

# A remark on the slicing problem

A. Giannopoulos, G. Paouris and B-H. Vritsiou

## Abstract

The purpose of this article is to describe a reduction of the slicing problem to the study of the parameter  $I_1(K, Z_q^\circ(K)) = \int_K \|\langle \cdot, x \rangle\|_{L_q(K)} dx$ . We show that an upper bound of the form  $I_1(K, Z_q^\circ(K)) \leq C_1 q^s \sqrt{n} L_K^2$ , with  $1/2 \leq s \leq 1$ , leads to the estimate

$$L_n \leq \frac{C_2 \sqrt[4]{n} \log n}{q^{\frac{1-s}{2}}},$$

where  $L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\}$ .

## 1 Introduction

A convex body  $K$  in  $\mathbb{R}^n$  is called isotropic if it has volume  $|K| = 1$ , it is *centered*, i.e. its center of mass is at the origin, and if its inertia matrix is a multiple of the identity. The last property is equivalent to the existence of a constant  $L_K > 0$  such that

$$(1.1) \quad \int_K \langle x, \theta \rangle^2 dx = L_K^2$$

for every  $\theta$  in the Euclidean unit sphere  $S^{n-1}$ . It is not hard to see that for every convex body  $K$  in  $\mathbb{R}^n$  there exists an affine transformation  $T$  of  $\mathbb{R}^n$  such that  $T(K)$  is isotropic. Moreover, this isotropic image is unique up to orthogonal transformations; consequently, one may define the isotropic constant  $L_K$  as an invariant of the affine class of  $K$ .

The isotropic constant is closely related to the hyperplane conjecture (also known as the slicing problem) which asks if there exists an absolute constant  $c > 0$  such that  $\max_{\theta \in S^{n-1}} |K \cap \theta^\perp| \geq c$  for every convex body  $K$  of volume 1 in  $\mathbb{R}^n$  with center of mass at the origin. This is because, by Brunn's principle, for any convex body  $K$  in  $\mathbb{R}^n$  and any  $\theta \in S^{n-1}$ , the function  $t \mapsto |K \cap (\theta^\perp + t\theta)|^{\frac{1}{n-1}}$  is concave on its support, and this is enough to show that

$$(1.2) \quad \int_K \langle x, \theta \rangle^2 dx \simeq |K \cap \theta^\perp|^{-2}.$$

Using this relation one can check that an affirmative answer to the slicing problem is equivalent to the following statement: "There exists an absolute constant  $C > 0$

such that  $L_K \leq C$  for every convex body  $K^n$ . We refer to the article [13] of Milman and Pajor for background information about isotropic convex bodies.

It is known that  $L_K \geq L_{B_2^n} \geq c > 0$  for every convex body  $K$  in  $\mathbb{R}^n$  (we use the letters  $c, c_1, C$  etc. to denote absolute constants). In the opposite direction, let us write  $L_n$  for the maximum of all isotropic constants of convex bodies in  $\mathbb{R}^n$ ,

$$(1.3) \quad L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\}.$$

Bourgain first proved in [4] that  $L_n \leq c\sqrt[n]{n} \log n$  and, a few years ago, Klartag [8] obtained the estimate  $L_n \leq c\sqrt[n]{n}$  (see also [9] for a second proof of this bound).

The purpose of this article is to describe a reduction of the slicing problem (or, equivalently, the question whether  $L_n$  can be bounded by a quantity independent of the dimension  $n$ ), to the study of the parameter

$$(1.4) \quad I_1(K, Z_q^\circ(K)) = \int_K \|\langle \cdot, x \rangle\|_{L_q(K)} dx$$

for isotropic convex bodies  $K$ . Generally, if  $K$  is a centered convex body of volume 1 in  $\mathbb{R}^n$ , then for every symmetric convex body  $C$  in  $\mathbb{R}^n$  and for every  $q \in (-n, \infty)$ ,  $q \neq 0$ , we define

$$(1.5) \quad I_q(K, C) := \left( \int_K \|x\|_C^q dx \right)^{1/q}.$$

The notation  $I_1(K, Z_q^\circ(K))$  is then justified by the fact that  $\|\langle \cdot, x \rangle\|_{L_q(K)}$  is the norm induced on  $\mathbb{R}^n$  by the polar body  $Z_q^\circ(K)$  of the  $L_q$ -centroid body of  $K$  (see the next section for background information on  $L_q$ -centroid bodies).

Our reduction can be viewed as a continuation of Bourgain's approach to the slicing problem in [4]: the bound  $O(\sqrt[n]{n} \log n)$  followed from the inequality

$$(1.6) \quad nL_K^2 \leq I_1(K, (T(K))^\circ),$$

after obtaining an upper bound for the quantity  $I_1(K, (T(K))^\circ)$ , where  $T \in SL(n)$  is a symmetric, positive definite matrix such that the mean width of  $T(K)$  satisfies the estimate  $w(T(K)) \leq c\sqrt{n} \log n$  (the existence of such a position for  $K$  is guaranteed by Pisier's estimate on the norm of the Rademacher projection; see [19]). In Section 4 we prove the following statement:

**Theorem 1.1.** *There exists an absolute constant  $\rho \in (0, 1)$  with the following property: given  $\kappa, \tau \geq 1$ , for every  $n \geq n_0(\tau)$  and every isotropic convex body  $K$  in  $\mathbb{R}^n$  which satisfies the following entropy estimate:*

$$(1.7) \quad \log N(K, tB_2^n) \leq \frac{\kappa n^2 \log^2 n}{t^2} \text{ for all } t \geq \tau \sqrt{n \log n},$$

*we have that, if  $q \geq 2$  satisfies*

$$(1.8) \quad 2 \leq q \leq \rho^2 n \text{ and } I_1(K, Z_q^\circ(K)) \leq \rho n L_K^2,$$

then

$$(1.9) \quad L_K^2 \leq C\kappa \sqrt{\frac{n}{q}} \log^2 n \max \left\{ 1, \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2} \right\}.$$

Theorem 1.1 can lead to an upper bound for  $L_n$ , provided that there exist  $(\kappa, \tau)$ -regular isotropic convex bodies in  $\mathbb{R}^n$ , i.e. bodies which satisfy the entropy estimate (1.7) for a pair of constants  $\kappa, \tau$ , and at the same time have maximal isotropic constant, i.e.  $L_K \simeq L_n$ . The existence of such bodies is essentially established by [5, Theorem 5.7]. In Section 5 we give a self-contained proof of this fact; see Theorem 5.1.

Observe that, for every isotropic convex body  $K$  in  $\mathbb{R}^n$ , we have that both conditions in (1.8) are satisfied with  $q = 2$ , since  $I_1(K, Z_2^\circ(K)) \leq \sqrt{n}L_K^2$ . Therefore, Theorem 1.1 will give us that

$$(1.10) \quad L_K^2 \leq C_1 \sqrt{n} \log^2 n$$

for any such body which is regular. Theorem 5.1 then guarantees that, for some absolute constants  $\kappa, \tau$  and  $\delta > 0$ , there exists a  $(\kappa, \tau)$ -regular isotropic convex body  $K$  in  $\mathbb{R}^n$  with  $L_K \geq \delta L_n$ , and hence (1.10) leads us to Bourgain's bound again:  $L_n \leq C_2 \sqrt[4]{n} \log n$ .

However, the behaviour of  $I_1(K, Z_q^\circ(K))$  may allow us to use much larger values of  $q$ . In Section 3 we discuss upper and lower bounds for this quantity. For every isotropic convex body  $K$  in  $\mathbb{R}^n$  we have some simple general estimates:

- (i) For every  $2 \leq q \leq n$ ,

$$c_1 \max \{ \sqrt{n}L_K^2, \sqrt{qn}, R(Z_q(K))L_K \} \leq I_1(K, Z_q^\circ(K)) \leq c_2 q \sqrt{n}L_K^2.$$

- (ii) If  $2 \leq q \leq \sqrt{n}$ , then

$$c_1 \max \{ \sqrt{n}L_K^2, \sqrt{qn}L_K \} \leq I_1(K, Z_q^\circ(K)) \leq c_2 q \sqrt{n}L_K^2.$$

Any improvement of the exponent of  $q$  in the upper bound  $I_1(K, Z_q^\circ(K)) \leq c q \sqrt{n}L_K^2$  would lead to an estimate  $L_n \leq Cn^\alpha$  with  $\alpha < \frac{1}{4}$ . It seems plausible that one could even have  $I_1(K, Z_q^\circ(K)) \leq c\sqrt{qn}L_K^2$ , at least when  $q$  is small, say  $2 \leq q \ll \sqrt{n}$ . Some evidence is given by the following facts:

- (iii) If  $K$  is an unconditional isotropic convex body in  $\mathbb{R}^n$ , then

$$c_1 \sqrt{qn} \leq I_1(K, Z_q^\circ(K)) \leq c_2 \sqrt{qn} \log n$$

for all  $2 \leq q \leq n$ .

- (iv) If  $K$  is an isotropic convex body in  $\mathbb{R}^n$  then, for every  $2 \leq q \leq \sqrt{n}$ , there exists a set  $A_q \subseteq O(n)$  with  $\nu(A_q) \geq 1 - e^{-q}$  such that  $I_1(K, Z_q^\circ(U(K))) \leq c_3 \sqrt{qn} L_K^2$  for all  $U \in A_q$ .

The proofs of (i)-(iv) are given in Section 3.

