

Multiple cover formula of generalized DT invariants I: parabolic stable pairs

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Abstract

In this paper, we introduce the notion of parabolic stable pairs on Calabi-Yau 3-folds and invariants counting them. By applying the wall-crossing formula developed by Joyce-Song, Kontsevich-Soibelman, we see that they are related to generalized Donaldson-Thomas invariants counting one dimensional semistable sheaves on Calabi-Yau 3-folds. Consequently, the conjectural multiple cover formula of generalized DT invariants is shown to be equivalent to a certain product expansion formula of the generating series of parabolic stable pair invariants. The application of this result to the multiple cover formula will be pursued in the subsequent paper.

1 Introduction

The purpose of this paper is to introduce the notion of parabolic stable pairs, construct their moduli spaces and counting invariants. This is a new notion relevant to curve counting invariants on Calabi-Yau 3-folds, and very similar to classical parabolic vector bundles on curves [14], coherent systems [17] and stable pairs [16], [7]. The wall-crossing formula established by Joyce-Song [7] and Kontsevich-Soibelman [10] are applied to the study of parabolic stable pairs. Applying the wall-crossing formula, we see that the parabolic stable pair invariants are related to generalized Donaldson-Thomas (DT) invariants counting one dimensional semistable sheaves on Calabi-Yau 3-folds. On the other hand, the generalized DT invariants are expected to satisfy a certain multiple cover formula, which is equivalent to the strong rationality conjecture of the generating series of Pandharipande-Thomas (PT) invariants. The conjectural multiple cover formula is shown to be equivalent to a certain product expansion of the generating series of parabolic stable pair invariants. In a subsequent paper [20], by applying the results in the present paper, we will show the multiple cover formula for generalized DT invariants in some cases.

1.1 Motivation

Let X be a smooth projective Calabi-Yau 3-fold over \mathbb{C} , i.e.

$$\bigwedge^3 T_X^\vee \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

The notion of DT invariants is introduced in [19] in order to give a holomorphic analogue of Casson invariants on real 3-manifolds. They are integer valued counting invariants of stable sheaves on a Calabi-Yau 3-fold X , and their rank one theory is conjectured to be equivalent to Gromov-Witten theory [13]. In recent years, the *generalized DT invariants* counting not only stable sheaves but also semistable sheaves are introduced by Joyce-Song [7], Kontsevich-Soibelman [10]. They are \mathbb{Q} -valued, and their definition involves sophisticated techniques on motivic Hall algebras. It is very difficult to compute or study the generalized DT invariants from their definition.

In this paper, we are interested in generalized DT invariants counting one dimensional semistable sheaves on X . Given data,

$$n \in \mathbb{Z}, \quad \beta \in H_2(X, \mathbb{Z}),$$

the generalized DT invariant is denoted by

$$N_{n,\beta} \in \mathbb{Q}. \tag{1}$$

The invariant (1) counts one dimensional semistable sheaves F on X satisfying

$$\chi(F) = n, \quad [F] = \beta.$$

If β and n are coprime, (e.g. $n = 1$), then the invariant $N_{n,\beta}$ is an integer, and can be defined as a holomorphic Casson invariant as in [19]. However if β and n are not coprime, then the invariant (1) may not be an integer, and its definition requires techniques on Hall algebras. Roughly speaking, it is defined by the integration of the Behrend function [1] on the ‘logarithm’ of the moduli stack of semistable sheaves in the Hall algebra. For the detail, see Definition 3.17.

A particularly important case is when $n = 1$. In this case, the invariant $N_{1,\beta}$ is nothing but Katz’s definition of *genus zero Gopakumar-Vafa invariant* [9]. In [7], [21], the following conjecture is proposed:

Conjecture 1.1. [7, Conjecture 6.20], [21, Conjecture 6.3] *We have the following formula,*

$$N_{n,\beta} = \sum_{k \geq 1, k|(n,\beta)} \frac{1}{k^2} N_{1,\beta/k}. \tag{2}$$

The formula (2) is called a *multiple cover formula* of $N_{n,\beta}$. The motivation of the above conjecture is that the formula (2) is equivalent to Pandharipande-Thomas’s conjecture [16, Conjecture 3.14] on the strong rationality of the generating series of stable pair invariants. (See Subsection 4.1 for some more discussions.) So far the above conjecture is checked in few cases, e.g. β is a multiple of a homology class of a $(-1, -1)$ -curve. (cf. [7, Example 4.14].) In general the above conjecture seems to be difficult to solve, especially due to the technical difficulties of the definition of (1).

In the present and the subsequent paper [20], we approach Conjecture 1.1 by the following ideas:

- Introduce the notion of *parabolic stable pairs* and their counting invariants. They are integer valued and we can relate them to the invariant $N_{n,\beta}$. Consequently the conjectural multiple cover formula (2) can be reduced to a certain formula relating parabolic stable pair invariants and the invariant $N_{1,\beta}$.
- There is also a local version of parabolic stable pair theory, and the local theory admits an action of the Jacobian group of the underlying curve. We study the formula (2) by the localization with respect to the actions of the Jacobian group to parabolic stable pair invariants.

The second idea on the Jacobian localizations will be pursued in [20]. The present paper is devoted to give a foundation on parabolic stable pairs.

1.2 Parabolic stable pairs

Recall that a parabolic structure on a vector bundle on a curve is given by a data of a filtration on some fiber together with a parabolic weight [14]. We apply this idea to a one dimensional sheaf F on a Calabi-Yau 3-fold X . In this case, we interpret the ‘fiber’ as a tensor product of the sheaf F with the structure sheaf of a fixed divisor H inside X .

Let $\mathcal{O}_X(1)$ be an ample line bundle on X , and we set $\omega = c_1(\mathcal{O}_X(1))$. We would like to take a divisor in X ,

$$H \in |\mathcal{O}_X(h)|, \quad h \gg 0,$$

so that H intersects with one dimensional sheaves F we are interested in transversally. In fact if we fix $d \in \mathbb{Z}_{>0}$, then we can find H so that any one cycle C on X with $\omega \cdot [C] \leq d$ satisfies, (cf. Lemma 2.9.)

$$\dim H \cap C = 0.$$

Below we fix such d and H . We introduce the notion of parabolic stable pairs on X to be pairs,

$$(F, s), \quad s \in F \otimes \mathcal{O}_H, \tag{3}$$

where F is a pure one dimensional sheaf on X with $\omega \cdot [F] \leq d$. The above pair (3) should satisfy the following stability condition:

- The sheaf F is an ω -semistable sheaf. We denote

$$\mu_\omega(F) := \frac{\chi(F)}{\omega \cdot [F]}.$$

- For any surjection $\pi: F \twoheadrightarrow F'$ with $\mu_\omega(F') = \mu_\omega(F)$, we have

$$(\pi \otimes \mathcal{O}_H)(s) \neq 0.$$

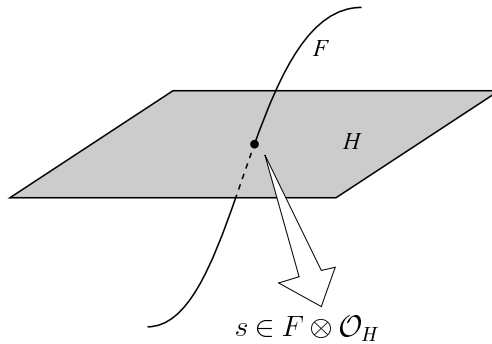


Figure 1: Parabolic stable pair

The reason we call a pair (3) as a parabolic stable pair is that it resembles both of some particular parabolic vector bundles on smooth projective curves [14], and stable pairs studied by Pandharipande-Thomas [16], Joyce-Song [7], based on the earlier work of coherent systems by Le Potier [17]. See Figure 1 for a geometric picture, and Subsection 2.3 for the discussion.

Let

$$\mathcal{M}_n^{\text{par}}(X, \beta) : \text{Sch}/\mathbb{C} \rightarrow \text{Set},$$

be the moduli functor which assigns an \mathbb{C} -scheme T to the set of flat families of parabolic stable pairs (F, s) over T satisfying $[F] = \beta$ and $\chi(F) = n$. Our first result is the following:

Theorem 1.2. [Theorem 2.10] *The moduli functor $\mathcal{M}_n^{\text{par}}(X, \beta)$ is represented by a projective scheme of finite type over \mathbb{C} , denoted by $M_n^{\text{par}}(X, \beta)$.*

1.3 Counting invariants of parabolic stable pairs

By Theorem 1.2, we are able to define the integer valued invariants counting parabolic stable pairs,

$$\text{DT}_{n,\beta}^{\text{par}} := \int_{M_n^{\text{par}}(X,\beta)} \nu_M d\chi,$$

where ν_M is Behrend's constructible function [1],

$$\nu_M : M_n^{\text{par}}(X, \beta) \rightarrow \mathbb{Z}.$$

For $\mu \in \mathbb{Q}$, the generating series of the invariants $\text{DT}_{n,\beta}^{\text{par}}$ is defined by

$$\text{DT}^{\text{par}}(\mu, d) := 1 + \sum_{\substack{0 < \beta \cdot \omega \leq d \\ n/\omega \cdot \beta = \mu}} \text{DT}_{n,\beta}^{\text{par}} q^{n t^\beta}.$$

By applying the wall-crossing formula [7], [10], we can express the generating series $\text{DT}^{\text{par}}(\mu, d)$ in terms of $N_{n,\beta}$, where $N_{n,\beta}$ is the generalized DT invariant (1). The equality of the generating series is described in the ring $\Lambda_{\leq d}$, which is a quotient of Λ ,

$$\Lambda := \bigoplus_{n \in \mathbb{Z}, \beta > 0} \mathbb{Q}q^n t^\beta,$$

by the ideal generated by $q^n t^\beta$ with $\beta \cdot \omega > d$. Here $\beta > 0$ means that β is a homology class of an effective one cycle on X .

Theorem 1.3. [Theorem 3.21] *We have the following formula in $\Lambda_{\leq d}$,*

$$\text{DT}^{\text{par}}(\mu, d) = \prod_{\substack{\beta > 0, \\ n/\omega \cdot \beta = \mu}} \exp \left((-1)^{\beta \cdot H - 1} N_{n,\beta} q^n t^\beta \right)^{\beta \cdot H}. \quad (4)$$

As a corollary, Conjecture 1.1 can be translated into a property on the generating series of the invariants of parabolic stable pairs.

Corollary 1.4. [Proposition 4.5] *The formula (2) holds for any (n, β) with $\beta \cdot \omega \leq d$ and $n/\beta \cdot \omega = \mu$ if and only if the following formula holds in $\Lambda_{\leq d}$,*

$$\text{DT}^{\text{par}}(\mu, d) = \prod_{\substack{\beta > 0, \\ n/\omega \cdot \beta = \mu}} \left(1 - (-1)^{\beta \cdot H} q^n t^\beta \right)^{(\beta \cdot H) N_{1,\beta}} \quad (5)$$

Note that the equality (5) is a relationship between \mathbb{Z} -valued invariants. There is also a local version of local parabolic stable pair theory and the result corresponding to Theorem 1.3. The detail will be discussed in Section 4.

1.4 Relation to existing works

The notion of parabolic structures on vector bundles on curves is introduced by Mehta and Seshadri [14]. It consists of a vector bundle F on a curve C and a filtration of some fiber of $F \rightarrow C$, satisfying some stability condition. Since its introduction, several generalizations have been discussed [12], [2]. In [12], Maruyama and Yokogawa generalize the notion of parabolic structures on vector bundles on curves to torsion free sheaves on arbitrary smooth projective variety, and construct their coarse moduli spaces. The result of Theorem 1.2 is interpreted as a version of their result for torsion one dimensional sheaves. In [2], Boden and Yokogawa studies parabolic Higgs bundles over smooth projective curves. Since a Higgs bundle is interpreted as a torsion sheaf on the total space of the canonical line bundle on a curve, our work is also interpreted as a 3-fold version of parabolic Higgs bundles. The setting of the above works are more general than ours in the sense that they work under an arbitrary choice of a filtration and a parabolic weight which determine parabolic structures. We stick to our situation, corresponding to a specific choice of a filtration type and a parabolic weight, as we will not need other choices. (See Subsection 2.3.) A complete generalization may be pursued elsewhere.

The idea of parabolic stable pair theory and the result of Theorem 1.3 are very similar to Joyce-Song's stable pair theory. In [7], Joyce-Song study stable pairs of the form,

$$\mathcal{O}_X(-n) \rightarrow F, \quad (6)$$

where F is a coherent sheaf on X and $n \gg 0$, satisfying a certain stability condition. Then Joyce-Song's stable pair invariants are shown to be related to their generalized DT invariants, in a way very similar to the formula (4). However there is an advantage of parabolic stable pair theory. First we note that, for each fixed reduced curve $C \subset X$, there is also a local version of parabolic stable pair invariants, and the local version of the formula (5) which is enough to show the global formula (5). (See Proposition 4.17.) The local parabolic stable pair theory admits an action of the Jacobian group $\text{Pic}^0(C)$, while the local stable pair theory (6) does not. Hence we can try to localize by this action. It is much easier to apply the Jacobian localization to the invariants $\text{DT}_{n,\beta}^{\text{par}}$ rather than $N_{n,\beta}$, as the definition of the latter invariants involve very complicated techniques on Hall algebras. The definition of the invariant $\text{DT}_{n,\beta}^{\text{par}}$ is much more elementary, and there is no technical problem in applying the localization to that invariant. This idea works well at least when C has at worst nodal singularities, and the details will be pursued in the subsequent paper [20]. (See Remark 4.19.)

1.5 Plan of the paper

The organization of the paper is as follows. In Section 2, we introduce parabolic stable pairs and construct their moduli spaces. In Section 3, we discuss categorical framework to discuss parabolic stable pairs, and the wall-crossing formula to show Theorem 1.3. In Section 4, we discuss a relationship between parabolic stable pair invariants and the conjectural multiple cover formula of generalized DT invariants.

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2 Moduli spaces of parabolic stable pairs

In this section, we introduce the notion of parabolic stable pairs and study their moduli spaces. In what follows, X is a smooth projective Calabi-Yau 3-fold over \mathbb{C} ,

$$\bigwedge^3 T_X^\vee \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

We fix an ample line bundle $\mathcal{O}_X(1)$ on X and set $\omega = c_1(\mathcal{O}_X(1))$.

2.1 Semistable sheaves

First we recall the notion of ω -semistable one dimensional sheaves on X . We set

$$\mathrm{Coh}_{\leq 1}(X) := \{F \in \mathrm{Coh}(X) : \dim \mathrm{Supp}(F) \leq 1\}.$$

For an object $F \in \mathrm{Coh}_{\leq 1}(X)$, its slope is defined by

$$\mu_\omega(F) := \frac{\chi(F)}{[F] \cdot \omega}.$$

Here $\chi(F)$ is the holomorphic Euler characteristic of F and $[F]$ is the fundamental one cycle associated to F , defined by

$$[F] = \sum_{\eta} (\mathrm{length}_{\mathcal{O}_{X,\eta}} F) \overline{\{\eta\}}. \quad (7)$$

In the above sum, η runs all the codimension two points in X . If $[F] \cdot \omega = 0$, i.e. F is a zero dimensional sheaf, then $\mu_\omega(F)$ is defined to be ∞ .

Definition 2.1. *An object $F \in \mathrm{Coh}_{\leq 1}(X)$ is ω -(semi)stable if for any subsheaf $F' \subset F$, we have the inequality,*

$$\mu_\omega(F') < (\leq) \mu_\omega(F).$$

Note that any one dimensional semistable sheaf F is pure, i.e. there is no zero dimensional subsheaf in F .

2.2 Definition of parabolic stable pairs

Let X be a Calabi-Yau 3-fold and $\omega = c_1(\mathcal{O}_X(1))$ as in the previous subsection. In this subsection, we also fix another divisor,

$$H \subset X. \quad (8)$$

We introduce the notion of parabolic stable pairs to be pairs of one dimensional sheaf F together with a data

$$s \in F \otimes_{\mathcal{O}_X} \mathcal{O}_H, \quad (9)$$

satisfying a certain stability condition. In what follows, we write $F \otimes_{\mathcal{O}_X} \mathcal{O}_H$ as $F \otimes \mathcal{O}_H$ for simplicity.

