SPECIAL DETERMINANTS IN HIGHER-RANK BRILL-NOETHER THEORY

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ABSTRACT. Continuing our previous study of modified expected dimensions for rank-2 Brill-Noether loci with prescribed special determinants, we introduce a general framework which applies *a priori* for arbitrary rank, and use it to prove modified expected dimension bounds in several new cases, applying both to rank 2 and to higher rank. The main tool is the introduction of generalized alternating Grassmannians, which are the loci inside Grassmannians corresponding to subspaces which are simultaneously isotropic for a family of multilinear alternating forms on the ambient vector space. In the case of rank 2 with 2-dimensional spaces of sections, we adapt arguments due to Teixidor i Bigas to show that our new modified expected dimensions are in fact sharp.

1. INTRODUCTION

The purpose of the present paper is to continue the systematic study of higher-rank Brill-Noether loci with fixed special determinant initiated in [10]. Given a smooth projective curve C of genus g, and a line bundle $\mathscr{L}$ on C, we set up a general framework for proving dimension lower bounds for Brill-Noether loci with fixed determinant $\mathscr{L}$, expressed in terms of $h^1(C, \mathscr{L})$. Although our immediate goal is a sharp understanding of the rank-2 case, the setup is carried out in full generality, including in arbitrary rank. We then apply it to obtain concrete results in several families of cases for which the dimension of the space of sections considered is relatively small compared to the rank.

Given k, r, denote by $\mathcal{G}^k_{r,\mathscr{L}}(C)$ the moduli stack of vector bundles on C of rank r and fixed determinant $\mathscr{L}$ together with a k-dimensional space of global sections. The naive expected dimension for $\mathcal{G}^k_{r,\mathscr{L}}(C)$ is

$$\rho - g := (r^2 - 1)(g - 1) - k(k - d + r(g - 1)).$$

Our first theorem is as follows.

Theorem 1.1. *Let C be a smooth, projective curve of genus g. Suppose $\mathscr{L} \in \mathrm{Pic}^d(C)$, and $h^1(C, \mathscr{L}) \geq m$. Given $r \geq 2$, let $\mathscr{E}$ be a vector bundle of rank r on C with determinant $\mathscr{L}$, and $V \subseteq H^0(C, \mathscr{E})$ a k-dimensional space of global sections. Suppose that in addition, one of the following conditions is satisfied.*

(I) *$k = r$, and V is not contained in any subbundle of $\mathscr{E}$ of rank $r - 2$.*
(II) *$k = r + 1$, $m = 1$, and no r-dimensional subspace of V is contained in any subbundle of $\mathscr{E}$ of rank $r - 2$.*

(III) $r = 3$, $k = 5$ or 6, $m = 1$, and no 2-dimensional subspace of V is contained in any subbundle of $\mathscr{E}$ of rank 1.

Then every component of $\mathcal{G}^k_{r,\mathscr{L}}(C)$ passing through the point corresponding to $(\mathscr{E}, V)$ has dimension at least

$$(1.1) \qquad\qquad \rho - g + m \binom{k}{r}.$$

Remarks 1.2. (i) Even though we are only proving a lower bound on dimension, a nondegeneracy hypothesis is required. This has not been the case for previous work on the subject, but is expected to be a feature of any further generalizations. The nondegeneracy hypothesis in cases (I) and (II) is essentially a generic version of what Mukai calls "semiirreducibility" in [9].

(ii) Only case (I) gives new results for rank 2, since the main results of [10] proved in particular the same dimension bound as above in the cases $r = 2, m \leq 2$, and with k arbitrary.

(iii) Although case (III) may appear special, recall that d and g are allowed to vary, so they still contain infinite families of rank-3 Brill-Noether loci, including some particularly interesting examples; see Example 5.3 below.

(iv) Our arguments also work for families of special determinants, and can thus be used to study the variable determinant case as well. See Theorem 3.4 below for a precise statement.

Under somewhat stronger nondegeneracy hypotheses than those imposed in Theorem 1.1, the dimension statements for cases (I) and (II) may be approached via direct analysis. This has been done in the literature for the case of varying determinant as follows: for $r = k = 2$ by Teixidor i Bigas in [12], and for the more general cases by Bradlow-García-Prada-Muñoz-Newstead [4], by Bradlow-García-Prada-Mercat-Muñoz-Newstead [3], and by Bhosle-Brambila-Paz-Newstead [1]. However, the same constructions may be applied to the fixed determinant case; see Grzegorczyk and Newstead [7]. Of particular note is that these constructions also show that the dimension lower bounds of Theorem 1.1 are sharp (still under the stronger nondegeneracy hypotheses).

We illustrate these methods by following and elaborating on the arguments of Teixidor to verify that case (I) of Theorem 1.1 is sharp for $r = 2$, even without additional nondegeneracy hypotheses. To state the theorem, we denote by $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$ the open substack of $\mathcal{G}^2_{2,\mathscr{L}}(C)$ on which the bundle is generically generated by the chosen space of global sections, and by $\mathcal{G}^{2,\mathrm{st}}_{2,\mathscr{L}}(C)$ the open substack on which the underlying bundle is stable.

Theorem 1.3. *Let C be a smooth, projective curve of genus g. Suppose $\mathscr{L} \in \mathrm{Pic}^d(C)$, and $h^1(C, \mathscr{L}) = m$.*

Then $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$ is nonempty if and only if $h^0(C, \mathscr{L}) > 0$, and in this case is irreducible of dimension $\rho - g + m$.

If C is Brill-Noether general with respect to $\mathfrak{g}^1_{d'}$'s for all d', then $\mathcal{G}^2_{2,\mathscr{L}}(C)$ has dimension $\rho - g + m$. If further $h^0(C, \mathscr{L}) > 0$, then $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$ is dense in $\mathcal{G}^2_{2,\mathscr{L}}(C)$, and in particular $\mathcal{G}^2_{2,\mathscr{L}}(C)$ is irreducible.

Finally, when $h^0(C, \mathscr{L}) > 0$, the stack $\mathcal{G}^{2,\mathrm{st}}_{2,\mathscr{L}}(C)$ contains a nonempty open substack of $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$ if C is nonhyperelliptic and $d \geq 3$ or if C is hyperelliptic and $d \geq 5$.

We conclude with a discussion of the prospects for further generalization, and speculation on the possible form of sharp dimension bounds in rank 2. In the process, we investigate several examples from the literature, and find that their constructions of Brill-Noether loci having greater than the expected dimension can be explained by our results.

As in [10], the techniques underlying Theorem 1.1 (and the more general framework) involve suitable generalizations of symplectic Grassmannians. Beyond introducing families of alternating forms as was already considered in [10], to treat the higher-rank case we consider multilinear forms instead of just bilinear forms. This adds additional complications, but due to some simplifications in the overall strategy we are able to prove Theorem 1.1. There is great potential for further generalization, but it will involve a more delicate analysis of how to translate the (multi)linear algebra into suitable nondegeneracy conditions.

In contrast, as in [12], Theorem 1.3 is proved using a careful study of extensions, and the proof is not expected to generalize. Systematic use of stack-theoretic dimension counting simplifies the arguments.

Others have previously considered the two directions of generalization of symplectic Grassmannians discussed above. Subspaces simultaneously isotropic for families of alternating forms have been studied by Buhler, Gupta and Harris [5] in the context of group theory, while Tevelev [13] has studied subspaces isotropic for generic multilinear alternating forms. However, in both cases the focus was on nonemptiness questions, whereas in our case we need to develop criteria for the spaces to be smooth of expected dimension at a particular point.

Acknowledgements

2. Generalized alternating Grassmannians

Let X be a scheme, and $\mathscr{E}$ a vector bundle on X of rank n. Recall that an r-linear alternating form on $\mathscr{E}$ is a morphism

$$\langle, \ldots, \rangle : \bigwedge^r \mathscr{E} \to \mathscr{O}_X.$$

A subbundle $\mathscr{F} \subseteq \mathscr{E}$ is isotropic for $\langle, \ldots, \rangle$ if the restriction of $\langle, \ldots, \rangle$ to $\bigwedge^r \mathscr{F}$ is equal to 0. The subbundle $\mathscr{F}$ is degenerate for $\langle, \ldots, \rangle$ if the induced morphism $\bigwedge^{r-1} \mathscr{F} \to \mathscr{E}^*$ is equal to 0.