We can make a final observation about the reduction of Theorem 1.1 on the basis that there exist  $(\kappa, \tau)$ -regular isotropic convex bodies  $K$  in  $\mathbb{R}^n$  with  $L_K \geq \delta L_n$  (where  $\kappa, \tau, \delta > 0$  are absolute constants) which, at the same time, have “small diameter”: they satisfy  $K \subseteq \gamma\sqrt{n}L_K B_2^n$ , where  $\gamma > 0$  is an absolute constant (see Theorem 5.9). In Section 6, we show that then it is enough to study the parameter  $I_1(K, Z_q^\circ(K))$  within the class  $\mathcal{IK}_{\text{sd}}$  of isotropic convex bodies which are  $O(\gamma)$ -close to the Euclidean ball  $D_n$  of volume 1 and have uniformly bounded isotropic constant. The precise statement which we prove is the following: if we have an isotropic symmetric convex body  $K$  in  $\mathbb{R}^n$  satisfying  $K \subseteq \gamma\sqrt{n}L_K B_2^n$ , then we can find an isotropic symmetric convex body  $C$  such that  $L_C \leq c_1, c_2 D_n \subseteq C \subseteq c_3 \gamma D_n$ , and

$$(1.11) \quad \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2} \leq c_4 \frac{I_1(C, Z_q^\circ(C))}{\sqrt{qn}}$$

for all  $1 \leq q \leq n$ , where  $c_1, c_2, c_3, c_4 > 0$  are absolute constants.

## 2 Notation and preliminaries

We work in  $\mathbb{R}^n$ , which is equipped with a Euclidean structure  $\langle \cdot, \cdot \rangle$ . We denote the corresponding Euclidean norm by  $\|\cdot\|_2$ , and write  $B_2^n$  for the Euclidean unit ball, and  $S^{n-1}$  for the unit sphere. Volume is denoted by  $|\cdot|$ . We write  $\omega_n$  for the volume of  $B_2^n$  and  $\sigma$  for the rotationally invariant probability measure on  $S^{n-1}$ . We also denote the Haar measure on  $O(n)$  by  $\nu$ . The Grassmann manifold  $G_{n,k}$  of  $k$ -dimensional subspaces of  $\mathbb{R}^n$  is equipped with the Haar probability measure  $\mu_{n,k}$ . Let  $k \leq n$  and  $F \in G_{n,k}$ . We will denote the orthogonal projection from  $\mathbb{R}^n$  onto  $F$  by  $P_F$ . We also define  $B_F := B_2^n \cap F$  and  $S_F := S^{n-1} \cap F$ .

The letters  $c, c', c_1, c_2$  etc. denote absolute positive constants whose value may change from line to line. Whenever we write  $a \simeq b$ , we mean that there exist absolute constants  $c_1, c_2 > 0$  such that  $c_1 a \leq b \leq c_2 a$ . Also if  $K, L \subseteq \mathbb{R}^n$  we will write  $K \simeq L$  if there exist absolute constants  $c_1, c_2 > 0$  such that  $c_1 K \subseteq L \subseteq c_2 K$ .

A convex body in  $\mathbb{R}^n$  is a compact convex subset  $C$  of  $\mathbb{R}^n$  with nonempty interior. We say that  $C$  is symmetric if  $x \in C$  implies that  $-x \in C$ . We say that  $C$  is centered if it has center of mass at the origin, i.e.  $\int_C \langle x, \theta \rangle dx = 0$  for every  $\theta \in S^{n-1}$ . The support function of a convex body  $C$  is defined by

$$(2.1) \quad h_C(y) := \max\{\langle x, y \rangle : x \in C\},$$

and the mean width of  $C$  is

$$(2.2) \quad w(C) := \int_{S^{n-1}} h_C(\theta) \sigma(d\theta).$$

For each  $-\infty < q < \infty, q \neq 0$ , we define the  $q$ -mean width of  $C$  by

$$(2.3) \quad w_q(C) := \left( \int_{S^{n-1}} h_C^q(\theta) \sigma(d\theta) \right)^{1/q}.$$

The radius of  $C$  is the quantity  $R(C) := \max\{\|x\|_2 : x \in C\}$ . Also, if the origin is an interior point of  $C$ , the polar body  $C^\circ$  of  $C$  is defined as follows:

$$(2.4) \quad C^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in C\}.$$

Finally, we write  $\bar{C}$  for the homothetic image of volume 1 of a convex body  $C \subseteq \mathbb{R}^n$ , i.e.  $\bar{C} := \frac{C}{|C|^{1/n}}$ .

Recall that if  $A$  and  $B$  are nonempty sets in  $\mathbb{R}^n$ , then the covering number  $N(A, B)$  of  $A$  by  $B$  is defined to be the smallest number of translates of  $B$  whose union covers  $A$ . In this paper,  $B$  will usually be a multiple of the Euclidean ball: in those cases we also require that the centres of the translates of  $B$  are taken from the set  $A$ ; one can easily check that this additional requirement does not crucially affect our entropy estimates.

## 2.1 $L_q$ -centroid bodies

Let  $K$  be a convex body of volume 1 in  $\mathbb{R}^n$ . For every  $q \geq 1$  and every  $y \in \mathbb{R}^n$  we set

$$(2.5) \quad h_{Z_q(K)}(y) := \left( \int_K |\langle x, y \rangle|^q dx \right)^{1/q}.$$

The  $L_q$ -centroid body  $Z_q(K)$  of  $K$  is the centrally symmetric convex body with support function  $h_{Z_q(K)}$ . Note that  $K$  is isotropic if and only if it is centered and  $Z_2(K) = L_K B_2^n$ . Also, if  $T \in GL(n)$  with  $\det T = \pm 1$ , then  $Z_p(T(K)) = T(Z_p(K))$ . From Hölder's inequality it follows that  $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$  for all  $1 \leq p \leq q \leq \infty$ , where  $Z_\infty(K) = \text{conv}\{K, -K\}$ . Using Borell's lemma (see [15, Appendix III]), one can check that inverse inclusions also hold:

$$(2.6) \quad Z_q(K) \subseteq \beta_1 q Z_1(K),$$

and more generally,

$$(2.7) \quad Z_q(K) \subseteq \beta_2 \frac{q}{p} Z_p(K)$$

for all  $1 \leq p < q$ . In particular, if  $K$  is isotropic, then  $R(Z_q(K)) \leq \beta_1 q L_K$ . One can also check that if  $K$  is centered, then  $Z_q(K) \supseteq \beta_3 K$  for all  $q \geq n$  (see [16]). All the constants  $\beta_i, \bar{\beta}_j$  that appear in this section are absolute positive constants which may be used again in the arguments of the next sections.

Let  $C$  be a symmetric convex body in  $\mathbb{R}^n$  and let  $\|\cdot\|_C$  denote the norm induced on  $\mathbb{R}^n$  by  $C$ . The parameter  $k_*(C)$  is the largest positive integer  $k \leq n$  with the property that the measure of  $F \in G_{n,k}$  for which we have  $\frac{1}{2}w(C)B_F \subseteq P_F(C) \subseteq 2w(C)B_F$  is greater than  $\frac{n}{n+k}$ . It is known that

$$(2.8) \quad \beta_4 n \frac{w(C)^2}{R(C)^2} \leq k_*(C) \leq \beta_5 n \frac{w(C)^2}{R(C)^2}.$$

The  $q$ -mean width  $w_q(C)$  is equivalent to  $w(C)$  as long as  $q \leq k_*(C)$ . Litvak, Milman and Schechtman proved in [11] that, for every symmetric convex body  $C$  in  $\mathbb{R}^n$ ,

- (i) If  $1 \leq q \leq k_*(C)$  then  $w(C) \leq w_q(C) \leq \beta_6 w(C)$ .
- (ii) If  $k_*(C) \leq q \leq n$  then  $\beta_7 \sqrt{q/n} R(C) \leq w_q(C) \leq \beta_8 \sqrt{q/n} R(C)$ .

Let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . Recall that, for every symmetric convex body  $C$  in  $\mathbb{R}^n$  and for every  $q \in (-n, \infty)$ ,  $q \neq 0$ , we define

$$(2.9) \quad I_q(K, C) := \left( \int_K \|x\|_C^q dx \right)^{1/q}.$$

When  $C = B_2^n$ , we write  $I_q(K) := I_q(K, B_2^n)$  for simplicity. In [17] and [18] it is proved that for every  $1 \leq q \leq n/2$ ,

$$(2.10) \quad I_q(K) \simeq \sqrt{n/q} w_q(Z_q(K)) \text{ and } I_{-q}(K) \simeq \sqrt{n/q} w_{-q}(Z_q(K)).$$

The parameter  $q_*(K)$  is also defined by

$$(2.11) \quad q_*(K) := \max\{q \leq n : k_*(Z_q(K)) \geq q\}.$$

Then, the main result of [18] states that, for every centered convex body  $K$  of volume 1 in  $\mathbb{R}^n$ , one has  $I_{-q}(K) \simeq I_q(K)$  for every  $1 \leq q \leq q_*(K)$ . In particular, for all  $q \leq q_*(K)$  one has  $I_q(K) \leq \beta_9 I_2(K)$ . If  $K$  is isotropic, one can check that  $q_*(K) \geq c\sqrt{n}$ , where  $c > 0$  is an absolute constant (for a proof, see [17]). Therefore,

$$(2.12) \quad I_q(K) \leq \beta_{10} \sqrt{n} L_K \text{ for every } q \leq \sqrt{n}.$$

In particular, from (2.10) and (2.12) we see that  $w(Z_q(K)) \simeq w_q(Z_q(K)) \simeq \sqrt{q} L_K$  for all  $q \leq \sqrt{n}$ .

