  $(\mathcal{X}_1, \mathcal{Y}_1)$ , we obtain

$$\begin{aligned} &\mathbb{E}L(\mu_1 + \mu_2) + D^p \sqrt{2|\mu_1 + \mu_2|} \\ &\geq \mathbb{E} \sup_{f \in \text{Lip}_{1,0}} \left( \sum_{x_1 \in \mathcal{X}_1} f(x_1) - \sum_{y_1 \in \mathcal{Y}_1} f(y_1) + \mathbb{E} \left( \sum_{x_2 \in \mathcal{X}_2} f(x_2) - \sum_{y_2 \in \mathcal{Y}_2} f(y_2) \right) \right) \\ &= \mathbb{E} \sup_{f \in \text{Lip}_{1,0}} \left( \sum_{x_1 \in \mathcal{X}_1} f(x_1) - \sum_{y_1 \in \mathcal{Y}_1} f(y_1) \right) \\ &\geq \mathbb{E}L(\mu_1) - D^p \sqrt{2|\mu_1|}, \end{aligned}$$

where we have noted that the inner expectation vanishes and used (26). The claim easily follows.  $\square$

### 6.3 Euclidean combinatorial optimization

Our proof for the lower bound for matchings extends to some combinatorial optimization functionals  $L$  defined by (13). In this paragraph, we explain how to adapt the above argument at the cost of ad-hoc assumptions on the collection of graphs  $(\mathcal{G}_n)_{n \in \mathbb{N}}$ . As motivating example, we will treat completely the case of the bipartite traveling salesperson tour.

### 6.3.1 Boundary functional

Let  $S \subset \mathbb{R}^d$  and  $\varepsilon, p \geq 0$ . Set  $q = 2^{p-1} \wedge 1$ . In what follows,  $p$  is fixed and will be omitted in most places where it would appear as an index. Given multisets  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_n\}$  included in  $\mathbb{R}^d$ , we first set

$$L_{\partial S, \varepsilon}^0(\mathcal{X}, \mathcal{Y}) = \min_{G \in \mathcal{G}_n} \left\{ \sum_{(i,j) \in [n]^2: \{i, n+j\} \in G} d_{S, \varepsilon, p}(X_i, Y_j) \right\},$$

where

$$d_{S, \varepsilon, p}(x, y) = \begin{cases} |x - y|^p & \text{if } x, y \in S, \\ 0 & \text{if } x, y \notin S, \\ q(\text{dist}(x, S^c)^p + \varepsilon^p) & \text{if } x \in S, y \notin S \\ q(\text{dist}(y, S^c)^p + \varepsilon^p) & \text{if } y \in S, x \notin S \end{cases} \quad (27)$$

Now, if  $\mathcal{X}$  and  $\mathcal{Y}$  are in  $S$ , we define the penalized boundary functional as

$$L_{\partial S, \varepsilon}(\mathcal{X}, \mathcal{Y}) = \min_{A, B \subset S^c} L_{\partial S, \varepsilon}^0(\mathcal{X} \cup A, \mathcal{Y} \cup B), \quad (28)$$

where the minimum is over all multisets  $A$  and  $B$  in  $S^c$  such that  $\text{card}(\mathcal{X} \cup A) = \text{card}(\mathcal{Y} \cup B) \geq \kappa_0$ . When  $\varepsilon = 0$  we simply write  $L_{\partial S}$ . The main idea of this definition is to consider all possible configurations outside the set  $S$  but not to count the distances outside of  $S$  (from a metric view point, all of  $S^c$  is identified to a point which is at distance  $\varepsilon$  from  $S$ ).

The existence of the minimum in (28) is due to the fact that  $L_{\partial S}^0(\mathcal{X} \cup A, \mathcal{Y} \cup B)$  can only take finitely many values less than any positive value (the quantities involved are just sums of distances between points of  $\mathcal{X}, \mathcal{Y}$  and of their distances to  $S^c$ ). Given an optimal choice of  $G, A, B$ , we say that the edge  $\{i, k + j\} \in G$  is a boundary edge if exactly one of its vertices corresponds to a point in  $S$ . We denote by  $N_{\partial S, \varepsilon}$  their number. Notice that definition (28) is consistent with the definition of the boundary functional for the matching functional  $M_p$ , given by (18). If  $\mathcal{X}$  and  $\mathcal{Y}$  are independent Poisson point processes with intensity  $\nu$  supported in  $S$  and with finite total mass, we write  $L_{\partial S, \varepsilon}(\nu)$  for the random variable  $L_{\partial S, \varepsilon}(\mathcal{X}, \mathcal{Y})$  and  $N_{\partial S, \varepsilon}(\nu)$  for  $N_{\partial S, \varepsilon}$ . Also note that  $d_{S, 0, p}(x, y) \leq |x - y|^p$ . Consequently if  $\text{card}(\mathcal{X}) = \text{card}(\mathcal{Y})$  then

$$L_{\partial S}^0(\mathcal{X}, \mathcal{Y}) \leq L(\mathcal{X}, \mathcal{Y}). \quad (29)$$

In order to perform our analysis, we now introduce two new assumptions on the collection of graphs  $\mathcal{G}_n$ . Namely, for all integer  $n \geq 1$ ,

- (A6) (*shuffling*) We start with a particular (and thus more demanding) version of this property, as it is easy to state and sufficient for simple applications. Let us call it strong shuffling property: For  $n \geq \kappa$  and  $E = \{\{i, n + i\}, i \in [2\kappa_0]\}$  and any  $G \in \mathcal{G}_n$  such that  $E \subset G$  and any two edges in  $E$  are at distance at least 2 in  $G$ , there exists a permutation  $\sigma$  on  $[2\kappa_0]$ , with  $\sigma([\kappa_0]) \not\subset [\kappa_0]$ , such that

$$G' = (G \setminus E) \cup \{\{i, n + \sigma(i)\}, i \in [2\kappa_0]\} \in \mathcal{G}_n.$$

This property means that given two small sets of well-separated edges in a graph of the family  $\mathcal{G}_n$ , it is possible to permute these edges in a way that mixes the two subsets, while staying in the admissible family  $\mathcal{G}_n$ .

The "real" shuffling property is more complicated, but its meaning will become transparent when applying it to the estimation of the number of boundary edges: given  $E \subset G \in \mathcal{G}_n$  as above there exist two applications  $f_1, f_2 : [\kappa_0] \rightarrow [2\kappa_0]$  such that  $f_1([\kappa_0]) \not\subset [\kappa_0]$  and there exists  $G' \in \mathcal{G}_n$  such that  $G' \subset (G \setminus E) \cup E'$ , where

$$E' = \left\{ \{i, n + f_1(i)\}; i \in [\kappa_0] \right\} \cup \left\{ \{f_2(j), n + \kappa_0 + j\}; j \in [\kappa_0] \right\} \cup \left\{ \{\kappa_0 + k, n + \ell\}; (k, \ell) \in [\kappa_0]^2 \right\}. \quad (30)$$

- (A7) (*strong restriction*) Let  $G \in \mathcal{G}_n$ ,  $\kappa_0 \leq m \leq n$  and let  $G'$  be the restriction of  $G$  to  $\mathcal{B}_m$ . Let  $a$  be the number of vertices in  $G'$  which have a lower degree in  $G'$  than they had in  $G$  (they have lost at least an edge in the restriction process). Then there exists  $G'' \in \mathcal{G}_m$  which differs from  $G'$  by at most  $\kappa a$  edges.

