GLOBAL NONLINEAR BRASCAMP–LIEB INEQUALITIES

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Abstract. We prove global versions of certain known nonlinear Brascamp–Lieb inequalities under a natural homogeneity assumption. We also establish a conditional theorem allowing one to generally pass from local to global nonlinear Brascamp–Lieb estimates under such a homogeneity assumption.

1. Introduction

The classical Brascamp–Lieb inequalities simultaneously generalise a number of fundamental inequalities in euclidean analysis such as the multilinear Hölder, Young convolution and Loomis–Whitney inequalities. To formulate these, suppose $m \geq 2$ and $d_1, \ldots, d_m$ are positive integers, $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ is a linear surjection and $p_j \in [0,1]$ for each $1 \leq j \leq m$. The associated Brascamp–Lieb inequality takes the form

$$\int_{\mathbb{R}^d} \prod_{j=1}^m (f_j \circ B_j)^{p_j} \leq C \prod_{j=1}^m \Big(\int_{\mathbb{R}^{d_j}} f_j \Big)^{p_j} \tag{1}$$

for all nonegative $f_j \in L^1(\mathbb{R}^{d_j})$. At this level of generality, there is of course no reason to expect that the constant C be finite, and in particular, an elementary scaling argument reveals that the condition $\sum p_j d_j = d$ is necessary for finiteness.

In order to present our results it will be convenient to adopt some notation from [3], and write

$$\mathrm{BL}(\mathbf{B}, \mathbf{p}; \mathbf{f}) = \frac{\int_{\mathbb{R}^d} \prod_{j=1}^m (f_j \circ B_j)^{p_j}}{\prod_{j=1}^m (\int_{\mathbb{R}^{d_j}} f_j)^{p_j}}, \tag{2}$$

where $\mathbf{B} = (B_j)$, $\mathbf{p} = (p_j)$ and $\mathbf{f} = (f_j)$. (It is of course implicit that (2) is only defined for inputs $\mathbf{f}$ satisfying $\int f_j \in (0, \infty)$ for each $1 \leq j \leq m$.) Using this functional notation the smallest constant $C \in (0, \infty]$ for which (1) holds is of course

$$\mathrm{BL}(\mathbf{B}, \mathbf{p}) := \sup_{\mathbf{f}} \mathrm{BL}(\mathbf{B}, \mathbf{p}; \mathbf{f}).$$

Lieb [6] proved that this supremum is exhausted by centred gaussian inputs. Further issues, including the finiteness of the Brascamp–Lieb constant $\mathrm{BL}(\mathbf{B}, \mathbf{p})$ and the shape of extremal inputs $\mathbf{f}$ when they exist, have been addressed by a number of authors; see for example [1], [5] and [3], and further references contained there for a fuller account.

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A rather expansive *nonlinear* generalisation of the Brascamp–Lieb inequality would be to take each B_j to be a smooth submersion in a neighbourhood of a point $x_0 \in \mathbb{R}^d$ and $\mathbf{p} \in [0,1]^m$ for which

$$\mathrm{BL}(\mathrm{d}\mathbf{B}(x_0), \mathbf{p}) < \infty,$$

and seek a neighbourhood U of this point and a finite constant C such that

$$\int_U \prod_{j=1}^m (f_j \circ B_j)^{p_j} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j} \tag{3}$$

for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$. Here $\mathrm{d}\mathbf{B}(x_0) = (\mathrm{d}B_j(x_0))$, where $\mathrm{d}B_j(x_0) : \mathbb{R}^d \to \mathbb{R}^{d_j}$ denotes the derivative of B_j at the point x_0. Such a generalisation has been shown to hold under the additional structural hypothesis that the kernels of the derivative maps $\mathrm{d}B_j(x_0)$ form a direct sum decomposition of $\mathbb{R}^d$; see [2] and earlier work in [4] upon which the former built, both being inspired by applications to problems in euclidean geometric and harmonic analysis. We refer the reader to these papers for further details.