## 2.2 The bodies $B_q(K, F)$

Another family of convex bodies associated with a centered convex body  $K \subset \mathbb{R}^n$  was introduced by Ball in [1] (see also [13]): to define them, let us consider a  $k$ -dimensional subspace  $F$  of  $\mathbb{R}^n$  and its orthogonal subspace  $E$ . For every  $\phi \in F \setminus \{0\}$  we denote by  $E^+(\phi)$  the halfspace  $\{x \in \text{span}\{E, \phi\} : \langle x, \phi \rangle \geq 0\}$ . Ball proved that, for every  $q \geq 0$ , the function

$$(2.13) \quad \phi \mapsto \|\phi\|_2^{1+\frac{q}{q+1}} \left( \int_{K \cap E^+(\phi)} \langle x, \phi \rangle^q dx \right)^{-\frac{1}{q+1}}$$

is the gauge function of a convex body  $B_q(K, F)$  on  $F$ . Several properties of these bodies can be found in [1], [13] and also in [17], [18]. In Section 5, we will make use of only two of those:

- (i) Let  $K \subset \mathbb{R}^n$  be isotropic, let  $1 \leq k < n$  and let  $F \in G_{n,k}$ . Then the body  $\overline{B}_{k+1}(K, F)$  is *almost isotropic*, namely it has (by definition) volume 1, and we can write  $\overline{B}_{k+1}(K, F) \simeq T(\overline{B}_{k+1}(K, F))$  where  $T(\overline{B}_{k+1}(K, F))$  is an isotropic (in the regular sense) linear image of  $\overline{B}_{k+1}(K, F)$ . In addition,

$$(2.14) \quad |K \cap F^\perp|^{1/k} \simeq \frac{L_{B_{k+1}(K, F)}}{L_K}.$$

- (ii) Let  $K, F$  and  $k < n$  be as above and consider any  $p \in [1, k]$ . Then

$$(2.15) \quad Z_p(\overline{B}_{k+1}(K, F)) \simeq |K \cap F^\perp|^{1/k} P_F(Z_p(K)).$$

### 2.3 Two related lemmas

We close this section with two lemmas that will be used in the sequel; they reveal some properties of the support function of the  $L_q$ -centroid bodies of a convex body with respect to subsets or certain integrals of maxima.

**Lemma 2.1.** *Let  $K$  be a convex body of volume 1 in  $\mathbb{R}^n$ , and consider any points  $z_1, z_2, \dots, z_N \in \mathbb{R}^n$ . If  $q \geq 1$  and  $p \geq \max\{\log N, q\}$ , then*

$$(2.16) \quad \left( \int_K \max_{1 \leq i \leq N} |\langle x, z_i \rangle|^q dx \right)^{1/q} \leq \overline{\beta}_1 \max_{1 \leq i \leq N} h_{Z_p(K)}(z_i),$$

where  $\overline{\beta}_1 > 0$  is an absolute constant.

*Proof.* Let  $p \geq \max\{\log N, q\}$  and  $\theta \in S^{n-1}$ . Markov's inequality shows that

$$(2.17) \quad |\{x \in K : |\langle x, \theta \rangle| \geq e^3 h_{Z_p(K)}(\theta)\}| \leq e^{-3p}.$$

Since  $x \mapsto |\langle x, \theta \rangle|$  is a seminorm, from Borell's lemma (see [15, Appendix III]) we get that

$$(2.18) \quad |\{x \in K : |\langle x, \theta \rangle| \geq e^3 t h_{Z_p(K)}(\theta)\}| \leq (1 - e^{-3p}) \left( \frac{e^{-3p}}{1 - e^{-3p}} \right)^{\frac{t+1}{2}} \leq e^{-pt}$$

for every  $t \geq 1$ . We set  $S := e^3 \max_{1 \leq i \leq N} h_{Z_p(K)}(z_i)$ . Then, for every  $t \geq 1$  we have that

$$(2.19) \quad |\{x \in K : \max_{1 \leq i \leq N} |\langle x, z_i \rangle| \geq St\}| \leq \sum_{i=1}^N |\{x \in K : |\langle x, z_i \rangle| \geq e^3 t h_{Z_p(K)}(z_i)\}| \leq N e^{-pt}.$$

It follows that

$$\begin{aligned}
(2.20) \quad \int_K \max_{1 \leq i \leq N} |\langle x, z_i \rangle|^q dx &= q \int_0^\infty s^{q-1} |\{x \in K : \max_{1 \leq i \leq N} |\langle x, z_i \rangle| \geq s\}| ds \\
&\leq S^q + q \int_S^\infty s^{q-1} |\{x \in K : \max_{1 \leq i \leq N} |\langle x, z_i \rangle| \geq s\}| ds \\
&= S^q \left( 1 + q \int_1^\infty t^{q-1} |\{x \in K : \max_{1 \leq i \leq N} |\langle x, z_i \rangle| \geq St\}| dt \right) \\
&\leq S^q \left( 1 + qN \int_1^\infty t^{q-1} e^{-pt} dt \right) \\
&= S^q \left( 1 + \frac{qN}{p^q} \int_p^\infty t^{q-1} e^{-t} dt \right) \\
&\leq S^q \left( 1 + \frac{qN}{p^q} e^{-p} p^q \right) \\
&\leq (3S)^q,
\end{aligned}$$

where we have also used the fact that, for every  $p \geq q \geq 1$ ,

$$(2.21) \quad \int_p^\infty t^{q-1} e^{-t} dt \leq e^{-p} p^q.$$

This finishes the proof (with  $\bar{\beta}_1 = 3e^3$ ).  $\square$

**Remark 2.2.** It is a well-known fact (see e.g. [6, Proposition 2.5.1]) that

$$(2.22) \quad \int_K \max_{1 \leq i \leq N} |\langle x, z_i \rangle| dx \leq C_1 \log N \max_{1 \leq i \leq N} h_{Z_1(K)}(z_i).$$

Through a variant of the argument in [6], and using (2.21) as well, one can also show that for  $q \leq \log N$ ,

$$(2.23) \quad \left( \int_K \max_{1 \leq i \leq N} |\langle x, z_i \rangle|^q dx \right)^{1/q} \leq C_2 \log N \max_{1 \leq i \leq N} h_{Z_1(K)}(z_i).$$

Now, both inequalities can be directly deduced from Lemma 2.1 combined with (2.6), however the lemma provides additional information on how well the quantities  $(\int_K \max_i |\langle x, z_i \rangle|^q dx)^{1/q}$  and  $\max_i (\int_K |\langle x, z_i \rangle|^q dx)^{1/q} \equiv \max_i h_{Z_q(K)}(z_i)$  can be compared: for  $q \leq \log N$ , using also (2.7), we have that

$$(2.24) \quad \left( \int_K \max_{1 \leq i \leq N} |\langle x, z_i \rangle|^q dx \right)^{1/q} \leq \bar{\beta}_1 \max_{1 \leq i \leq N} h_{Z_{\log N}(K)}(z_i) \leq C \frac{\log N}{q} \max_{1 \leq i \leq N} h_{Z_q(K)}(z_i),$$

whereas for  $q > \log N$ ,

$$(2.25) \quad \left( \int_K \max_{1 \leq i \leq N} |\langle x, z_i \rangle|^q dx \right)^{1/q} \simeq \max_{1 \leq i \leq N} \left( \int_K |\langle x, z_i \rangle|^q dx \right)^{1/q}.$$



We now turn our attention to the  $L_q$ -centroid bodies of subsets of  $K$ .

**Lemma 2.3.** *Let  $K$  be a convex body of volume 1 in  $\mathbb{R}^n$  and let  $1 \leq q, r \leq n$ . There exists an absolute constant  $\bar{\beta}_2 > 0$  such that if  $A$  is a subset of  $K$  with  $|A| \geq 1 - e^{-\bar{\beta}_2 q}$ , then*

$$(2.26) \quad Z_p(K) \subseteq 2Z_p(\bar{A})$$

for all  $1 \leq p \leq q$ . Also, for the opposite inclusion, it suffices to have  $|A| \geq 2^{-\frac{r}{2}}$  to conclude that

$$(2.27) \quad Z_p(\bar{A}) \subseteq 2Z_p(K)$$

for all  $r \leq p \leq n$ .

*Proof.* Let  $\theta \in S^{n-1}$ . Note that

$$(2.28) \quad h_{Z_p(\bar{A})}(\theta) = \left( \int_{\bar{A}} |\langle x, \theta \rangle|^p dx \right)^{1/p} = \frac{1}{|A|^{\frac{1}{p} + \frac{1}{n}}} \left( \int_A |\langle x, \theta \rangle|^p dx \right)^{1/p}.$$

We first prove (2.27): since  $A \subseteq K$  and assuming that  $|A| \geq 2^{-\frac{r}{2}}$ , we have

$$(2.29) \quad \begin{aligned} h_{Z_p(K)}(\theta) &= \left( \int_K |\langle x, \theta \rangle|^p dx \right)^{1/p} \geq \left( \int_A |\langle x, \theta \rangle|^p dx \right)^{1/p} \\ &\geq 2^{-\frac{r}{2p} - \frac{r}{2n}} \left( \int_{\bar{A}} |\langle x, \theta \rangle|^p dx \right)^{1/p} \geq \frac{1}{2} h_{Z_p(\bar{A})}(\theta) \end{aligned}$$

for all  $r \leq p \leq n$ . On the other hand, assuming that  $|A| \geq 1 - e^{-\bar{\beta}_2 q}$  and using the fact that  $\|\langle \cdot, \theta \rangle\|_{2p} \leq 2\bar{\beta}_2 \|\langle \cdot, \theta \rangle\|_p$ , we have

$$(2.30) \quad \begin{aligned} \int_K |\langle x, \theta \rangle|^p dx &= \int_A |\langle x, \theta \rangle|^p dx + \int_{K \setminus A} |\langle x, \theta \rangle|^p dx \\ &\leq |A|^{1 + \frac{p}{n}} \int_{\bar{A}} |\langle x, \theta \rangle|^p dx + |K \setminus A|^{1/2} \left( \int_K |\langle x, \theta \rangle|^{2p} dx \right)^{1/2} \\ &\leq \int_{\bar{A}} |\langle x, \theta \rangle|^p dx + e^{-\bar{\beta}_2 q/2} (2\bar{\beta}_2)^p \int_K |\langle x, \theta \rangle|^p dx \\ &\leq \int_{\bar{A}} |\langle x, \theta \rangle|^p dx + \frac{1}{2} \int_K |\langle x, \theta \rangle|^p dx \end{aligned}$$

for every  $p \leq q$ , if  $\bar{\beta}_2 > 0$  is chosen large enough. This proves (2.26).  $\square$

### 3 Simple estimates for $I_1(K, Z_q^\circ(K))$

In this section we give some upper and lower bounds for  $I_1(K, Z_q^\circ(K))$  which hold true for every isotropic convex body  $K$  in  $\mathbb{R}^n$  and any  $1 \leq q \leq n$ . In fact, our

arguments are quite direct and make use of estimates for simple parameters of the bodies  $Z_q(K)$ , such as their radius or their volume, so that it is straightforward to reach analogous upper and lower bounds for  $I_1(K, Z_q^\circ(M))$  in the more general case when  $K$  and  $M$  are not necessarily the same isotropic convex body.