Property (A6) relies on a kind of local stability of the collection  $\mathcal{G}_n$ . For example, for the travelling salesperson tour, we may take  $\kappa_0 = 2$  and the shuffling property is easily seen to hold. Indeed, consider a cycle  $G$  in  $\mathcal{B}_n$ , with  $n \geq 4$  which contains  $E = \{\{i, n+i\}, i \in [4]\}$ . We may orient the cycle  $G$  in oriented edges  $(e_1, \dots, e_{2n})$  such that  $e_1 = (1, n+1)$  and for  $i \in [2n]$ ,  $e_i = (u_i, u_{i+1})$  with  $u_{2n+1} = 1$ . We have  $\{3, n+3\} = \{u_k, u_{k+1}\}$  and  $\{4, n+4\} = \{u_\ell, u_{\ell+1}\}$ . Without loss of generality, we may assume that  $1 < k < \ell < 2n$ . If  $(u_k, u_{k+1}) = (n+3, 3)$  then we define the permutation  $\sigma = (1\ 3)$  (in cycle notation), otherwise, if  $(u_\ell, u_{\ell+1}) = (n+4, 4)$ , we set  $\sigma = (1\ 4)$ . Finally, if  $(u_k, u_{k+1}) = (3, n+3)$  and  $(u_\ell, u_{\ell+1}) = (4, n+4)$ , then we set  $\sigma = (1\ 3\ 4)$ . Then, with the notation of property (A6), setting,  $E' = \{\{i, n+\sigma(i)\}, i \in [2\kappa_0]\}$ , it is easy to check that the graph  $G'$  is a bipartite cycle (see Figure 3).

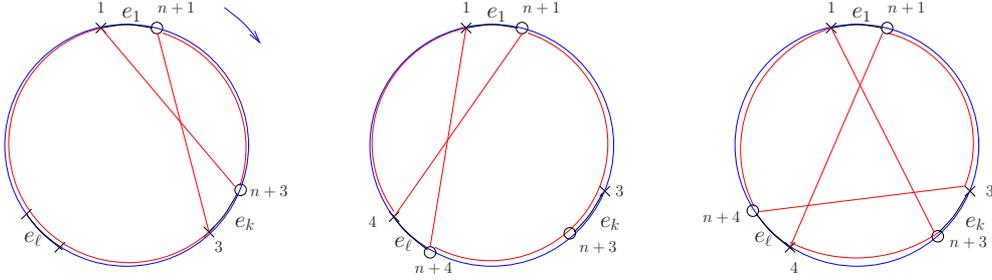


Figure 3: The three possibilities for the oriented cycle (in blue) and the corresponding graph  $G'$  (in red).

The same type of argument applies also for bipartite  $r$ -regular graphs with  $\kappa_0 = 2$ , provided  $r$  is even. Indeed, assume we may find  $\kappa_0 + 1 \leq i_2 < i_3 \leq 2\kappa_0$  such that the edges of  $E$ ,  $\{1, n+1\}, \{i_2, n+i_2\}, \{i_3, n+i_3\}$ , are at distance at least 2 in  $G$  from each other. Since  $G$  has all its vertices of even degree and is connected, it admits an Eulerian cycle. We are back to the situation of the travelling salesperson problem, with the difference that  $G$  may not be a proper cycle. Nevertheless the same permutations work: for a good choice of  $\sigma \in \{(1\ i_3), (1\ i_4), (1\ i_3\ i_4)\}$  and  $E' = \{\{i, n+\sigma(i)\}, i \in [2\kappa_0]\}$ , we obtain a new graph which is connected and  $r$ -regular.

The strong restriction property reinforces the restriction property (A5). Indeed, repeated applications of the (A5) property would give (under the assumptions of (A7)), a graph  $G''$  which differs from  $G'$  by at most  $C(n-m)$  edges. But the bounded degree property ensures that the number of edges erased in the restriction is at most  $2\kappa(n-m)$ . Hence the number of affected vertices verifies  $a \leq 2\kappa(n-m)$ . However on our examples (minimal  $r$ -regular connected graph, or minimal spanning tree with bounded degree), the argument used to prove the restriction property also works for the strong restriction property.

The next lemma will be used to reduce to uniform distributions on squares.

**Lemma 34.** *Assume (A1-A5). Let  $\mu, \mu'$  be two probability measures on  $\mathbb{R}^d$  with supports in  $Q$  and  $n > 0$ . Then, for some constant  $c$  depending only on  $\kappa, \kappa_0$ ,*

$$\mathbb{E}L_{\partial Q}(n\mu) \leq \mathbb{E}L_{\partial Q}(n\mu') + 2cn \operatorname{diam}(Q)^p d_{\text{TV}}(\mu, \mu').$$

Consequently, if  $f$  is a non-negative locally integrable function on  $\mathbb{R}^d$ , setting  $\alpha = \int_Q f / \operatorname{vol}(Q)$ , it holds

$$\mathbb{E}L_{\partial Q}(nf \mathbb{1}_Q) \leq \mathbb{E}L_{\partial Q}(n\alpha \mathbb{1}_Q) + cn \operatorname{diam}(Q)^p \int_Q |f(x) - \alpha| dx.$$

*Proof.* The functional  $L_{\partial Q}$  satisfies a slight modification of property  $(\mathcal{R}_p)$ : for all multisets  $\mathcal{X}, \mathcal{Y}, \mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$  in  $Q$ , it holds

$$L_{\partial Q}(\mathcal{X} \cup \mathcal{X}_1, \mathcal{Y} \cup \mathcal{Y}_1) \leq L_{\partial Q}(\mathcal{X} \cup \mathcal{X}_2, \mathcal{Y} \cup \mathcal{Y}_2) + C \operatorname{diam}(Q)^p (\operatorname{card}(\mathcal{X}_1) + \operatorname{card}(\mathcal{X}_2) + \operatorname{card}(\mathcal{Y}_1) + \operatorname{card}(\mathcal{Y}_2)), \quad (31)$$

with  $C = C(\kappa, \kappa_0)$ . The above inequality is established as in the proof of Lemma 23. Indeed, by linearity and symmetry we should check (15) and (17) for  $L_{\partial Q}$ . To prove (15), we consider an optimal triplet  $(G, A, B)$  for  $(\mathcal{X}, \mathcal{Y})$  and apply the merging property (A4) to  $G$  with the empty graph and  $m = 1$ : we obtain a graph  $G''$  and get a triplet  $(G'', A, B \cup \{b\})$  for  $(\mathcal{X} \cup \{a\}, \mathcal{Y})$ , where  $b$  is any point in  $\partial Q$ . To

prove (17), we now consider an optimal triplet  $(G, A, B)$  for  $(\mathcal{X} \cup \{a\}, \mathcal{Y})$  and move the point  $a$  to the  $a'$  in  $\partial Q$  in order to obtain a triplet  $(G, A \cup \{a'\}, B)$  for  $(\mathcal{X}, \mathcal{Y})$ .