In order to make the notation (9) well-defined, we need a transversality condition of the support of F . Namely for a one cycle C on X , we say C intersects with H transversally if $H \cap C$ is zero dimensional, or equivalently, any irreducible component $C' \subset C$ is not contained in H . (We allow the multiplicity of $H \cap C$.) For a one dimensional sheaf F on X , we say that F intersects with H transversally if the one cycle $[F]$ given by (7) intersects with H transversally. Then $F \otimes \mathcal{O}_H$ is a zero dimensional sheaf on X , and we identify $F \otimes \mathcal{O}_H$ with its global section $\Gamma(X, F \otimes \mathcal{O}_H)$ as a finite dimensional \mathbb{C} -vector space.

Definition 2.2. For a fixed divisor H on X as above, a parabolic stable pair is defined to be a pair

$$(F, s), \quad s \in F \otimes \mathcal{O}_H, \quad (10)$$

such that the following conditions are satisfied.

- The sheaf F is a one dimensional ω -semistable sheaf on X .
- The one cycle $[F]$ intersects with H transversally.
- For any surjection $F \xrightarrow{\pi} F'$ with $\mu_\omega(F) = \mu_\omega(F')$, we have

$$(\pi \otimes \mathcal{O}_H)(s) \neq 0.$$

For two parabolic stable pairs,

$$(F_i, s_i), \quad i = 1, 2,$$

we say they are *isomorphic* if there is an isomorphism of sheaves,

$$g: F_1 \xrightarrow{\sim} F_2,$$

satisfying $(g \otimes \mathcal{O}_H)(s_1) = s_2$. In the following local $(-1, -1)$ -curve example, the isomorphism classes of parabolic stable pairs can be easily classified.

Example 2.3. Let $f: X \rightarrow Y$ be a birational contraction which contracts a $(-1, -1)$ -curve $C \subset X$, i.e.

$$C \cong \mathbb{P}^1, \quad N_{C/X} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1).$$

Suppose that a divisor $H \subset X$ intersects with C at a one point $p \in C$. Let (F, s) be a parabolic stable pair on X with F supported on C . It is easy to see that any semistable sheaf on X supported on C is an \mathcal{O}_C -module, hence we have

$$F \cong \mathcal{O}_C(a)^{\oplus r},$$

for some $a \in \mathbb{Z}$ and $r \in \mathbb{Z}_{\geq 1}$. By applying an action of $\text{Aut}(F) \cong \text{GL}(r, \mathbb{C})$, we may assume that $s \in F \otimes \mathcal{O}_H \cong \mathcal{O}_p(a)^{\oplus r}$ is of the form,

$$s = (s_1, 0, \dots, 0) \in \mathcal{O}_p(a) \oplus \dots \oplus \mathcal{O}_p(a),$$

with $s_1 \neq 0$. However if $r \geq 2$, then the surjection

$$F \cong \mathcal{O}_C(a)^{\oplus r} \ni (x_1, \dots, x_r) \mapsto x_r \in \mathcal{O}_C(a),$$

violates the third condition of Definition 2.2. Therefore parabolic stable pairs supported on C are given by

$$(\mathcal{O}_C(a), s), \quad a \in \mathbb{Z}, \quad s \in \mathcal{O}_p(a) \setminus \{0\}.$$

The next example shows that there is a parabolic stable pair (F, s) with F strictly ω -semistable.

Example 2.4. Let $i: C \hookrightarrow X$ be an irreducible curve whose arithmetic genus is one, i.e.

$$H^1(C, \mathcal{O}_C) = \mathbb{C}.$$

Suppose that there is a divisor $H \subset X$ which intersects with C at a one point $p \in C$. There is a non-trivial exact sequence of sheaves on C ,

$$0 \rightarrow \mathcal{O}_C \xrightarrow{j_1} E \xrightarrow{j_2} \mathcal{O}_C \rightarrow 0.$$

By taking the pushforward i_* and the tensor product with \mathcal{O}_H , we obtain the exact sequence of vector spaces,

$$0 \rightarrow \mathcal{O}_p \xrightarrow{j_{1,p}} i_*E \otimes \mathcal{O}_H \xrightarrow{j_{2,p}} \mathcal{O}_p \rightarrow 0.$$

Suppose that $s \in i_*E \otimes \mathcal{O}_H$ satisfies $j_{2,p}(s) \neq 0$. Then it is easy to see that the pair

$$(i_*E, s),$$

is a parabolic stable pair. Note that i_*E is ω -semistable but not ω -stable.

2.3 Relation to parabolic vector bundles, PT stable pairs

The notion of parabolic stable pairs resembles particular parabolic vector bundles on smooth projective curves [14]. Recall that for a smooth projective curve C and a vector bundle E on C , a quasi-parabolic structure on E at $p \in C$ is given by a filtration,

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = E_p, \quad (11)$$

where E_p is the fiber of $E \rightarrow C$ at p . A parabolic structure is given by choosing a parabolic weight,

$$0 < \alpha_n < \alpha_{n-1} < \cdots < \alpha_1 < 1,$$

giving a stability condition on quasi-parabolic vector bundles.

Suppose that the curve C is embedded into a Calabi-Yau 3-fold X by $i: C \hookrightarrow X$, and a divisor $H \subset X$ scheme theoretically intersects with C at a point $p \in C$. Then we have the isomorphism of \mathbb{C} -vector spaces,

$$i_*E \otimes \mathcal{O}_H \cong E_p,$$

and $s \in i_*E \otimes \mathcal{O}_H$ determines a one dimensional subspace in E_p . Thus a parabolic stable pair

$$(i_*E, s), \quad s \in i_*E \otimes \mathcal{O}_H,$$

determines a filtration (11) with F_1 one dimensional and $F_2 = E_p$. The stability condition in Definition 2.2 corresponds to a choice of a parabolic weight $0 < \alpha_2 < \alpha_1 \ll 1$.

On the other hand, if F is a pure one dimensional sheaf whose support intersects with H transversally, a pair (10) is also interpreted to be a pair,

$$(F, s), \quad N_{H/X}[-1] \xrightarrow{s} F. \quad (12)$$

Here $N_{H/X}$ is the normal bundle of H to X , and $[-1]$ is a (-1) -shift in the derived category of coherent sheaves on X . The above observation follows from the following lemma.

Lemma 2.5. *Let F be a pure one dimensional sheaf whose support intersects with H transversally. Then we have the canonical isomorphisms,*

$$\mathrm{Ext}_X^i(N_{H/X}, F) \cong \begin{cases} F \otimes \mathcal{O}_H, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

Proof. We have the following local to global spectral sequence,

$$E_2^{p,q} := H^p(X, \mathcal{E}xt_X^q(N_{H/X}, F)) \Rightarrow \mathrm{Ext}_X^i(N_{H/X}, F). \quad (13)$$

By the exact sequence,

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(H) \rightarrow N_{H/X} \rightarrow 0,$$

we have $\mathcal{E}xt_X^i(N_{H/X}, F) = 0$ for $i \neq 0, 1$ and the exact sequence,

$$0 \rightarrow \mathcal{H}om(N_{H/X}, F) \rightarrow F(-H) \xrightarrow{\tau} F \rightarrow \mathcal{E}xt_X^1(N_{H/X}, F) \rightarrow 0.$$

By the transversality assumption, the map τ is an isomorphism in dimension one. Also since F is pure, the morphism τ is injective, hence we have

$$\mathcal{H}om(N_{H/X}, F) = 0, \quad \mathcal{E}xt_X^1(N_{H/X}, F) \cong F \otimes \mathcal{O}_H.$$

Since $F \otimes \mathcal{O}_H$ is a zero dimensional sheaf, the spectral sequence (13) degenerates and the assertion holds. \square

Remark 2.6. *Note that we have the canonical isomorphism,*

$$N_{H/X} \xrightarrow{\sim} \mathcal{O}_H(H),$$

since H is a divisor. By taking a trivialization of $\mathcal{O}_H(H)$ near the intersection $\mathrm{Supp}(F) \cap H$, the pair (12) is also interpreted to be a pair,

$$(F, s), \quad \mathcal{O}_H[-1] \xrightarrow{s} F. \quad (14)$$

However the correspondence between (10) and (14) is not canonical, so we interpret (10) as a pair (12), not (14).

The pair (12) also resembles stable pairs discussed in [16], [7], or more generally coherent systems [17]. For instance, a PT stable pair [16] consists of a pair,

$$\mathcal{O}_X \xrightarrow{s} F, \quad (15)$$

where F is a pure one dimensional sheaf and s is surjective in dimension one. Our pair (12) replaces \mathcal{O}_X by $N_{H/X}[-1]$. (But the stability conditions are different.)

In subsection 3.2, we will interpret the stability condition in Definition 2.2 in terms of a stability condition in the category of pairs of the form

$$N_{H/X}^{\oplus r}[-1] \rightarrow F, \quad (16)$$

and proceed the arguments as given in [7, Section 13]. In fact using the description of parabolic stable pairs in terms of stable objects in the category of pairs (16), we can show the following.

Lemma 2.7. *For a parabolic stable pair (F, s) , let $\text{Aut}(F, s)$ be*

$$\text{Aut}(F, s) := \{g \in \text{Aut}(F) : (g \otimes \mathcal{O}_H)(s) = s\}.$$

Then we have $\text{Aut}(F, s) = \{\text{id}_F\}$.

Proof. The proof will be given in Corollary 3.10. □

2.4 Families of parabolic stable pairs

The moduli space of parabolic stable pairs is defined to be the scheme representing the functor,

$$\mathcal{M}_n^{\text{par}}(X, \beta) : \text{Sch}/\mathbb{C} \rightarrow \text{Set}, \quad (17)$$

which assigns an \mathbb{C} -scheme T to the isomorphism classes of families of parabolic stable pairs (F, s) satisfying a numerical condition,

$$[F] = \beta, \quad \chi(F) = n. \quad (18)$$

Here $\beta \in H_2(X, \mathbb{Z})$ and, by an abuse of notation, $[F]$ is the homology class of the fundamental one cycle (7).

More precisely, the functor (17) is defined as follows. The set of T -valued points of $\mathcal{M}_n^{\text{par}}(X, \beta)$ consists of isomorphism classes of pairs,

$$(\mathcal{F}, s),$$

satisfying the following conditions.

- $\mathcal{F} \in \text{Coh}(X \times T)$ is a flat family of one dimensional coherent sheaves on X satisfying (18), the first and the second conditions in Definition 2.2.

- s is a global section,

$$s \in \Gamma(T, \pi_{T*}(\mathcal{F} \otimes \pi_X^* \mathcal{O}_H)),$$

such that for each closed point $t \in T$, the pair (\mathcal{F}_t, s_t) is a parabolic stable pair on X . Here π_X and π_T are projections from $X \times T$ onto the corresponding factors.

Here we explain the above second condition. Note that the support of $\mathcal{F} \otimes \pi_X^* \mathcal{O}_H$ is finite over T by the first condition, hence we have

$$R^i \pi_{T*}(\mathcal{F} \otimes \pi_X^* \mathcal{O}_H) = 0,$$

for $i > 0$. By the base change theorem, $\pi_{T*}(\mathcal{F} \otimes \pi_X^* \mathcal{O}_H)$ is a vector bundle on T , satisfying

$$\pi_{T*}(\mathcal{F} \otimes \pi_X^* \mathcal{O}_H) \otimes k(t) \cong \mathcal{F}_t \otimes \mathcal{O}_H, \quad (19)$$

for any closed point $t \in T$. The above second condition is now clear from (19).

Remark 2.8. *The moduli functor $\mathcal{M}_n^{\text{par}}(X, \beta)$ also depends on ω and H . We omit these in $\mathcal{M}_n^{\text{par}}(X, \beta)$ in order to simplify the notation.*

In general, the moduli space of parabolic stable pairs may not be compact, since the semistable sheaf F may degenerate to a sheaf which lies in H . In order to avoid such a case, we need to choose a suitable divisor

$$H \in |\mathcal{O}_X(h)|, \quad h \gg 0,$$

which intersects with one dimensional semistable sheaves we are interested in transversally. (Recall that $\mathcal{O}_X(1)$ is an ample line bundle on X with $c_1(\mathcal{O}_X(1)) = \omega$.) Such a divisor H can be found, once we fix a positive integer $d \in \mathbb{Z}_{\geq 1}$ and consider only one dimensional semistable sheaves F satisfying $\omega \cdot [F] \leq d$. In fact we have the following lemma.

Lemma 2.9. *For each $d \in \mathbb{Z}_{>0}$, there is a divisor $H \in |\mathcal{O}_X(h)|$ for $h \gg 0$, (depending on d ,) which intersects with any one cycle C on X satisfying $\omega \cdot C \leq d$, transversally.*

Proof. Let $\text{Chow}_{\leq d}(X)$ be the Chow variety parameterizing one cycles C on X with $\omega \cdot C \leq d$. Let $\mathcal{O}_X(1)$ be a very ample line bundle on X . We would like to find $h \gg 0$ and an element $H \in |\mathcal{O}_X(h)|$ which satisfies the desired condition. We define the set Z to be the subset,

$$Z \subset \text{Chow}_{\leq d}(X) \times |\mathcal{O}_X(h)|,$$

consisting of $([C], H)$ such that there is an irreducible component $C' \subset C$ with $C' \subset H$. A desired H can be found if the projection

$$Z \rightarrow |\mathcal{O}_X(h)|, \quad (20)$$

is not a dominant map.

By the boundedness of the Chow variety, we may assume that

$$H^1(X, I_{C'}(h)) = H^1(C', \mathcal{O}_{C'}(h)) = 0, \quad (21)$$

for any one cycle $[C] \in \text{Chow}_{\leq d}(X)$ and an irreducible component $C' \subset C$. Also the Euler characteristic $\chi(\mathcal{O}_{C'})$ for the above C' is bounded below, say $\chi(\mathcal{O}_{C'}) \geq D$. Let e be the smallest number of $\deg \mathcal{O}_C(1)$ among curves C on X , and $Z_{[C]}$ the fiber of the projection

$$Z \rightarrow \text{Chow}_{\leq d}(X),$$

at a one cycle $[C]$. If $C' \subset C$ is an irreducible component, we have the exact sequence,

$$0 \rightarrow I_{C'} \otimes \mathcal{O}_X(h) \rightarrow \mathcal{O}_X(h) \rightarrow \mathcal{O}_{C'}(h) \rightarrow 0. \quad (22)$$

Using (21), (22) and the Riemann-Roch theorem, the dimension of $Z_{[C]}$ is evaluated as

$$\begin{aligned} \dim Z_{[C]} &= \dim |\mathcal{O}_X(h)| - \dim H^0(C', \mathcal{O}_{C'}(h)) \\ &\leq \dim |\mathcal{O}_X(h)| - he - D. \end{aligned}$$

Therefore we have

$$\dim Z \leq \dim \text{Chow}_{\leq d}(X) + \dim |\mathcal{O}_X(h)| - he - D. \quad (23)$$

Suppose that the map (20) is a dominant map. Then we have $\dim Z \geq \dim |\mathcal{O}_X(h)|$, hence the inequality (23) implies

$$he \leq \dim \text{Chow}_{\leq d}(X) - D. \quad (24)$$

The above inequality is not satisfied for $h \gg 0$. If $h > 0$ does not satisfy the above inequality, then we can find a desired $H \in |\mathcal{O}_X(h)|$. \square

The next subsections are devoted to show the following theorem.

Theorem 2.10. *For $d \in \mathbb{Z}_{>0}$, choose a divisor $H \in |\mathcal{O}_X(h)|$ satisfying the condition of Lemma 2.9. Then for $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$ with $\omega \cdot \beta \leq d$, the functor $\mathcal{M}_n^{\text{par}}(X, \beta)$ is represented by a projective \mathbb{C} -scheme $M_n^{\text{par}}(X, \beta)$.*

2.5 Construction of the moduli space

In this subsection, we give a construction of the moduli space of parabolic stable pairs. As we discussed in Subsection 2.3, the notion of parabolic stable pairs resemble both of parabolic vector bundles on curves and PT stable pairs (coherent systems). The moduli space of parabolic vector bundles is constructed in [14], and its generalization to parabolic sheaves for torsion free sheaves on arbitrary algebraic varieties is discussed in [12]. On the other hand, the moduli space of coherent systems is constructed in [17], and its construction is simplified for PT stable pairs in [18]. Our strategy is to imitate the construction of the moduli space of PT stable pairs by applying the arguments in [18, Section 3].