Since  $h_{Z_q(K)}(x) \leq R(Z_q(K))\|x\|_2$ , we have that

$$(3.1) \quad I_1(K, Z_q^\circ(K)) \leq R(Z_q(K)) \int_K \|x\|_2 dx \leq R(Z_q(K))\sqrt{n}L_K,$$

which, in combination with the fact that  $R(Z_q(K)) \leq \beta_1 q L_K$  (a direct consequence of (2.6)), leads to the bound

$$(3.2) \quad I_1(K, Z_q^\circ(K)) \leq \beta_1 q \sqrt{n} L_K^2.$$

More generally, we have that

$$(3.3) \quad I_1(K, Z_q^\circ(M)) \leq R(Z_q(M)) \int_K \|x\|_2 dx \leq \beta_1 q \sqrt{n} L_K L_M.$$

However, in the case that  $M$  is an orthogonal transformation of  $K$ , the next lemma shows that the average of the quantity  $I_1(K, Z_q^\circ(M))$  can be bounded much more effectively than in (3.3).

**Lemma 3.1.** *Let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . For every  $2 \leq q \leq n$ ,*

$$(3.4) \quad \left( \int_{O(n)} I_1^q(K, Z_q^\circ(U(K))) d\nu(U) \right)^{1/q} \leq C \sqrt{q/n} I_q^2(K),$$

where  $C > 0$  is an absolute constant.

*Proof.* We write

$$(3.5) \quad \begin{aligned} \int_{O(n)} I_1^q(K, Z_q^\circ(U(K))) d\nu(U) &\leq \int_{O(n)} I_q^q(K, Z_q^\circ(U(K))) d\nu(U) \\ &= \int_{O(n)} \int_K \int_{U(K)} |\langle x, y \rangle|^q dy dx d\nu(U) \\ &= \int_K \int_K \int_{O(n)} |\langle x, Uy \rangle|^q d\nu(U) dy dx \\ &= \int_K \int_K \|y\|_2^q \int_{S^{n-1}} |\langle x, \theta \rangle|^q d\sigma(\theta) dy dx \\ &= c_{n,q}^q \int_K \int_K \|y\|_2^q \|x\|_2^q dy dx \\ &= c_{n,q}^q I_q^{2q}(K), \end{aligned}$$

where  $c_{n,q} \simeq \sqrt{q/n}$ . □

Recall that in the case that  $K$  is isotropic, one has from [17] that  $I_q(K) \simeq \max\{\sqrt{n}L_K, R(Z_q(K))\}$ . Then, Lemma 3.1 shows that, for every  $2 \leq q \leq n$ ,

$$(3.6) \quad \left( \int_{O(n)} I_1^q(K, Z_q^\circ(U(K))) d\nu(U) \right)^{1/q} \leq C_1 \max\{\sqrt{qn}, q^2\sqrt{q/n}\} L_K^2,$$

where  $C_1 > 0$  is an absolute constant. Therefore, for every  $2 \leq q \leq \sqrt{n}$ , there exists a set  $A_q \subseteq O(n)$  with  $\nu(A_q) \geq 1 - e^{-q}$  such that  $I_1(K, Z_q^\circ(U(K))) \leq C_2\sqrt{qn}L_K^2$  for all  $U \in A_q$ . It is thus conceivable that there are properties of the bodies  $Z_q(K)$  which we can exploit to also bound  $I_1(K, Z_q^\circ(K))$  more effectively than in (3.1) and (3.2).

We now pass to lower bounds; we will present three simple arguments. For the first one we do not have to assume that  $K$  or  $M$  are in the isotropic position, only that they are centered and have volume 1: from [13, Corollary 2.2.a] we have that

$$(3.7) \quad I_1(K, Z_q^\circ(M)) = \int_K h_{Z_q(M)}(x) dx \geq \frac{n}{n+1} \frac{1}{|Z_q^\circ(M)|^{1/n}}.$$

Then, by the Blaschke–Santaló inequality, we get that

$$(3.8) \quad I_1(K, Z_q^\circ(M)) \geq c_1 n |Z_q(M)|^{1/n} \geq c_2 \sqrt{qn} L_M$$

for all  $2 \leq q \leq \sqrt{n}$ , because  $|Z_q(M)|^{1/n} \geq c_3 \sqrt{q/n} L_M$  for this range of values of  $q$  by a recent result of Klartag and E. Milman (see [9]). When  $\sqrt{n} \leq q \leq n$ , we have the weaker lower bound  $|Z_q(M)|^{1/n} \geq c_4 \sqrt{q/n}$ , which is due to Lutwak, Yang and Zhang (see [12]). It follows that  $I_1(K, Z_q^\circ(M)) \geq c_5 \sqrt{qn}$  for this range of  $q$ .

For the second argument, we require that  $K$  is isotropic and we write

$$(3.9) \quad \begin{aligned} I_1(K, Z_q^\circ(M)) &= \int_K h_{Z_q(M)}(x) dx = \int_K \max_{z \in Z_q(M)} |\langle x, z \rangle| dx \\ &\geq \max_{z \in Z_q(M)} \int_K |\langle x, z \rangle| dx \geq c \max_{z \in Z_q(M)} \|z\|_2 L_K \\ &= c R(Z_q(M)) L_K. \end{aligned}$$

Finally, if  $M$  is isotropic as well, we can use Hölder's inequality to get

$$(3.10) \quad \begin{aligned} I_1(K, Z_q^\circ(M)) &= \int_K h_{Z_q(M)}(x) dx \\ &\geq \int_K h_{Z_2(M)}(x) dx = \int_K \|x\|_2 L_M dx \geq c\sqrt{n} L_K L_M. \end{aligned}$$

All the estimates presented above are gathered in the next proposition.

**Proposition 3.2.** *Let  $K$  and  $M$  be isotropic convex bodies in  $\mathbb{R}^n$ . For every  $2 \leq q \leq n$ ,*

$$(3.11) \quad c_1 \max\{\sqrt{n}L_K L_M, \sqrt{qn}, R(Z_q(M))L_K\} \leq I_1(K, Z_q^\circ(M)) \leq c_2 q \sqrt{n} L_K L_M.$$

In addition, if  $2 \leq q \leq \sqrt{n}$  then

$$(3.12) \quad c_1 \max \{ \sqrt{n} L_K L_M, \sqrt{qn} L_M \} \leq I_1(K, Z_q^\circ(M)) \leq c_2 q \sqrt{n} L_K L_M.$$

The situation is more or less clear in the unconditional case. Recall that a convex body  $K$  in  $\mathbb{R}^n$  is called unconditional if it is symmetric with respect to all coordinate hyperplanes (for some orthonormal basis of  $\mathbb{R}^n$ ). Then, it is easy to check that one can bring  $K$  to the isotropic position by applying an operator which is diagonal with respect to this basis. It is also not hard to prove that the isotropic constant of  $K$  satisfies  $L_K \simeq 1$ . The upper bound follows from the Loomis–Whitney inequality; see also [2]. It is known (from [3]) that, for every  $q \geq 2$ , one has  $h_{Z_q^\circ(K)}(y) \leq c \sqrt{qn} \|y\|_\infty$ , where  $c > 0$  is an absolute constant. This leads us to the estimates

$$(3.13) \quad c_1 \sqrt{qn} \leq I_1(K, Z_q^\circ(K)) \leq c \sqrt{qn} \int_K \|x\|_\infty dx \leq c_2 \sqrt{qn} \log n$$

for all  $2 \leq q \leq n$  (the same estimates hold true for the quantity  $I_1(K, Z_q^\circ(M))$  when  $M$  is too an unconditional isotropic convex body).