With (31) at hand, the statements follow from the proofs of Proposition 8 and Corollary 9.  $\square$

The next lemma gives a lower bound on  $L$  in terms of its boundary functional and states an important superadditive property of  $L_{\partial S}$ .

**Lemma 35.** *Assume (A1-A5). Let  $\nu$  be a finite measure on  $\mathbb{R}^d$  and consider a partition  $Q = \cup_{P \in \mathcal{P}} P$  of a bounded subset of  $\mathbb{R}^d$ . Then, if  $c = 4\kappa(1 + \kappa_0)$ , we have*

$$c\sqrt{\nu(\mathbb{R}^d)} \text{diam}(Q)^p + \mathbb{E}L(\nu) \geq \mathbb{E}L_{\partial Q}(\mathbb{1}_Q \cdot \nu) \geq \sum_{P \in \mathcal{P}} \mathbb{E}L_{\partial P}(\mathbb{1}_P \cdot \nu).$$

*Proof.* We start with the first inequality. Let  $\mathcal{X} = \{X_1, \dots, X_m\}, \mathcal{Y} = \{Y_1, \dots, Y_n\}$  be multisets included in  $Q$  and  $\mathcal{X}' = \{X_{m+1}, \dots, X_{m+m'}\}, \mathcal{Y}' = \{Y_{n+1}, \dots, Y_{n+n'}\}$  be multisets included in  $Q^c$ . First, let us show that

$$c_1|m + m' - n - n'| \text{diam}(Q)^p + L(\mathcal{X} \cup \mathcal{X}', \mathcal{Y} \cup \mathcal{Y}') \geq L_{\partial Q}(\mathcal{X}, \mathcal{Y}), \quad (32)$$

with  $c_1 = \kappa(1 + \kappa_0)$ . To do so, let us consider an optimal graph  $G$  for  $L(\mathcal{X} \cup \mathcal{X}', \mathcal{Y} \cup \mathcal{Y}')$ . It uses all the points but  $|m + m' - n - n'|$  points in excess. We consider the subsets  $\mathcal{X}_0 \subset \mathcal{X}$  and  $\mathcal{Y}_0 \subset \mathcal{Y}$  of points that are used in  $G$  and belong to  $Q$ . By definition there exist subsets  $A, B \subset Q^c$  such that  $\text{card}(\mathcal{X}_0 \cup A) = \text{card}(\mathcal{Y}_0 \cup B)$  and  $L(\mathcal{X} \cup \mathcal{X}', \mathcal{Y} \cup \mathcal{Y}') = L(\mathcal{X}_0 \cup A, \mathcal{Y}_0 \cup B)$ . By definition of the boundary functional and using (29),

$$L_{\partial Q}(\mathcal{X}_0, \mathcal{Y}_0) \leq L_{\partial Q}^0(\mathcal{X}_0 \cup A, \mathcal{Y}_0 \cup B) \leq L(\mathcal{X}_0 \cup A, \mathcal{Y}_0 \cup B) = L(\mathcal{X} \cup \mathcal{X}', \mathcal{Y} \cup \mathcal{Y}').$$

Finally, since there are at most  $|n + n' - m - m'|$  points in  $\mathcal{X} \cup \mathcal{Y}$  which are not in  $\mathcal{X}_0 \cup \mathcal{Y}_0$  (i.e. points of  $Q$  not used for the optimal  $G$ ), the modified  $(\mathcal{R}_p)$  property given by Equation (31) yields (32). We apply the latter inequality to  $\mathcal{X}, \mathcal{Y}$  independent Poisson processes of intensity  $\mathbf{1}_Q \cdot \nu$ , and  $\mathcal{X}', \mathcal{Y}'$ , two independent Poisson processes of intensity  $\mathbf{1}_{Q^c} \cdot \nu$ , independent of  $(\mathcal{X}, \mathcal{Y})$ . Then  $\mathcal{X} \cup \mathcal{X}', \mathcal{Y} \cup \mathcal{Y}'$  are independent Poisson processes of intensity  $\nu$ . Taking expectation, we obtain the first inequality, with  $c = 4c_1$ .

We now prove the second inequality. As above, let  $\mathcal{X} = \{X_1, \dots, X_m\}, \mathcal{Y} = \{Y_1, \dots, Y_n\}$  be multisets included in  $Q$ . Let  $G \in \mathcal{G}_k$  be an optimal graph for  $L_{\partial Q}(\mathcal{X}, \mathcal{Y})$  and  $A = \{X_{m+1}, \dots, X_k\}, B = \{Y_{n+1}, \dots, Y_k\}$  be optimal sets in  $Q^c$ . Given this graph  $G$  and a set  $S$ , we denote by  $E_S^0$  the set of edges  $\{i, k+j\}$  of  $G$  such that  $X_i \in S$  and  $Y_j \in S$ , by  $E_S^1$  the set of edges  $\{i, k+j\}$  of  $G$  such that  $X_i \in S$  and  $Y_j \in S^c$ , and by  $E_S^2$  the set of edges  $\{i, k+j\}$  of  $G$  such that  $X_i \in S^c$  and  $Y_j \in S$ . Then by definition of the boundary functional

$$\begin{aligned} L_{\partial Q}(\mathcal{X}, \mathcal{Y}) &= L_{\partial Q}^0(\mathcal{X} \cup A, \mathcal{Y} \cup B) \\ &= \sum_{\{i, k+j\} \in E_Q^0} |X_i - Y_j|^p + \sum_{\{i, k+j\} \in E_Q^1} q d(X_i, Q^c)^p + \sum_{\{i, k+j\} \in E_Q^2} q d(Y_j, Q^c)^p. \end{aligned}$$

Next, we bound these sums from below by considering the cells of the partition  $\mathcal{P}$ . If  $x \in Q$ , we denote by  $P(x)$  the unique  $P \in \mathcal{P}$  that contains  $x$ .

If an edge  $e = \{i, k+j\} \in G$  is such that  $X_i, Y_j$  belong to the same cell  $P$ , we observe that  $e \in E_P^0$  and we leave the quantity  $|X_i - Y_j|^p$  unchanged.

If on the contrary,  $X_i$  and  $Y_j$  belong to different cells, from Hölder inequality,

$$|X_i - Y_j|^p \geq q d(X_i, P(X_i)^c)^p + q d(Y_j, P(Y_j)^c)^p.$$

Eventually, for any boundary edge in  $E_Q^1$ , we lower bound the contribution  $d(X_i, Q^c)^p$  by  $d(X_i, P(X_i)^c)^p$  and we do the same for  $E_Q^2$ . Combining these inequalities and grouping the terms according to the cell  $P \in \mathcal{P}$  to which the points belong,

$$L_{\partial Q}(\mathcal{X}, \mathcal{Y}) \geq \sum_{P \in \mathcal{P}} \left( \sum_{\{i, k+j\} \in E_P^0} |X_i - Y_j|^p + \sum_{\{i, k+j\} \in E_P^1} q d(X_i, \partial P)^p + \sum_{\{i, k+j\} \in E_P^2} q d(Y_j, \partial P)^p \right).$$

For a given cell  $P$ , set  $A' = (\mathcal{X} \cup A) \cap P^c$  and  $B' = (\mathcal{Y} \cup B) \cap P^c$ . We get

$$\begin{aligned} & \sum_{\{i,k+j\} \in E_p^0} |X_i - Y_j|^p + \sum_{\{i,k+j\} \in E_p^1} q d(X_i, \partial P)^p + \sum_{\{i,k+j\} \in E_p^2} q d(Y_j, \partial P)^p \\ &= L_{P^c}^0((\mathcal{X} \cap P) \cup A', (\mathcal{Y} \cap P) \cup B') \geq L_{\partial P}(\mathcal{X} \cap P, \mathcal{Y} \cap P). \end{aligned}$$

So applying these inequalities to  $\mathcal{X}$  and  $\mathcal{Y}$  two independent Poisson point processes with intensity  $\nu \mathbf{1}_Q$  and taking expectation, we obtain the claim.  $\square$

As before, it is crucial to estimate the number of boundary edges, in order to reallocate them.