Let us take $d \in \mathbb{Z}$, $H \in |\mathcal{O}_X(h)|$ and n, β as in the statement of Theorem 2.10. We first note that, by the boundedness of semistable sheaves, there is $m > 0$ such that $m' \geq m$ implies $H^i(X, F(m')) = 0$ for $i > 0$ and $F(m')$ is globally generated for any ω -semistable one dimensional sheaf F with $[F] = \beta$ and $\chi(F) = n$. However for later use, we take $m > 0$ satisfying the following stronger condition. Note that we have

$$\mu_\omega(F(1)) = 1 + \mu_\omega(F).$$

Also the set of ω -semistable one dimensional sheaves F' satisfying

$$\omega \cdot [F'] \leq d, \quad \frac{n}{\omega \cdot \beta} \leq \mu_\omega(F') < 1 + \frac{n}{\omega \cdot \beta}, \quad (25)$$

is bounded. Hence we can take $m > 0$ such that

$$H^i(X, F'(m)) = 0, \quad i > 0, \quad (26)$$

and $F'(m)$ is globally generated for any ω -semistable sheaves F' satisfying $\omega \cdot [F'] \leq d$ and $\mu_\omega(F') \geq n/\omega \cdot \beta$. Below we fix such $m > 0$.

Let F be an ω -semistable one dimensional sheaf with $[F] = \beta$ and $\chi(F) = n$. By the above choice of m , we have

$$\begin{aligned} \chi_F(m) &:= \dim H^0(X, F(m)) \\ &= m(\omega \cdot \beta) + n, \end{aligned} \quad (27)$$

by the Riemann-Roch theorem. Below we write

$$\chi_{n,\beta}(m) := m(\omega \cdot \beta) + n,$$

and set V to be the \mathbb{C} -vector space of dimension $\chi_{n,\beta}(m)$. Then by the vanishing (26), for such an ω -semistable sheaf F , we have a surjection,

$$V \otimes \mathcal{O}_X(-m) \twoheadrightarrow F, \quad (28)$$

such that the induced morphism

$$V \rightarrow H^0(X, F(m)) \quad (29)$$

is an isomorphism.

We consider the Quot-scheme,

$$Q := \text{Quot}(V \otimes \mathcal{O}_X(-m), n, \beta), \quad (30)$$

which parameterizes quotients $V \otimes \mathcal{O}_X(-m) \twoheadrightarrow F$ with F one dimensional sheaf satisfying $[F] = \beta$ and $\chi(F) = n$. The scheme Q contains the open subset,

$$U \subset Q, \quad (31)$$

which corresponds to quotients (28) such that F is ω -semistable and the induced morphism (29) is an isomorphism. Note that outside U the sheaf F may no longer be semistable nor a pure sheaf.

Let

$$(V \otimes \mathcal{O}_X(-m)) \boxtimes \mathcal{O}_Q \twoheadrightarrow \mathbb{F},$$

be the universal quotient on $X \times Q$ and we denote by \mathbb{F}_U the restriction of \mathbb{F} to $X \times U$. By the arguments in the previous subsection and Lemma 2.9, we have the vector bundle R_U on U ,

$$R_U := \pi_{U*}(\mathbb{F}_U \otimes \mathcal{O}_{H \times U}) \rightarrow U. \quad (32)$$

Note that pairs (F, s) satisfying the first and the second conditions in Definition 2.2 and $[F] = \beta$, $\chi(F) = n$ bijectively correspond to closed points in R_U up to the action of $\mathrm{GL}(V)$ on R_U . Let

$$R_U^s \subset R_U,$$

be the subset corresponding to parabolic stable pairs. Suppose for instance that R_U^s is an open subset of R_U . Then the resulting moduli space can be constructed to be the quotient space,

$$R_U^s / \mathrm{GL}(V) = R_U'^s / \mathrm{SL}(V), \quad (33)$$

where $R_U'^s$ is the quotient space of R_U^s by the diagonal subgroup $\mathbb{C}^* \subset \mathrm{GL}(V)$. Note that we have

$$R_U'^s \subset \mathbb{P}(R_U), \quad (34)$$

where $\mathbb{P}(R_U)$ is the projectivization of R_U .

By Lemma 2.7, the action of $\mathrm{SL}(V)$ on $R_U'^s$ is free, hence the quotient space (33) is at least an algebraic space of finite type, once we prove the openness of parabolic stable pair locus. In fact we show that $R_U'^s$ coincides with a GIT stable locus in a certain projective compactification of $\mathbb{P}(R_U)$, hence in particular R_U^s is an open subset of $\mathbb{P}(R_U)$.

2.6 Compactification of $\mathbb{P}(R_U)$

In order to interpret $R_U'^s$ as a GIT stable locus, we embedded $\mathbb{P}(R_U)$ into a product of a Grassmannian and the Quot scheme Q . Let $V \otimes \mathcal{O}_X(-m) \twoheadrightarrow F$ be a quotient corresponding to a point in Q , and $V' \subset V$ a linear subspace. Let $F' \subset F$ be the subsheaf generated by V' . We take an exact sequence,

$$0 \rightarrow G' \rightarrow V' \otimes \mathcal{O}_X(-m) \rightarrow F' \rightarrow 0.$$

By the boundedness of Quot scheme and the subspaces $V' \subset V$, we can take $m' > 2m$ satisfying

$$H^1(H, G' \otimes \mathcal{O}_H(m')) = 0, \quad (35)$$

and $\mathcal{O}_X(m')$ is globally generated. Below such $m' > 2m$ is also fixed.

Suppose that $V \otimes \mathcal{O}_X(-m) \rightarrow F$ corresponds to a point in U . By the vanishing (35) for $V' = V$, we have the surjection,

$$V \otimes K \rightarrow F \otimes \mathcal{O}_H(m'), \quad (36)$$

where $K = H^0(H, \mathcal{O}_H(m' - m))$. Also since $\mathcal{O}_X(m')$ is globally generated, the natural morphism,

$$F \rightarrow F(m') \otimes H^0(X, \mathcal{O}_X(m'))^\vee, \quad (37)$$

is injective whose cokernel is a pure one dimensional sheaf. By setting $M = H^0(X, \mathcal{O}_X(m'))^\vee$, tensoring M, \mathcal{O}_H with (36), (37) respectively, and composing them, we obtain the surjections,

$$\begin{aligned} V \otimes K \otimes M &\rightarrow F \otimes \mathcal{O}_H(m') \otimes M \\ &\rightarrow (F \otimes \mathcal{O}_H(m') \otimes M) / (F \otimes \mathcal{O}_H). \end{aligned}$$

Hence a one dimensional subspace in $F \otimes \mathcal{O}_H$ corresponds to a $d_M(\beta \cdot H) - 1$ -dimensional quotient of $V \otimes K \otimes M$, where $d_M = \dim M$.

The above argument yields the embedding,

$$\mathbb{P}(R_U) \hookrightarrow G(V \otimes K \otimes M, d_M(\beta \cdot H) - 1) \times Q.$$

Here for a vector space V , we denote by $G(V, m)$ the Grassmannian parameterizing quotients $V \rightarrow V'$ with $\dim V' = m$. We define R to be the Zariski closure,

$$R := \overline{\mathbb{P}(R_U)} \subset G(V \otimes K \otimes M, d_M(\beta \cdot H) - 1) \times Q.$$

The next purpose is to give a polarization of R . Let $V \otimes \mathcal{O}_X(-m) \rightarrow F$ be a quotient giving a closed point in Q . By taking $l \gg 0$, we have the surjection,

$$V \otimes W \rightarrow H^0(X, F(l)),$$

where $W = H^0(X, \mathcal{O}_X(l - m))$. The above surjections induce the embedding,

$$Q \hookrightarrow G(V \otimes W, \chi_{n,\beta}(l)).$$

Recall that any Grassmannian $G(V, m)$ is embedded into $\mathbb{P}(\wedge^m V)$ via Plücker embedding. Pulling back $\mathcal{O}(1)$ on $\mathbb{P}(\wedge^m V)$, we have a $\mathrm{GL}(V)$ -equivariant polarization $\mathcal{O}_{G(V,m)}(1)$ on $G(V, m)$.

For simplicity we write

$$G_1 = G(V \otimes K \otimes M, d_M(\beta \cdot H) - 1), \quad G_2 = G(V \otimes W, \chi_{n,\beta}(l)).$$

By the above argument, we have $R \subset G_1 \times G_2$, hence there is a following $\mathrm{SL}(V)$ -equivariant ample line bundle \mathcal{L} on R ,

$$\mathcal{L} := \mathcal{O}_{G_1}(1) \boxtimes \mathcal{O}_{G_2}(1)|_R.$$

In the next subsection, for $l \gg 0$, we see that the GIT stable locus in R w.r.t. \mathcal{L} coincides with R_U^s .

2.7 Hilbert-Mumford criterion

We apply the Hilbert-Mumford criterion [15] to investigate the GIT stable locus with respect to (R, \mathcal{L}) . Let λ be a one parameter subgroup,

$$\lambda: \mathbb{C}^* \rightarrow \mathrm{SL}(V).$$

Then λ corresponds to the grading of V ,

$$V = \bigoplus_{k \in \mathbb{Z}} V_k,$$

where V_k is the space of weight k . Let us take a closed point in R ,

$$(V \otimes K \otimes M \xrightarrow{\phi} A, V \otimes \mathcal{O}_X(-m) \xrightarrow{\rho} F) \in R \subset G_1 \times Q.$$

For simplicity, we write the above point by (ϕ, ρ) . We set $V_{\leq k} := \bigoplus_{j \leq k} V_j$ and $A_{\leq k}, F_{\leq k}$ to be

$$\begin{aligned} A_{\leq k} &:= \phi(V_{\leq k} \otimes K \otimes M), \\ F_{\leq k} &:= \rho(V_{\leq k} \otimes \mathcal{O}_X(-m)). \end{aligned}$$

Let $A_k := A_{\leq k}/A_{\leq k-1}$ and $F_k := F_{\leq k}/F_{\leq k-1}$. By the construction, we have the surjections,

$$\begin{aligned} \phi_k: V_k \otimes K \otimes M &\twoheadrightarrow A_k, \\ \rho_k: V_k \otimes \mathcal{O}_X(-m) &\twoheadrightarrow F_k. \end{aligned}$$

The following proposition follows from an argument of the application of Hilbert-Mumford criterion to the moduli of sheaves. For instance, see [4, Lemma 4.4.3, Lemma 4.4.4], [18, Lemma 3.12].

Proposition 2.11. *In the above situation, we have*

$$\lim_{t \rightarrow 0} \lambda(t) \cdot (\phi, \rho) = (\bigoplus_k \phi_k, \bigoplus_k \rho_k).$$

The Hilbert-Mumford weight $\mu^{\mathcal{L}}((\phi, \rho), \lambda)$ is given by

$$\frac{1}{\dim V} \sum_k \{ \dim V(\chi_{F_{\leq k}}(l) + \dim A_{\leq k}) - \dim V_{\leq k}(\chi_F(l) + \dim A) \}.$$

Here $\chi_F(l)$ is the Hilbert polynomial (27).

By the Hilbert-Mumford criterion, a point $(\phi, \rho) \in R$ is GIT (semi)stable if for any non-trivial one parameter subgroup λ as above, we have $\mu^{\mathcal{L}}((\phi, \rho), \lambda) > (\geq) 0$. This condition is equivalent to that, for any proper subspace $V' \subset V$, we have

$$\dim V(\chi_{F'}(l) + \dim A') - \dim V'(\chi_F(l) + \dim A) > (\geq) 0. \quad (38)$$

Here $A' \subset A$ and $F' \subset F$ are subspace and the subsheaf generated by $V' \otimes K \otimes M$ and $V' \otimes \mathcal{O}_X(-m)$ respectively. In fact if we have such a subspace $V' \subset V$, then there is a one parameter subgroup λ whose induced grading on V is $V_{\leq -1} = 0$, $V_{\leq 0} = V'$ and $V_{\leq 1} = V$.

Let $R^{\mathcal{L}\text{-s}}$ and $R^{\mathcal{L}\text{-ss}}$ be the GIT stable and semistable locus in R , and R_U^{s} the subspace of $\mathbb{P}(R_U)$ given in (34). We prove the following proposition.

Proposition 2.12. For $(\phi, \rho) \in R$, the following three conditions are equivalent.

- (i) We have $(\phi, \rho) \in R'_U{}^s$.
- (ii) We have $(\phi, \rho) \in R^{\mathcal{L}^s}$.
- (iii) We have $(\phi, \rho) \in R^{\mathcal{L}^{-ss}}$.

Proof. (i) \Rightarrow (ii) : Suppose that $(\phi, \rho) \in R'_U{}^s$, i.e. it determines a parabolic stable pair. We show that the LHS of (38) is positive for any proper subspace $V' \subset V$. For simplicity, we write $\chi_F(l) = rl + n$ and $\chi_{F'}(l) = r'l + n'$. Note that $r = \beta \cdot \omega$. Since we are taking $l \gg 0$, the assertion holds if either

$$r' \dim V > r \dim V', \quad (39)$$

or $r' \dim V = r \dim V'$ and

$$\dim V(n' + \dim A') > \dim V'(n + \dim A), \quad (40)$$

holds. Also since V' is a subspace of $H^0(X, F'(m))$, we may assume that $V' = H^0(X, F'(m))$. Then we have

$$\frac{r \dim V'}{r' \dim V} = \frac{m + \mu_\omega(F')}{m + \mu_\omega(F)} \leq 1,$$

by the ω -semistability of F . Therefore (39) does not hold only if $\mu_\omega(F') = \mu_\omega(F)$ and $r \dim V' = r' \dim V$.

Suppose that (39) does not hold. Then by the above argument, we have $n' \dim V = n \dim V'$, hence it is enough to show that

$$\dim V \cdot \dim A' > \dim V' \cdot \dim A. \quad (41)$$

Since we have the surjection $V' \otimes K \otimes M \twoheadrightarrow F' \otimes \mathcal{O}_H(m') \otimes M$ by the vanishing (35), we have

$$A' = \text{Im}(F' \otimes \mathcal{O}_H(m') \otimes M \rightarrow F \otimes \mathcal{O}_H(m') \otimes M \rightarrow A).$$

There is a commutative diagram,

$$\begin{array}{ccccc} 0 & \longrightarrow & F' \otimes \mathcal{O}_H & \longrightarrow & F \otimes \mathcal{O}_H & \\ & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & F' \otimes \mathcal{O}_H(m') \otimes M & \longrightarrow & F \otimes \mathcal{O}_H(m') \otimes M & \\ & & \downarrow \psi' & & \downarrow \psi & \\ 0 & \longrightarrow & A' & \longrightarrow & A. & \end{array} \quad (42)$$

By the assumption, $\text{Ker}(\psi)$ is one dimensional, contained in $F \otimes \mathcal{O}_H$, and any non-zero element $s \in \text{Ker}(\psi)$ gives a parabolic stable pair (F, s) . Since $\text{Ker}(\psi') \subset \text{Ker}(\psi) = \mathbb{C} \cdot s$, and the top square of (42) is Cartesian, the stability condition in Definition 2.2 implies that $\text{Ker}(\psi') = 0$. Hence $\dim A' = d_M h r'$, and together with $r \dim V' = r' \dim V$, $\dim A = d_M h r - 1$, we obtain the inequality (41).

(ii) \Rightarrow (iii) : Obvious.

(iii) \Rightarrow (i) : Suppose that (ϕ, ρ) is a GIT semistable point. First we show that $(\phi, \rho) \in \mathbb{P}(R_U) \subset R$. Since $\mathbb{P}(R_U)$ is projective over U , it is enough to show that $\rho: V \otimes \mathcal{O}_X(-m) \rightarrow F$ is a point in U , i.e. F is ω -semistable and the map

$$V \rightarrow H^0(X, F(m)), \quad (43)$$

is an isomorphism. By the GIT stability, for any proper subspace $V' \subset V$, the LHS of (38) is positive for $l \gg 0$. By looking at the leading coefficients, we have

$$r' \dim V \geq r \dim V'. \quad (44)$$

In order to show the map (43) is an isomorphism, it is enough to show the injectivity of (43). If (43) is not injective, then there is a proper subspace $V' \subset V$ which generates a zero sheaf in F , which contradicts to (44). Therefore the map (43) is an isomorphism.