Suppose we are given a collection

$$\underline{\langle, \ldots, \rangle} = \{\langle, \ldots, \rangle_1, \ldots, \langle, \ldots, \rangle_m\}$$

of m r-linear alternating forms on $\mathscr{E}$. Then we make the following definition:

Definition 2.1. Given $k < n$, we have the **generalized alternating Grassmannian** $GAG(k, \mathscr{E}, \underline{\langle, \ldots, \rangle})$ defined as the closed subscheme of $G(k, \mathscr{E})$ whose points correspond to subbundles which are simultaneously isotropic for every $\langle, \ldots, \rangle_i \in \underline{\langle, \ldots, \rangle}$.

If X is a point and the forms are sufficiently general, the generalized alternating Grassmannian has codimension $m\binom{k}{r}$ in $G(k,\mathscr{E})$. However, the case of interest for us is not completely general, so we have to carry out a closer analysis. The case $r = 2, m \leq 2$ was handled in [10]. We will see that the same criterion considered in *loc. cit.* (which does not hold in general) also holds when $k = r$, or when $m = 1$ and $k = r + 1$. We first give a general description translating smoothness into (multi)linear algebra.

Lemma 2.2. *Suppose $\mathscr{E}$ is a vector bundle of rank n on a scheme X, and $\langle,\ldots,\rangle_i$ for $i = 1,\ldots,m$ are r-linear alternating forms on $\mathscr{E}$. Given a field K, and a K-valued point x of X, suppose we have $V \subseteq \mathscr{E}|_x$ corresponding to a K-valued point z of $GAG(k, \mathscr{E}|_x, \langle,\ldots,\rangle)$. Then at the (image of the) point z, we have $GAG(k, \mathscr{E}, \langle,\ldots,\rangle)$ smooth over X of codimension $m\binom{k}{r}$ inside $G(k,\mathscr{E})$ if and only if the induced map of K-vector spaces*

$$(2.1) \qquad (\bigwedge^r V)^{\oplus m} \to \mathrm{Hom}(V, \mathscr{E}|_x/V)^*$$

is injective, where the map is determined by

$$\sum_{i=1}^m v_{1,i} \wedge \cdots \wedge v_{r,i} \mapsto \left(\varphi \mapsto \sum_{i=1}^m \sum_{j=1}^r \langle v_{1,i},\ldots,\varphi(v_{j,i}),\ldots,v_{r,i}\rangle_i\right).$$

The following lemma is standard, but we state it for convenience of notation:

Lemma 2.3. *Let $X \to S$ be smooth of relative dimension d, and $Z \subseteq X$ a closed subscheme. Suppose that for some $z \in Z$, with image $s \in S$, we have that the ideal sheaf $\mathscr{I}_Z$ is generated by c elements locally near z, and that the fiber Z_s is smooth at z over $\mathrm{Spec}\,\kappa(s)$, of codimension c in X_s. Then Z is smooth at z of relative dimension $d - c$ over S.*

Proof. This follows essentially immediately from Proposition 2.2.7 of [2]. Indeed, if $f_1,\ldots,f_c$ are local generators for $\mathscr{I}_Z$ near z, then applying part (c) of *loc. cit.* to the fibers X_s and Z_s we find that the differentials $df_1,\ldots,df_c$ must be linearly independent in $\Omega^1_{X/S}|_z$. But then applying part (d) of *loc. cit.* to X and Z, we find that Z is smooth at z of relative dimension $d - c$, as desired. $\qquad\square$

Proof of Lemma 2.2. Recall that if E is a K-vector space, and $V \subseteq E$ corresponds to a K-valued point z of the classical Grassmannian $G(k,E)$, then the tangent space to $G(k,E)$ at z is given by $\mathrm{Hom}(V,E/V)$. Now, if $\langle,\ldots,\rangle$ is an r-linear alternating form on E, and V is isotropic for $\langle,\ldots,\rangle$, then every tangent vector of $G(k,E)$ at z gives us an r-linear alternating form $\langle,\ldots,\rangle^\varphi$ as follows: if the tangent vector is given by $\varphi \in \mathrm{Hom}(V,E/V)$, the associated form is determined by sending $v_1 \wedge \cdots \wedge v_r \in \bigwedge^r V$ to

$$\sum_{i=1}^r \langle v_1,\ldots,\varphi(v_i),\ldots,v_r\rangle.$$

This gives us a map

$$\mathrm{Hom}(V,E/V) \to \left(\bigwedge^r V\right)^*.$$

Thus, the given alternating forms induce a map

$$\mathrm{Hom}(V, E/V) \to \bigoplus_{i=1}^{m} \left(\bigwedge^{r} V \right)^{*}.$$

It is easy to see that the tangent space to $GAG(k, E, \langle, \ldots, \rangle)$ is precisely the kernel of this map. Note also that this map is dual to (2.1) (with E in place of $\mathscr{E}|_x$). Now, we know that $GAG(k, E, \langle, \ldots, \rangle)$ is locally cut out by $m\binom{k}{r}$ equations inside $G(k, E)$, so every component of $GAG(k, E, \langle, \ldots, \rangle)$ has codimension at most $m\binom{k}{r}$ in $G(k, E)$, and $GAG(k, E, \langle, \ldots, \rangle)$ is smooth at z of pure dimension $k(n-k) - m\binom{k}{r}$ if and only if the tangent space at z has dimension is $k(n - k) - m\binom{k}{r}$, if and only if the above map is surjective. This in turn is equivalent to the injectivity of (2.1), again with E in place of $\mathscr{E}|_x$.

Considering the situation of the lemma statement, if we set $E = \mathscr{E}|_x$, recalling that smoothness of a fiber may be checked after extending the base field we conclude from the above that the fiber over x of $GAG(k, \mathscr{E}, \langle, \ldots, \rangle)$ is smooth of codimension $m\binom{k}{r}$ in $G(k, \mathscr{E})$ at the point z if and only if (2.1) is injective. Finally, we conclude the statement of the lemma by applying Lemma 2.3. $\square$

We thus conclude the following general statement on loci of subbundles contained in two given subbundles.

Proposition 2.4. *Suppose $\mathscr{E}$ is a vector bundle of rank n on an algebraic stack $\mathcal{X}$ of finite type over a universally catenary scheme S, and $\langle, \ldots, \rangle_i$ for $i = 1, \ldots, m$ are r-linear alternating forms on $\mathscr{E}$. Let $\mathscr{F}$ and $\mathscr{G}$ be subbundles of $\mathscr{E}$ of ranks s and t, both isotropic with respect to all of the $\langle, \ldots, \rangle_i$. Let $\mathcal{G}(k, \mathscr{F} \cap \mathscr{G})$ denote the closed substack of $\mathcal{G}(k, \mathscr{E})$ parametrizing rank-k subbundles of $\mathscr{E}$ contained in both $\mathscr{F}$ and $\mathscr{G}$. Suppose that for some field K and some K-valued point x of $\mathcal{X}$, we have $V \subseteq \mathscr{F}|_x \cap \mathscr{G}|_x$ such that the map (2.1) is injective. Then every component of $\mathcal{G}(k, \mathscr{F} \cap \mathscr{G})$ passing through the point corresponding to V has codimension at most*

$$k(2n - s - t) - m \binom{k}{r}$$

in $\mathcal{G}(k, \mathscr{E})$.

Proof. We first reduce from the case of an algebraic stack $\mathcal{X}$ to the case of a scheme X by letting $X \to \mathcal{X}$ be a smooth cover, and pulling back the bundles, as in the argument for Corollary 3.7 of [10]. Then X is of finite type over S, and hence universally catenary.