In this note, we consider *global* nonlinear Brascamp–Lieb inequalities under the additional assumption that the submersions B_j are homogeneous of degree one. This is a natural level of homogeneity to consider since of course it encompasses the case of linear mappings. The difficulty in passing from local statements such as (3) to global ones where the neighbourhood U is replaced by $\mathbb{R}^d$ comes from the fact that in general functions which are homogeneous of degree 1 do not possess derivatives at the origin. Thus typically the neighbourhood U in (3) will not contain the origin, excluding the possibility of generating global estimates via an elementary scaling and limiting argument. In this paper we present a method for passing from local to global statements through the identification of a certain orthogonality present in even the most general of settings.

In the first of our two main theorems, we prove a global analogue of the aforementioned local result from [2]. Secondly, we obtain a conditional theorem which states that given a local nonlinear Brascamp–Lieb inequality and assuming the necessary scaling condition on the exponents p_j, one can always generate a global extension if the B_j are homogeneous of degree one.

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2. The unconditional theorem

In this section we shall prove the following.

Theorem 1. *Suppose that for each $1 \leq j \leq m$ the mappings $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ are homogeneous of degree one and $C^{1,\beta}$ submersions on $\mathbb{S}^{d-1}$ for some $\beta > 0$. Suppose further that*

$$\bigoplus_{j=1}^m \ker \mathrm{d}B_j(\omega) = \mathbb{R}^d \tag{4}$$

for each $\omega \in \mathbb{S}^{d-1}$. Then there exists a constant C such that

$$\int_{\mathbb{R}^d} \prod_{j=1}^m (f_j \circ B_j)^{\frac{1}{m-1}} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{\frac{1}{m-1}}$$

for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$.

It will be clear that our proof of Theorem 1 permits a more quantitative statement, since we do a certain amount of book-keeping.

We begin by recalling the quantitative local nonlinear Brascamp–Lieb inequalities from [2]. If $1 \leq d_1, \ldots, d_m \leq d-1$ satisfy $\sum_{j=1}^m d_j = (m-1)d$, or equivalently $\sum_{j=1}^m d_j^* = d$ where $d_j^* := d - d_j$, then we set

$$\mathcal{K}_j = \{d_1 + \cdots + d_{j-1} + 1, \ldots, d_1 + \cdots + d_{j-1} + d_j\}$$

and

$$\mathcal{K}_j^* = \{d - (d_1^* + \cdots + d_{j-1}^* + d_j^*) + 1, \ldots, d - (d_1^* + \cdots + d_{j-1}^*)\}$$

for $1 \leq j \leq m$. Thus, each $\mathcal{K}_j$ has d_j elements and together form a partition of $\{1, \ldots, (m-1)d\}$, and each $\mathcal{K}_j^*$ has d_j^* elements and together partition $\{1, \ldots, d\}$.

As in [2], for linear mappings $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$, let $X_j(B_j) \in \Lambda^{d_j}(\mathbb{R}^d)$, the d_jth exterior algebra of $\mathbb{R}^d$, be given by

$$X_j(B_j) = \bigwedge_{k=1}^{d_j} B_j^*(e_k),$$

where e_k is the kth standard basis vector of $\mathbb{R}^{d_j}$. Here, $B_j^* : \mathbb{R}^{d_j} \to \mathbb{R}^d$ denotes the adjoint of B_j. Also, we shall use $\star : \Lambda^n(\mathbb{R}^d) \to \Lambda^{n-d}(\mathbb{R}^d)$ to denote the Hodge star operator.

Theorem 2. [2] *Let $\beta, \varepsilon, \kappa > 0$. Suppose that for each $1 \leq j \leq m$ the submersions $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ satisfy $\|B_j\|_{C^{1,\beta}} \leq \kappa$ in a neighbourhood of a point $x_0 \in \mathbb{R}^d$. Suppose further that*

$$\left| \star \bigwedge_{j=1}^m \star X_j(\mathrm{d}B_j(x_0)) \right| \geq \varepsilon. \tag{5}$$

Then there exists a neighbourhood U of x_0, depending on at most $\beta, \varepsilon, \kappa$ and d, and a constant C, depending on at most d, such that

$$\int_U \prod_{j=1}^m (f_j \circ B_j)^{\frac{1}{m-1}} \leq C \varepsilon^{-\frac{1}{m-1}} \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{\frac{1}{m-1}}$$

for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$.