Next we show that F is ω -semistable. Let us take an exact sequence in $\text{Coh}_{\leq 1}(X)$,

$$0 \rightarrow F' \rightarrow F \xrightarrow{\pi} F'' \rightarrow 0, \quad (45)$$

and suppose that $\mu_\omega(F') > \mu_\omega(F)$. Let V' be the \mathbb{C} -vector space $H^0(X, F'(m))$, which is considered to be a subspace of V via the isomorphism (43). Then by a choice of m in Subsection 2.5, $V' \otimes \mathcal{O}_X(-m)$ generates F' . Applying (44), we obtain $\mu_\omega(F) \geq \mu_\omega(F')$, a contradiction. Hence F is ω -semistable.

Finally we show that (ϕ, ρ) determines a parabolic stable pair. Since $(\phi, \rho) \in \mathbb{P}(R_U)$, the surjection $\phi: V \otimes K \otimes M \twoheadrightarrow A$ factors through $F \otimes \mathcal{O}_H(m) \otimes M$,

$$V \otimes K \otimes M \rightarrow F \otimes \mathcal{O}_H(m') \otimes M \xrightarrow{\psi} A,$$

such that $\text{Ker}(\psi)$ is one dimensional and contained in $F \otimes \mathcal{O}_H$. We would like to show that, for any non-zero element $s \in \text{Ker}(\psi)$, the pair (F, s) is a parabolic stable pair. Suppose by contradiction that there is an exact sequence of the form (45) with $\mu_\omega(F') = \mu_\omega(F) = \mu_\omega(F'')$ and $(\pi \otimes \mathcal{O}_H)(s) = 0$. Setting $V' = H^0(X, F'(m))$, our choices of m and m' yield the commutative diagram,

$$\begin{array}{ccccc} V' \otimes K \otimes M & \longrightarrow & F' \otimes \mathcal{O}_H(m') \otimes M & \xrightarrow{\psi'} & A' \\ \downarrow & & \downarrow & & \downarrow \\ V \otimes K \otimes M & \longrightarrow & F \otimes \mathcal{O}_H(m') \otimes M & \xrightarrow{\psi} & A. \end{array}$$

Here all the vertical arrows are injections and horizontal arrows are surjections. By the assumption $(\pi \otimes \mathcal{O}_H)(s) = 0$, we have $\text{Ker}(\psi') = \text{Ker}(\psi) = \mathbb{C} \cdot s$, hence $\dim A' = d_M h r' - 1$. On the other hand, (ρ, ϕ) is GIT semistable by the assumption, hence the LHS of (38) should be non-negative. Also since $\mu_\omega(F) = \mu_\omega(F')$, we have

$$\frac{r \dim V'}{r' \dim V} = \frac{m + \mu_\omega(F')}{m + \mu_\omega(F)} = 1.$$

Hence we have

$$\frac{\dim V'}{\dim V} = \frac{r'}{r} = \frac{n'}{n}. \quad (46)$$

Therefore we have

$$\frac{\dim V' \cdot \dim A}{\dim V \cdot \dim A'} = \frac{r'}{r} \cdot \frac{d_M hr - 1}{d_M hr' - 1} > 1. \quad (47)$$

The equalities (46), (47) imply that the LHS of (38) is negative, a contradiction. \square

Proof of Theorem 2.10:

Proof. By Proposition 2.12, if we take $l \gg 0$, then we have

$$R_U^s / \mathrm{SL}(V) = R^{\mathcal{L}^{-s}} / \mathrm{SL}(V) = R^{\mathcal{L}^{-ss}} // \mathrm{SL}(V). \quad (48)$$

Then by a general theory of GIT quotient [15], the space (48) is a projective scheme. By Lemma 2.7, the automorphism group of any parabolic stable pair is trivial, hence there is a universal parabolic stable pairs on (48). Hence the scheme (48) is the desired fine moduli space. \square

Remark 2.13. *If we take $H \in |\mathcal{O}_X(h)|$ which does not satisfy the condition in Lemma 2.9, then the moduli functor $\mathcal{M}_n^{\mathrm{par}}(X, \beta)$ may not be represented by a projective scheme. However in the argument of Theorem 2.10, let us replace $U \subset Q$ in (31) by the open subscheme*

$$U^\circ \subset U \subset Q,$$

corresponding to quotients $V \otimes \mathcal{O}_X(-m) \twoheadrightarrow F$ where F intersects with H transversally. Then following the same arguments, we can show that $\mathcal{M}_n^{\mathrm{par}}(X, \beta)$ is represented by a quasi-projective scheme over \mathbb{C} .

Remark 2.14. *It is a natural question whether there is a symmetric perfect obstruction theory on $M_n^{\mathrm{par}}(X, \beta)$ or not. This question seems to be not obvious by the following reason. For instance in PT stable pair case [16], for a PT stable pair (F, s) as in (15), we have the associated object in the derived category, $I^\bullet = (\mathcal{O}_X \rightarrow F)$. The perfect obstruction theory can be constructed by taking the cone of the trace morphism,*

$$\mathbf{R} \mathrm{Hom}(I^\bullet, I^\bullet) \xrightarrow{\mathrm{tr}} \mathbf{R} \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X).$$

In our case, by regarding a parabolic stable pair (F, s) as a pair $N_{H/X}[-1] \xrightarrow{s} F$, we can associate $E \in \mathrm{Coh}(X)$ which fits into the exact sequence,

$$0 \rightarrow F \rightarrow E \rightarrow N_{H/X} \rightarrow 0,$$

whose extension class is s . One might expect that, as an analogy of the trace map, there may be a natural morphism,

$$\mathbf{R} \mathrm{Hom}(E, E) \rightarrow \mathbf{R} \mathrm{Hom}(N_{H/X}, N_{H/X}), \quad (49)$$

and taking its cone may give a perfect obstruction theory. Unfortunately there is no such a map (49), so we cannot discuss as in the PT stable pair case.

3 Wall-crossing formula

In this section, we introduce invariants counting parabolic stable pairs, and show that they are related to generalized DT invariants introduced in [7], [10]. As in the previous section, X is a smooth projective Calabi-Yau 3-fold over \mathbb{C} .

3.1 Counting invariants

Let us take $d \in \mathbb{Z}_{>0}$ and a divisor $H \subset X$ satisfying the condition in Lemma 2.9. By Theorem 2.10, for $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$ with $\omega \cdot \beta \leq d$, there is a fine moduli space $M_n^{\text{par}}(X, \beta)$ which parameterizes parabolic stable pairs (F, s) with $[F] = \beta$ and $\chi(F) = n$.

Recall that, for any \mathbb{C} -scheme M , Behrend [1] constructs a canonical constructible function

$$\nu: M \rightarrow \mathbb{Z},$$

satisfying the following properties.

- For $p \in M$, suppose that there is an analytic open neighborhood $p \in U$, a complex manifold V and a holomorphic function $f: V \rightarrow \mathbb{C}$ such that $U \cong \{df = 0\}$. Then $\nu(p)$ is given by

$$\nu(p) = (-1)^{\dim V} (1 - \chi(M_p(f))).$$

Here $M_p(f)$ is the Milnor fiber of f at p .

- If there is a symmetric perfect obstruction theory on M , we have

$$\begin{aligned} \deg[M]^{\text{vir}} &= \int_M \nu d\chi \\ &:= \sum_{m \in \mathbb{Z}} m \chi(\nu^{-1}(m)). \end{aligned}$$

We define the invariant $\text{DT}_{n,\beta}^{\text{par}}$ as follows.

Definition 3.1. We define $\text{DT}_{n,\beta}^{\text{par}} \in \mathbb{Z}$ to be

$$\text{DT}_{n,\beta}^{\text{par}} = \int_{M_n^{\text{par}}(X,\beta)} \nu_M d\chi.$$

Here ν_M is the Behrend function on $M_n^{\text{par}}(X, \beta)$.

Remark 3.2. We remark that the invariant $\text{DT}_{n,\beta}^{\text{par}}$ also depends on the choice of ω and H . We omit these notation in $\text{DT}_{n,\beta}^{\text{par}}$ for the simplification.

In the local $(-1, -1)$ -curve example, the above invariant can be computed very easily.

Example 3.3. Let $f: X \rightarrow Y$, $C \subset X$ be as in Example 2.3. Suppose that there is $H \in |\mathcal{O}_X(1)|$ which intersects with C at a one point. Then by the classification of parabolic stable pairs in Example 2.3, we have

$$M_n^{\text{par}}(X, m[C]) = \begin{cases} \text{Spec } \mathbb{C}, & m = 1, \\ \emptyset, & m \geq 2. \end{cases}$$

Therefore we have

$$\text{DT}_{n,m[C]}^{\text{par}} = \begin{cases} 1, & m = 1, \\ 0, & m \geq 2. \end{cases}$$

We introduce the generating series of $\text{DT}_{n,\beta}^{\text{par}}$ as follows. For $\mu \in \mathbb{Q}$, we set

$$\text{DT}^{\text{par}}(\mu, d) := 1 + \sum_{\substack{0 < \beta \cdot \omega \leq d \\ n/\omega \cdot \beta = \mu}} \text{DT}_{n,\beta}^{\text{par}} q^{n t^\beta}. \quad (50)$$

The above series is contained in the ring $\Lambda_{\leq d}$ defined as follows. First the ring Λ is defined by

$$\Lambda := \bigoplus_{n \in \mathbb{Z}, \beta > 0} \mathbb{Q} q^{n t^\beta}.$$

Here $\beta > 0$ means that β is a numerical class of an effective one cycle on X . The ring Λ is defined by the quotient ring of Λ by the ideal generated by $q^{n t^\beta}$ with $\omega \cdot \beta > d$. We have

$$\text{DT}^{\text{par}}(\mu, d) \in \Lambda_{\leq d}.$$

3.2 Category of parabolic pairs

Let (F, s) be a parabolic stable pair as in Definition 2.2. As we observed in Subsection 2.3, the pair (F, s) can be also interpreted as a pair,

$$N_{H/X}[-1] \xrightarrow{s} F. \quad (51)$$

In this subsection and next subsection, we construct the category of pairs as above, and interpret the stability condition in Definition 2.2 in terms of the pair (51).

Let (X, ω, d, H) be as in Lemma 2.9. For $\mu \in \mathbb{Q}$, the category $\mathcal{A}(\mu, d)$ is defined as follows.

Definition 3.4. We define the category $\mathcal{A}(\mu, d)$ to be the category of pairs,

$$N_{H/X}^{\oplus r}[-1] \xrightarrow{s} F,$$

where $r \in \mathbb{Z}_{\geq 0}$ and F is a one dimensional ω -semistable sheaf satisfying

$$\mu_\omega(F) = \mu, \quad \omega \cdot [F] \leq d.$$

For two objects $E_i = (N_{H/X}^{\oplus r_i}[-1] \xrightarrow{s_i} F_i) \in \mathcal{A}(\mu, d)$ with $i = 1, 2$, the set of morphisms $\text{Hom}(E_1, E_2)$ is given by the commutative diagram,

$$\begin{array}{ccc} N_{H/X}^{\oplus r_1}[-1] & \xrightarrow{s_1} & F_1 \\ \phi \otimes \text{id}_{N_{H/X}} \downarrow & & \downarrow g \\ N_{H/X}^{\oplus r_2}[-1] & \xrightarrow{s_2} & F_2, \end{array} \quad (52)$$

for $\phi \in M(r_1, r_2)$ and $g \in \text{Hom}(F_1, F_2)$.

Let $g: F_1 \rightarrow F_2$ be a morphism of one dimensional ω -semistable sheaves with $\mu_\omega(F_i) = \mu$. It is easy to see that $\text{Ker}(g)$, $\text{Im}(g)$ and $\text{Cok}(g)$ are all ω -semistable sheaves with $\mu_\omega(*) = \mu$. Also let us take an exact sequence of ω -semistable one dimensional sheaves,

$$0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0.$$

If $\omega \cdot [F_2] \leq d$, then we have the exact sequence by Lemma 2.5,

$$0 \rightarrow \text{Ext}_X^1(N_{H/X}, F_1) \rightarrow \text{Ext}_X^1(N_{H/X}, F_2) \rightarrow \text{Ext}_X^1(N_{H/X}, F_3) \rightarrow 0. \quad (53)$$

The condition $\omega \cdot [F_2] \leq d$ is required for the divisor H to intersect with F_i transversally. Hence for a morphism (52), we can define its kernel, image and cokernel in $\mathcal{A}(\mu, d)$. For instance, the kernel is given by

$$\text{Ker}(\phi) \otimes N_{H/X}[-1] \rightarrow \text{Ker}(g).$$

The notion of monomorphisms, epimorphisms and exact sequences in $\mathcal{A}(\mu, d)$ can be defined in an usual way using the above kernel, image and the cokernel.

However for two objects

$$E_i = (N_{H/X}^{\oplus r_i}[-1] \xrightarrow{s_i} F_i) \in \mathcal{A}(\mu, d), \quad (54)$$

with $i = 1, 2$, the set of extensions in $\mathcal{A}(\mu, d)$,

$$0 \rightarrow E_1 \rightarrow (N_{H/X}^{\oplus r}[-1] \xrightarrow{s} F) \rightarrow E_2 \rightarrow 0, \quad (55)$$

can be defined *only if* the following condition holds:

$$\omega \cdot ([F_1] + [F_2]) \leq d. \quad (56)$$

In particular $E_1 \oplus E_2$ cannot be defined without the condition (56). This implies that $\mathcal{A}(\mu, d)$ is not an abelian category.

For our purpose, we only use extensions (55) satisfying the condition (56). Except the above restriction for the possible extensions, the category $\mathcal{A}(\mu, d)$ behaves as if it is a \mathbb{C} -linear abelian category. For instance if the condition (56) is satisfied, then the set of isomorphism classes of extensions (55), $\text{Ext}^1(E_2, E_1)$, is a finite dimensional \mathbb{C} -vector space. Furthermore the following lemma holds.

Lemma 3.5. *For two objects (54), suppose that the condition (56) holds. Then we have the following exact sequence of \mathbb{C} -vector spaces,*

$$0 \rightarrow \text{Hom}(E_2, E_1) \rightarrow M(r_2, r_1) \oplus \text{Hom}(F_2, F_1) \quad (57)$$

$$\rightarrow \text{Hom}(N_{H/X}^{\oplus r_2}[-1], F_1) \rightarrow \text{Ext}^1(E_2, E_1) \rightarrow \text{Ext}^1(F_2, F_1) \rightarrow 0. \quad (58)$$

Proof. Let $E = (N_{H/X}^{\oplus r}[-1] \xrightarrow{s} F)$ be an object in $\mathcal{A}(\mu, d)$ which fits into an exact sequence (55). Then we have the exact sequence of sheaves,

$$0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0, \quad (59)$$

hence we obtain a linear map,

$$\gamma_1: \text{Ext}^1(E_2, E_1) \rightarrow \text{Ext}^1(F_2, F_1),$$

sending E to F . The map γ_1 is surjective by the exact sequence (53) applied to (59).

Let us look at the kernel of γ_1 . The kernel of γ_1 consists of isomorphism classes of extensions in $\mathcal{A}(\mu, d)$ of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_{H/X}^{\oplus r_1}[-1] & \xrightarrow{i_1} & N_{H/X}^{\oplus r_1+r_2}[-1] & \xrightarrow{i_2} & N_{H/X}^{\oplus r_2}[-1] \longrightarrow 0, \\ & & \downarrow s_1 & & \downarrow s & & \downarrow s_2 \\ 0 & \longrightarrow & F_1 & \xrightarrow{j_1} & F_1 \oplus F_2 & \xrightarrow{j_2} & F_2 \longrightarrow 0. \end{array}$$

Here i_1, j_1 are embedding into corresponding factors, and i_2, j_2 are projections onto corresponding factors. The above extension is given if we give a $\text{Hom}(N_{H/X}^{\oplus r_2}[-1], F_1)$ -factor of s , hence we obtain a surjection,

$$\gamma_2: \text{Hom}(N_{H/X}^{\oplus r_2}[-1], F_1) \twoheadrightarrow \text{Ker}(\gamma_1).$$

Next we look at the kernel of γ_2 . An element $s' \in \text{Hom}(N_{H/X}^{\oplus r_2}[-1], F_1)$ is contained in $\text{Ker}(\gamma_2)$ if and only if there are split projections i'_1, j'_1 of i_1, j_1 respectively such that the following diagram commutes,

$$\begin{array}{ccc} N_{H/X}^{\oplus r_1+r_2}[-1] & \xrightarrow{i'_1} & N_{H/X}^{\oplus r_1}[-1], \\ \downarrow s & & \downarrow s_1 \\ F_1 \oplus F_2 & \xrightarrow{j'_1} & F_1, \end{array} \quad (60)$$

where s is given by the matrix,

$$s = \begin{pmatrix} s_1 & s' \\ 0 & s_2 \end{pmatrix}.$$

Let ϕ and ψ be $\text{Hom}(N_{H/X}^{\oplus r_2}[-1], N_{H/X}^{\oplus r_1}[-1])$ and $\text{Hom}(F_2, F_1)$ -components of i'_1 and j'_1 respectively. Then the diagram (60) commutes if and only if the following equality holds:

$$s' = s_1 \circ \phi - \psi \circ s_2.$$

Hence we obtain the surjection,

$$\gamma_3: M(r_2, r_1) \oplus \text{Hom}(F_2, F_1) \twoheadrightarrow \text{Ker}(\gamma_2),$$

sending (ϕ, ψ) to $s_1 \circ \phi - \psi \circ s_2$.