**Lemma 36.** *Assume (A1-A6) and  $p > 0$ . Let  $Q \subset \mathbb{R}^d$  be a cube of side length 1. Let  $n > 1$ . If  $\varepsilon \in (0, 1]$  then*

- If  $d = 1$ , then  $\mathbb{E}N_{\partial Q}(n\mathbf{1}_Q) \leq c(1)\sqrt{n}$ .
- If  $d = 2$ , then  $\mathbb{E}N_{\partial Q}(n\mathbf{1}_Q) \leq c(2)\sqrt{n} \ln n$ .
- If  $d \geq 3$ , then

$$\mathbb{E}N_{\partial Q}(n\mathbf{1}_Q) \leq c(d)n^{1-\frac{1}{d}} \quad \text{and} \quad \mathbb{E}N_{\partial Q, \varepsilon}(n\mathbf{1}_Q) \leq c(d)\varepsilon^{1-\frac{d}{2}}\sqrt{n}.$$

*Proof.* The proof is close to the proof of Lemma 27. By translation invariance, we may assume that  $Q = [-1/2, 1/2]^d$  is a unit cube with center at the origin and  $\mathcal{X} \cap Q$  is a Poisson process with intensity  $n\mathbf{1}_Q$ . We define  $\mathcal{X}_{\partial Q, \varepsilon}$  and  $\mathcal{Y}_{\partial Q, \varepsilon}$  the sets of points adjacent to a boundary edge in  $\mathcal{X}$  and  $\mathcal{Y}$  respectively.

Assume first that  $\varepsilon = 0$ . Let  $\ell \geq 1$  be the unique integer such that

$$2^{(\ell-1)d} \leq n < 2^{\ell d}.$$

Let  $U_0 = \frac{1}{3}Q$  and for any integer  $0 \leq j \leq \ell$ , let

$$U_j = \left(1 - \frac{2^{-j+1}}{3}\right) \cdot Q = \left[-\frac{1}{2} + \frac{2^{-j}}{3}, \frac{1}{2} - \frac{2^{-j}}{3}\right]^d.$$

In the proof of Lemma 27, we have built a partition  $\mathcal{R}_j$  of  $U_j \setminus U_{j-1}$  into at most  $c_2 2^{j(d-1)}$  subcubes with  $c_1 = 2d3^{d-1}2^{md}$ . We do the same here, with the following slight change: we make a finer final refinement by choosing  $m$  the minimal integer such that  $2^m > q^{-1/p}\sqrt{d}$ . The important property of this partition is that for each  $R \in \mathcal{R}_j$ ,  $0 \leq j \leq \ell$ ,

$$q^{-\frac{1}{p}} \text{diam}(R) < \text{dist}(R, \partial Q) = \inf_{x \in R} \text{dist}(x, \partial Q).$$

If  $N_{\partial Q}(R)$  is the number of boundary edges with an adjacent point in  $R$ , from the bounded degree property (A3),

$$N_{\partial Q}(R) \leq \kappa(\mathcal{X}_{\partial Q}(R) + \mathcal{Y}_{\partial Q}(R)).$$

We now check that  $\mathcal{X}_{\partial Q}(R)$  and  $\mathcal{Y}_{\partial Q}(R)$  cannot be both larger than  $\kappa\kappa_0$ . Otherwise, from the bounded degree property, for each species, we could find  $\kappa_0$  boundary edges with distinct adjacent points in the optimal graph  $G$  corresponding to the boundary functional. That is, there would exist two sets  $\{x_1, \dots, x_{\kappa_0}, x'_1, \dots, x'_{\kappa_0}\}$  and  $\{y'_1, \dots, y'_{\kappa_0}, y_1, \dots, y_{\kappa_0}\}$ , such that for  $1 \leq i \leq \kappa_0$ ,  $x_i \in \mathcal{X} \cap R$ ,  $y_i \in \mathcal{Y} \cap R$ ,  $x'_i, y'_i$  are in  $Q^c$ , and  $\{x_i, y'_i\}$  and  $\{x'_i, y_i\}$  are boundary edges. In this case, the shuffling property (A6) would provide a new graph  $G'$  with a strictly lower cost, which would lead to a contradiction since  $G$  was minimal. Indeed, the new graph  $G'$  defined by (30) is obtained by the following operations:

- Each boundary edge of the form  $\{x_i, y'_i\}$ ,  $i \leq \kappa_0$ , is replaced by an edge of the form  $\{x_i, b_i\}$  for some  $b_i \in \{y_1, \dots, y_{\kappa_0}, y'_1, \dots, y'_{\kappa_0}\}$ . If  $b_i \in \{y'_1, \dots, y'_{\kappa_0}\}$  then the new edge  $\{x_i, b_i\}$  is still a boundary edge and its cost is the same as for the initial edge, namely  $q d(x_i, Q^c)^p$ . If  $b_i \in \{y_1, \dots, y_{\kappa_0}\}$  then the new edge is an inner edge and it has a strictly smaller cost than the initial boundary edge since

$$|x_i - b_i|^p \leq \text{diam}(R)^p < q \text{dist}(R, Q^c)^p \leq q d(x_i, Q^c)^p.$$

The property (A6) ensures that at least one of the new edges is an inner edge and thus has strictly lower cost.

- Each boundary edge of the form  $\{x'_i, y_i\}$ ,  $i \leq \kappa_0$ , is replaced by an edge of the form  $\{a_i, y_i\}$  for some  $a_i \in \{y_1, \dots, y_{\kappa_0}, y'_1, \dots, y'_{\kappa_0}\}$ . By the same reasoning, this operation cannot increase the cost.
- Outer edges can be added (i.e. connecting points in  $Q^c$ ). They do not contribute in the calculation of the boundary functional.
- Eventually some edges may be deleted (this cannot increase the cost of course).