Now, we can realize $G(k, \mathscr{F} \cap \mathscr{G})$ as follows: note that $G(k, \mathscr{F})$ is smooth over S, and has pure codimension $k(n - s)$ everywhere in $G(k, \mathscr{E})$. Because $\mathscr{F}$ and $\mathscr{G}$ are isotropic for the $\langle, \ldots, \rangle_i$, the universal subbundle on $G(k, \mathscr{F})$, together with the pullback of $\mathscr{G}$, induce a morphism

$$G(k, \mathscr{F}) \to GAG(k, \mathscr{E}, \langle, \ldots, \rangle) \times_S GAG(t, \mathscr{E}, \langle, \ldots, \rangle).$$

Denote the latter product by P, and let $I \subseteq P$ be the closed subscheme determined by the incidence correspondence. Then $G(k, \mathscr{F} \cap \mathscr{G})$ is precisely the preimage of the incidence correspondence, so because X is universally catenary it suffices to show that I is cut out locally at x by $k(n - t) - m\binom{k}{r}$ equations inside P. But we can construct I as a relative Grassmannian of subbundles of the universal bundle on

the second factor $GAG(t, \mathscr{E}, \langle, \ldots, \rangle)$; thus, I is smooth over $GAG(t, \mathscr{E}, \langle, \ldots, \rangle)$ of relative dimension $k(t - k)$. On the other hand, by hypothesis and Lemma 2.2 we have that P is smooth over $GAG(t, \mathscr{E}, \underline{\langle, \ldots, \rangle})$ of relative dimension $k(n-k) - m\binom{k}{r}$ at the point z corresponding to V. Thus, by Proposition 2.2.7 of [2], we have that locally near z, the scheme I is cut out by $k(n - t) - m\binom{k}{r}$ equations, as desired. $\square$

We now consider in some special cases what it means for (2.1) to have a nontrivial kernel. We observe that one way in which (2.1) can fail to be injective is if there is some r-dimensional subspace $W \subseteq V$ and some nonzero K-linear combination of the $\langle, \ldots, \rangle_i|_x$ for which W is degenerate. In [10], we saw that the converse holds when $r = 2$ and $m = 1, 2$. However, the converse does not hold in general. Nonetheless, we now observe that the converse holds in two other situations, as follows.

Proposition 2.5. *Let E be a K-vector space, $\langle, \ldots, \rangle$ an m-dimensional space of r-linear alternating forms on E, and $V \subseteq E$ an k-dimensional subspace. Suppose either that $k = r$, or that $k = r + 1$ and $m = 1$. Then the map*

$$(2.2) \qquad (\bigwedge^r V)^{\oplus m} \to \mathrm{Hom}(V, E/V)^*$$

induced as in (2.1) is injective if and only if there is no nonzero $\langle, \ldots, \rangle \in \underline{\langle, \ldots, \rangle}$ which is degenerate on an r-dimensional subspace of V.

Proof. As remarked above, if some nonzero $\langle, \ldots, \rangle \in \underline{\langle, \ldots, \rangle}$ is degenerate on an r-dimensional subspace of V, then (2.2) fails to be injective much more generally.

Conversely, first suppose $k = r$, and let $\langle, \ldots, \rangle_i$ for $i = 1, \ldots, m$ be a basis for $\underline{\langle, \ldots, \rangle}$ and $v_1, \ldots, v_r$ a basis for V. Then $\bigwedge^r V$ is 1-dimensional, with basis $v_1 \wedge \cdots \wedge v_r$. A nonzero element of the kernel of (2.2) may thus be written as as $\sum_i c_i (v_1 \wedge \cdots \wedge v_r)_i$, where the subscript i denotes the ith place in the direct sum, and not all c_i are 0. By definition, this means that for all $\varphi \in \mathrm{Hom}(V, E/V)^*$, we have

$$\sum_{i=1}^m \sum_{j=1}^r c_i \langle v_1, \ldots, \varphi(v_j), \ldots, v_r \rangle_i = 0.$$

Since we may choose $\varphi(v_j) = 0$ for all but one j, and $\varphi(v_j)$ arbitrary for the remaining index, this implies that the span of any $r - 1$ of the v_j is degenerate for $\sum_{i=1}^m c_i \langle, \ldots, \rangle_i$. We thus conclude that V is likewise degenerate, proving the desired statement.

On the other hand, if $m = 1$ and $k = r + 1$, it is still true that every nonzero element of $\bigwedge^r V$ is of the form $v_1 \wedge \cdots \wedge v_r$ for some linearly independent $v_i \in V$, so an element of the kernel of (2.2) is simply of the form $v_1 \wedge \cdots \wedge v_r$, and arguing as above we conclude that the span of the v_i is degenerate, as desired. $\square$

The following proposition uses a variant approach to treat some additional cases when $r = 3$ and $m = 1$, as in Case (III) of Theorem 1.1.

Proposition 2.6. *Let E be a K-vector space, $\langle, , \rangle$ a 3-linear alternating form on E, and $V \subseteq E$ a k-dimensional subspace, with $k \leq 6$. Then the map*

$$(2.3) \qquad \bigwedge^3 V \to \mathrm{Hom}(V, E/V)^*$$

induced as in (2.1) *is injective if there is no 2-dimensional subspace* $W \subseteq V$ *on which* $\langle , , \rangle$ *is degenerate.*

Proof. The significance of the restriction to $r = 3$ and $k \leq 6$ is that for any element of $\bigwedge^3 V$, there exists a basis $v_1, \ldots, v_k$ of V such that the given element may be expressed as

$$v_1 \wedge v_2 \wedge v_3 + (\text{terms not involving } v_1).$$

Indeed, this follows from the classification of $\mathrm{GL}(V)$ orbits of $\bigwedge^3 V$ as described for instance in §1.4 and §2.2 of [11]. Now, suppose that (2.3) is not injective, and choose a basis of V so that an element of the kernel has the above form. Then we can choose $\varphi \in \mathrm{Hom}(V, E/V)^*$ sending v_i to 0 for $i > 0$, and v_1 to an arbitrary element of E. By definition of (2.3), we see that $\langle v, v_2, v_3 \rangle = 0$ for all $v \in E$, or equivalently, that $\mathrm{span}(v_2, v_3)$ is degenerate for $\langle , , \rangle$. We thus conclude the statement of the proposition. $\qquad\square$

Putting together Propositions 2.4, 2.5, and 2.6, we immediately obtain the following corollary:

Corollary 2.7. *Suppose $\mathscr{E}$ is a vector bundle of rank n on an algebraic stack $\mathcal{X}$ of finite type over a universally catenary scheme S, and $\langle , \ldots , \rangle_i$ for $i = 1, \ldots, m$ are alternating r-linear forms on $\mathscr{E}$. Let $\mathscr{F}$ and $\mathscr{G}$ be subbundles of $\mathscr{E}$ of ranks s and t, both isotropic with respect to all of the $\langle , \ldots , \rangle_i$. Let $\mathcal{G}(k, \mathscr{F} \cap \mathscr{G})$ denote the closed substack of $\mathcal{G}(k, \mathscr{E})$ parametrizing rank-k subbundles of $\mathscr{E}$ contained in both $\mathscr{F}$ and $\mathscr{G}$. Suppose that for some $x \in \mathcal{X}$, we have $V \subseteq \mathscr{F}|_x \cap \mathscr{G}|_x$ satisfying one of the following conditions:*

- (I) *$k = r$, and the subspace V is not degenerate for any nonzero linear combination $\langle , \ldots , \rangle$ of the $\langle , \ldots , \rangle_i|_x$;*
- (II) *$k = r + 1$, $m = 1$, and no r-dimensional subspace of $W \subseteq V$ is degenerate for $\langle , \ldots , \rangle_1$;*
- (III) *$r = 3$, $k = 5$ or 6, $m = 1$, and no 2-dimensional subspace $W \subseteq V$ is degenerate for $\langle , \ldots , \rangle_1$.*

Then every component of $\mathcal{G}(k, \mathscr{F} \cap \mathscr{G})$ passing through the point corresponding to V has codimension at most

$$k(2n - s - t) - m \binom{k}{r}$$

in $\mathcal{G}(k, \mathscr{E})$.