Finally the kernel of γ_3 coincides with $\text{Hom}(E_2, E_1)$ by its definition. Therefore we obtain the exact sequence (57). \square

3.3 Weak stability conditions on $\mathcal{A}(\mu, d)$

In this subsection, we construct weak stability conditions on $\mathcal{A}(\mu, d)$ and investigate their relationship to parabolic stability. First we define the slope function on $\mathcal{A}(\mu, d)$.

Definition 3.6. For a non-zero object $E = (N_{H/X}^{\oplus r}[-1] \xrightarrow{s} F) \in \mathcal{A}(\mu, d)$ and $\alpha \in \mathbb{Q}$, we set $\widehat{\mu}_\alpha(E)$ to be

$$\widehat{\mu}_\alpha(E) = \begin{cases} \alpha, & \text{if } r \neq 0, \\ \mu(= \mu_\omega(F)), & \text{if } r = 0. \end{cases}$$

The above $\widehat{\mu}_\alpha$ -slope function satisfies the following weak seesaw property. Let $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ be an exact sequence in $\mathcal{A}(\mu, d)$ with $E_1, E_3 \neq 0$ in the sense explained in the previous subsection. Then either one of the following conditions hold:

$$\begin{aligned} \widehat{\mu}_\alpha(E_1) &\geq \widehat{\mu}_\alpha(E_2) \geq \widehat{\mu}_\alpha(E_3), \\ \widehat{\mu}_\alpha(E_1) &\leq \widehat{\mu}_\alpha(E_2) \leq \widehat{\mu}_\alpha(E_3). \end{aligned}$$

The $\widehat{\mu}_\alpha$ -stability on $\mathcal{A}(\mu, d)$ is defined as follows.

Definition 3.7. An object $E \in \mathcal{A}(\mu, d)$ is $\widehat{\mu}_\alpha$ -(semi)stable if for any exact sequence $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ in $\mathcal{A}(\mu, d)$ with $E_1, E_3 \neq 0$, we have the inequality,

$$\widehat{\mu}_\alpha(E_1) < (\leq) \widehat{\mu}_\alpha(E_3).$$

The μ_α -stability satisfies the usual Harder-Narasimhan (HN) property.

Lemma 3.8. For any $E \in \mathcal{A}(\mu, d)$, there is a filtration in $\mathcal{A}(\mu, d)$,

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N = E, \tag{61}$$

with $N \leq 2$ such that each $F_i = E_i/E_{i-1}$ is $\widehat{\mu}_\alpha$ -semistable satisfying $\widehat{\mu}_\alpha(F_i) > \widehat{\mu}_\alpha(F_{i+1})$ for all i . The exact sequence (61) is unique up to isomorphism.

Proof. Note that for any non-zero object $(N_{H/X}^{\oplus r}[-1] \rightarrow F) \in \mathcal{A}(\mu, d)$, we have $r \geq 0$, $\omega \cdot [F] \geq 0$ and either one of them is non-zero. Hence there are no infinite sequences in $\mathcal{A}(\mu, d)$,

$$\begin{aligned} E &= E_0 \supset E_1 \supset \cdots \supset E_n \supset \cdots, \\ E &= E_0 \twoheadrightarrow E_1 \twoheadrightarrow \cdots \twoheadrightarrow E_n \twoheadrightarrow \cdots. \end{aligned}$$

Therefore the criterion in [22, Proposition 2.12] is satisfied, and there are HN filtrations with respect to $\widehat{\mu}_\alpha$ -stability. Since $\widehat{\mu}_\alpha(*) \in \{\alpha, \mu\}$, the HN filtrations are at most 2-steps, i.e. $N \leq 2$. \square

Next we see the relationship between $\widehat{\mu}_\alpha$ -stability and parabolic stability.

Proposition 3.9. *For an object*

$$E = (N_{H/X}[-1] \rightarrow F) \in \mathcal{A}(\mu, d), \quad (62)$$

we have the following.

- (i) If $\alpha < \mu$, then E is $\widehat{\mu}_\alpha$ -semistable if and only if $F = 0$.
- (ii) If $\alpha = \mu$, then any object E as in (62) is $\widehat{\mu}_\alpha$ -semistable.
- (iii) If $\alpha > \mu$, then E is $\widehat{\mu}_\alpha$ -semistable if and only if it is $\widehat{\mu}_\alpha$ -stable, if and only if the pair

$$(F, s), \quad s \in F \otimes \mathcal{O}_H, \quad (63)$$

determined by (62) and Lemma 2.5 is a parabolic stable pair.

Proof. (i) Suppose that E is $\widehat{\mu}_\alpha$ -semistable and $F \neq 0$. Then we have the exact sequence in $\mathcal{A}(\mu, d)$,

$$0 \rightarrow (0 \rightarrow F) \rightarrow E \rightarrow (N_{H/X}[-1] \rightarrow 0) \rightarrow 0.$$

Since we have

$$\widehat{\mu}_\alpha(0 \rightarrow F) = \mu, \quad \widehat{\mu}_\alpha(N_{H/X}[-1] \rightarrow 0) = \alpha,$$

the above sequence destabilize E , a contradiction.

Conversely if $E = (N_{H/X}[-1] \rightarrow 0)$, then there is no exact sequence $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ with $E_1, E_2 \neq 0$. In particular E is $\widehat{\mu}_\alpha$ -stable.

(ii) Obvious.

(iii) Suppose that E is $\widehat{\mu}_\alpha$ -semistable and assume that there is a surjection $F \xrightarrow{\pi} F'$ with $\mu_\omega(F') = \mu$ such that $(\pi \otimes \mathcal{O}_H)(s) = 0$. Then by Lemma 2.5, there is an exact sequence in $\mathcal{A}(\mu, d)$ of the form,

$$0 \rightarrow (N_{H/X}[-1] \rightarrow F'') \rightarrow E \rightarrow (0 \rightarrow F') \rightarrow 0. \quad (64)$$

Since $\alpha > \mu$, the sequence (64) destabilizes E , a contradiction.

Conversely suppose that the pair (63) is parabolic stable and take an exact sequence in $\mathcal{A}(\mu, d)$,

$$0 \rightarrow (N_{H/X}^{\oplus a}[-1] \rightarrow F') \rightarrow E \rightarrow (N_{H/X}^{\oplus b}[-1] \rightarrow F'') \rightarrow 0.$$

Since $a + b = 1$, we have the two possibilities, $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$. When $(a, b) = (1, 0)$, then the surjection $F \rightarrow F''$ takes s to 0 by taking $\otimes \mathcal{O}_H$, hence it contradicts to the parabolic stability. When $(a, b) = (0, 1)$, then we have

$$\widehat{\mu}_\alpha(0 \rightarrow F') = \mu < \alpha = \widehat{\mu}_\alpha(N_{H/X}[-1] \rightarrow F'').$$

Hence E is $\widehat{\mu}_\alpha$ -stable. □

As a corollary, we have the following.

Corollary 3.10. *Let (F, s) be a parabolic stable pair and $E = (N_{H/X}[-1] \rightarrow F)$ the associated object in $\mathcal{A}(\mu, d)$ by Lemma 2.5. Then we have*

$$\mathrm{Hom}(E, E) = \mathbb{C}. \quad (65)$$

In particular the group $\mathrm{Aut}(F, s)$ defined in Lemma 2.7 is $\{\mathrm{id}_F\}$.

Proof. By Proposition 3.9, the object E is $\widehat{\mu}_\alpha$ -stable, hence (65) follows from a general argument of stable objects. (cf. [4, Corollary 1.2.8].) The group $\mathrm{Aut}(F, s)$ is identified with the subset of $\mathrm{Hom}(E, E)$, consisting of commutative diagrams,

$$\begin{array}{ccc} N_{H/X}[-1] & \longrightarrow & F \\ \mathrm{id} \downarrow & & \downarrow g \\ N_{H/X}[-1] & \longrightarrow & F, \end{array} \quad (66)$$

where g is an isomorphism. Hence by (65), g must be an identity. \square

As another corollary, we can explicitly describe the HN filtrations of objects of the form $(N_{H/X}[-1] \rightarrow F)$.

Corollary 3.11. *For an object $E = (N_{H/X}[-1] \rightarrow F) \in \mathcal{A}(\mu, d)$, we have the following.*

(i) *If $\alpha < \mu$, then the HN filtration of E is either $0 = E_0 \subset E_1 = E$ or the exact sequence,*

$$0 \rightarrow (0 \rightarrow F) \rightarrow E \rightarrow (N_{H/X}[-1] \rightarrow 0) \rightarrow 0.$$

(ii) *If $\alpha = \mu$, then the HN filtration of E is $0 = E_0 \subset E_1 = E$.*

(iii) *If $\alpha > \mu$, then the HN filtration of E is either $0 = E_0 \subset E_1 = E$ or an exact sequence of the form,*

$$0 \rightarrow (N_{H/X}[-1] \xrightarrow{s'} F') \rightarrow E \rightarrow (0 \rightarrow F'') \rightarrow 0,$$

where $(N_{H/X}[-1] \xrightarrow{s'} F')$ is determined by a parabolic stable pair (F', s') .

Proof. The result is obvious from Lemma 3.8 and Proposition 3.9. \square

3.4 Stack of objects in $\mathcal{A}(\mu, d)$

In this subsection, we study stack of objects in the category $\mathcal{A}(\mu, d)$. For the introduction to stacks, see [11].

Let $\mathcal{A}(\mu, d)$ be the 2-functor,

$$\mathcal{A}(\mu, d): \mathrm{Sch}/\mathbb{C} \rightarrow \text{groupoid},$$

sending an \mathbb{C} -scheme T to the groupoid of pairs,

$$\mathcal{N}[-1] \rightarrow \mathcal{F}, \quad (67)$$

where \mathcal{N} and \mathcal{F} are T -flat coherent sheaves on $X \times T$, such that for any closed point $t \in T$, there is a commutative diagram,

$$\begin{array}{ccc} \mathcal{N}_t[-1] & \longrightarrow & \mathcal{F}_t \\ \downarrow \psi & & \downarrow h \\ N_{H/X}^{\oplus r}[-1] & \longrightarrow & F. \end{array}$$

Here the bottom pair is an object in $\mathcal{A}(\mu, d)$, and ψ, h are isomorphisms of sheaves on X .

The morphisms in the groupoid $\mathcal{A}(\mu, d)(T)$ are given by the commutative diagrams,

$$\begin{array}{ccc} \mathcal{N}_1[-1] & \longrightarrow & \mathcal{F}_1 \\ \downarrow \phi & & \downarrow g \\ \mathcal{N}_2[-1] & \longrightarrow & \mathcal{F}_2, \end{array} \quad (68)$$

where ϕ and g are isomorphisms of sheaves on $X \times T$.

The stack $\mathcal{A}(\mu, d)$ can be easily shown to be a global quotient stack of some scheme locally of finite type over \mathbb{C} , in particular, it is an Artin stack of finite type over \mathbb{C} . In order to see this, we decompose $\mathcal{A}(\mu, d)$ into components,

$$\mathcal{A}(\mu, d) = \coprod_{\substack{(r, \beta, n) \in \Gamma(\mu), \\ r \geq 0, 0 \leq \omega \cdot \beta \leq d}} \mathcal{A}_{r, \beta, n},$$

where $\mathcal{A}_{r, \beta, n}$ is the stack of pairs $N_{H/X}^{\oplus r}[-1] \rightarrow F$ with $[F] = \beta$ and $\chi(F) = n$, and $\Gamma(\mu)$ is the abelian group defined by

$$\Gamma(\mu) := \{(r, \beta, n) \in \mathbb{Z} \oplus H_2(X, \mathbb{Z}) \oplus \mathbb{Z} : n = \mu(\omega \cdot \beta)\}. \quad (69)$$

Below we use the notation in Subsection 2.5. For fixed β and n as above, we take $m \gg 0$ and the Quot scheme Q as in (30). Let $U \subset Q$ be the open subscheme as in (31). We have the following coherent sheaf on U ,

$$R_U^{(r)} := \mathcal{E}xt_{\pi_U}^1(N_{H/X}^{\oplus r} \boxtimes \mathcal{O}_U, \mathbb{F}_U), \quad (70)$$

where $\mathcal{E}xt_{\pi_U}^i(*, *)$ is the i -th derived functor with respect to the functor $\pi_{U*} \mathcal{H}om(*, *)$. By Lemma 2.5, the sheaf $R_U^{(r)}$ is a vector bundle and we have the canonical isomorphism,

$$R_U^{(r)} \cong \pi_{U*}(\mathbb{F}_U \otimes \mathcal{O}_{H \times U}^{\oplus r}).$$

In particular $R_U^{(1)}$ coincides with R_U given in (32). We also denote the total space of the vector bundle (70) as $R_U^{(r)}$.

The space $R_U^{(r)}$ parameterizes data,

$$V \otimes \mathcal{O}_X(-m) \rightarrow F \leftarrow N_{H/X}^{\oplus r}[-1],$$

where the left arrow represents a point in U . The groups $\mathrm{GL}(V)$, $\mathrm{GL}(r, \mathbb{C})$ act on $V \otimes \mathcal{O}_X(-m)$, $N_{H/X}^{\oplus r}[-1]$ respectively. Hence we obtain the action of $\mathrm{GL}(V) \times \mathrm{GL}(r, \mathbb{C})$ on $R_U^{(r)}$. The stack $\mathcal{A}_{r, \beta, n}$ is constructed to be the quotient stack,

$$\mathcal{A}_{r, \beta, n} = \left[R_U^{(r)} / (\mathrm{GL}(V) \times \mathrm{GL}(r, \mathbb{C})) \right]. \quad (71)$$

3.5 Hall algebra

In this subsection, we recall the Hall type algebra of the category $\mathcal{A}(r, d)$ based on the work [5], using the Artin stack $\mathcal{A}(\mu, d)$.

The \mathbb{Q} -vector space $H(\mu, d)$ is spanned by the isomorphism classes of pairs,

$$(\mathcal{X}, \rho), \quad (72)$$

where \mathcal{X} is an Artin stack of finite type over \mathbb{C} with affine stabilizers, and ρ is a 1-morphism,

$$\rho: \mathcal{X} \rightarrow \mathcal{A}(\mu, d).$$

Two pairs (\mathcal{X}_i, ρ_i) for $i = 1, 2$ are isomorphic if there is an 1-isomorphism $f: \mathcal{X}_1 \rightarrow \mathcal{X}_2$ which 2-commutes with ρ_i . The \mathbb{Q} -vector space $H(\mu, d)$ is decomposed as

$$H(\mu, d) = \bigoplus_{\substack{(r, \beta, n) \in \Gamma(\mu) \\ r \geq 0, 0 \leq \omega \cdot \beta \leq d}} H_{r, \beta, n},$$

where $H_{r, \beta, n}$ is the subvector space of $H(\mu, d)$, spanned by (72) such that ρ factors through the substack $\mathcal{A}_{r, \beta, n} \subset \mathcal{A}(\mu, d)$.