Since the number of points in  $R$  cannot be larger than  $\kappa\kappa_0$  for both species,

$$\mathcal{X}_{\partial Q}(R) + \mathcal{Y}_{\partial Q}(R) \leq |\mathcal{X}(R) - \mathcal{Y}(R)| + \kappa\kappa_0(\mathbf{1}_{\mathcal{X}(R)>0} + \mathbf{1}_{\mathcal{Y}(R)>0}).$$

We deduce that

$$\begin{aligned} N_{\partial Q} &\leq N_{\partial Q}(Q \setminus U_\ell) + \sum_{j=0}^{\ell} \sum_{R \in \mathcal{R}_j} N_{\partial Q}(R) \\ &\leq \kappa(\mathcal{X}(Q \setminus U_\ell) + \mathcal{Y}(Q \setminus U_\ell)) + \kappa^2 \kappa_0 \sum_{j=0}^{\ell} \sum_{R \in \mathcal{R}_j} (|\mathcal{X}(R) - \mathcal{Y}(R)| + \mathbf{1}_{\mathcal{X}(R)>0} + \mathbf{1}_{\mathcal{Y}(R)>0}). \end{aligned}$$

As already pointed,  $1 - e^{-t} \leq \sqrt{t}$  for all  $t \geq 0$ . The moat  $Q \setminus U_\ell$  has width  $(2/3)2^{-\ell}$ . Thus taking expectation, for some positive constants  $c_2, c_3$  depending only on  $\kappa, \kappa_0$  and on the dimension,  $d \geq 3$ ,

$$\begin{aligned} \mathbb{E}N_{\partial Q} &\leq \frac{8d\kappa}{3}n2^{-\ell} + \sum_{j=0}^{\ell} c_2 2^{j(d-1)} \sqrt{n2^{-jd}}. \\ &\leq c_3 \left( n2^{-\ell} + \sqrt{n} 2^{\ell(\frac{d}{2}-1)} \right), \end{aligned}$$

while for  $d = 2$ , we get  $\mathbb{E}N_{\partial Q}(Q) \leq c_3(n2^{-\ell} + \ell\sqrt{n})$ . Finally, if  $d = 1$  the above series is convergent and  $\mathbb{E}N_{\partial Q}(Q) \leq c_3(n2^{-\ell} + \sqrt{n})$ . With our choice of  $\ell$ , we obtain the required statement for  $\varepsilon = 0$ .

We now assume that  $0 < \varepsilon \leq 1$  and  $d \geq 3$ . We redefine  $\ell$  as the unique integer such that

$$\frac{2^{-\ell+1}}{3} \leq \varepsilon < \frac{2^{-\ell+2}}{3}.$$

For  $0 \leq j \leq \ell$ , we consider the same partition  $\mathcal{R}_j$  as above. The moat  $Q \setminus U_\ell$  is partitioned into subcubes of side-length  $2^{-\ell-1}/3$ . Each of these subcubes is further partitioned into  $2^{md}$  subcubes where  $m$  is minimal subject to the constraint  $2^m > q^{-1/p}\sqrt{d}$ . We then obtain a partition  $\mathcal{R}_{\ell+1}$  of  $Q \setminus U_\ell$  into at most  $2c_1 2^{(\ell+1)(d-1)}$  subcubes. Also, due to our choice of  $\ell$ , each  $R \in \mathcal{R}_j$ ,  $0 \leq j \leq \ell + 1$ , enjoys the property

$$q^{-\frac{1}{p}} \text{diam}(R) < \max(\text{dist}(R, \partial Q), \varepsilon).$$

Hence, arguing as above, from the local stability property,  $\mathcal{X}_{\partial Q}(R)$  and  $\mathcal{Y}_{\partial Q}(R)$  cannot be both larger than  $\kappa\kappa_0$ . It follows

$$N_{\partial Q, \varepsilon} \leq \kappa^2 \kappa_0 \sum_{j=0}^{\ell+1} \sum_{R \in \mathcal{R}_j} (|\mathcal{X}(R) - \mathcal{Y}(R)| + \mathbf{1}_{\mathcal{X}(R)>0} + \mathbf{1}_{\mathcal{Y}(R)>0}).$$

Taking expectation, for some positive constants  $c_4$ ,

$$\mathbb{E}N_{\partial Q, \varepsilon} \leq \sum_{j=0}^{\ell+1} 2c_2 2^{j(d-1)} \sqrt{n2^{-jd}} = c_4 \sqrt{n} 2^{\ell(\frac{d}{2}-1)}.$$

With our choice of  $\ell$ , we obtain the required statement.  $\square$

Let  $Q = [0, 1]^d$  and denote

$$\bar{L}_{\partial Q}(n) = \mathbb{E}L_{\partial Q}(n\mathbf{1}_Q).$$

**Lemma 37.** *Assume (A1-A7). Let  $Q \subset \mathbb{R}^d$  be a cube of side-length 1. If either  $d \geq 3$  and  $0 < p < (\sqrt{2d-3} + 1)/2$ , or  $d \in \{1, 2\}$  and  $0 < 2p < d$ , then*

$$\liminf_{n \rightarrow \infty} \frac{\bar{L}_{\partial Q}(n)}{n^{1-\frac{p}{d}}} \geq \beta_L,$$

where  $\beta_L$  is the constant from Theorem 15.

*Proof.* In order to prove this lemma, we shall lower bound  $L_{\partial Q, \varepsilon}$  by an expression involving  $L$ . Let  $\varepsilon \in [0, 1]$  and  $\mathcal{X} = \{X_1, \dots, X_\ell\}$ ,  $\mathcal{Y} = \{Y_1, \dots, Y_m\}$  be contained in  $Q$ . We choose an optimal triplet  $(G, A, B)$  in the definition of the boundary cost  $L_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) = L_{\partial Q, \varepsilon}^0(\mathcal{X} \cup A, \mathcal{Y} \cup B)$ . Let  $A = \{X_{\ell+1}, \dots, X_k\}$ ,  $B = \{Y_{m+1}, \dots, Y_k\}$ . We define  $E_Q^0$  as the set of non-boundary edges, and  $E_Q^1, E_Q^2$  as the sets of boundary edges with an incident point from the first and second species respectively. We get

$$\begin{aligned} L_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) &= \sum_{e=\{i, k+j\} \in E_Q^0} |X_i - Y_j|^p + q \sum_{e=\{i, k+j\} \in E_Q^1} (d(X_i, \partial Q)^p + \varepsilon^p) + q \sum_{e=\{i, k+j\} \in E_Q^2} (d(Y_j, \partial Q)^p + \varepsilon^p) \\ &\geq \sum_{e=\{i, k+j\} \in E_Q^0} |X_i - Y_j|^p. \end{aligned}$$

Assume for example  $\ell \leq m$ . Let  $G'$  be the restriction of  $G$  to  $\mathcal{B}_\ell$ . This means that the points outside  $Q$  as well as the excess points  $Y_{\ell+1}, \dots, Y_m$  have been erased, as well as their edges. In particular, the number of vertices of  $G'$  which have a lower degree than in  $G$  is at most  $N_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) + \kappa|\ell - m|$  (the second term comes from removing the points in excess, and the bounded degree property (A3)). The above reasoning shows that  $L_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) \geq \sum_{e=\{i, \ell+j\} \in G'} |X_i - Y_j|^p$ . This is where the strong restriction property (A7) is used: it ensures the existence of a graph  $G'' \in \mathcal{G}_\ell$  which differs from  $G'$  by at most  $\kappa(N_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) + \kappa|\ell - m|)$  edges. Consequently

$$L_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) \geq L(\mathcal{X}, \mathcal{Y}) - \kappa \text{diam}(Q)^p (N_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) + \kappa|\ell - m|).$$