3. APPLICATION TO VECTOR BUNDLES ON CURVES

We consider the following situation. Let S be a scheme, $\mathcal{X}$ an algebraic S-stack, and $\pi : \mathcal{C} \to \mathcal{X}$ a smooth, projective relative curve of genus g over $\mathcal{X}$. Let $\mathscr{L}$ be a line bundle on $\mathcal{C}$ of relative degree d, and $\mathscr{E}$ a vector bundle of rank r, together with an isomorphism $\psi : \det \mathscr{E} \xrightarrow{\sim} \mathscr{L}$.

We describe how to construct r-linear alternating forms on $\pi_*(\mathscr{E}(D)/\mathscr{E}(-(r-1)D))$.

Proposition 3.1. *In the above situation, suppose we also have a morphism $\xi : \mathscr{L} \to \Omega^1_{\mathcal{C}/\mathcal{X}}$, and $P_1, \ldots, P_N : \mathcal{X} \to \mathcal{C}$ disjoint sections of π, and set $D = \sum_i P_i$.*

Then we construct an alternating r-linear form $\langle,\dots,\rangle_\xi$ on $\pi_*(\mathscr{E}(D)/\mathscr{E}(-(r-1)D))$ defined locally on $\mathcal{X}$ by

$$\langle s_1, \dots, s_r \rangle = \sum_i^n \operatorname{res}_{P_i}(\xi \circ \psi)(\tilde{s}_1 \wedge \cdots \wedge \tilde{s}_r),$$

where each $\tilde{s}_j$ is a representative in $\mathscr{E}(D)$ of s_j in a neighborhood of P_i (more precisely, in a neighborhood of the point of P_i lying over a given point of $\mathcal{X}$). Moreover, this form is compatible with base change.

Proof. The argument is largely the same the first half of the proof of Lemma 5.1 of [10]. The main distinction is that we are forced to use $\mathscr{E}(-(r-1)D))$ as the appropriate generalization of $\mathscr{E}(-D)$, to ensure that if we take a wedge product with $r-1$ local sections of $\mathscr{E}(P_i)$, the result will still be regular at P_i, and thus will have residue equal to 0. □

Note that for $r > 2$, the form constructed in Proposition 3.1 is highly degenerate: in particular, the subbundle $\pi_*(\mathscr{E}(-D)/(\mathscr{E}(-(r-1)D)))$ is always degenerate. Nonetheless, we see that we can make these forms behave in a rather nondegenerate manner when we restrict our attention to their values on global sections.

Proposition 3.2. *In the situation of Proposition 3.1, given a field K and a K-valued point x of $\mathcal{X}$, and*

$$V \subseteq \Gamma(\mathcal{C}|_x, \mathscr{E}|_x)$$

a k-dimensional space of global sections of $\mathscr{E}|_x$, suppose that ξ is not identically zero on the fiber over x, and that for some $n \leq k$, we have that no n-dimensional subspace of V is contained in a subbundle of $\mathscr{E}|_x$ of rank $r-2$.

Then there exists some $N > 0$ such that for all $N' > N$, and any disjoint sections $P_1, \dots, P_{N'}$ of π, we have that the form $\langle, \dots, \rangle_\xi$ has the property that no subspace $W \subseteq V$ of dimension n is degenerate for $\langle, \dots, \rangle_\xi|_x$.

Note that the statement of the proposition depends only on the fiber of $\mathscr{E}$ at x; the base stack $\mathcal{X}$ plays no role.

Proof. First, observe that the Grassmannanian of n-dimensional subspaces of V is of finite type, and the loci on which the subspaces have rank at most $r-2$ is likewise of finite type. It then follows that there is some N'' such that any n-dimensional subspace of V can have rank less than or equal to $r-2$ at at most N'' points of $\mathcal{C}|_x$. Choose $N' > N'' + 2g - 2 - d$ (note that $2g - 2 - d \geq 0$ since $\pi_* \mathscr{H}om(\mathscr{L}, \Omega^1_{\mathcal{C}/\mathcal{X}}) \neq 0$).

It is clear that we have the decomposition

$$\pi_*(\mathscr{E}(D)/\mathscr{E}(-(r-1)D)) \cong \bigoplus_{i=1}^{N'} \pi_*(\mathscr{E}(P_i)/\mathscr{E}(-(r-1)P_i)).$$

By definition, the form $\langle, \dots, \rangle_\xi|_x$ is compatible with this direct sum decomposition, so to show that a subspace is not degenerate, it suffices to show that there exists some i such that its image in $\pi_*(\mathscr{E}(P_i)/\mathscr{E}(-(r-1)P_i))$ is not degenerate for the restriction of $\langle, \dots, \rangle_\xi|_x$ to P_i. Calculating locally at P_i, we see moreover that if P_i is not a zero of the map ξ, then $\langle, \dots, \rangle_\xi|_x$ induces an isomorphism

$$\bigwedge^{r-1} \pi_*(\mathscr{E}/\mathscr{E}(-P_i)) \xrightarrow{\sim} \pi_*(\mathscr{E}(P_i)/\mathscr{E})^*.$$

Let $W \subseteq V$ be an n-dimensional subspace; then by hypothesis the restriction of W to P_i is contained in $\pi_*(\mathscr{E}/\mathscr{E}(-(r-1)P_i))$, so by the above isomorphism, to show that W is nondegenerate, it is enough to see that the map

$$\bigwedge^{r-1} W \to \bigwedge^{r-1} \pi_*(\mathscr{E}/\mathscr{E}(-P_i))$$

is nonzero, or equivalently, that the sections comprising W span a subspace of dimension at least $r-1$ in the fiber of $\mathscr{E}$ at P_i.

Now, we can have at most $2g - 2 - d$ points at which ξ vanishes, and at most N'' points at which the sections of W span a subspace of $\mathscr{E}|_{P_i}$ having dimension strictly less than $r-1$, so by construction there is necessarily some i such that $\bigwedge^{r-1} W$ has nonzero image in $\bigwedge^{r-1} \pi_*(\mathscr{E}/\mathscr{E}(-P_i))$, and we conclude that W is not degenerate for $\langle, \ldots, \rangle_\xi|_x$, as desired. $\qquad \square$

We can now prove Theorem 1.1. Indeed, we prove a more general form of the theorem, allowing the determinant to vary in families. We first recall the definition of the moduli stack in question:

Definition 3.3. Let S be a scheme, and C/S a smooth, projective relative curve. Given also, $d, r, k \in \mathbb{Z}$, with $r \geq 2$, $k \geq 1$, and $\mathscr{L} \in \operatorname{Pic}^d(C)$, the stack $\mathcal{G}^k_{r,\mathscr{L}}(C/S)$ parametrizes triples $(\mathscr{E}, \psi, V)$ over every S-scheme T, where $\mathscr{E}$ is a vector bundle of rank r on $C_T := C \times_S T$, $\psi : \det \mathscr{E} \xrightarrow{\sim} \mathscr{L}|_{C_T}$ is an isomorphism, and V is a rank-k subbundle of $p_{2*}\mathscr{E}$, in the sense that it is a locally free subsheaf such that for all $T' \to T$, we have that the induced map $V|_{T'} \to p'_{2*}(\mathscr{E}|_{C'_T})$ remains injective, where $p'_2 : C_{T'} \to T'$ is the projection morphism.

In the case that S is the spectrum of a field, we write simply $\mathcal{G}^k_{r,\mathscr{L}}(C)$.

The construction of $\mathcal{G}^k_{r,\mathscr{L}}(C/S)$ as a closed substack of a relative Grassmannian over the moduli stack of vector bundles of rank r and determinant $\mathscr{L}$ proceeds exactly as in the classical rank-1 case.