Let $\mathcal{E}x(\mu, d)$ be the stack of exact sequences in $\mathcal{A}(\mu, d)$,

$$0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0. \quad (73)$$

As in the previous subsection, it is not difficult to show that $\mathcal{E}x(\mu, d)$ is an Artin stack locally of finite type over \mathbb{C} , and we omit the detail. We have the 1-morphisms,

$$p_i: \mathcal{E}x(\mu, d) \rightarrow \mathcal{A}(\mu, d),$$

sending an exact sequence (73) to the object E_i .

For two elements $\alpha_i = (\mathcal{X}_i, \rho_i) \in H(\mu, d)$ with $i = 1, 2$, its $*$ -product $\alpha_1 * \alpha_2$ is defined in the following way. We have the diagram,

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{\rho} & \mathcal{X} & \xrightarrow{p_2} & \mathcal{A}(\mu, d), \\ \downarrow & & \downarrow (p_1, p_3) & & \\ \mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{(\rho_1, \rho_2)} & \mathcal{A}(\mu, d)^{\times 2} & & \end{array}$$

Here the left diagram is a Cartesian square. The product $\alpha_1 * \alpha_2$ is given by

$$\alpha_1 * \alpha_2 = [(\mathcal{Y}, p_2 \circ \rho)] \in H(\mu, d).$$

We have the following proposition.

Proposition 3.12. *The $*$ -product on $H(\mu, d)$ is an associative product on $H(\mu, d)$, with unit given by $[\text{Spec } \mathbb{C} \rightarrow \mathcal{A}(\mu, d)]$, corresponding to $0 \in \mathcal{A}(\mu, d)$.*

Proof. Although the category $\mathcal{A}(\mu, d)$ is not an abelian category, the same proof as in [5, Theorem 5.2] works. The only thing to notice is that, for $\alpha_i \in H_{r_i, \beta_i, n_i}$ with $i = 1, 2, 3$ and

$$\omega \cdot (\beta_1 + \beta_2 + \beta_3) > d,$$

we have the equalities,

$$\alpha_1 * (\alpha_2 * \alpha_3) = (\alpha_1 * \alpha_2) * \alpha_3 = 0.$$

□

3.6 Elements $\delta_{r, \beta, n}^\alpha, \epsilon_{r, \beta, n}^\alpha$

For $\alpha \in \mathbb{Q}$, we have the substack,

$$\mathcal{M}_{r, \beta, n}^\alpha \subset \mathcal{A}_{r, \beta, n},$$

parameterizing $\widehat{\mu}_\alpha$ -semistable objects $(N_{H/X}^{\oplus r}[-1] \rightarrow F) \in \mathcal{A}(\mu, d)$ with $[F] = \beta, \chi(F) = n$.

It is not difficult to check that the stack $\mathcal{M}_{r, \beta, n}^\alpha$ is an open finite type substack of $\mathcal{A}_{r, \beta, n}$, hence it is an Artin stack of finite type over \mathbb{C} . We will only need this fact for $r = 0, 1$, the cases which follow from Proposition 3.9 and Theorem 2.10 immediately. For instance, suppose that $r = 1$ and $\alpha > \mu$. Then in the notation of the previous section, we have

$$\mathcal{M}_{1, \beta, n}^\alpha = [R_U^s / (\mathbb{C}^* \times \mathrm{GL}(V))] \quad (74)$$

by (71) and Proposition 3.9 (iii). By Proposition 2.12, the RHS of (74) is an open substack of $\mathcal{A}_{r, \beta, n}$, and we have

$$\mathcal{M}_{1, \beta, n}^\alpha \cong [M_n^{\mathrm{par}}(X, \beta) / \mathbb{C}^*]. \quad (75)$$

Here \mathbb{C}^* acts on $M_n^{\mathrm{par}}(X, \beta)$ trivially.

Remark 3.13. (i) For any α , the \mathbb{C} -valued point of the stack $\mathcal{M}_{r, 0, 0}^\alpha$ consists of $(N_{H/X}^{\oplus r}[-1] \rightarrow 0)$, hence

$$\mathcal{M}_{r, 0, 0} \cong [\mathrm{Spec} \mathbb{C} / \mathrm{GL}(r, \mathbb{C})]. \quad (76)$$

In particular the stack (76) does not depend on α .

(ii) For any α , the \mathbb{C} -valued points of the stack $\mathcal{M}_{0, \beta, n}^\alpha$ consist of the objects of the form $(0 \rightarrow F)$. In particular the stack $\mathcal{M}_{0, \beta, n}^\alpha$ does not depend on α , and isomorphic to the moduli stack of one dimensional μ_ω -semistable sheaves F on X with $[F] = \beta, \chi(F) = n$.

Following [6, Definition 3.1.8], we define $\delta_{r, \beta, n}^\alpha$ and $\epsilon_{r, \beta, n}^\alpha \in H_{r, \beta, n}$ to be

$$\delta_{r, \beta, n}^\alpha = [\mathcal{M}_{r, \beta, n}^\alpha \hookrightarrow \mathcal{A}_{r, \beta, n}], \quad (77)$$

$$\epsilon_{r, \beta, n}^\alpha = \sum_{\substack{l \geq 1, (r_i, \beta_i, n_i) \in \Gamma(\mu), 1 \leq i \leq l, \\ (r_1, \beta_1, n_1) + \dots + (r_l, \beta_l, n_l) = (r, \beta, n), \\ \widehat{\mu}_\alpha(r_i, \beta_i, n_i) = \widehat{\mu}_\alpha(r, \beta, n)}} \frac{(-1)^{l-1}}{l} \delta_{r_1, \beta_1, n_1}^\alpha * \dots * \delta_{r_l, \beta_l, n_l}^\alpha. \quad (78)$$

Here $\widehat{\mu}_\alpha(r, \beta, n)$ is defined to be α if $r \neq 0$ and μ if $r = 0$. In other words, the element $\epsilon_{r, \beta, n}^\alpha$ is given by

$$\sum_{\widehat{\mu}_\alpha(r, \beta, n) = \mu'} \epsilon_{r, \beta, n}^\alpha = \log \left(1 + \sum_{\widehat{\mu}_\alpha(r, \beta, n) = \mu'} \delta_{r, \beta, n}^\alpha \right),$$

for $\mu' \in \{\alpha, \mu\}$. Or equivalently,

$$1 + \sum_{\widehat{\mu}_\alpha(r, \beta, n) = \mu'} \delta_{r, \beta, n}^\alpha = \exp \left(\sum_{\widehat{\mu}_\alpha(r, \beta, n) = \mu'} \epsilon_{r, \beta, n}^\alpha \right). \quad (79)$$

Note that, by Remark 3.13, the elements $\delta_{r, 0, 0}^\alpha$, $\delta_{0, \beta, n}^\alpha$, $\epsilon_{0, \beta, n}^\alpha$ do not depend on α . Below we omit α in the notation for these elements. Also for the convention, we set $\delta_{0, 0, 0} = 1$.

Using the results of Corollary 3.11, the following lemma obviously follows.

Proposition 3.14. *Take $\alpha > \mu$ and $(0, \beta, n) \in \Gamma(\mu)$. Then we have the following identity in $H(\mu, d)$,*

$$\begin{aligned} \delta_{1, \beta, n}^\mu &= \delta_{0, \beta, n} * \delta_{1, 0, 0} \\ &= \sum_{\substack{(\beta_1, n_1) + (\beta_2, n_2) = (\beta, n), \\ n_i = \mu(\omega \cdot \beta_i)}} \delta_{1, \beta_1, n_1}^\alpha * \delta_{0, \beta_2, n_2}. \end{aligned}$$

Proof. The result obviously follows from Corollary 3.11 and the arguments given in [6, Theorem 5.11]. \square

3.7 Lie algebra homomorphism

Following [7], we can construct a Lie algebra homomorphism from a certain Lie subalgebra of $H(\mu, d)$ to a Lie algebra defined by the Euler pairing on $\mathcal{A}(\mu, d)$. Applying the Lie algebra homomorphism to the formula in Proposition 3.14, we obtain a formula relating invariants counting parabolic stable pairs to generalized DT invariants introduced in [7], [10].

First by [5, Definition 5.13], there is a Lie subalgebra,

$$H^{\text{Lie}}(\mu, d) \subset H(\mu, d),$$

consisting of elements $[\rho: \mathcal{X} \rightarrow \mathcal{A}(\mu, d)]$, supported on ‘virtual indecomposable objects’. The definition of the virtual indecomposable objects is very complicated and we omit its detail. The only property we need is that,

$$\epsilon_{r, \beta, n}^\alpha \in H^{\text{Lie}}(\mu, d),$$

for any element $\epsilon_{r, \beta, n}^\alpha$ constructed in the previous subsection.

Next we define the Lie algebra $C(\mu, d)$. Let $\Gamma(\mu)$ be the abelian group given by (69). We have the map,

$$\text{cl}: \mathcal{A}(\mu, d) \rightarrow \Gamma(\mu),$$

defined by

$$\text{cl}(N_{H/X}^{\oplus r}[-1] \rightarrow F) = (r, [F], \chi(F)).$$

Let $\chi: \Gamma(\mu) \times \Gamma(\mu) \rightarrow \mathbb{Z}$ be a bilinear anti-symmetric pairing given by

$$\chi((r_1, \beta_1, n_1), (r_2, \beta_2, n_2)) = r_2(\beta_1 \cdot H) - r_1(\beta_2 \cdot H).$$

We have the following lemma.

Lemma 3.15. *For $E_i \in \mathcal{A}(\mu, d)$ with $i = 1, 2$ satisfying*

$$\text{cl}(E_i) = (r_i, \beta_i, n_i), \quad \omega \cdot (\beta_1 + \beta_2) \leq d,$$

we have

$$\begin{aligned} \chi(E_1, E_2) &= \dim \text{Hom}(E_1, E_2) - \dim \text{Ext}^1(E_1, E_2) \\ &\quad - \dim \text{Hom}(E_2, E_1) + \dim \text{Ext}^1(E_2, E_1). \end{aligned} \quad (80)$$

Proof. By Lemma 3.5, we have

$$\begin{aligned} &\dim \text{Hom}(E_2, E_1) - \dim \text{Ext}^1(E_2, E_1) \\ &= \dim \text{Hom}(F_2, F_1) - \dim \text{Ext}^1(F_2, F_1) + r_1 r_2 - r_1(\beta_2 \cdot H). \end{aligned} \quad (81)$$

On the other hand, by the Serre duality and Riemann-Roch theorem on X , we have

$$\begin{aligned} &\dim \text{Hom}(F_1, F_2) - \dim \text{Ext}^1(F_1, F_2) \\ &\quad + \dim \text{Ext}^1(F_2, F_1) - \dim \text{Hom}(F_2, F_1) = 0. \end{aligned} \quad (82)$$

The formula (80) follows from (81) and (82). \square

Let us consider the following \mathbb{Q} -vector space,

$$C(\mu) := \bigoplus_{(r, \beta, n) \in \Gamma(\mu)} \mathbb{Q}_{c_{(r, \beta, n)}}. \quad (83)$$

There is the Lie algebra structure on the \mathbb{Q} -vector space $C(\mu)$ by

$$[c_{v_1}, c_{v_2}] = (-1)^{\chi(v_1, v_2)} \chi(v_1, v_2) c_{v_1 + v_2},$$

for $v_1, v_2 \in \Gamma(\mu)$. Let $I_d \subset C(\mu)$ be the ideal of $C(\mu)$, generated by $c_{(r, \beta, n)}$ with $\omega \cdot \beta > d$. The Lie algebra $C(\mu, d)$ is defined to be the quotient,

$$C(\mu, d) := C(\mu)/I_d.$$

Finally we recall that the Behrend function [1] on \mathbb{C} -schemes are naturally generalized to those on Artin stacks over \mathbb{C} . (cf. [7, Theorem 5.12].) Let

$$\nu_{\mathcal{A}}: \mathcal{A}(\mu, d) \rightarrow \mathbb{Z},$$

be the Behrend function on $\mathcal{A}(\mu, d)$. The argument of Joyce-Song [7] is applied in our situation, and the following result holds.

Theorem 3.16. *There is a Lie algebra homomorphism,*

$$\Upsilon: H^{\text{Lie}}(\mu, d) \rightarrow C(\mu, d),$$

such that for an element

$$u = \left[[M/\mathbb{C}^*] \xrightarrow{\rho} \mathcal{A}_{r,\beta,n} \right] \in H^{\text{Lie}}(\mu, d),$$

where M is a \mathbb{C} -scheme with a trivial \mathbb{C}^ -action, we have*

$$\Upsilon(u) = \left(\int_M \rho^* \nu_{\mathcal{A}} d\chi \right) c_{(r,\beta,n)}. \quad (84)$$

Proof. Although $\mathcal{A}(\mu, d)$ is not an abelian category, the same proof of [7, Theorem 5.12] works. We only have to notice the following two things.

First for $v_i = (r_i, \beta_i, n_i) \in \Gamma(\mu)$ with $i = 1, 2$ and $\omega \cdot (\beta_1 + \beta_2) > d$, we have

$$[u_1, u_2] = 0, \quad [c_{v_1}, c_{v_2}] = 0,$$

for $u_i \in H_{r_i, \beta_i, n_i}$. Therefore we only need to consider the Lie brackets in the case of $\omega \cdot (\beta_1 + \beta_2) \leq d$. In that case, we can discuss as if $\mathcal{A}(\mu, d)$ were an abelian category, and use the argument in [7, Theorem 5.12].

The next thing is to show the version of [7, Theorem 5.3] in our situation. Namely we need to check that the moduli stack $\mathcal{A}(\mu, d)$ is analytically locally written as a critical locus of some holomorphic function, up to some group action.

It is enough to check this for the substack $\mathcal{A}_{r,\beta,n} \subset \mathcal{A}(\mu, d)$. Let $\mathcal{M}_{\beta,n}$ be the moduli stack of one dimensional semistable sheaves F on X with $[F] = \beta$, $\chi(F) = n$. We have the forgetting 1-morphism,

$$\mathbf{f}: \mathcal{A}_{r,\beta,n} \rightarrow \mathcal{M}_{\beta,n},$$

sending $(N_{H/X}^{\oplus r}[-1] \rightarrow F)$ to F . On the other hand, in the notation of Subsection 2.5, Subsection 3.4, the stack $\mathcal{M}_{\beta,n}$ is written as the quotient stack,

$$\mathcal{M}_{\beta,n} \cong [U / \text{GL}(V)].$$

Combined with the description (71) of $\mathcal{A}_{r,\beta,n}$, the 1-morphism \mathbf{f} is induced from the natural projection $R_U^{(r)} \rightarrow U$,

$$\mathbf{f}: \left[R_U^{(r)} / (\text{GL}(V) \times \text{GL}(r, \mathbb{C})) \right] \rightarrow [U / \text{GL}(V)].$$

The morphism $R_U^{(r)} \rightarrow U$ is a Zariski-locally trivial fibration with fiber $(\mathbb{C}^{\beta \cdot H})^{\times r}$. Thus the morphism \mathbf{f} is also a Zariski locally trivial fibration with fiber the quotient stack

$$[(\mathbb{C}^{\beta \cdot H})^{\times r} / \text{GL}(r, \mathbb{C})].$$

By [7, Theorem 5.3], the stack $\mathcal{M}_{\beta,n}$ is analytically locally written as a critical locus of some holomorphic function up to some group action. Hence the same property holds for the stack $\mathcal{A}_{r,\beta,n}$ by the above argument. \square

3.8 Product expansion formula

Let Υ be the Lie algebra homomorphism discussed in Theorem 3.16. If $\alpha \in \mathbb{Q}$ satisfies $\alpha > \mu$, then we have $\delta_{1,\beta,n}^\alpha = \epsilon_{1,\beta,n}^\alpha$ and

$$\Upsilon(\epsilon_{1,\beta,n}^\alpha) = -\mathrm{DT}_{n,\beta}^{\mathrm{par}C(1,\beta,n)}, \quad (85)$$

by Definition 3.1, the isomorphism (75) and the formula (84). Here we need to change the sign since the Behrend function on a scheme M and that on $[M/\mathbb{C}^*]$ differ by a sign.

By applying Υ to the element $\epsilon_{0,\beta,n} \in H^{\mathrm{Lie}}(\mu, d)$, we obtain the invariant $N_{n,\beta} \in \mathbb{Q}$.