## 4 The reduction

Let  $\kappa, \tau > 0$ . Throughout this paper, we say that an isotropic convex body  $K$  in  $\mathbb{R}^n$  is  $(\kappa, \tau)$ -regular if

$$(4.1) \quad \log N(K, tB_2^n) \leq \frac{\kappa n^2 \log^2 n}{t^2} \text{ for all } t \geq \tau \sqrt{n \log n}.$$

The purpose of this section is to present a reduction of the slicing problem to the study of the quantity  $I_1(K, Z_q^\circ(K))$  for  $(\kappa, \tau)$ -regular isotropic convex bodies: we show that any upper bound for  $I_1(K, Z_q^\circ(K))$  immediately leads to an upper bound for the isotropic constant  $L_K$  of a regular convex body  $K$ . Note that the dependence seems to be nontrivial, in the sense that using the simple estimates of Section 3 we can already recover the currently known bound for  $L_K$  with a loss of a logarithmic factor, while a small (although not necessarily easy) improvement to those estimates will also result in new bounds for  $L_K$ . In a sense, we will have fully presented our reduction by the end of the next section, where we provide a self-contained proof of the fact that there exist regular isotropic convex bodies  $K$  in  $\mathbb{R}^n$  with  $L_K \simeq L_n$ . First, let us see how the quantity  $I_1(K, Z_q^\circ(K))$  and the isotropic constant of a regular convex body  $K$  are connected.

**Theorem 4.1.** *There exists an absolute constant  $\rho \in (0, 1)$  with the following property: given  $\kappa, \tau \geq 1$ , for every  $n \geq n_0(\tau)$  and every  $(\kappa, \tau)$ -regular isotropic convex body  $K$  in  $\mathbb{R}^n$  we have that, if  $q \geq 2$  satisfies*

$$(4.2) \quad 2 \leq q \leq \rho^2 n \text{ and } I_1(K, Z_q^\circ(K)) \leq \rho n L_K^2,$$

then

$$(4.3) \quad L_K^2 \leq C\kappa \sqrt{\frac{n}{q}} \log^2 n \max \left\{ 1, \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2} \right\}.$$

*Proof.* We define a convex body  $W$  in  $\mathbb{R}^n$ , setting

$$(4.4) \quad W := \{x \in K : h_{Z_q(K)}(x) \leq C_1 I_1(K, Z_q^\circ(K))\},$$

where  $C_1 = e^{2\bar{\beta}_2}$  and  $\bar{\beta}_2 > 0$  is the constant which appears in Lemma 2.3. From Markov's inequality we have that  $|W| \geq 1 - e^{-2\bar{\beta}_2}$  and also trivially that  $|W| \geq 2^{-1} \geq 2^{-\frac{n}{2}}$  (as long as  $\bar{\beta}_2 \geq 1$ ). Then we set

$$(4.5) \quad K_1 := \overline{W}.$$

Applying both cases of Lemma 2.3 to the set  $W$  with  $p = 2$ , we see that

$$(4.6) \quad \frac{1}{2}Z_2(K_1) \subseteq Z_2(K) \subseteq 2Z_2(K_1).$$

This implies that

$$(4.7) \quad \frac{1}{4}L_K^2 = \frac{1}{4} \int_K \langle x, \theta \rangle^2 dx \leq \int_{K_1} \langle x, \theta \rangle^2 dx \leq 4 \int_K \langle x, \theta \rangle^2 dx = 4L_K^2$$

for every  $\theta \in S^{n-1}$ , and hence

$$(4.8) \quad \frac{nL_K^2}{4} \leq \sum_{i=1}^n \int_{K_1} \langle x, e_i \rangle^2 dx = \int_{K_1} \|x\|_2^2 dx \leq 4nL_K^2.$$

We also have

$$(4.9) \quad K_1 = |W|^{-1/n} W \subseteq 2W \subseteq 2K,$$

thus for every  $x \in K_1$  we have  $x/2 \in W$ , and using (2.27) of Lemma 2.3 again, we can write

$$(4.10) \quad h_{Z_q(K_1)}(x) \leq 2h_{Z_q(K)}(x) = 4h_{Z_q(K)}(x/2) \leq 4C_1 I_1(K, Z_q^\circ(K)).$$

Finally,

$$(4.11) \quad \log N(K_1, tB_2^n) \leq \log N(2K, tB_2^n) \leq \frac{4\kappa n^2 \log^2 n}{t^2},$$

for all  $t \geq 2\tau\sqrt{n \log n}$ . We now write

$$(4.12) \quad nL_K^2 \leq 4 \int_{K_1} \|x\|_2^2 dx \leq 4 \int_{K_1} \max_{z \in K_1} |\langle x, z \rangle| dx.$$

(4.11) tells us that for every  $t \geq 2\tau\sqrt{n\log n}$ , we can find  $z_1, \dots, z_{N_t} \in K_1$  such that  $K_1 \subseteq \bigcup_{i=1}^{N_t} (z_i + tB_2^n)$ , and  $|N_t| \leq \exp\left(\frac{4\kappa n^2 \log^2 n}{t^2}\right)$ . It follows that

$$(4.13) \quad \max_{z \in K_1} |\langle x, z \rangle| \leq \max_{1 \leq i \leq N_t} |\langle x, z_i \rangle| + \max_{w \in tB_2^n} |\langle x, w \rangle| = \max_{1 \leq i \leq N_t} |\langle x, z_i \rangle| + t\|x\|_2,$$

and hence

$$(4.14) \quad \begin{aligned} nL_K^2 &\leq 4 \int_{K_1} \max_{1 \leq i \leq N_t} |\langle x, z_i \rangle| dx + 4t \int_{K_1} \|x\|_2 dx \\ &\leq 4 \int_{K_1} \max_{1 \leq i \leq N_t} |\langle x, z_i \rangle| dx + 8t\sqrt{n}L_K. \end{aligned}$$

We choose

$$(4.15) \quad t_0^2 = 16C_2\kappa \max\left\{1, \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\} \frac{n^{3/2}}{\sqrt{q}} \log^2 n,$$

where  $C_2 = 16C_1\beta_2\bar{\beta}_1$  with  $\beta_2$  the constant appearing in (2.7) and  $\bar{\beta}_1$  the constant from Lemma 2.1. With this choice of  $t_0$ , we have

$$(4.16) \quad t_0^2 \geq 16C_2\kappa \sqrt{\frac{n}{q}} n \log^2 n \geq \frac{16C_2\kappa}{\rho} n \log^2 n,$$

as long as  $q$  satisfies (4.2), and

$$(4.17) \quad t_0^2 \geq 16C_2\kappa \frac{I_1(K, Z_q^\circ(K))}{qL_K^2} n \log^2 n.$$

From (4.16) it is clear that

$$(4.18) \quad t_0^2 \geq 16C_2\kappa \frac{n \log^2 n}{\rho} \geq 4\tau^2 n \log n,$$

provided that  $n \geq n_0(\tau, \kappa, \rho)$ , so the above argument, leading up to (4.14), remains valid for  $t = t_0$ . We also set  $p_0 := \frac{4\kappa n^2 \log^2 n}{t_0^2}$ . Observe that  $p_0 \geq q$  (as long as  $q$  is assumed to satisfy (4.2)), if  $\rho$  is chosen properly: indeed, we have  $\max\left\{1, \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\} \leq \rho\sqrt{n/q}$ , and hence

$$(4.19) \quad t_0^2 \leq 16C_2\kappa\rho \frac{n^2 \log^2 n}{q}.$$

If we choose  $\rho < 1/(4C_2)$ , then we have

$$(4.20) \quad p_0 = \frac{4\kappa n^2 \log^2 n}{t_0^2} \geq \frac{4\kappa n^2 q \log^2 n}{16C_2\kappa\rho n^2 \log^2 n} = \frac{q}{4C_2\rho} \geq q.$$

We therefore see that, using Lemma 2.1 with  $q' = 1$ , we can write

$$(4.21) \quad \int_{K_1} \max_{1 \leq i \leq N_{t_0}} |\langle x, z_i \rangle| dx \leq \bar{\beta}_1 \max_{1 \leq i \leq N_{t_0}} h_{Z_{p_0}(K_1)}(z_i) \leq \bar{\beta}_1 \beta_2 \frac{p_0}{q} \max_{1 \leq i \leq N_{t_0}} h_{Z_q(K_1)}(z_i).$$

Combining the above with (4.14), (4.10) and the definition of  $C_2$ , we get

$$(4.22) \quad nL_K^2 \leq C_2 \frac{p_0}{q} I_1(K, Z_q^\circ(K)) + 8t_0 \sqrt{n} L_K.$$

Also, from (4.17) and the definition of  $p_0$ , we have

$$(4.23) \quad C_2 \frac{p_0}{q} I_1(K, Z_q^\circ(K)) = \frac{4C_2 \kappa I_1(K, Z_q^\circ(K))}{qt_0^2} n^2 \log^2 n \leq \frac{1}{4} nL_K^2.$$

Therefore, (4.22) becomes

$$(4.24) \quad nL_K^2 \leq C_3 t_0 \sqrt{n} L_K.$$

This gives us that

$$(4.25) \quad L_K^2 \leq C_4 \frac{t_0^2}{n} = C\kappa \max \left\{ 1, \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn} L_K^2} \right\} \sqrt{\frac{n}{q}} \log^2 n,$$

as we desired.  $\square$

## 5 Regular convex bodies with maximal isotropic constant

Recall that  $L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\}$ . In order to be able to use the argument of the previous section to bound  $L_n$ , we need to establish the existence of  $(\kappa, \tau)$ -regular convex bodies, namely bodies satisfying (4.1), whose isotropic constant is as “close” to  $L_n$  as possible. The following theorem, formulated in the more general setting of log-concave measures, was proven in [5].

**Theorem 5.1.** *There exist absolute constants  $\kappa, \tau$  and  $\delta > 0$  such that, for every  $n \in \mathbb{N}$ , there exists an isotropic convex body  $K$  in  $\mathbb{R}^n$  with the following properties:*

- (i)  $L_K \geq \delta L_n$ .
- (ii)  $\log N(K, tB_2^n) \leq \frac{\kappa n^2 \log^2 n}{t^2}$ , for all  $t \geq \tau \sqrt{n \log n}$ .