Theorem 3.4. *Let S be an equidimensional scheme of finite type over a field, and C be a smooth, projective relative curve over S of genus g. Suppose $\mathscr{L} \in \operatorname{Pic}^d(C)$, and $h^1(C_s, \mathscr{L}|_s)$ is constant as $s \in S$ varies, and is at least m. Given $r \geq 2$, and $s \in S$, let $\mathscr{E}$ be a vector bundle of rank r on C_s with determinant $\mathscr{L}|_s$, and $V \subseteq H^0(C_s, \mathscr{E})$ a k-dimensional space of global sections. Suppose that in addition, one of the following conditions is satisfied.*

(I) *$k = r$, and V is not contained in any subbundle of $\mathscr{E}$ of rank $r-2$.*

(II) *$k = r + 1$, $m = 1$, and no r-dimensional subspace of V is contained in any subbundle of $\mathscr{E}$ of rank $r-2$.*

(III) *$r = 3$, $k = 5$ or 6, $m = 1$, and no 2-dimensional subspace of V is contained in any subbundle of $\mathscr{E}$ of rank 1.*

Then every component of $\mathcal{G}^k_{r,\mathscr{L}}(C/S)$ passing through the point corresponding to $(\mathscr{E}, V)$ has dimension at least

$$(3.1) \qquad \dim S + \rho - g + m \binom{k}{r}.$$

Proof. First observe that since the statement is purely dimension-theoretic, we may assume that S is reduced. Then, by Grauert's theorem and Serre duality, the pushforward of $\mathscr{H}om(\mathscr{L}, \Omega^1_{C/S})$ is locally free of rank at least m, and pushforward

commutes with base change. Since the statement is local on S, we may suppose we have m linearly independent sections of this pushforward which remain linearly independent under base change, and we use these together with Proposition 3.1 to construct m alternating forms. Furthermore, since etale base change does not affect dimension, we may assume we have disjoint sections $P_1, \ldots, P_{N'}$ as in Proposition 3.2. Using Corollary 2.7, the argument then proceeds almost identically to the proof of Theorem 1.3 in §5 of [10]. In the notation of *loc. cit.*, the only difference is that the sheaves $\mathscr{E}_{\mathcal{U}}(D')/\mathscr{E}_{\mathcal{U}}(-D')$ and $\mathscr{E}_{\mathcal{U}}/\mathscr{E}_{\mathcal{U}}(-D')$ should be replaced by $\mathscr{E}_{\mathcal{U}}(D')/\mathscr{E}_{\mathcal{U}}(-(r-1)D')$ and $\mathscr{E}_{\mathcal{U}}/\mathscr{E}_{\mathcal{U}}(-(r-1)D')$ respectively, and the resulting ranks and dimension counts modified appropriately. $\qquad\square$

Theorem 1.1 then follows as the special case for which the base S is a point.

Remark 3.5. Note that the condition that $h^1(C, \mathscr{L})$ be constant in fibers is an important one. Without it, not only does the argument fail, but the theorem fails as well. See Example 5.4 below.

Remark 3.6. The case of varying but special determinants is important when one wants to study components of the stack $\mathcal{G}_{r,d}^k(C)$; see for instance Example 5.2, and Example 5.4. According to the theorem we may also let the curve vary in families, but this seems less important outside the context of degeneration arguments.

4. The case of rank 2

The basic strategy of our analysis in the case of rank 2 is to carry out dimension counts via a detailed analysis of the possibilities for extensions of line bundles. By virtue of Theorem 1.1, we will only have to compute upper bounds on dimensions to get the desired result.

Definition 4.1. Let $\mathcal{S}_{\mathscr{L}}$ denote the stack over $\mathcal{G}_{2,\mathscr{L}}^2(C)$ consisting of tuples $(\mathscr{E}, \psi, V, s_1, s_2)$, where $(\mathscr{E}, \psi, V)$ are as in Definition 3.3, and s_1, s_2 are a basis of V. Let $\mathcal{S}_{\mathscr{L}}^{\mathrm{gg}}$ denote the open substack obtained as the preimage of $\mathcal{G}_{2,\mathscr{L}}^{2;\mathrm{gg}}(C)$ in $\mathcal{S}_{\mathscr{L}}$. Given $d' \geq 0$, denote by $\mathcal{S}_{d',\mathscr{L}}^{\mathrm{gg}}$ the locally closed substack of $\mathcal{S}_{\mathscr{L}}^{\mathrm{gg}}$ on which s_1 vanishes along a divisor of degree d'.

Then $\mathcal{S}_{\mathscr{L}}$ is a GL_2-torsor over $\mathcal{G}_{2,\mathscr{L}}^2(C)$, and is in particular smooth of relative dimension 4 over $\mathcal{G}_{2,\mathscr{L}}^2(C)$. As d' varies, the stacks $\mathcal{S}_{d',\mathscr{L}}^{\mathrm{gg}}$ give a stratification of $\mathcal{S}_{\mathscr{L}}^{\mathrm{gg}}$. If D is the divisor of vanishing of s_1 for a point of $\mathcal{S}_{d',\mathscr{L}}^{\mathrm{gg}}$, we see that s_2 gives a nonzero section of $\mathscr{L}(-D)$, so in particular we must have $d' \leq d$.

Proposition 4.2. *Suppose $d \geq 0$, and $0 \leq d' \leq d$. Then*

$$\dim \mathcal{S}_{d',\mathscr{L}}^{\mathrm{gg}} \leq 2d + 1 - g + m - d'.$$

Moreover, $\mathcal{S}_{0,\mathscr{L}}^{\mathrm{gg}}$ is irreducible.

Proof. We have a morphism from $\mathcal{S}_{d',\mathscr{L}}^{\mathrm{gg}}$ to the symmetric product $S^{d'}C$ by taking the vanishing divisor of s_1, and we denote by $\mathcal{S}_{D,\mathscr{L}}^{\mathrm{gg}}$ the fiber of this morphism over the point corresponding to a given effective divisor D. For a given choice of D, consider the stack $\mathcal{E}_D$ parametrizing pairs (η, s), where η is an extension of $\mathscr{L}(-D)$ by $\mathscr{O}(D)$, and $s \in H^0(C, \mathscr{L}(-D))$ is nonzero and lifts inside η. We then have a morphism

$$(4.1) \qquad\qquad\qquad \mathcal{S}_{D,\mathscr{L}}^{\mathrm{gg}} \to \mathcal{E}_D$$

obtained by letting η be the extension induced by s_1, and letting s be the image of s_2 under the extension.

Write $\ell := h^0(C, \mathscr{O}(D))$. We see that the fibers of (4.1) are determined by the choice of s_2 lifting s, and thus the morphism has relative dimension ℓ. Now, an extension of $\mathscr{L}(-D)$ by $\mathscr{O}(D)$ corresponds to an element of $H^1(\mathscr{L}(-2D))$. The infinitesimal automorphisms of such an extension are in correspondence with $H^0(\mathscr{L}^{-1}(2D))$, so the dimension of the stack of extensions is $-\chi(\mathscr{L}^{-1}(2D)) = d - 2d' + g - 1$. Using Serre duality, an extension corresponds to an element of $H^0(C, \Omega_C^1 \otimes \mathscr{L}(-2D))^*$, which we still denote by η. The section $s \in H^0(C, \mathscr{L}(-D))$ lifts under η if and only if the kernel of η contains the image of the (injective) map

$$\otimes s : H^0(C, \Omega_C^1(-D)) \to H^0(C, \Omega_C^1 \otimes \mathscr{L}(-2D)).$$

We have

$$h^0(C, \Omega_C^1(-D)) = h^1(C, \mathscr{O}(D)) = \ell - d' - 1 + g,$$

so for a given $s \in H^0(C, \mathscr{L}(-D))$, the dimension of the choices for η is

$$d - 2d' + g - 1 - (\ell - d' - 1 + g) = d - d' - \ell.$$

On the other hand, set $d'' = d' - h^0(C, \mathscr{L}) + h^0(C, \mathscr{L}(-D))$. There are $h^0(C, \mathscr{L}(-D)) = d'' - d' + d + 1 - g + m$ dimensions for s, so we conclude that $\mathcal{E}_D$ has dimension $2d + 1 - g + m - 2d' + d'' - \ell$, and $\mathcal{S}_{D,\mathscr{L}}^{\mathrm{gg}}$ has dimension $2d + 1 - g + m - 2d' + d''$. Finally, for a given value of d'', the corresponding stratum of $S^{d'}C$ has dimension at most $d' - d''$, so we find that $\mathcal{S}_{d',\mathscr{L}}^{\mathrm{gg}}$ has dimension at most $2d + 1 - g + m - d'$, as desired.