Definition 3.17. *For $(0, \beta, n) \in \Gamma(\mu)$ with $\omega \cdot \beta \leq d$, the invariant $N_{n,\beta} \in \mathbb{Q}$ is defined by*

$$\Upsilon(\epsilon_{0,\beta,n}) = -N_{n,\beta} c_{(0,\beta,n)}. \quad (86)$$

The invariant $N_{n,\beta} \in \mathbb{Q}$ is nothing but a generalized DT invariant introduced in [7], [10], counting ω -semistable one dimensional sheaves F satisfying $[F] = \beta$, $\chi(F) = n$. In fact, by Remark 3.13 (ii), the stack defining $\delta_{0,\beta,n}$ is the moduli stack of the above ω -semistable sheaves, and it is obvious that the invariant $N_{n,\beta}$ defined via (86) coincides with the invariant given in [7]. Also see [21, Section 4] for the explanation of the construction of $N_{n,\beta} \in \mathbb{Q}$.

Remark 3.18. *The restriction $\omega \cdot \beta \leq d$ is not necessary in defining $N_{n,\beta}$ in [7]. The restriction $\omega \cdot \beta \leq d$ is put in Definition 3.17 since $\epsilon_{0,\beta,n} = 0$ if $\omega \cdot \beta > 0$ so $N_{n,\beta}$ cannot be defined via (86) without the above restriction.*

Remark 3.19. *As explained in [7, Theorem 6.16], the invariant $N_{n,\beta}$ does not depend on a choice of ω . Of course it also does not depend on H , as we do not use H to define ω -semistable sheaves.*

Remark 3.20. *Suppose that n and β are coprime. Then there is a \mathbb{Q} -ample divisor ω such that the moduli space of ω -stable one dimensional sheaves F with $[F] = \beta$, $\chi(F) = n$ is a projective scheme. (i.e. there is no strictly ω -semistable sheaves.) If $M_n(X, \beta)$ is such a moduli space, then $N_{n,\beta}$ is given by*

$$N_{n,\beta} = \int_{M_n(X,\beta)} \nu_M d\chi,$$

where ν_M is the Behrend function on $M_n(X, \beta)$.

Recall that we defined the series $\mathrm{DT}^{\mathrm{par}}(\mu, d)$ as an element in $\Lambda_{\leq d}$ in Subsection 3.1. Applying the Lie algebra homomorphism Υ in Theorem 3.16, we obtain a formula relating $\mathrm{DT}_{n,\beta}^{\mathrm{par}}$ and $N_{n,\beta}$. The following result follows from the same arguments as in [7, Theorem 5.24], [23, Theorem 4.7], [22, Theorem 5.8]. For the reader's convenience, we provide the argument.

Theorem 3.21. *We have the following formula in $\Lambda_{\leq d}$,*

$$\text{DT}^{\text{par}}(\mu, d) = \prod_{\substack{\beta > 0, \\ n/\omega \cdot \beta = \mu}} \exp \left((-1)^{\beta \cdot H - 1} N_{n, \beta} q^n t^\beta \right)^{\beta \cdot H}. \quad (87)$$

Here $\beta > 0$ means that β is a homology class of an effective one cycle on X .

Proof. Let us take $\alpha \in \mathbb{Q}$ satisfying $\alpha > \mu$. We set $\delta^\alpha \in H^{\text{Lie}}(\mu, d)$ to be

$$\delta^\alpha := \sum_{n=\mu(\omega \cdot \beta)} \delta_{1, \beta, n}^\alpha = \sum_{n=\mu(\omega \cdot \beta)} \epsilon_{1, \beta, n}^\alpha.$$

Note that $\delta_{1, \beta, n}^\alpha = \epsilon_{1, \beta, n}^\alpha = \delta_{0, \beta, n} = \epsilon_{0, \beta, n} = 0$ if $\omega \cdot \beta > d$. By Proposition 3.14, we have the identity in $H(\mu, d)$,

$$\left(\sum_{n=\mu(\omega \cdot \beta)} \delta_{0, \beta, n} \right) * \delta_{1, 0, 0} = \delta^\alpha * \left(\sum_{n=\mu(\omega \cdot \beta)} \delta_{0, \beta, n} \right). \quad (88)$$

We set $\mathfrak{E} \in H^{\text{Lie}}(\mu, d)$ to be

$$\mathfrak{E} = \sum_{n=\mu(\omega \cdot \beta)} \epsilon_{0, \beta, n}.$$

Then by (79) and (88), we have

$$\begin{aligned} \delta^\alpha &= \exp(\mathfrak{E}) * \delta_{1, 0, 0} * \exp(\mathfrak{E})^{-1} \\ &= \sum_{k \geq 0} \frac{1}{k!} (\text{Ad}_{\mathfrak{E}})^k (\delta_{1, 0, 0}). \end{aligned} \quad (89)$$

Here we have used the Baker-Campbell-Hausdorff formula in (89). Applying Υ to (89), using (85), (86) and setting $\text{DT}_{0,0}^{\text{par}} = 1$, we have the following identities in $C(\mu, d)$,

$$\begin{aligned} & \sum_{n=\mu(\omega \cdot \beta)} \text{DT}_{n, \beta}^{\text{par}} c_{(1, \beta, n)} \\ &= \sum_{k \geq 0} \frac{(-1)^k}{k!} \sum_{\substack{\beta_1, \dots, \beta_k \in H_2(X, \mathbb{Z}), \\ n_1, \dots, n_k \in \mathbb{Z}, \\ n_i = \mu(\omega \cdot \beta_i)}} \prod_{i=1}^k N_{n_i, \beta_i} \cdot \text{Ad}_{c_{(0, \beta_1, n_1)}} \circ \dots \circ \text{Ad}_{c_{(0, \beta_k, n_k)}} (c_{(1, 0, 0)}) \\ &= \sum_{k \geq 0} \frac{1}{k!} \sum_{\substack{\beta_1, \dots, \beta_k \in H_2(X, \mathbb{Z}), \\ n_1, \dots, n_k \in \mathbb{Z}, \\ n_i = \mu(\omega \cdot \beta_i)}} \prod_{i=1}^k (-1)^{\beta_i \cdot H - 1} (\beta_i \cdot H) N_{n_i, \beta_i} \cdot c_{(1, \sum_{i=1}^k \beta_i, \sum_{i=1}^k n_i)}. \end{aligned} \quad (90)$$

The formula (87) obviously follows from (90). \square

4 Multiple cover formula

In this section, we discuss relationship between Theorem 3.21 and the conjectural multiple cover formula of the generalized DT invariants $N_{n,\beta}$.

4.1 Conjectural multiple cover formula

In this subsection, we recall a conjectural multiple cover formula of the invariants $N_{n,\beta}$. The statement of the conjecture is as follows.

Conjecture 4.1. [7, Conjecture 6.20], [21, Conjecture 6.3] *We have the following formula,*

$$N_{n,\beta} = \sum_{k \geq 1, k|(n,\beta)} \frac{1}{k^2} N_{1,\beta/k}. \quad (91)$$

The conjecture is motivated by the strong rationality conjecture of the generating series of PT invariants [16]. As we recalled in Subsection 2.3, a PT stable pair consists of data,

$$(F, s), \quad s: \mathcal{O}_X \rightarrow F, \quad (92)$$

where F is a pure one dimensional sheaf and s is surjective in dimension one. For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, The moduli space of PT stable pairs (92) with $[F] = \beta$ and $\chi(F) = n$ is denoted by $P_n(X, \beta)$. The PT invariant is defined by

$$P_{n,\beta} := \int_{P_n(X,\beta)} \nu_P d\chi \in \mathbb{Z}.$$

Here ν_P is the Behrend function on $P_n(X, \beta)$.

Let $\text{PT}_\beta(X)$ and $\text{PT}(X)$ be the generating series,

$$\begin{aligned} \text{PT}_\beta(X) &:= \sum_{n \in \mathbb{Z}} P_{n,\beta} q^n, \\ \text{PT}(X) &:= \sum_{\beta \in H_2(X, \mathbb{Z})} \text{PT}_\beta(X) t^\beta. \end{aligned} \quad (93)$$

The main conjecture by Pandharipande-Thomas [16] is an equivalence between the generating series of Gromov-Witten invariants and the generating series of PT invariants $\text{PT}(X)$ after a suitable variable change. In order to make the variable change well-defined, the series $\text{PT}_\beta(X)$ should satisfy a (weak) rationality property: i.e. $\text{PT}_\beta(X)$ should be the Laurent expansion of a rational function of q , invariant under $q \leftrightarrow 1/q$.

On the other hand, if we believe GW/PT correspondence, then the series $\text{PT}(X)$ is expected to be written as a certain infinite product expansion, called *Gopakumar-Vafa form*. The conjecture is formulated in the following way. (cf. [8, Equation (18)], [21, Conjecture 6.2].)

Conjecture 4.2. *There are integers*

$$n_g^\beta \in \mathbb{Z}, \text{ for } g \geq 0, \beta \in H_2(X, \mathbb{Z}),$$

such that we have

$$\text{PT}(X) = \prod_{\beta > 0} \prod_{j=1}^{\infty} (1 - (-q)^j t^\beta)^{jn_0^\beta} \cdot \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} (1 - (-q)^{g-1-k} t^\beta)^{(-1)^{k+gn_g^\beta} \binom{2g-2}{k}}. \quad (94)$$

The above conjecture also implies the weak rationality of $\text{PT}_\beta(X)$, and in fact it is nothing but Pandharipande-Thomas's strong rationality conjecture [16, Conjecture 3.14].

Now in [23], [3], the weak rationality of $\text{PT}_\beta(X)$ is solved. The idea is to relate the invariants $P_{n,\beta}$ with $N_{n,\beta}$ and other invariants $L_{n,\beta}$, and use some properties of the latter invariants. More precisely we have the following theorem.

Theorem 4.3. [23] (**Euler characteristic version**), [3] *There are invariants $L_{n,\beta} \in \mathbb{Q}$ satisfying*

$$L_{n,\beta} = L_{-n,\beta}, \quad L_{n,\beta} = 0, \text{ for } |n| \gg 0,$$

such that the following formula holds:

$$\text{PT}(X) = \prod_{n > 0, \beta > 0} \exp \left((-1)^{n-1} N_{n,\beta} q^n t^\beta \right)^n \left(\sum_{n,\beta} L_{n,\beta} q^n t^\beta \right). \quad (95)$$

The above theorem implies the weak rationality of $\text{PT}_\beta(X)$. (cf. [23, Corollary 4.8].) The formula (95) is weaker than the formula (94), but it reduces Conjecture 4.2 to Conjecture 4.1. Namely we have the following corollary.

Corollary 4.4. [3, Theorem 1.1], [21, Theorem 6.4]

(i) *The series $\text{PT}_\beta(X)$ is the Laurent expansion of a rational function of q , invariant under $q \leftrightarrow 1/q$.*

(ii) *Conjecture 4.2 holds if and only if Conjecture 4.1 holds. In this case, we have*

$$n_{0,\beta} = N_{1,\beta}.$$

4.2 Multiple cover formula via parabolic stable pairs

Recall that we have given a formula relating invariants counting parabolic stable pairs to the invariants $N_{n,\beta}$. In this subsection, we see that conjecture 4.1 is also equivalent to a conjectural product expansion formula of the series $\text{DT}^{\text{par}}(\mu, d)$. Namely we have the following proposition.

Proposition 4.5. *The formula (91) holds for any (n, β) with $\beta \cdot \omega \leq d$ and $n/\beta \cdot \omega = \mu$ if and only if the following formula holds in $\Lambda_{\leq d}$,*

$$\text{DT}^{\text{par}}(\mu, d) = \prod_{\substack{\beta > 0, \\ n/\omega \cdot \beta = \mu}} (1 - (-1)^{\beta \cdot H} q^n t^\beta)^{(\beta \cdot H) N_{1,\beta}} \quad (96)$$

Proof. By taking the logarithm of both sides of (87), we have

$$\log \text{DT}^{\text{par}}(\mu, d) = \sum_{\substack{\beta > 0, \\ n/\omega \cdot \beta = \mu}} (-1)^{\beta \cdot H - 1} (\beta \cdot H) N_{n, \beta} q^n t^\beta. \quad (97)$$

On the other hand, the logarithm of the RHS of (96) is

$$\begin{aligned} & \sum_{\substack{\beta > 0, \\ n/\omega \cdot \beta = \mu}} (\beta \cdot H) N_{1, \beta} \log(1 - (-1)^{\beta \cdot H} q^n t^\beta) \\ &= \sum_{\substack{\beta > 0, \\ n/\omega \cdot \beta = \mu}} \sum_{k \geq 1, k | (n, \beta)} \frac{(-1)^{\beta \cdot H - 1}}{k^2} (\beta \cdot H) N_{1, \beta/k} q^n t^\beta. \end{aligned} \quad (98)$$

Comparing (97) with (98), we obtain the result. \square

Remark 4.6. *The proof of Proposition 4.5 shows that, in order to check the formula (91) for a specific (β, n) , it is enough to check the equality of the coefficients of $q^n t^\beta$ of $\log \text{DT}^{\text{par}}(\mu, d)$ and the logarithm of the RHS of (96).*

By the above proposition, the following conjecture is equivalent to both of Conjecture 4.1 and Conjecture 4.2.

Conjecture 4.7. *We have the following formula in $\Lambda_{\leq d}$,*

$$\text{DT}^{\text{par}}(\mu, d) = \prod_{\substack{\beta > 0, \\ n/\omega \cdot \beta = \mu}} (1 - (-1)^{\beta \cdot H} q^n t^\beta)^{(\beta \cdot H) N_{1, \beta}}.$$

By Remark 4.6, the above conjecture is equivalent to the following: if we set $\widehat{\text{DT}}_{n, \beta}^{\text{par}} \in \mathbb{Q}$ by

$$\log \text{DT}^{\text{par}}(\mu, d) = \sum_{n, \beta} \widehat{\text{DT}}_{n, \beta}^{\text{par}} q^n t^\beta, \quad (99)$$

then we have the formula,

$$\widehat{\text{DT}}_{n, \beta}^{\text{par}} = (-1)^{\beta \cdot H - 1} (\beta \cdot H) \sum_{k \geq 1, k | (n, \beta)} \frac{1}{k^2} N_{1, \beta/k}.$$

4.3 Local parabolic stable pair invariants

In the following subsections, we study the local version of parabolic stable pairs and relevant results. All the arguments are similar to the global case, and we omit several details.

In what follows, we fix a reduced subscheme,

$$i: C \subset X,$$

with $\dim C = 1$. We also fix a divisor in X ,

$$H \in |\mathcal{O}_X(h)|, \quad h > 0, \quad (100)$$

which intersects with C transversally. Of course, any one cycle on X supported on C intersects with H transversally. On the other hand, we do not assume the transversality for the intersection of H with curves of bounded degree other than C . In this sense, the way we choose for (100) is different from the way for $H \subset X$ in Lemma 2.9. The former one depends on the curve C , while the latter one depends on the degree $d \in \mathbb{Z}_{>0}$.

Let $M_n^{\text{par}}(X, \beta)$ be the moduli space of parabolic stable pairs with respect to the above choice of H . Since H may not satisfy the condition in Lemma 2.9, the moduli space $M_n^{\text{par}}(X, \beta)$ may not be projective, but it is at least a quasi-projective scheme as we mentioned in Remark 2.13. Let $\text{Chow}_\beta(X)$ be the Chow variety parameterizing one cycles $C' \subset X$ with $[C'] = \beta$. There is a Hilbert-Chow type morphism,

$$\pi_\beta: M_n^{\text{par}}(X, \beta) \rightarrow \text{Chow}_\beta(X),$$

sending a parabolic stable pair (F, s) to the one cycle $[F]$.

Let C_1, \dots, C_N be the set of irreducible components of C . We have

$$H_2(C, \mathbb{Z}) \cong \bigoplus_{i=1}^N \mathbb{Z}[C_i],$$

and we can identify $H_2(C, \mathbb{Z})$ with the group of one cycles on X supported on C . The effective cone is defined by,

$$H_2(C, \mathbb{Z})_{>0} := \left\{ \sum_{i=1}^N a_i [C_i] : a_i \geq 0 \right\} \setminus \{0\} \subset H_2(C, \mathbb{Z}).$$

For each $\gamma \in H_2(C, \mathbb{Z})_{>0}$, we can regard it as

$$\gamma \in \text{Chow}_{i_*\gamma}(X).$$

We define the local parabolic stable pair invariant in the following way.