Note that if $\bigoplus_{j=1}^m \ker \mathrm{d}B_j(x_0) = \mathbb{R}^d$ then $|\star \bigwedge_{j=1}^m \star X_j(\mathrm{d}B_j(x_0))| \neq 0$ and therefore (5) is a quantification of this form of transversality.

Proof of Theorem 1. Let $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ satisfy the hypotheses of Theorem 1. By the smoothness hypothesis and (4), the function

$$\omega \mapsto \star \bigwedge_{j=1}^m \star X_j(\mathrm{d}B_j(\omega))$$

is continuous and nonvanishing on $\mathbb{S}^{d-1}$. Hence there exists $\varepsilon > 0$ such that

$$\left| \star \bigwedge_{j=1}^m \star X_j(\mathrm{d}B_j(\omega)) \right| \geq \varepsilon \tag{6}$$

for all $\omega \in \mathbb{S}^{d-1}$. If we let $\kappa > 0$ dominate the $C^{1,\beta}$ norm of each B_j, for each $\omega \in \mathbb{S}^{d-1}$ we may apply Theorem 2 to obtain a neighbourhood $B(\omega, \delta)$, where $0 < \delta < 1$ depends on at most $\beta, \varepsilon, \kappa$ and d, and a constant C depending on ε and d such that

$$\int_{B(\omega,\delta)} \prod_{j=1}^m (f_j \circ B_j)^{\frac{1}{m-1}} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{\frac{1}{m-1}}$$

for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$. For notational convenience, let $U = B(\omega, \delta)$.

By isotropic scaling, using the homogeneity of the B_j, we obtain

$$\int_{\lambda U} \prod_{j=1}^m (f_j \circ B_j)^{\frac{1}{m-1}} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{\frac{1}{m-1}} \tag{7}$$

uniformly in $\lambda > 0$. If $\Gamma(\omega, \delta)$ denotes the cone in $\mathbb{R}^d$ with axis $\omega \in \mathbb{S}^{d-1}$ and aperture $\delta > 0$, then by elementary considerations,

$$\Gamma(\omega, \delta) \subseteq \bigcup_{k \in \mathbb{Z}} (1 + c\delta)^k U$$

for a suitable absolute constant $c > 0$ [1]. With $\lambda = 1 + c\delta$, by the homogeneity of the B_j and (7), it follows that

$$\begin{aligned}
\int_{\Gamma(\omega,\delta)} \prod_{j=1}^m (f_j \circ B_j)^{\frac{1}{m-1}} &\leq \sum_{k \in \mathbb{Z}} \int_{\lambda^k U} \prod_{j=1}^m (f_j \circ B_j)^{\frac{1}{m-1}} \\
&= \sum_{k \in \mathbb{Z}} \int_{\lambda^k U} \prod_{j=1}^m \left((f_j \chi_{B_j(\lambda^k U)}) \circ B_j \right)^{\frac{1}{m-1}} \\
&\leq C \sum_{k \in \mathbb{Z}} \prod_{j=1}^m \left(\int_{B_j(\lambda^k U)} f_j \right)^{\frac{1}{m-1}} \\
&= C \sum_{k \in \mathbb{Z}} \prod_{j=1}^m \left(\int_{\lambda^k B_j(U)} f_j \right)^{\frac{1}{m-1}}.
\end{aligned}$$

Lemma 3. *The sets $\{\lambda^k B_j(U) : k \in \mathbb{Z}\}$ have bounded overlap, with bound depending on at most ε, κ and d, for at least $m-1$ indices $j \in \{1, \ldots, m\}$.*

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[1] If we assume, as we may, that $\delta < 1/3$, then we may take $c = 2$.