Applying when  $\mathcal{X}, \mathcal{Y}$  are independent Poisson point processes of intensity  $n \mathbb{1}_Q(x) dx$  yields

$$\mathbb{E}L_{\partial Q, \varepsilon}(n \mathbb{1}_Q) \geq \mathbb{E}L(n \mathbb{1}_Q) - c_1 \text{diam}(Q)^p (\mathbb{E}N_{\partial Q, \varepsilon}(n \mathbb{1}_Q) + \sqrt{n}). \quad (33)$$

*Case :  $0 < p < 1$ .* Then, for  $\varepsilon = 0$ , if  $d \geq 3$ , taking expectations, we get from Lemma 36 that

$$\mathbb{E}L_{\partial Q}(n \mathbb{1}_Q) \geq \mathbb{E}L(n \mathbb{1}_Q) - c_2(n^{1-\frac{1}{d}} + \sqrt{n}).$$

If  $d = 2$ , we get instead  $\mathbb{E}L_{\partial Q}(n \mathbb{1}_Q) \geq \mathbb{E}L(n \mathbb{1}_Q) - c_2\sqrt{n} \ln n$ , while if  $d = 1$ , we obtain  $\mathbb{E}L_{\partial Q}(n \mathbb{1}_Q) \geq \mathbb{E}L(n \mathbb{1}_Q) - c_2\sqrt{n}$ . We divide by  $n^{1-\frac{p}{d}}$  and take the limit. Thanks to Theorem 15 and the assumption  $0 < p < 1$ , we obtain the claim of the lemma.

*Case :  $1 \leq p < (\sqrt{2d-3} + 1)/2$ .* Note that in this case,  $d \geq 3$  and  $q = 1$ . We shall now use the penalized boundary functional  $L_{\partial Q, \varepsilon}$  for some well chosen  $\varepsilon > 0$ . Again, let  $\mathcal{X}, \mathcal{Y}$  be independent Poisson point processes with intensity  $n \mathbb{1}_Q$  and  $\varepsilon > 0$ . We choose an optimal triplet  $(G, A, B)$  in the definition of the boundary cost  $L_{\partial Q}(\mathcal{X}, \mathcal{Y}) = L_{\partial Q}^0(\mathcal{X} \cup A, \mathcal{Y} \cup B)$ . With the above notation, we may write,

$$\begin{aligned} L_{\partial Q}(\mathcal{X}, \mathcal{Y}) &= \sum_{e=\{i, k+j\} \in E_Q^0} |X_i - Y_j|^p + \sum_{e=\{i, k+j\} \in E_Q^1} d(X_i, \partial Q)^p + \sum_{e=\{i, k+j\} \in E_Q^2} d(Y_j, \partial Q)^p \\ &= \sum_{e=\{i, k+j\} \in E_Q^0} |X_i - Y_j|^p + \sum_{e=\{i, k+j\} \in E_Q^1} (d(X_i, \partial Q)^p + \varepsilon^p) \\ &\quad + \sum_{e=\{i, k+j\} \in E_Q^2} (d(Y_j, \partial Q)^p + \varepsilon^p) - \varepsilon^p N_{\partial Q} \\ &\geq L_{\partial Q, \varepsilon}(\mathcal{X}, \mathcal{Y}) - \varepsilon^p N_{\partial Q}. \end{aligned}$$

Taking expectations and using Lemma 36 gives

$$\mathbb{E}L_{\partial Q}(n \mathbb{1}_Q) \geq \mathbb{E}L_{\partial Q, \varepsilon}(n \mathbb{1}_Q) - \varepsilon^p \mathbb{E}N_{\partial Q}(n \mathbb{1}_Q) \geq \mathbb{E}L_{\partial Q, \varepsilon}(n \mathbb{1}_Q) - c\varepsilon^p n^{\frac{d-1}{d}}.$$

Hence, if  $\varepsilon_n := \left(\delta n^{\frac{1-p}{d}}\right)^{\frac{1}{p}}$ , with  $0 < \delta < 1$ ,

$$\liminf_{n \rightarrow \infty} \frac{\bar{L}_{\partial Q}(n)}{n^{1-\frac{1}{d}}} \geq \liminf_{n \rightarrow \infty} \frac{\mathbb{E}L_{\partial Q, \varepsilon_n}(n\mathbb{1}_Q)}{n^{1-\frac{1}{d}}} - 2c\delta. \quad (34)$$

We combine (33) and Lemma 36 to get

$$\mathbb{E}L_{\partial Q, \varepsilon}(n\mathbb{1}_Q) \geq \mathbb{E}L(n\mathbb{1}_Q) - c_2\sqrt{n}\varepsilon^{1-\frac{d}{2}} - c_2\sqrt{n}.$$

Applied to  $\varepsilon_n$ , dividing by  $n^{1-\frac{d}{2}}$  and taking the limit we obtain thanks to Theorem 15 that

$$\liminf_n \frac{\mathbb{E}L_{\partial Q, \varepsilon_n}(n\mathbb{1}_Q)}{n^{1-\frac{1}{d}}} \geq \beta_L$$

is valid for any  $\delta > 0$ . Combining this inequality with (34) and letting  $\delta$  tend to zero gives the claim of the lemma.  $\square$

### 6.3.2 General absolutely continuous measures with unbounded support

**Theorem 38.** *Assume (A1-A7). Assume that either  $d \in \{1, 2\}$  and  $0 < 2p < d$ , or  $d \geq 3$  and  $0 < p < (\sqrt{2d-3} + 1)/2$ . Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$  be an integrable function. We assume that for some  $\alpha > \frac{2dp}{d-2p}$ ,  $\int |x|^\alpha f < +\infty$ . Then*

$$\lim_n \frac{\mathbb{E}L(nf)}{n^{1-\frac{p}{d}}} = \beta_L \int_{\mathbb{R}^d} f^{1-\frac{p}{d}}.$$

*Proof.* First, from Theorem 17, it is sufficient to check that

$$\liminf_n \frac{\mathbb{E}L(nf)}{n^{1-\frac{p}{d}}} \geq \beta_L \int f^{1-\frac{p}{d}}.$$

The proof is now formally the same than the proof of Theorem 29, invoking Lemmas 34, 35 and 37 in place of Lemmas 26, 25 and 28 respectively.  $\square$

*Remark 7.* Finding good lower bounds for a general bipartite functional  $L$  on  $\mathbb{R}^d$  satisfying the properties  $(\mathcal{H}_p)$ ,  $(\mathcal{R}_p)$ ,  $(\mathcal{S}_p)$  could be significantly more difficult. It is far from obvious to define a proper boundary functional  $L_{\partial Q}$  at this level of generality. However if there exists a bipartite functional  $L_{\partial Q}$  on  $\mathbb{R}^d$  indexed on sets  $Q \subset \mathbb{R}^d$  such that for any  $t > 0$ ,  $\mathbb{E}L_{\partial(tQ)}(n\mathbb{1}_{tQ}) = t^p \mathbb{E}L_{\partial Q}(nt^d \mathbb{1}_Q)$  and such that the statements of Lemmas 35, 34, 37 hold, then the statement of Theorem 29 also holds for the functional  $L$ . Thus, the caveat of this kind of techniques lies in the good definition of a boundary functional  $L_{\partial Q}$ .