For the reader’s convenience, we will give an outline of the proof in the setting of convex bodies. First, we recall the following theorem by Pisier which will be used in several steps of the argument (see [19] for a proof in the symmetric case; this can easily be extended to the general case):

**Theorem 5.2.** Let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . For every  $\alpha \in (0, 2)$  there exists an ellipsoid  $\mathcal{E}_\alpha$  with  $|\mathcal{E}_\alpha| = 1$  such that, for every  $t \geq 1$ ,

$$(5.1) \quad \log N(K, t\mathcal{E}_\alpha) \leq \frac{\kappa(\alpha)}{t^\alpha} n,$$

where  $\kappa(\alpha) > 0$  is a constant depending only on  $\alpha$ .

**Remark 5.3.** One can take  $\kappa(\alpha) \leq \frac{\kappa_1}{2-\alpha}$ , where  $\kappa_1 > 0$  is an absolute constant. An ellipsoid  $\mathcal{E}_\alpha$  which satisfies (5.1) is called an  $\alpha$ -regular  $M$ -ellipsoid for  $K$ .

Secondly, let us gather some useful facts about ellipsoids in  $\mathbb{R}^n$  that we are going to need for the proof of Theorem 5.1 (proofs for these facts can be found in [5], [10] and [22]).

**Lemma 5.4.** Let  $\mathcal{E}$  be an ellipsoid in  $\mathbb{R}^n$ , then  $\mathcal{E} = T(B_2^n)$  for some  $T \in GL(n)$ . We denote the eigenvalues of the matrix  $\sqrt{T^*T}$  by  $\lambda_1 \geq \dots \geq \lambda_n > 0$  (recall that  $T^*T$  is a symmetric, positive definite matrix). Then, for all  $1 \leq k \leq n-1$ ,

$$(5.2) \quad \max_{F \in G_{n,k}} |\mathcal{E} \cap F| = \max_{F \in G_{n,k}} |P_F(\mathcal{E})| = \omega_k \prod_{i=1}^k \lambda_i$$

and

$$(5.3) \quad \min_{F \in G_{n,k}} |\mathcal{E} \cap F| = \min_{F \in G_{n,k}} |P_F(\mathcal{E})| = \omega_k \prod_{i=n-k+1}^n \lambda_i.$$

Also, if the dimension  $n$  is even, we can find a subspace  $F \in G_{n,n/2}$  such that  $P_F(\mathcal{E}) = \lambda_{n/2} B_F$  ( $= \lambda_{n/2} B_2^n \cap F$ ).

In view of the last part of Lemma 5.4, we choose to restrict ourselves to the cases that the dimension  $n$  is even,  $n = 2m$  for some  $m \geq 1$ , and prove Theorem 5.1 for those. However, as we will see in Remark 5.7, it is not hard to then extend the theorem to all dimensions.

**Proof of Theorem 5.1.** We start with an isotropic convex body  $K_1$  with  $L_{K_1} \geq \delta_1 L_{2m}$ , where  $\delta_1 \in (0, 1)$ . Then, one has the following upper bound for the volume of sections of  $K_1$ .

**Lemma 5.5.** For every  $k$ -codimensional subspace  $E$  of  $\mathbb{R}^{2m}$ ,  $|K_1 \cap E|^{1/k} \leq c_1(\delta_1)$ , where  $c_1(\delta_1) > 0$  depends only on  $\delta_1$ .

*Proof.* Let  $E$  be a  $k$ -codimensional subspace of  $\mathbb{R}^{2m}$ , and denote its orthogonal subspace by  $F$ . We consider the body  $B_{k+1}(K_1, F)$ , a convex body in the subspace  $F$  defined as in Subsection 2.2, and we recall that

$$(5.4) \quad c_1 \frac{L_{B_{k+1}(K_1, F)}}{L_{K_1}} \leq |K_1 \cap E|^{1/k} \leq c_2 \frac{L_{B_{k+1}(K_1, F)}}{L_{K_1}}$$



for some absolute constants  $c_1, c_2$  independent of  $m$  or  $k$ . On the other hand, it is not hard to check that if  $k \leq j$  then  $L_k \leq c_3 L_j$  (see e.g. [6, Theorem 4.2.2]). Thus,

$$(5.5) \quad L_{B_{k+1}(K_1, F)} \leq L_k \leq c_3 L_{2m} = (c_3/\delta_1) L_{K_1},$$

and the lemma follows with  $c_1(\delta_1) = c_2 c_3 / \delta_1$ .  $\square$

We will now invoke Pisier's theorem to also give a lower bound for the volume of  $m$ -dimensional sections of  $K_1$  that contain its barycenter.

**Lemma 5.6.** *For every  $F \in G_{2m, m}$  we have  $|K_1 \cap F|^{1/m} \geq c_2(\delta_1)$ , where  $c_2(\delta_1) > 0$  depends only on  $\delta_1$ .*

*Proof.* We consider an  $\alpha$ -regular  $M$ -ellipsoid  $\mathcal{E}_\alpha$  for  $K_1$  (for the proof of this lemma we could have fixed  $\alpha = 1$ ; however, some steps of this more general argument will be needed again later). Set  $t_\alpha = \max\{1, [\kappa(\alpha)]^{1/\alpha}\}$ . Then,

$$(5.6) \quad |P_F(K_1)| \leq N(K_1, t_\alpha \mathcal{E}_\alpha) |P_F(t_\alpha \mathcal{E}_\alpha)| \leq e^{2m} |P_F(t_\alpha \mathcal{E}_\alpha)|$$

for every  $F \in G_{2m, m}$ . We also need the Rogers-Shephard inequality (see [20]) for both  $K_1$  and  $\mathcal{E}_\alpha$ : since  $|K_1| = |\mathcal{E}_\alpha| = 1$ , we know that

$$(5.7) \quad 1 = c_1 \leq |K_1 \cap F|^{1/m} |P_{F^\perp}(K_1)|^{1/m} \leq c_2,$$

and similar estimates hold true for  $\mathcal{E}_\alpha$  (see [21] or [14] for the left hand side inequality). The idea of the argument is the following: inequality (5.7) helps us relate the volume of  $m$ -dimensional sections of  $K_1$  (or  $\mathcal{E}_\alpha$ ) to that of  $m$ -dimensional projections of  $K_1$  (or  $\mathcal{E}_\alpha$  respectively); an upper bound for the former will give us a lower bound for the latter and vice versa. Also, inequality (5.6) allows us to compare the maximum (or minimum) volume of the  $m$ -dimensional projections of  $K_1$  to the maximum (or minimum) volume of the corresponding projections of  $\mathcal{E}_\alpha$ . However, as we recalled in Lemma 5.4, the maximum volume of the  $m$ -dimensional projections of an ellipsoid is the same as the maximum volume of its  $m$ -dimensional sections, so we can use inequalities (5.6) and (5.7) once more to get from upper bounds for the volume of sections of  $K_1$  to lower bounds.

We now give the precise argument: combining (5.7) with the conclusion of Lemma 5.5, we see that  $\min_{F \in G_{2m, m}} |P_{F^\perp}(K_1)|^{1/m} \geq c_3(\delta_1)$ . We then get from (5.6) that  $\min_{F \in G_{2m, m}} |P_F(t_\alpha \mathcal{E}_\alpha)|^{1/m} \geq c_4(\delta_1)$ . Now, using (5.7) for  $\mathcal{E}_\alpha$  we get  $|\mathcal{E}_\alpha \cap F|^{1/m} \leq c_5(\delta_1) t_\alpha$  for every  $F \in G_{2m, m}$ . But from (5.2) we have that

$$(5.8) \quad \max_{F \in G_{2m, m}} |P_F(\mathcal{E}_\alpha)|^{1/m} = \max_{F \in G_{2m, m}} |\mathcal{E}_\alpha \cap F|^{1/m} \leq c_5(\delta_1) t_\alpha.$$

Using (5.6) once again, we get  $|P_F(K_1)|^{1/m} \leq c_6(\delta_1) t_\alpha^2$  for every  $F \in G_{2m, m}$ . Inserting this estimate into (5.7), we see that  $|K_1 \cap F|^{1/m} \geq c_7(\delta_1) / t_\alpha^2$  for every  $F \in G_{2m, m}$ . We may choose  $\alpha = 1$  now, and complete the proof with  $c_2(\delta_1) = c_7(\delta_1) / t_1^2$ .  $\square$

**Conclusion of the proof of Theorem 5.1.** Let  $\alpha \in (1, 2)$  and let  $\mathcal{E}_\alpha$  be an  $\alpha$ -regular  $M$ -ellipsoid for  $K$ . Recall that  $|\mathcal{E}_\alpha| = 1$ . Also, if  $\mathcal{E}_\alpha = T(B_2^n) = T(B_2^{2m})$ , let  $\lambda_1 \geq \dots \geq \lambda_{2m} > 0$  be the eigenvalues of the matrix  $\sqrt{T^*T}$ ; observe from Lemma 5.4 that

$$(5.9) \quad |B_2^m| \prod_{i=m+1}^{2m} \lambda_i = \min_{F \in G_{2m,m}} |P_F(\mathcal{E}_\alpha)| \leq \max_{F \in G_{2m,m}} |P_F(\mathcal{E}_\alpha)| = |B_2^m| \prod_{i=1}^m \lambda_i.$$