Assuming this lemma is true, let us see how Theorem 1 follows. If $\omega \in \mathbb{S}^{d-1}$ then by Lemma 3 we know that there exists $j(\omega) \in \{1, \ldots, m\}$ such that the sets $\{\lambda^k B_j(U) : k \in \mathbb{Z}\}$ for $j \neq j(\omega)$ have bounded overlap, depending on at most ε, κ and d. Hence, by the $(m-1)$-linear discrete Hölder inequality,

$$\begin{aligned}
\int_{\Gamma(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{\frac{1}{m-1}} &\leq C \sum_{k\in\mathbb{Z}} \prod_{j=1}^{m} \left(\int_{\lambda^k B_j(U)} f_j \right)^{\frac{1}{m-1}} \\
&\leq C \left(\int_{\mathbb{R}^{d_{j(\omega)}}} f_{j(\omega)} \right)^{\frac{1}{m-1}} \sum_{k\in\mathbb{Z}} \prod_{j\neq j(\omega)} \left(\int_{\lambda^k B_j(U)} f_j \right)^{\frac{1}{m-1}} \\
&\leq C \left(\int_{\mathbb{R}^{d_{j(\omega)}}} f_{j(\omega)} \right)^{\frac{1}{m-1}} \prod_{j\neq j(\omega)} \left(\sum_{k\in\mathbb{Z}} \int_{\lambda^k B_j(U)} f_j \right)^{\frac{1}{m-1}} \\
&\leq C' \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{\frac{1}{m-1}},
\end{aligned}$$

where C is the same constant from (2), and C' is a constant depending on at most ε, κ and d. Theorem 1 follows by piecing together enough cones $\Gamma(\omega, \delta)$ to fill $\mathbb{R}^d$. $\square$

Proof of Lemma 3. It suffices to prove that the distance from $B_j(U)$ to the origin in $\mathbb{R}^{d_j}$ is bounded below by a positive constant, depending on at most ε, κ and d, for at least $m-1$ indices $j \in \{1, \ldots, m\}$. By the smoothness of the B_j, this follows if we can show that there exists such a constant $\eta > 0$ so that for each $\omega \in \mathbb{S}^{d-1}$ we have $|B_j(\omega)| \geq \eta$ for at least $m-1$ indices $j \in \{1, \ldots, m\}$. For this, we may need to shrink δ, as we may, to a level which depends on at most ε, κ and d.

Suppose $\omega \in \mathbb{S}^{d-1}$. Let $\{a_1, \ldots, a_d\}$ be a basis for $\mathbb{R}^d$ where $\{a_k : k \in \mathcal{K}_j^*\}$ is an orthonormal basis for $\ker \mathrm{d}B_j(\omega)$, and let $\{a_k^\perp : k \in \mathcal{K}_j\}$ be an orthonormal basis for the orthogonal complement of $\ker \mathrm{d}B_j(\omega)$. Then we may write $\omega = \omega_j + \sum_{k\in\mathcal{K}_j} \lambda_k a_k^\perp$ where ω_j is the projection of ω onto $\ker \mathrm{d}B_j(\omega)$, and $\lambda_k \in \mathbb{R}$ for $k \in \mathcal{K}_j$. Now, using the homogeneity of the B_j,

$$\sum_{k\in\mathcal{K}_j} \lambda_k \mathrm{d}B_j(\omega) a_k^\perp = \mathrm{d}B_j(\omega)\omega = B_j(\omega). \tag{8}$$

If M_j is the matrix whose columns are $\mathrm{d}B_j(\omega)a_k^\perp$ for $k \in \mathcal{K}_j$ then

$$\begin{aligned}
|\det(M_j)| &= \left| \left\langle X_j(\mathrm{d}B_j(\omega)), \bigwedge_{k\in\mathcal{K}_j} a_k^\perp \right\rangle_{\Lambda^{d_j}(\mathbb{R}^d)} \right| \\
&= \|X_j(\mathrm{d}B_j(\omega))\|_{\Lambda^{d_j}(\mathbb{R}^d)} \left| \left\langle \star \bigwedge_{k\in\mathcal{K}_j^*} a_k, \bigwedge_{k\in\mathcal{K}_j} a_k^\perp \right\rangle_{\Lambda^{d_j}(\mathbb{R}^d)} \right|
\end{aligned}$$