### 6.3.3 Dealing with the singular component. Example of the traveling salesperson problem.

Let  $p \in (0, 1]$ . We shall say that a bipartite functional  $L$  on  $\mathbb{R}^d$  satisfies the inverse subadditivity property  $(\mathcal{I}_p)$  if there is a constant  $C$  such that for all finite multisets  $\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$  included in a bounded set  $Q \subset \mathbb{R}^d$ ,

$$L(\mathcal{X}_1, \mathcal{Y}_1) \leq L(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) + L(\mathcal{X}_2, \mathcal{Y}_2) + C \text{diam}(Q)^p (1 + |\mathcal{X}_1(Q) - \mathcal{Y}_1(Q)| + |\mathcal{X}_2(Q) - \mathcal{Y}_2(Q)|).$$

Although it makes sense for all  $p$ , we have been able to check this property on examples only for  $p \in (0, 1]$ . Also we could have added a constant in front of  $L(\mathcal{X}_2, \mathcal{Y}_2)$ .

It is plain that the argument of Section 6.2.3 readily adapts to a functional satisfying  $(\mathcal{I}_p)$ , for which one already knows a general upper limit result and a limit result for absolutely continuous laws. It therefore provides a limit result for general laws. In the remainder of this section, we show that the traveling salesperson bipartite tour functional  $L = T_p$ ,  $p \in (0, 1]$  enjoys the inverse subadditivity property. This allows to prove the following result:

**Theorem 39.** *Assume that either  $d \in \{1, 2\}$  and  $0 < 2p < d$ , or  $d \geq 3$  and  $p \in (0, 1]$ . Let  $L = T_p$  be the traveling salesperson bipartite tour functional. Let  $\mu$  be a finite measure such that for some  $\alpha > \frac{2dp}{d-2p}$ ,  $\int |x|^\alpha d\mu < +\infty$ . Then, if  $f$  is a density function for the absolutely continuous part of  $\mu$ ,*

$$\lim_n \frac{\mathbb{E}L(n\mu)}{n^{1-\frac{p}{d}}} = \beta_L \int_{\mathbb{R}^d} f^{1-\frac{p}{d}}.$$

All we have to do is to check property  $(\mathcal{I}_p)$ . More precisely:

**Lemma 40.** *Assume  $p \in (0, 1]$  and  $L = T_p$ . For any set  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$  in a bounded set  $Q$*

$$L(\mathcal{X}_1, \mathcal{Y}_1) \leq L(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) + L(\mathcal{X}_2, \mathcal{Y}_2) \\ + 2 \operatorname{diam}(Q)^p (1 + |\operatorname{card}(\mathcal{X}_1) - \operatorname{card}(\mathcal{Y}_1)| + |\operatorname{card}(\mathcal{X}_2) - \operatorname{card}(\mathcal{Y}_2)|).$$

*Proof.* We may assume without loss of generality that  $\operatorname{card}(\mathcal{X}_1) \wedge \operatorname{card}(\mathcal{Y}_1) \geq 2$ , otherwise,  $L(\mathcal{X}_1, \mathcal{Y}_1) = 0$  and there is nothing to prove. Consider an optimal cycle  $G_{1,2}$  for  $L(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2)$ . In  $G_{1,2}$ ,  $m = |\operatorname{card}(\mathcal{X}_1) + \operatorname{card}(\mathcal{Y}_1) - \operatorname{card}(\mathcal{X}_2) - \operatorname{card}(\mathcal{Y}_2)| \leq |\operatorname{card}(\mathcal{X}_1) - \operatorname{card}(\mathcal{Y}_1)| + |\operatorname{card}(\mathcal{X}_2) - \operatorname{card}(\mathcal{Y}_2)|$  points have been left aside. We shall build a bipartite tour  $G_1$  on  $(\mathcal{X}'_1, \mathcal{Y}'_1)$ , the points of  $(\mathcal{X}_1, \mathcal{Y}_1)$  that have not been left aside by  $G_{1,2}$ .

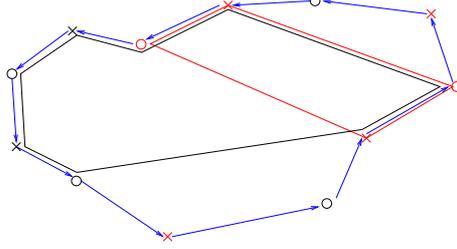


Figure 4: In blue, the oriented cycle  $G_{1,2}$ , in red  $G_2$ , in black  $G'_{1,2}$ . The points in  $\mathcal{X}_1 \cup \mathcal{X}_2$  are represented by a cross, points in  $\mathcal{Y}_1 \cup \mathcal{Y}_2$  by a circle.

We consider an optimal cycle  $G_2$  for  $L(\mathcal{X}'_2, \mathcal{Y}'_2)$ , where  $(\mathcal{X}'_2, \mathcal{Y}'_2)$  are the points of  $(\mathcal{X}_2, \mathcal{Y}_2)$  that have not been left aside by  $G_{1,2}$ . We define  $(\mathcal{X}''_2, \mathcal{Y}''_2) \subset (\mathcal{X}'_2, \mathcal{Y}'_2)$  as the sets of points that are in  $G_2$ . Since  $\operatorname{card}(\mathcal{X}'_1) + \operatorname{card}(\mathcal{X}'_2) = \operatorname{card}(\mathcal{Y}'_1) + \operatorname{card}(\mathcal{Y}'_2)$ , we get  $\operatorname{card}(\mathcal{X}'_1) - \operatorname{card}(\mathcal{Y}'_1) = -\operatorname{card}(\mathcal{X}'_2) + \operatorname{card}(\mathcal{Y}'_2)$ . It implies that the same number of points from the opposite type need to be removed in  $(\mathcal{X}'_1, \mathcal{Y}'_1)$  and  $(\mathcal{X}'_2, \mathcal{Y}'_2)$  in order to build a bipartite tour. We fix an orientation on  $G_{1,2}$ . Assume for example that  $\operatorname{card}(\mathcal{X}'_2) \geq \operatorname{card}(\mathcal{Y}'_2)$ , if a point  $x \in \mathcal{X}'_2 \setminus \mathcal{X}''_2$ , we then remove the next point  $y$  on the oriented cycle  $G_{1,2}$  of  $\mathcal{Y}'_1$ . Doing so, this defines a couple of sets  $(\mathcal{X}''_1, \mathcal{Y}''_1) \subset (\mathcal{X}'_1, \mathcal{Y}'_1)$  of cardinality  $\operatorname{card}(\mathcal{X}'_1) \wedge \operatorname{card}(\mathcal{Y}'_1)$  and

$$L(\mathcal{X}''_1, \mathcal{Y}''_1) \leq L(\mathcal{X}'_1, \mathcal{Y}'_1).$$

We define  $G'_{1,2}$  as the cycle on  $(\mathcal{X}''_1 \cup \mathcal{X}''_2, \mathcal{Y}''_1 \cup \mathcal{Y}''_2)$  obtained from  $G_{1,2}$  by saying that the point after  $x \in (\mathcal{X}''_1 \cup \mathcal{X}''_2, \mathcal{Y}''_1 \cup \mathcal{Y}''_2)$  in the oriented cycle  $G'_{1,2}$  is the next point  $y \in (\mathcal{X}''_1 \cup \mathcal{X}''_2, \mathcal{Y}''_1 \cup \mathcal{Y}''_2)$  in  $G_{1,2}$ . By construction,  $G'_{1,2}$  is a bipartite cycle. Also, since  $p \in (0, 1]$ , we may use the triangle inequality : the distance between two successive points in the circuit  $G'_{1,2}$  is bounded by the sum of the length of the intermediary edges in  $G_{1,2}$ . We get

$$L(\mathcal{X}''_1 \cup \mathcal{X}''_2, \mathcal{Y}''_1 \cup \mathcal{Y}''_2) \leq L(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2).$$