Finally, to see that $\mathcal{S}_{0,\mathscr{L}}^{\mathrm{gg}}$ is irreducible, in the case $d' = 0$ we necessarily have $D = 0$ and our stratification is the trivial one corresponding to $\ell = 1$. We observe that the space of choices for $s \in H^0(C, \mathscr{L})$ are irreducible, and given a choice of s, the spaces of extensions η is also irreducible. Now, the preimage of $\mathcal{E}_0$ is an open substack of $\mathcal{S}_{\mathscr{L}}^{\mathrm{gg}}$, and hence we have a dimension lower bound as well, concluding that $\mathcal{E}_0$ is of dimension $2d - g + m$. It then follows that every component of $\mathcal{E}_0$ must dominate the space of choices for s, and thus that $\mathcal{E}_0$ is irreducible. By the same argument, we then conclude that $\mathcal{S}_{0,\mathscr{L}}^{\mathrm{gg}}$, being smooth with connected fibers over $\mathcal{E}_0$, is likewise irreducible. $\qquad\square$

Definition 4.3. Let $\mathcal{G}_{2,\mathscr{L}}^{2,\mathrm{dg}}(C)$ be the closed substack of $\mathcal{G}_{2,\mathscr{L}}^2(C)$ on which V is not generically generating.

Proposition 4.4. *If C is Brill-Noether general with respect to $\mathfrak{g}_{d'}^1$'s for all $d' > 0$, we have*

$$\dim \mathcal{G}_{2,\mathscr{L}}^{2,\mathrm{dg}}(C) \leq 2d - 3 - g + m,$$

and equality holds if and only if $h^0(C, \mathscr{L}) = 0$.

Proof. First observe that if $h^0(C, \mathscr{L}) = 0$, then $\mathcal{G}_{2,\mathscr{L}}^{2,\mathrm{dg}}(C) = \mathcal{G}_{2,\mathscr{L}}^2(C)$, which has dimension at least $2d - 3 - g + m$, so it is enough to show that $\dim \mathcal{G}_{2,\mathscr{L}}^{2,\mathrm{dg}}(C) \leq 2d - 3 - g - m$, with strict inequality when $h^0(C, \mathscr{L}) > 0$.

Given $d' > 0$, consider the stack $\mathcal{E}_{d'}'$ parametrizing triples $(\mathscr{M}, W, \eta)$, where $(\mathscr{M}, W)$ is a $\mathfrak{g}_{d'}^1$ on C, and η is an extension of $\mathscr{L} \otimes \mathscr{M}^{-1}$ by $\mathscr{M}$. For a given $\mathscr{M}$, the stack of extensions η has dimension calculated as before:

$$-\chi(\mathscr{L}^{-1} \otimes \mathscr{M}^2) = d - 2d' + g - 1.$$

Since C is Brill-Noether general with respect to $\mathfrak{g}^1_{d'}$'s, the dimension of the stack of pairs $(\mathscr{M}, W)$ is $2d' - 2 - g - 1$ (note that this is 1 less than the classical number because we have to take the stack dimension). We conclude that the dimension of $\mathcal{E}'_{d'}$ is $d - 4$.

Now, noting that the form of the extension η induces an isomorphism $\det \mathscr{E} \xrightarrow{\sim} \mathscr{L}$, we obtain a forgetful morphism

$$(4.2) \qquad\qquad \mathcal{E}'_{d'} \to \mathcal{G}^{2,\mathrm{dg}}_{2,\mathscr{L}}(C).$$

Moreover, any nontrivial automorphism of an object of $\mathcal{E}'_{d'}$ induces a nontrivial automorphism of $\mathscr{E}$ and hence maps to a nontrivial automorphism in $\mathcal{G}^{2,\mathrm{dg}}_{2,\mathscr{L}}(C)$, so we conclude that (4.2) is relatively representable by algebraic spaces, and in particularly has nonnegative relative dimension.

Now, as d' varies over all positive values, the union of the images of the morphisms (4.2) surjects onto $\mathcal{G}^{2,\mathrm{dg}}_{2,\mathscr{L}}(C)$, so we conclude that the dimension of $\mathcal{G}^{2,\mathrm{dg}}_{2,\mathscr{L}}(C)$ is at most the supremum of the dimensions of the $\mathcal{E}'_{d'}$, which is $d - 4$. Finally,

$$2d - 3 - g + m - (d - 4) = d + 1 - g + m = h^0(C, \mathscr{L}),$$

so we obtain the desired statement. $\square$

Definition 4.5. Suppose $h^0(C, \mathscr{L}) > 0$ and $d > 0$. Let $\mathcal{S}^{\mathrm{ns}}_{\mathscr{L}}$ be the closed substack of $\mathcal{S}^{\mathrm{gg}}_{0,\mathscr{L}}$ on which $\mathscr{E}$ is not stable. Given $d' \geq \frac{d}{2}$, denote by $\mathcal{S}^{\mathrm{ns}}_{d',\mathscr{L}}$ the locally closed substack of $\mathcal{S}^{\mathrm{ns}}_{\mathscr{L}}$ on which $\mathscr{E}$ has a destabilizing line subbundle of degree d'.

Thus, as d' varies, the $\mathcal{S}^{\mathrm{ns}}_{d',\mathscr{L}}$ give a stratification of $\mathcal{S}^{\mathrm{ns}}_{\mathscr{L}}$.

Proposition 4.6. *We have*

$$\dim \mathcal{S}^{\mathrm{ns}}_{d',\mathscr{L}} < 2d + 1 - g + m$$

if C is not hyperelliptic and $d > 2$ or if C is hyperelliptic and $d > 4$.

Proof. Consider the stack $\mathcal{E}''_{d'}$ parametrizing tuples $(\eta, s, \mathscr{M}, \iota)$, where η is an extension of $\mathscr{L}$ by $\mathscr{O}$, $s \in H^0(C, \mathscr{L})$ lifts in η, $\mathscr{M}$ is a line bundle of degree d', and $\iota : \mathscr{M} \to \mathscr{E}$ imbeds $\mathscr{M}$ as a line subbundle. Given such a tuple, this yields a map $\mathscr{M} \to \mathscr{L}$, which must be nonzero since $d' > 0$. Thinking of this map as a section $t \in H^0(C, \mathscr{L} \otimes \mathscr{M}^{-1})$, the condition that it came from a map $\mathscr{M} \to \mathscr{E}$ is equivalent to the condition that t lifts in the extension $\eta \otimes \mathscr{M}^{-1}$

$$0 \to \mathscr{M}^{-1} \to \mathscr{E} \otimes \mathscr{M}^{-1} \to \mathscr{L} \otimes \mathscr{M}^{-1} \to 0.$$

As before, this in turn is equivalent to asking that the image of $H^0(C, \Omega^1_C \otimes \mathscr{M})$ under the map

$$\otimes t : H^0(C, \Omega^1_C \otimes \mathscr{M}) \to H^0(C, \Omega^1_C \otimes \mathscr{L})$$

be contained in the kernel of the extension $\eta \otimes \mathscr{M}^{-1}$, considered as an element of $H^0(C, \Omega^1_C \otimes \mathscr{L})^*$. We thus need to determine how this condition interacts with the condition that s must lift to $\mathscr{E}$ as well.