and therefore

$$|\det(M_j)| = \|X_j(\mathrm{d}B_j(\omega))\|_{\Lambda^{d_j}(\mathbb{R}^d)}.$$

If A is the matrix whose columns are a_j for $1 \leq j \leq d$ then

$$\left| \star \bigwedge_{j=1}^{m} X_j(\mathrm{d}B_j(\omega)) \right| = |\det(A)| \prod_{j=1}^{m} \|X_j(\mathrm{d}B_j(\omega))\|_{\Lambda^{d_j}(\mathbb{R}^d)}, \tag{9}$$

and therefore each $\|X_j(\mathrm{d}B_j(\omega))\|_{\Lambda^{d_j}(\mathbb{R}^d)}$ is bounded above and below by a constant depending on at most ε, κ and d. It follows from this and (8) that if $|B_j(\omega)| < \eta$ for a particular index j, then $|\lambda_k| \leq C_{\varepsilon,\kappa,d}\eta$ for each $k \in \mathcal{K}_j$ and therefore

$$|\omega - \omega_j| \leq C_{\varepsilon,\kappa,d}\eta. \tag{10}$$

If we also have $|B_{j'}(\omega)| < \eta$ for some $j' \neq j$ then by the same argument we also have

$$|\omega - \omega_{j'}| \leq C_{\varepsilon,\kappa,d}\eta, \tag{11}$$

and together (10) and (11) imply that $|\det(A)| \leq C_{\varepsilon,\kappa,d}\eta$, as long as η is sufficiently small depending on ε, κ and d. However, because of (9), this leads to a contradiction, again, by choosing η to be sufficiently small depending on ε, κ and d. Hence, for such a choice of $\eta > 0$, for each $\omega \in \mathbb{S}^{d-1}$ we have $|B_j(\omega)| \geq \eta$ occurring for at least $m-1$ indices $j \in \{1, \ldots, m\}$. This completes our claim, and consequently the proof of the lemma. $\square$

3. The conditional theorem

Based on the argument in the previous section, we may obtain an abstract conditional theorem which allows one to promote a local inequality to a global inequality in the following sense.

Theorem 4. *Suppose $\sum_{j=1}^m p_j d_j = d$ and that for each $1 \leq j \leq m$ the mappings $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ are homogeneous of degree one and $C^{1,\beta}$ submersions on a neighbourhood of some point $\omega \in \mathbb{R}^d$ for some $\beta > 0$. Suppose further that for some $\delta > 0$ and constant C, the inequality*

$$\int_{B(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j} \tag{12}$$

holds for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$. Then there is a constant C' such that

$$\int_{\Gamma(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} \leq C' \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}$$

holds for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$.

The following lemma is key to the proof of Theorem 4, for which the homogeneity of the B_j is not required.

Lemma 5. *Suppose $\sum_{j=1}^m p_j d_j = d$ and that for each $1 \leq j \leq m$ the mappings $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ are $C^{1,\beta}$ submersions on a neighbourhood of some point $\omega \in \mathbb{R}^d$ for some $\beta > 0$. Suppose further that for some $\delta > 0$ and constant C, the inequality*

$$\int_{B(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j} \tag{13}$$

holds for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$*,* $1 \le j \le m$*. Then* $\mathrm{BL}(\mathrm{d}\mathbf{B}(\omega), \mathbf{p}) < \infty$*.*

Proof. By translation invariance, we may assume that $\omega = 0$ and $B_j(\omega) = 0$ for each $1 \le j \le m$. Suppose κ dominates the $C^{1,\beta}$ norm of each B_j on $B(0,\delta)$. Then it follows that

$$|B_j(x) - \mathrm{d}B_j(0)x| \le C_d \kappa |x|^{1+\beta} \tag{14}$$

for all $x \in B(0,\delta)$.

Recall from [2] the class $L^1_M(\mathbb{R}^{d_j})$ of nonnegative functions $f \in L^1(\mathbb{R}^{d_j})$ satisfying $f(y_1) \le 2f(y_2)$ whenever y_1 and y_2 are in the support of f and $|y_1 - y_2| \le M^{-1}$. Such f are effectively constant at the scale M^{-1}. By density and scaling arguments, it suffices to prove that there is a neighbourhood U of the origin in $\mathbb{R}^d$, depending on β, κ and d, and a constant C', depending on m and C, such that whenever $f_j \in L^1_M(\mathbb{R}^{d_j})$,

$$\int_U \prod_{j=1}^m (f_j \circ \mathrm{d}B_j(0))^{p_j} \le C' \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}$$

uniformly in M larger than some threshold depending on δ and κ.