Now consider the (multi) graph  $G = G'_{1,2} \cup G_2$  obtained by adding all edges of  $G'_{1,2}$  and  $G_2$ . This graph is bipartite, connected, and points in  $(\mathcal{X}''_1, \mathcal{Y}''_1)$  have degree 2 while those in  $(\mathcal{X}''_2, \mathcal{Y}''_2)$  have degree 4. Let  $k$  be the number of edges in  $G$ , we recall that an eulerian circuit in  $G$  is a sequence  $E = (e_1, \dots, e_k)$  of adjacent edges in  $G$  such that  $e_k$  is also adjacent to  $e_1$  and all edges of  $G$  appears exactly once in the sequence  $E$ . By the Euler's circuit theorem, there exists an eulerian circuit in  $G$ . Moreover, this eulerian circuit can be chosen so that if  $e_i = \{u_{i-1}, u_i\} \in G_2$  then  $e_{i+1} = \{u_{i+1}, u_i\} \in G'_{1,2}$  with the convention that  $e_{k+1} = e_1$ .

This sequence  $E$  defines an oriented circuit of points. Now we define an oriented circuit on  $(\mathcal{X}''_1, \mathcal{Y}''_1)$ , by connecting a point  $x$  of  $(\mathcal{X}''_1, \mathcal{Y}''_1)$  to the next point  $y$  in  $(\mathcal{X}''_1, \mathcal{Y}''_1)$  visited by the oriented circuit  $E$ . Due to the property that  $e_i \in G_2$  implies  $e_{i+1} \in G'$ , if  $x \in \mathcal{X}''_1$  then  $y \in \mathcal{Y}''_1$  and conversely, if  $x \in \mathcal{Y}''_1$  then  $y \in \mathcal{X}''_1$ . Hence, this oriented circuit defines a bipartite cycle  $G_1$  in  $(\mathcal{X}''_1, \mathcal{Y}''_1)$ .

By the triangle inequality, the distance between two successive points in the circuit  $G_1$  is bounded by the sum of the length of the intermediary edges in  $E$ . Since each edge of  $G$  appears exactly once in  $E$ , it follows that

$$L(\mathcal{X}''_1, \mathcal{Y}''_1) \leq L(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) + L(\mathcal{X}''_2, \mathcal{Y}''_2).$$

To conclude, we merge arbitrarily to the cycle  $G_1$  the remaining points of  $(\mathcal{X}_1, \mathcal{Y}_1)$ , there are at most  $m$  of them (regularity  $(\mathcal{R}_p)$  property).  $\square$

## 7 Variants and final comments

As a conclusion, we briefly discuss variants and possible extensions of Theorem 2. For  $d > 2p$  and when  $\mu$  is the uniform distribution on the cube  $[0, 1]^d$ , there exists a constant  $\beta_p(d) > 0$  such that almost surely

$$\lim_{n \rightarrow \infty} n^{\frac{d}{2}-1} M_p(\{X_1, \dots, X_n\}, \{Y_1, \dots, Y_n\}) = \beta_p(d).$$

A natural question is to understand what happens below the critical line  $d = 2p$ , i.e. when  $d \leq 2p$ . For example for  $d = 2$  and  $p = 1$ , a similar convergence is also expected in dimension 2 with scaling  $\sqrt{n \ln n}$ , but this is a difficult open problem. The main result in this direction goes back to Ajtai, Komlós and Tusnády [1]. See also the improved upper bound of Talagrand and Yukich in [17]. In dimension 1, there is no such stabilization to a constant.

Recall that

$$\left( \frac{1}{n} M_p(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n) \right)^{\frac{1}{p}} = W_p \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \frac{1}{n} \sum_{i=1}^n \delta_{Y_i} \right).$$

where  $W_p$  is the  $L_p$ -Wasserstein distance. A variant of Theorem 3 can be obtained along the same lines, concerning the convergence of

$$n^{\frac{1}{d}} W_p \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \mu \right),$$

where  $\mu$  is the common distribution of the  $X_i$ 's. Such results are of fundamental importance in statistics. Also note that combining the triangle inequality and Jensen inequality, it is not hard to see that

$$\mathbb{E} W_1 \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \mu \right) \leq \mathbb{E} W_1 \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \frac{1}{n} \sum_{i=1}^n \delta_{Y_i} \right) \leq 2 \mathbb{E} W_1 \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \mu \right),$$

(similar inequalities hold for  $p \geq 1$ ). Hence it is clear that the behaviour of this functional is quite close to the one of the two-sample optimal matching. However, the extension of Theorem 2 would require some care in the definition of the boundary functional.

Finally, it is worthy to note that the case of uniform distribution for  $L = M_p$  has a connection with stationary matchings of two independent Poisson point processes of intensity 1, see Holroyd, Pemantle, Peres and Schramm [6]. Indeed, consider mutually independent random variables  $(X_i)_{i \geq 1}$  and  $(Y_j)_{j \geq 1}$  having uniform distribution on  $Q = [-1/2, 1/2]^d$ . It is well known that for any  $x$  in the interior of  $Q$ , the pair of point processes

$$\left( \frac{1}{n} \sum_{i=1}^n \delta_{n^{\frac{1}{d}}(X_i - x)}, \frac{1}{n} \sum_{i=1}^n \delta_{n^{\frac{1}{d}}(Y_i - x)} \right)$$

converges weakly for the topology of vague convergence to  $(\Xi_1, \Xi_2)$ , where  $\Xi_1$  and  $\Xi_2$  are two independent Poisson point processes of intensity 1. Also, we may write

$$n^{\frac{d}{2}-1} \mathbb{E} M_p(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n) = \frac{1}{n} \mathbb{E} \sum_{i=1}^n \left| n^{\frac{1}{d}}(X_i - x) - n^{\frac{1}{d}}(Y_{\sigma_n(i)} - x) \right|^p.$$

where  $\sigma_n$  is an optimal matching. Now, the fact that for  $0 < p < 2d$ ,  $\lim_n n^{\frac{d}{2}-1} \mathbb{E} M_p(\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n) = \beta_p(d)$  implies the tightness of the sequence of matchings  $\sigma_n$  and it can be used to define a stationary matching  $\sigma$  on  $(\Xi_1, \Xi_2)$ , see the proof of Theorem 1 (iii) in [6] for the details of such an argument. In particular, this matching  $\sigma$  will enjoy a local notion of minimality for the  $L_p$ -norm, as defined by Holroyd in [5] (for the  $L_1$ -norm). See also related work of Huesmann and Sturm [7].

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