Using (5.6) and the conclusion of Lemma 5.6, we get

$$(5.10) \quad |B_2^m|^{1/m} \lambda_m \geq \min_{F \in G_{2m,m}} |P_F(\mathcal{E}_\alpha)|^{1/m} \geq \frac{e^{-2}}{t_\alpha} \min_{F \in G_{2m,m}} |P_F(K_1)|^{1/m} \\ \geq \frac{e^{-2}}{t_\alpha} \min_{F \in G_{2m,m}} |K_1 \cap F|^{1/m} \geq \frac{c_8(\delta_1)}{t_\alpha},$$

and hence

$$(5.11) \quad \lambda_m \geq \frac{c_9(\delta_1)}{t_\alpha} \sqrt{n}.$$

In a similar way, using (5.8), we see that  $|B_2^m|^{1/m} \lambda_m \leq \max_{F \in G_{2m,m}} |P_F(\mathcal{E}_\alpha)|^{1/m} \leq c_5(\delta_1)t_\alpha$ , and hence  $\lambda_m \leq c_{10}(\delta_1)t_\alpha\sqrt{n}$ . But from the last part of Lemma 5.4 we know that there exists a subspace  $F_0 \in G_{2m,m}$  such that  $P_{F_0}(\mathcal{E}_\alpha) = \lambda_m B_{F_0}$ , therefore,

$$(5.12) \quad \frac{c_9(\delta_1)}{t_\alpha} \sqrt{n} B_{F_0} \subseteq P_{F_0}(\mathcal{E}_\alpha) \subseteq c_{10}(\delta_1)t_\alpha\sqrt{n} B_{F_0}.$$

Let  $W := \overline{B}_{m+1}(K_1, F_0)$  and  $K := W \times U(W)$ , where  $U \in O(2m)$  satisfies  $U(F_0) = F_0^\perp$ . Since  $W$  is almost isotropic and  $L_{U(W)} = L_W$ , from [6, Lemma 1.6.6] we see that  $K = W \times U(W)$  is an almost isotropic convex body in  $\mathbb{R}^n \cong \mathbb{R}^{2m}$  with  $L_K = L_W$ . We will show that  $K$  satisfies (i) and (ii); the same conclusion will then immediately follow (perhaps with slightly different constants for property (ii)) for any isotropic linear image  $T(K)$  of  $K$  satisfying  $T(K) \simeq K$ .

*Proof of (i):* Since  $L_K = L_W$ , from (5.4) we get

$$(5.13) \quad L_K = L_W \geq c_2^{-1} L_{K_1} |K_1 \cap F_0^\perp|^{1/m} \geq c_2^{-1} c_2(\delta_1) L_{K_1} \geq \delta L_n,$$

where  $\delta = \delta_1 c_2(\delta_1)/c_2$ . For the last two inequalities we have used Lemma 5.6 and the fact that  $L_{K_1} \geq \delta_1 L_n$ .

*Proof of (ii):* Using the fact that  $N(A \times A, B \times B) \leq N(A, B)^2$  for any two nonempty sets  $A, B$ , and also the fact that  $B_2^m \times B_2^m \subseteq \sqrt{2} B_2^{2m}$ , we may write for any  $s > 0$ ,

$$(5.14) \quad N(K, s\sqrt{2n}B_2^n) \leq N(W \times U(W), s\sqrt{n}(B_{F_0} \times B_{F_0^\perp})) \leq N(W, s\sqrt{n}B_{F_0})^2.$$

From (2.15) we know that

$$(5.15) \quad Z_m(\overline{B}_{m+1}(K_1, F_0)) \simeq |K_1 \cap F_0^\perp|^{1/m} P_{F_0}(Z_m(K_1)),$$

therefore, using Lemmas 5.5, 5.6 and the fact that  $\text{conv}(C, -C) \simeq Z_m(C)$  for every centered convex body  $C$  of volume 1 in  $F_0$  or in  $\mathbb{R}^n$ , we get

$$(5.16) \quad \begin{aligned} \text{conv}(W, -W) &\simeq Z_m(\overline{B}_{m+1}(K_1, F_0)) \simeq |K_1 \cap F_0^\perp|^{1/m} P_{F_0}(Z_m(K_1)) \\ &\simeq_{\delta_1} P_{F_0}(\text{conv}(K_1, -K_1)). \end{aligned}$$

But then, recalling also (5.12), we have for every  $r > 0$ ,

$$(5.17) \quad \begin{aligned} N(W, c_{10}(\delta_1)t_\alpha r \sqrt{n} B_{F_0}) &\leq N(\text{conv}(W, -W), c_{10}(\delta_1)t_\alpha r \sqrt{n} B_{F_0}) \\ &\leq N(\text{conv}(W, -W), r P_{F_0}(\mathcal{E}_\alpha)) \\ &\leq N(c_{11}(\delta_1) P_{F_0}(\text{conv}(K_1, -K_1)), r P_{F_0}(\mathcal{E}_\alpha)) \\ &\leq N(c_{11}(\delta_1) \text{conv}(K_1, -K_1), r \mathcal{E}_\alpha) \\ &\leq N(K_1 - K_1, c_{12}(\delta_1) r (\mathcal{E}_\alpha - \mathcal{E}_\alpha)) \\ &\leq N(K_1, c_{13}(\delta_1) r \mathcal{E}_\alpha)^2 \end{aligned}$$

(note that for the last two inequalities we have also used that  $\mathcal{E}_\alpha$  is convex and symmetric, so  $\mathcal{E}_\alpha - \mathcal{E}_\alpha = 2\mathcal{E}_\alpha$ , that  $K_1$  is convex and contains the origin, so  $\text{conv}(K_1, -K_1) \subset K_1 - K_1$ , as well as the fact that  $N(A - A, B - B) \leq N(A, B)^2$ ). It follows that

$$(5.18) \quad N(K, t\sqrt{n} B_2^n) \leq N\left(K_1, \frac{c_{13}(\delta_1)t}{\sqrt{2}c_{10}(\delta_1)t_\alpha} \mathcal{E}_\alpha\right)^4$$

for every  $t > 0$ . Since  $\mathcal{E}_\alpha$  is an  $\alpha$ -regular  $M$ -ellipsoid for  $K_1$ , it remains to consider large enough  $t \geq \tau(\delta_1, \alpha)$ , where

$$(5.19) \quad \tau(\delta_1, \alpha) := \sqrt{2}c_{10}(\delta_1)t_\alpha/c_{13}(\delta_1) = t_\alpha/c_{14}(\delta_1),$$

to deduce from (5.1) and (5.18) that

$$(5.20) \quad \log N(K, t\sqrt{n} B_2^n) \leq 4 \log N\left(K_1, \frac{c_{14}(\delta_1)t}{t_\alpha} \mathcal{E}_\alpha\right) \leq \frac{4\kappa(\alpha)t_\alpha^\alpha n}{c_{14}^\alpha(\delta_1) t^\alpha}.$$

Choosing  $\alpha = 2 - \frac{1}{\log n}$ , we have  $\kappa(\alpha) \leq \kappa_1 \log n$  and  $t^\alpha \simeq t^2$  as long as, say,  $t \leq n^2$ . This completes the proof.  $\square$

**Remark 5.7.** Now that we have proven the existence of an isotropic body  $K$  in  $\mathbb{R}^{2m}$  which has properties (i) and (ii) of Theorem 5.1, we can easily prove the existence of such bodies in  $\mathbb{R}^{2m-1}$  as well: just note that for every subspace  $F \in G_{2m, 2m-1}$  we have that  $2L_K \leq |K \cap F^\perp| \leq 2R(K)$ . Combining this with the properties

(2.14), (2.15) for the almost isotropic convex body  $\overline{B}_{2m}(K, F)$  in the  $(2m - 1)$ -dimensional subspace  $F$ , we get that

$$(5.21) \quad L_{\overline{B}_{2m}(K, F)} \simeq |K \cap F^\perp|^{\frac{1}{2m-1}} L_K \simeq L_K \geq \delta L_{2m} \geq c\delta L_{2m-1},$$

and also that

$$(5.22) \quad \overline{B}_{2m}(K, F) \simeq Z_{2m-1}(\overline{B}_{2m}(K, F)) \simeq |K \cap F^\perp|^{\frac{1}{2m-1}} P_F(Z_{2m-1}(K)) \simeq P_F(K).$$

Since for every  $t > 0$ ,  $N(P_F(K), tB_F) = N(P_F(K), tP_F(B_2^{2m})) \leq N(K, tB_2^{2m})$ , we conclude that the body  $\overline{B}_{2m}(K, F)$  will also satisfy properties (i) and (ii) of Theorem 5.1 with perhaps slightly different, but still independent of the dimension, constants  $\kappa, \tau$  and  $\delta$ .

In the statement of Theorem 5.1, we can add one more property about the radius of the body  $K$  that we look for: we can require that  $R(K) \leq \gamma\sqrt{n}L_K$  where  $\gamma > 0$  is an absolute constant. The first step towards this is to use Bourgain's argument [4] which reduces the slicing problem to the study of bodies with small diameter; one can prove the following fact (see e.g. [6, Proposition 2.3.1]).

**Lemma 5.8.** *There exists an isotropic convex body  $K_1$  in  $\mathbb{R}^n$  with  $L_{K_1} \geq \delta_1 L_n$  and  $R(K_1) \leq \gamma_1\sqrt{n}L_{K_1}$ , where  $\delta_1, \gamma_1 > 0$  are absolute constants.*

Then, we can repeat the proof of Theorem 5.1 starting with the body  $K_1 \subset \mathbb{R}^n = \mathbb{R}^{2m}$  given by Lemma 5.8. One has now that  $R(W) \leq c(\delta_1)\gamma_1\sqrt{n}L_{K_1}$ : to see this, write

$$(5.23) \quad \begin{aligned} R(W) &= R(\overline{B}_{m+1}(K_1, F_0)) \leq c_1 |K_1 \cap F_0^\perp|^{1/m} R(P_{F_0}(Z_m(K_1))) \\ &\leq c_2(\delta_1) R(\text{conv}(K_1, -K_1)) = c_2(\delta_1) R(K_1) \leq c_2(\delta_1)\gamma_1\sqrt{n}L_{K_1}. \end{aligned}$$

It is also easy to check that  $R(K) = R(W \times U(W)) \simeq R(W)$ , hence  $R(K) \leq \gamma\sqrt{n}L_K$  for some absolute constant  $\gamma > 0$ . Similarly for the odd dimensions, we see that for every  $F \in G_{2m, 2m-1}$ ,

$$(5.24) \quad R(\overline{B}_{2m}(K, F)) \simeq R(P_F(K)) \leq R(K) \leq c\gamma\sqrt{2m-1} L_{\overline{B}_{2m}(K, F)},$$

where we have made use of (5.21), (5.22). Thus, we can state the following version of Theorem 5.1.