Let $D = \operatorname{div} s$, and $D' = \operatorname{div} t$, so that $\deg D' = d - d'$. Let $D'' = \gcd(D, D')$, and set $d'' = \deg D''$. Observe that the condition that a nonzero element $s' \in H^0(C, \Omega^1_C \otimes \mathscr{L})$ be in the image of $H^0(C, \Omega^1_C \otimes \mathscr{M})$ under $\otimes t$ is precisely equivalent to having $D' \leq \operatorname{div} s'$, and similarly s' is in the image of $H^0(C, \Omega^1_C)$ under $\otimes s$ if and only if $D \leq \operatorname{div} s'$. For a given s and t, we want to compute the dimension of the stack of extensions whose kernels contain both images, so we need to compute the dimension of the span of the images. Since we know the dimensions of each

image, it suffices to compute the dimension of the intersection. We have that s' is in the intersection of the images if and only if $\mathrm{lcm}(D, D') = D + D' - D'' \leq \mathrm{div}\, s'$, so the intersection of the images is given by

$$H^0(C, \Omega_C^1 \otimes \mathscr{L}(-D - D' + D'')) \cong H^0(C, \Omega_C^1(-D' + D'')).$$

First suppose that $\deg \Omega_C^1(-D' + D'') \geq 0$. Then Clifford's theorem implies that

$$\frac{2g - 2 - d + d' + d''}{2} + 1 = g - \frac{d - d' - d''}{2},$$

with equality possible only if $\Omega_C^1(-D' + D'') \cong \mathscr{O}_C$, if $D' = D''$, or if C is hyperelliptic. Thus, the span of the images has dimension at least

$$g + (g - 1 + d') - (g - \frac{d - d' - d''}{2}) = g - 1 + \frac{d + d' - d''}{2},$$

and the dimension of the choices of extensions for a given $s \in H^0(C, \mathscr{L})$ and $t \in H^0(C, \mathscr{L} \otimes \mathscr{M}^{-1})$ is at most

$$d + g - 1 - (g - 1 + \frac{d + d' - d''}{2}) = \frac{d - d' + d''}{2}.$$

As before, the choices for s add $d + 1 - g + m$ dimensions, while choosing the pair $(\mathscr{M}, t)$ is just equivalent to choosing any effective divisor of degree $d - d'$ containing D'', so adds $d - d' - d''$ dimensions. Since $d' \geq \frac{d}{2}$, the total dimension of $\mathcal{E}_{d'}''$ is at most

$$\frac{5d - 3d' - d''}{2} + 1 - g + m \leq \frac{7d}{4} + 1 - g + m.$$

Considering the stack of tuples $(\eta, s, \mathscr{M}, \iota, s_2)$, where $(\eta, s, \mathscr{M}, \iota)$ is as in the definition of $\mathcal{E}_{d'}''$, and s_2 is a lift of s, we obtain a correspondence between $\mathcal{S}_{d', \mathscr{L}}^{\mathrm{ns}}$ and $\mathcal{E}_{d'}''$. A fiber of the map to $\mathcal{E}_{d'}''$ corresponds to the choices of s_2, which form a torsor over $H^0(C, \mathscr{O})$. Thus, the fibers are 1-dimensional. On the other hand, the morphism to $\mathcal{S}_{d', \mathscr{L}}^{\mathrm{ns}}$ is surjective with fibers representable by algebraic spaces, so we conclude that $\dim \mathcal{S}_{d', \mathscr{L}}^{\mathrm{ns}} \leq \dim \mathcal{E}_{d'}'' + 1$, yielding

$$\dim \mathcal{S}_{d', \mathscr{L}}^{\mathrm{ns}} \leq \frac{7d}{4} + 2 - g + m.$$

This already yields the desired inequality in the hyperelliptic case. In the nonhyperelliptic case, if we had strict inequality in Clifford's theorem, the two sides had to differ by at least $\frac{1}{2}$, so the relevant open substack of $\mathcal{S}_{d', \mathscr{L}}^{\mathrm{ns}}$ has dimension bounded by $\frac{7d+2}{4} + 1 - g + m$, which is also enough to obtain the desired inequality. On the other hand, if either $\Omega_C^1(-D' + D'') \cong \mathscr{O}_C$ or $D' = D''$, we can calculate directly, and in both cases obtain that the corresponding strata of $\mathcal{S}_{d', \mathscr{L}}^{\mathrm{ns}}$ have dimension bounded by

$$2d - d' + 2 - g + m \leq \frac{3d}{2} + 2 - g + m,$$

again yielding the desired inequality. Finally, if $\deg \Omega_C^1(-D' + D'') < 0$, we can again calculate directly, finding

$$\dim \mathcal{S}_{d', \mathscr{L}}^{\mathrm{ns}} \leq 2d + 2 - 2g + m,$$

which gives the asserted inequality for $g \geq 2$. □

Proof of Theorem 1.3. It is clear that if $h^0(C, \mathscr{L}) = 0$, then $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$ must be empty. Conversely, if $h^0(C, \mathscr{L}) > 0$, then taking the trivial extension of $\mathscr{L}$ by $\mathscr{O}$ shows that $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$ is nonempty. By Theorem 1.1, it has dimension at least $\rho - g + m = 2d - 3 - g + m$. Then, recalling that $\mathcal{S}^{\mathrm{gg}}_{d',\mathscr{L}}$ is smooth of relative dimension 4 over $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$, Proposition 4.2 gives us the necessary upper bound to conclude the dimension is equal to $\rho - g + m$. This argument also shows that the open substack of pairs containing a nowhere vanishing global section is dense in $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$, and in particular we conclude that $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$ is irreducible from the statement of Proposition 4.2 that $\mathcal{S}^{\mathrm{gg}}_{0,\mathscr{L}}$ is irreducible.

Next, if C is Brill-Noether general with respect to $\mathfrak{g}^1_{d'}$'s for all d', then Proposition 4.4 implies that the complement of $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$ in $\mathcal{G}^2_{2,\mathscr{L}}(C)$ can have dimension at most $\rho - g + m$, with equality if and only if $h^0(C, \mathscr{L}) = 0$, so we conclude that $\mathcal{G}^2_{2,\mathscr{L}}(C)$ is pure of dimension $\rho - g + m$, and that $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$ is dense in $\mathcal{G}^2_{2,\mathscr{L}}(C)$ when $h^0(C, \mathscr{L}) > 0$.

Finally, when $h^0(C, \mathscr{L}) > 0$, we see from Proposition 4.6 that the complement of the stable locus in $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C)$ must have strictly smaller dimension under the stated hypotheses, so we conclude that $\mathcal{G}^{2,\mathrm{st}}_{2,\mathscr{L}}(C)$ contains a nonempty open substack, as desired.   □

We also obtain a corollary for varying determinant loci in the case of rank 2, which refines the main dimension result of Teixidor i Bigas in [12].

Corollary 4.7. *Fix $d, g, m \geq 0$, let C be a Brill-Noether general curve, and suppose that $\ell = d + 1 - g + m$ is nonnegative. Let S be the locally closed subvariety*

$$W^{\ell-1}_d \smallsetminus W^\ell_d \subseteq \mathrm{Pic}^d(C),$$

and let $\mathscr{L}$ be the restriction of the Poincare line bundle to $S \times C$. Then $\mathcal{G}^2_{2,\mathscr{L}}(C/S)$ is nonempty if and only if $m\ell \leq g$, and in this case it has pure dimension

$$(4.3) \qquad\qquad\qquad\qquad \rho - (\ell - 1)m.$$

Similarly, the open substack $\mathcal{G}^{2,\mathrm{gg}}_{2,\mathscr{L}}(C/S)$ is nonempty (necessarily of the same dimension) if and only if $\ell > 0$ and $m\ell \leq g$.

Proof. The classical Brill-Noether theorem gives that S is non-empty and

$$\dim S = g - m\ell$$

if and only if $m\ell \leq g$. We then obtain the desired dimensional lower bound from Theorem 1.1, and the corresponding upper bound from Theorem 1.3, working fiber by fiber. The nonemptiness assertion likewise follows from Theorem 1.3.   □

Remark 4.8. We observe from (4.3) that there are two possibilities for getting dimension exactly ρ: either $\ell = 1$, or $m = 0$. For any given d, g, one of these is always possible. We also see that if we consider the degenerate locus, we can allow $\ell = 0$ and $m > 0$ and find that in this varying determinant situation, we actually obtain dimension strictly greater than ρ on the degenerate locus. This occurs if $m = g - d - 1 > 0$, so $g > d + 1$. In this case, we check that a degenerate pair must have an unstable underlying bundle, so this does not contradict [12].