To this end, fix

$$M > \frac{1}{(C_d\kappa)^{\frac{\beta}{1+\beta}} \delta^\beta} \tag{15}$$

and $f_j \in L^1_M(\mathbb{R}^{d_j})$. Now let $g_j : \mathbb{R}^{d_j} \to [0,\infty)$ be the rescaled functions given by $g_j(y) = f_j(M^{1/\beta} y)$. Using (14) it follows that whenever $x \in B(0,\delta_M)$ we have

$$g_j(\mathrm{d}B_j(0)x) \le 2 g_j(B_j(x)),$$

where

$$\delta_M = \frac{1}{(C_d\kappa)^{\frac{1}{1+\beta}} M^{\frac{1}{\beta}}}.$$

By (15) and (13) it follows that

$$\int_{B(0,\delta_M)} \prod_{j=1}^m (g_j \circ \mathrm{d}B_j(0))^{p_j} \le 2^m C \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} g_j \right)^{p_j},$$

and using the scaling hypothesis, $\sum_{j=1}^m p_j d_j = d$, we obtain

$$\int_{B(0,M^{1/\beta}\delta_M)} \prod_{j=1}^m (f_j \circ \mathrm{d}B_j(0))^{p_j} \le 2^m C \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j},$$

which is the desired conclusion. □

Proof of Theorem 4. As in the proof of Theorem 1, from the local inequality (12) and by scaling, making use of the homogeneity of the B_j,

$$\int_{\Gamma(\omega,\delta)} \prod_{j=1}^m (f_j \circ B_j)^{p_j} \le C \sum_{k\in\mathbb{Z}} \prod_{j=1}^m \left(\int_{\lambda^k B_j(U)} f_j \right)^{p_j},$$

where $U = B(\omega,\delta)$ and $\lambda = 1 + c\delta$ for some suitably chosen constant $c > 0$.

By Lemma 5, $\mathrm{BL}(\mathrm{d}\mathbf{B}(\omega), \mathbf{p}) < \infty$ and so, by the necessary conditions for finiteness of the Brascamp–Lieb constant in [3], we have

$$1 \leq \sum_{j=1}^{m} p_j \dim(\mathrm{d}B_j(\omega)V),$$

where V is the one-dimensional subspace of $\mathbb{R}^d$ spanned by ω. By the homogeneity of the B_j we have $\mathrm{d}B_j(\omega)\omega = B_j(\omega)$ and therefore

$$\sum_{j:B_j(\omega)\neq 0} p_j \geq 1.$$

Hence, by discrete Hölder's inequality,

$$\begin{aligned}\int_{\Gamma(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} &\leq C \prod_{j:B_j(\omega)=0} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j} \sum_{k\in\mathbb{Z}} \prod_{j:B_j(\omega)\neq 0} \left(\int_{\lambda^k B_j(U)} f_j \right)^{p_j} \\ &\leq C \prod_{j:B_j(\omega)=0} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j} \prod_{j:B_j(\omega)\neq 0} \left(\sum_{k\in\mathbb{Z}} \int_{\lambda^k B_j(U)} f_j \right)^{p_j}.\end{aligned}$$

If $B_j(\omega) \neq 0$ then the distance from $B_j(U)$ to the origin in $\mathbb{R}^{d_j}$ is strictly positive. This follows from the smoothness of the B_j and, if necessary, shrinking δ below some threshold (which may depend on ω). Consequently, for $1 \leq j \leq m$ such that $B_j(\omega) \neq 0$, the sets $\{\lambda^k B_j(U) : k \in \mathbb{Z}\}$ have bounded overlap (with overlapping multiplicity which may depend on ω). Hence

$$\int_{\Gamma(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} \leq C' \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}$$

as claimed. □

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