**Theorem 5.9.** *There exist absolute constants  $\kappa, \tau, \gamma$  and  $\delta > 0$  such that, for every  $n \in \mathbb{N}$ , there can be found an isotropic convex body  $K$  in  $\mathbb{R}^n$  with  $R(K) \leq \gamma\sqrt{n}L_K$ ,  $L_K \geq \delta L_n$ , and the property that*

$$(5.25) \quad \log N(K, tB_2^n) \leq \frac{\kappa n^2 \log^2 n}{t^2} \text{ for all } t \geq \tau\sqrt{n \log n}.$$

**Definition 5.10.** Let  $\mathcal{IK}(\kappa, \tau, \gamma, \delta)$  denote the class of isotropic convex bodies whose existence is established in Theorem 5.9. Let  $\rho > 0$  be the absolute constant in Theorem 4.1. Then, we define  $A(n, \kappa, \tau, \gamma, \delta)$  to be the set of all  $q \in [2, \rho^2 n]$  for which there exists  $K \in \mathcal{IK}(\kappa, \tau, \gamma, \delta)$  such that  $I_1(K, Z_q^\circ(K)) \leq \rho n L_K^2$ . Observe that already, by (3.2),  $A(n, \kappa, \tau, \gamma, \delta)$  can be shown to contain an interval of the form  $[2, c\sqrt{n}]$  where  $c > 0$  is an absolute constant. Clearly, any improvement to the upper bound in (3.2) will automatically give us that  $A(n, \kappa, \tau, \gamma, \delta)$  contains an even larger part of  $[2, \rho^2 n]$ . For those  $q$  we set

$$(5.26) \quad B(q) = \inf \left\{ \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn} L_K^2} : K \in \mathcal{IK}(\kappa, \tau, \gamma, \delta) \right\}.$$

Then, Theorem 4.1 implies the following: for every  $q \in A(n, \kappa, \tau, \gamma, \delta)$ ,

$$(5.27) \quad \delta^2 L_n^2 \leq C \kappa \sqrt{n/q} \log^2 n \max\{1, B(q)\}.$$

In other words, we have:

**Theorem 5.11.** *There exist absolute constants  $\kappa, \tau, \gamma$  and  $\delta > 0$  such that, for every  $n \in \mathbb{N}$ ,*

$$(5.28) \quad L_n^2 \leq \min \left\{ \frac{C \kappa}{\delta^2} \sqrt{n/q} \log^2 n \max\{1, B(q)\} : q \in A(n, \kappa, \tau, \gamma, \delta) \right\}.$$

The estimate  $L_n \leq c \sqrt[4]{n} \log n$  is a direct consequence of Theorem 5.11: observe that  $B(2) \simeq 1$ .

## 6 Isotropic convex bodies with small diameter

In [7, Section 3] it is proven that for every isotropic convex body  $K$  there exists a second isotropic convex body  $C$  with bounded isotropic constant and the “same behaviour” as  $K$  with respect to linear functionals.

**Theorem 6.1.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . There exists an isotropic convex body  $C$  in  $\mathbb{R}^n$  with the following properties:*

- (i)  $L_C \leq c_1$ .
- (ii)  $c_2 Z_q(C) \subseteq \frac{Z_q(K)}{L_K} + \sqrt{q} B_2^n \subseteq c_3 Z_q(C)$  for all  $1 \leq q \leq n$ .
- (iii)  $c_4 I_q(C, W) \leq \frac{I_q(K, W)}{L_K} + I_q(D_n, W) \leq c_5 I_q(C, W)$  for all  $1 \leq q \leq n$  and every symmetric convex body  $W$  in  $\mathbb{R}^n$ .

The constants  $c_i$ ,  $i = 1, \dots, 5$  are absolute positive constants.

The body  $C$  is defined as the “convolution” of  $K$  with a multiple of  $B_2^n$ . If we also assume that  $K$  is symmetric, then using the fact that  $Z_n(C) \simeq C$  and  $Z_n(K) \simeq K$ , we see that

$$(6.1) \quad C \simeq \frac{K}{L_K} + D_n.$$

From the previous section, we know that for our purposes it is enough to study the quantity  $I_1(K, Z_q^\circ(K))$  in the cases that  $K$  is an isotropic symmetric convex body with small diameter; that is, we can assume that  $R(K) \leq \gamma\sqrt{n}L_K$  for some  $\gamma \simeq 1$ . The next proposition, which makes use of Theorem 6.1, shows us that it even suffices to consider isotropic convex bodies which are  $c(\gamma)$ -isomorphic to a ball.

**Proposition 6.2.** *Let  $K$  be an isotropic symmetric convex body in  $\mathbb{R}^n$  with  $R(K) \leq \gamma\sqrt{n}L_K$ . Then, there exists an isotropic symmetric convex body  $C$  such that:*

- (i)  $L_C \leq c_6$ ,
- (ii)  $c_7D_n \subseteq C \subseteq c_8\gamma D_n$ , and
- (iii)  $I_1(K, Z_q^\circ(K)) \leq c_9I_1(C, Z_q^\circ(C))L_K^2$  for all  $1 \leq q \leq n$ ,

where  $c_6, c_7, c_8, c_9 > 0$  are absolute constants.

*Proof.* We will use the fact that  $w_q(Z_q(K)) \simeq \sqrt{q/n}I_q(K)$ , and hence

$$(6.2) \quad c_1\sqrt{q}L_K \leq w_q(Z_q(K)) \leq \gamma\sqrt{q}L_K$$

for all  $1 \leq q \leq n$ . We consider the body  $C$  defined by Theorem 6.1. It is clear that  $L_C \leq c_6$  for some absolute constant  $c_6 > 0$ . Since  $\frac{1}{L_K}Z_q(K) \subseteq c_3Z_q(C)$ , we have  $c_3L_KZ_q^\circ(K) \supseteq Z_q^\circ(C)$ , and hence

$$(6.3) \quad I_1(C, Z_q^\circ(C)) \geq I_1(C, c_3L_KZ_q^\circ(K)) = \frac{1}{c_3L_K}I_1(C, Z_q^\circ(K)).$$

Applying the inequality  $c_5I_1(C, W) \geq \frac{I_1(K, W)}{L_K}$  with  $W = Z_q^\circ(K)$ , we get

$$(6.4) \quad I_1(C, Z_q^\circ(C)) \geq c_9 \frac{I_1(K, Z_q^\circ(K))}{L_K^2},$$

with  $c_9 = (c_3c_5)^{-1}$ . Finally, from (6.1) and the fact that  $\frac{K}{L_K} \subseteq \gamma\sqrt{n}B_2^n$ , we see that  $c_7D_n \subseteq C \subseteq c_8\gamma D_n$ .  $\square$

In view of this result, we can give one more version of the ‘‘reduction theorem’’ of Section 4.

**Definition 6.3.** Let  $\mathcal{IK}_{\text{sd}}(\gamma)$  denote the class of isotropic convex bodies that satisfy

- (i)  $L_C \leq c_6$  and
- (ii)  $c_7D_n \subseteq C \subseteq c_8\gamma D_n$ ,

where  $c_i > 0$  are absolute constants (e.g. the ones in Proposition 6.2). For every  $2 \leq q \leq n$ , set

$$(6.5) \quad \Gamma(q) = \sup \left\{ \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}} : K \in \mathcal{IK}_{\text{sd}}(\gamma) \right\}.$$

Then, Theorem 5.11 and Proposition 6.2 imply the following:

**Theorem 6.4.** *There exist absolute constants  $\kappa, \tau, \gamma$  and  $\delta > 0$  such that, for every  $n \in \mathbb{N}$ ,*

$$(6.6) \quad L_n^2 \leq \min \left\{ \frac{C\kappa}{\delta^2} \sqrt{n/q} \log^2 n \Gamma(q) : q \in A(n, \kappa, \tau, \gamma, \delta) \right\}.$$

In other words, studying the behaviour of  $\frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}}$  within the class  $\mathcal{IK}_{\text{sd}}(\gamma)$  is enough in order to understand the behaviour of the parameter  $B(q)$  of Section 5 as well as whether that behaviour can lead to improved upper bounds for  $L_n$ .

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APOSTOLOS GIANNOPOULOS: Department of Mathematics, University of Athens, Panepistimioupolis 157-84, Athens, Greece.

*E-mail:* [apgiannop@math.uoa.gr](mailto:apgiannop@math.uoa.gr)

GRIGORIS PAOURIS: Department of Mathematics, Texas A & M University, College Station, TX 77843 U.S.A.

*E-mail:* [grigoris\\_paouris@yahoo.co.uk](mailto:grigoris_paouris@yahoo.co.uk)

BEATRICE-HELEN VRITSIOU: Department of Mathematics, University of Athens, Panepistimioupolis 157-84, Athens, Greece.

*E-mail:* [bevritsi@math.uoa.gr](mailto:bevritsi@math.uoa.gr)