5. Further discussion

The arguments used to prove Theorem 1.1 show that for any k, r, m, if $h^1(C, \mathscr{L}) \geq m$, there is an open substack of $\mathcal{G}^k_{r,\mathscr{L}}(C)$ satisfying the dimension lower bound of (1.1). The difficulty is in translating the criterion of Proposition 2.4 into a concrete nondegeneracy criterion describing this open substack, as for instance in the statement of Theorem 1.1. *A priori*, we have no way of knowing even whether the open substack in question is ever nonempty. We observe that for $m \geq 3$ or $r \geq 3$, the formula of (1.1) is in fact increasing in k for k sufficiently large. This underlines the likelihood that nondegeneracy hypotheses will be necessary in these cases.

We now consider several examples, examining the necessity of nondegeneracy hypotheses, and evaluating our predicted bounds in examples from the literature of larger-dimensional Brill-Noether loci.

Example 5.1. Although the nondegeneracy hypothesis of Theorem 1.1 is vacuous in rank 2, and there are also no nondegeneracy hypotheses in the main results of [10], we mention that as soon as $m \geq 3$, even in rank 2 we will need some nondegeneracy hypotheses in order for the lower bound (1.1) to be valid. Specifically, for any fixed d, choose k very large with respect to d and g, so that the only pairs $(\mathscr{E}, V)$ with $V \subseteq H^0(C, \mathscr{E})$, $\deg \mathscr{E} = d$, and $\dim V = k$ must be degenerate, with V contained in some high-degree line subbundle of $\mathscr{E}$. Fix any line bundle $\mathscr{L}$ of degree d. Set $d' = k + g - 1$, so that every line bundle $\mathscr{M}$ of degree d' has $h^0(C, \mathscr{M}) = k$, but no line bundle of smaller degree has a k-dimensional space of global sections. Let $\mathcal{U}_{d'} \subseteq \mathcal{G}^k_{2,\mathscr{L}}(C)$ be the open substack on which the vector bundles have sublinebundles of degree at most d'.

We then see that $\mathcal{U}_{d'}$ is pure of dimension

$$2g - 2 - 2d' + d = d - 2k.$$

Indeed, by construction it consists entirely of bundles of form $\mathscr{M} \oplus (\mathscr{L} \otimes \mathscr{M}^{-1})$, with $\mathscr{M}$ a line bundle of degree d'. Note that since d' is large with respect to d and g, there are no nontrivial extensions of $\mathscr{L} \otimes \mathscr{M}^{-1}$ by $\mathscr{M}$. There is a g-dimensional space of choices for such a vector bundle, and taking into account the fixed determinant condition, the dimension of the automorphism group of each is

$$1 + h^0(C, \mathscr{M}^2 \otimes \mathscr{L}^{-1}) = 2d' - d + 2 - g.$$

This gives the claimed formula for the dimension of $\mathcal{U}_{d'}$.

In particular, we see that the dimension of $\mathcal{U}_{d'}$ is decreasing in k. On the other hand, we have already observed that for $m \geq 3$ and k large, our lower bound

$$\rho - g + m \binom{k}{2}$$

is increasing in k. We thus see that whatever nondegeneracy condition is required for this lower bound to hold, it must be violated by the present examples.

In the next two examples, we see that in two interesting examples of Brill-Noether loci of larger than expected dimension, the discrepancy is explained by our techniques, and that moreover our lower bound is sharp in these cases.

Example 5.2. In [6], Farkas and Ortega study the case of odd genus $2a + 1$, with rank 2, degree $2a + 4$ and $k = 4$. They find that while in this case $\rho = 1$, the dimension of the coarse moduli space of $\mathcal{G}^{4,\mathrm{st}}_{2,2a+4}(C)$ is 2. As far as we are aware,

this is the only known example of larger-than-expected dimension in rank 2 other than those described explicitly by special determinants. We note however that this example is nonetheless explained by our work in [10]: indeed, their analysis shows that $\mathcal{G}_{2,2a+4}^{4,\mathrm{st}}(C)$ is supported entirely over the locus of determinants $\mathscr{L}$ having $h^1(C,\mathscr{L}) > 0$. This locus has dimension $g-5$, and for a fixed such $\mathscr{L}$ we know that the dimension of $\mathcal{G}_{2,\mathscr{L}}^4(C)$ is at least

$$\rho - g + \binom{k}{2} = \rho - g + 6,$$

so if we allow ρ to vary we conclude that the dimension of $\mathcal{G}_{2,2a+4}^{4,\mathrm{st}}(C)$ should be at least $\rho + 1 = 2$, as oberved by Farkas and Ortega.

Example 5.3. As discussed in [8], Mukai has shown that for a general curve of genus 9, there exists a unique stable vector bundle of rank 3 and degree 16 with a 6-dimensional space of global sections. In this case, $\rho = -11$. On the other hand, this vector bundle has canonical determinant, and the modified expected dimension arising from Theorem 1.1 is

$$\rho - g + \binom{k}{r} = -11 - 9 + \binom{6}{3} = 0.$$

In addition, one checks using the generality of the curve that in this case, stability of the vector bundle implies the nondegeneracy hypothesis of the theorem. Thus, we see that in at least one interesting example, not only does (1.1) give a valid lower bound for the dimension, but it is in fact sharp, at least on the stable locus.

Also in [8], Lange, Mercat and Newstead show that on a general curve of genus 11, there exist stable bundles of rank 3 and degree 20 with a 6-dimensional space of global sections, although in this case $\rho = -5$. These bundles also have canonical determinant, so we again find that our modified expected dimension is nonnegative, in this case equal to 4.

Finally, we discuss the necessity of restricting to determinant loci with constant h^1 in Theorem 3.4.

Example 5.4. In Example 6.1 of [10], we consider the case $r = 2$, $k = 2$, and $d = g - 2$. If we let S be all of $\mathrm{Pic}^d(C)$, and $\mathscr{L}$ the Poincare line bundle, then every fiber has nonzero h^1, so if Theorem 3.4 remained valid without the hypothesis that h^1 is constant, we could use the $m = 1$ case to conclude that if the relative stack $\mathcal{G}_{2,g-2}^2(C/S)$ is nonempty, every component has dimension at least $\rho + 1$. However, the stable locus of $\mathcal{G}_{2,g-2}^2(C/S)$ is in fact nonempty of dimension ρ. The explanation is that this stable locus is supported over the locus of $\mathrm{Pic}^d(C)$ on which fibers of $\mathscr{L}$ have h^1 at least 2.

Note however that $\mathcal{G}_{2,g-2}^2(C/S)$ is in fact nonempty over all of $\mathrm{Pic}^d(C)$ on the degenerate locus, so if we let S be the locus of $\mathrm{Pic}^d(C)$ on which fibers of $\mathscr{L}$ have h^1 exactly equal to 1, the theorem does imply that $\mathcal{G}_{2,g-2}^2(C/S)$ has dimension at least $\rho + 1$. We conclude that while a nondegeneracy hypothesis is not necessary for Theorem 1.3, it is necessary in the varying determinant case treated by Teixidor i Bigas.

Of course, the ultimate goal of the program is to produce modified expected dimensions which are actually sharp. Theorem 1.3 together with the more general

situation for cases (I) and (II) of Theorem 1.1 discussed in the introduction provide some simple cases where our bounds are already sharp, but are undoubtedly extremely special. It seems likely that there is a degree of inductive structure to the problem, and thus that it makes sense to focus attention initially on rank 2. In light of Theorem 1.1 of [10], it is evident that even if we prove dimension bounds as discussed above in full generality, we will not have sharp results. It is natural to speculate that given a determinant line bundle $\mathscr{L}$, there should be a sequence of expected dimensions, associated to the sequence $\delta_1, \delta_2, \ldots$, where δ_m is the minimal degree of an effective divisor D_m such that $h^1(C, \mathscr{L}(-D_m)) \geq m$. It is then possible that the correct expected dimension would be furnished by the maximum value of this sequence. While it seems likely that Theorem 1.1 of [10] gives the correct value for δ_1, the analysis for δ_i with $i > 1$ is subtler, and we do not hazard a guess as to the correct value.

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