Definition 4.8. For each $\gamma \in H_2(C, \mathbb{Z})_{>0}$, we define $\text{DT}_{n,\gamma}^{\text{par}} \in \mathbb{Z}$ to be

$$\text{DT}_{n,\gamma}^{\text{par}} := \int_{\pi_{i_*\gamma}^{-1}(\gamma)} \nu_M d\chi.$$

Here we note that ν_M is a Behrend function on $M_n^{\text{par}}(X, i_*\gamma)$, not on the fiber $\pi_{i_*\gamma}^{-1}(\gamma)$.

4.4 Local generalized DT invariants

We can also define the local version of generalized DT invariants in a way similar to Definition 3.17. Namely we replace the category $\mathcal{A}(\mu, d)$ by the category of pairs,

$$N_{H/X}^{\oplus r}[-1] \rightarrow F, \quad (101)$$

where F is an ω -semistable sheaf supported on C , satisfying $\mu_\omega(F) = \mu$. The category consisting of the above pairs is denoted by $\mathcal{A}(\mu, C)$. The moduli stack of objects in $\mathcal{A}(\mu, C)$ can be constructed as in Subsection 3.4. Namely let $\mathcal{A}(\mu)$ be the stack of pairs $N_{H/X}^{\oplus r}[-1] \rightarrow F$, where F is μ_ω -semistable with $\mu_\omega(F) = \mu$, but not necessary supported on C . By the arguments in Subsection 3.4 and Remark 2.13, the stack $\mathcal{A}(\mu)$ can be shown to be an Artin stack locally of finite type over \mathbb{C} . The desired stack of objects in $\mathcal{A}(\mu, C)$ is the closed substack,

$$\mathcal{A}(\mu, C) \subset \mathcal{A}(\mu).$$

The stack $\mathcal{A}(\mu, C)$ decomposes as

$$\mathcal{A}(\mu, C) = \coprod_{(r, \gamma, n) \in \Gamma(\mu, C)} \mathcal{A}_{r, \gamma, n},$$

where $\mathcal{A}_{r, \gamma, n}$ is the stack of pairs (101) with $[F] = \gamma$ as a one cycle supported on C , and $\Gamma(\mu, C)$ is defined similarly to (69), by replacing $H_2(X, \mathbb{Z})$ by $H_2(C, \mathbb{Z})$.

Hence we have the Hall type algebra $H(\mu, C)$ and the Lie subalgebra of virtual indecomposable objects,

$$H^{\text{Lie}}(\mu, C) \subset H(\mu, C).$$

The Lie algebra $C(\mu, C)$ is also defined similarly to (83), just by replacing $H_2(X, \mathbb{Z})$ by $H_2(C, \mathbb{Z})$,

$$C(\mu, C) = \bigoplus_{(r, \gamma, n) \in \Gamma(\mu, C)} \mathbb{Q}_{C(r, \gamma, n)}.$$

Here we do not take a quotient as in defining $C(\mu, d)$.

The Lie algebra homomorphism

$$\Upsilon : H^{\text{Lie}}(\mu, C) \rightarrow C(\mu, C),$$

can be similarly constructed as in Theorem 3.16. The only point we have to notice is that we use the Behrend function on $\mathcal{A}(\mu)$, not on $\mathcal{A}(\mu, C)$. As in the proof of Theorem 3.16, the stack $\mathcal{A}(\mu)$ is analytic locally written as a critical locus of some holomorphic function, hence the same argument in Theorem 3.16 can be applied. Also for an element

$$u = \left[[M/\mathbb{C}^*] \xrightarrow{\rho} \mathcal{A}_{r, \gamma, n} \right] \in H^{\text{Lie}}(\mu, C),$$

where M is a \mathbb{C} -scheme with a trivial \mathbb{C}^* -action, we have

$$\Upsilon(u) = \left(\int_M \rho^* \nu_{\mathcal{A}} d\chi \right) c_{(r, \gamma, n)}.$$

Here $\nu_{\mathcal{A}}$ is the Behrend function on $\mathcal{A}(\mu)$ restricted to $\mathcal{A}_{r, \gamma, n}$, which may be different from that on $\mathcal{A}_{r, \gamma, n}$.

The elements,

$$\delta_{r,\gamma,n}^\alpha \in H(\mu, C), \quad \epsilon_{r,\gamma,n}^\alpha \in H^{\text{Lie}}(\mu, C),$$

can be defined similarly to (77), (78), using similar $\widehat{\mu}_\alpha$ -semistability on $\mathcal{A}(\mu, C)$ and the moduli stack of $\widehat{\mu}_\alpha$ -semistable pairs (101). The local generalized DT invariant is defined as follows.

Definition 4.9. For $(0, \gamma, n) \in \Gamma(\mu, C)$, the invariant $N_{n,\gamma} \in \mathbb{Q}$ is defined by

$$\Upsilon(\epsilon_{0,\gamma,n}) = -N_{n,\gamma} c_{(0,\gamma,n)}.$$

Remark 4.10. Similarly to Remark 3.19, the invariant $N_{n,\gamma}$ does not depend on ω .

Remark 4.11. Suppose that n and γ are coprime. Then there is an ample \mathbb{Q} -divisor ω such that the moduli space of ω -stable sheaves F with $\chi(F) = n$ and $[F] = \gamma$ as a one cycle is a closed subscheme,

$$M_n(C, \gamma) \subset M_n(X, i_*\gamma).$$

As in Remark 3.20, the invariant $N_{n,\gamma}$ is given by

$$N_{n,\gamma} = \int_{M_n(C,\gamma)} \nu_M d\chi, \quad (102)$$

where ν_M is the Behrend function on $M_n(X, i_*\gamma)$, not on $M_n(C, \gamma)$.

4.5 Generating series of local invariants

Let $\text{DT}^{\text{par}}(\mu, C)$ be the generating series,

$$\text{DT}^{\text{par}}(\mu, C) := 1 + \sum_{\substack{n \in \mathbb{Z}, \gamma \in H_2(C, \mathbb{Z}) > 0, \\ n/\omega \cdot \gamma = \mu}} \text{DT}_{n,\gamma}^{\text{par}} q^n t^\gamma \in \Lambda_C.$$

Here Λ_C is defined by

$$\Lambda_C := \prod_{n \in \mathbb{Z}, \gamma \in H_2(C, \mathbb{Z}) > 0} \mathbb{Q} q^n t^\gamma.$$

As an analogy of Theorem 3.21, the following result holds.

Theorem 4.12. We have the following formula in Λ_C ,

$$\text{DT}^{\text{par}}(\mu, C) = \prod_{\substack{n \in \mathbb{Z}, \gamma \in H_2(C, \mathbb{Z}) > 0, \\ n/\omega \cdot \gamma = \mu}} \exp\left((-1)^{\gamma \cdot H - 1} N_{n,\gamma} q^n t^\gamma\right)^{\gamma \cdot H}. \quad (103)$$

Proof. The same proof of Theorem 3.21 is applied. \square

As an analogy of Conjecture 4.1 and Conjecture 4.7, we propose the following conjecture.

Conjecture 4.13. For $(n, \gamma) \in \mathbb{Z} \oplus H_2(C, \mathbb{Z})$, we have the following formula,

$$N_{n, \gamma} = \sum_{k \geq 1, k | (n, \gamma)} \frac{1}{k^2} N_{1, \gamma/k}. \quad (104)$$

Or equivalently, we have the following formula for any $\mu \in \mathbb{Q}$,

$$\mathrm{DT}^{\mathrm{par}}(\mu, C) = \prod_{\substack{\gamma \in H_2(C, \mathbb{Z})_{>0}, \\ n/\omega \cdot \gamma = \mu}} (1 - (-1)^{\gamma \cdot H} q^n t^\gamma)^{(\gamma \cdot H) N_{1, \gamma}}.$$

Example 4.14. Let $f: X \rightarrow Y$ and $C \subset X$ be as in Example 2.3, Example 3.3. As in Example 2.3, suppose that there is a divisor $H \in |\mathcal{O}_X(1)|$ which intersects with C at one point $p \in C$. Then by Example 3.3, we have

$$\mathrm{DT}^{\mathrm{par}}(\mu, C) = \begin{cases} 1 + q^\mu t, & \mu \in \mathbb{Z}, \\ 1, & \text{otherwise.} \end{cases}$$

Hence Conjecture 4.13 holds in this case with $N_{1, [C]} = 1$ and $N_{1, m[C]} = 0$ for $m \geq 2$. In particular for $(m, n) \in \mathbb{Z}^{\oplus 2}$ with $m \geq 1$, we have $N_{n, m[C]} \neq 0$ only if $m | n$, and in this case, we have

$$N_{n, m[C]} = \frac{1}{m^2}. \quad (105)$$

Remark 4.15. In Example 4.14, there may not exist a divisor $H \subset X$ which intersects with C at a one point in general. However such a divisor always exists on an analytic neighborhood of $C \subset U \subset X$, and we can deduce (105) by the arguments on U .

Remark 4.16. The formula (105) is well-known, c.f. [7, Example 6.2]. However the argument in Example 4.14 seems to be the easiest argument to deduce (105).

4.6 From local theory to global theory

Finally in this section, we show that Conjecture 4.13 implies Conjecture 4.1 using parabolic stable pair invariants.

Proposition 4.17. For given $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, suppose that the formula (104) holds for any reduced curve $C \xrightarrow{i} X$ and $\gamma \in H_2(C, \mathbb{Z})$ with $i_* \gamma = \beta$. Then $N_{n, \beta}$ satisfies the formula (91).

Proof. We take $d > \omega \cdot \beta$ and a divisor $H \subset X$ as in Lemma 2.9. We set $\mu = n/\omega \cdot \beta$, and consider the invariant $\widehat{\mathrm{DT}}_{n, \beta}^{\mathrm{par}} \in \mathbb{Q}$ as in (99). Also we set $\widehat{N}_{n, \beta} \in \mathbb{Q}$ to be

$$\widehat{N}_{n, \beta} := \sum_{k \geq 1, k | (n, \beta)} \frac{1}{k^2} N_{1, \beta/k} \quad (106)$$

By Remark 4.6, it is enough to show the formula,

$$\widehat{\text{DT}}_{n,\beta}^{\text{par}} = (-1)^{\beta \cdot H - 1} (\beta \cdot H) \widehat{N}_{n,\beta}. \quad (107)$$

Let $\gamma \in \text{Chow}_\beta(X)$ be a one cycle on X and $C_\gamma \subset X$ the reduced curve defined by $C_\gamma = \text{Supp}(\gamma)$. By our choice of H , the curve C_γ intersects with H transversally. Let us consider the generating series $\text{DT}^{\text{par}}(\mu, C_\gamma)$ and its logarithm. We can similarly define $\widehat{\text{DT}}_{n,\gamma}^{\text{par}} \in \mathbb{Q}$ by the $q^n t^\gamma$ -coefficient of $\log \text{DT}^{\text{par}}(\mu, C_\gamma)$. Also by replacing β by γ in (106), we can similarly define $\widehat{N}_{n,\gamma}$.

Let us consider the assignments $\gamma \mapsto \widehat{\text{DT}}_{n,\gamma}^{\text{par}}, \widehat{N}_{n,\gamma}$. They determine constructible functions on $\text{Chow}_\beta(X)$,

$$\begin{aligned} \widehat{\text{DT}}_{n,*}^{\text{par}} : \text{Chow}_\beta(X) \ni \gamma &\mapsto \widehat{\text{DT}}_{n,\gamma}^{\text{par}} \in \mathbb{Q}, \\ \widehat{N}_{n,*} : \text{Chow}_\beta(X) \ni \gamma &\mapsto \widehat{N}_{n,\gamma} \in \mathbb{Q}. \end{aligned}$$

We consider the integrations of these constructible functions over $\text{Chow}_\beta(X)$. First note that, by the definition of local parabolic stable pair invariant in Definition 4.8, we have

$$\text{DT}_{n,\beta}^{\text{par}} = \int_{\gamma \in \text{Chow}_\beta(X)} \text{DT}_{n,\gamma}^{\text{par}} d\chi. \quad (108)$$

Therefore we have

$$\begin{aligned} &\int_{\gamma \in \text{Chow}_\beta(X)} \widehat{\text{DT}}_{n,\gamma}^{\text{par}} d\chi \\ &= \int_{\gamma \in \text{Chow}_\beta(X)} \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{(\gamma_i, n_i) \in H_2(C_\gamma, \mathbb{Z}) \oplus \mathbb{Z}, 1 \leq i \leq l, \\ (\gamma_1, n_1) + \dots + (\gamma_l, n_l) = (\gamma, n), \\ n_i = \mu(\omega \cdot \gamma_i)}} \prod_{i=1}^l \text{DT}_{n_i, \gamma_i}^{\text{par}} d\chi \\ &= \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{(\beta_i, n_i) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z}, 1 \leq i \leq l, \\ (\beta_1, n_1) + \dots + (\beta_l, n_l) = (\beta, n), \\ n_i = \mu(\omega \cdot \beta_i)}} \int_{\{\gamma_i\}_{i=1}^l \in \prod_{i=1}^l \text{Chow}_{\beta_i}(X)} \prod_{i=1}^l \text{DT}_{n_i, \gamma_i}^{\text{par}} d\chi \\ &= \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{(\beta_i, n_i) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z}, 1 \leq i \leq l, \\ (\beta_1, n_1) + \dots + (\beta_l, n_l) = (\beta, n), \\ n_i = \mu(\omega \cdot \beta_i)}} \prod_{i=1}^l \text{DT}_{n_i, \beta_i}^{\text{par}} \\ &= \widehat{\text{DT}}_{n,\beta}^{\text{par}}. \end{aligned} \quad (109)$$

Here we have used (108) in the third equation.

Next, we consider the integration of the function $\widehat{N}_{n,*}$. For $a \in \mathbb{Z}_{\geq 1}$, we set $\text{Chow}_\beta^{(a)}(X) \subset \text{Chow}_\beta(X)$ to be

$$\text{Chow}_\beta^{(a)}(X) := \{\gamma \in \text{Chow}_\beta(X) : \text{div}(\gamma, n) = a\},$$

where $\text{div}(\ast)$ is the divisibility of the vector \ast . By setting $e = \text{div}(\beta, n)$, we have

$$\begin{aligned}
& \int_{\gamma \in \text{Chow}_{\beta}(X)} \widehat{N}_{n,\gamma} d\chi \\
&= \sum_{a \geq 1} \int_{\gamma \in \text{Chow}_{\beta}^{(a)}(X)} \sum_{k \geq 1, k|a} \frac{1}{k^2} N_{1,\gamma/k} d\chi \\
&= \sum_{k \geq 1, k|e} \frac{1}{k^2} \sum_{a \geq 1, k|a|e} \int_{\gamma \in \text{Chow}_{\beta}^{(a)}(X)} N_{1,\gamma/k} d\chi \\
&= \sum_{k \geq 1, k|e} \frac{1}{k^2} \int_{\gamma' \in \text{Chow}_{\beta/k}} N_{1,\gamma'} d\chi \tag{110}
\end{aligned}$$

$$= \sum_{k \geq 1, k|e} \frac{1}{k^2} N_{1,\beta/k} \tag{111}$$

$$= \widehat{N}_{n,\beta}. \tag{112}$$

Here (110) follows from the set theoretic bijection for $k|e$,

$$\text{Chow}_{\beta/k}(X) \ni \gamma' \mapsto k\gamma' \in \bigcup_{a \geq 1, k|a|e} \text{Chow}_{\beta}^{(a)}(X).$$

Also (111) follows from (102).

Now we use the assumption for the local theory. It just implies the equality,

$$\widehat{\text{DT}}_{n,\gamma}^{\text{par}} = (-1)^{\gamma \cdot H - 1} (\gamma \cdot H) \widehat{N}_{n,\gamma}, \tag{113}$$

for any $\gamma \in \text{Chow}_{\beta}(X)$. By (113), (109), (112) and noting $\gamma \cdot H = \beta \cdot H$, the equality (107) follows. \square

As a corollary, we have the following.

Corollary 4.18. *Suppose that Conjecture 4.13 is true for any reduced curve $C \subset X$. Then Conjecture 4.1 holds.*

Remark 4.19. *For $\gamma \in \text{Chow}_{\beta}(X)$, let us take the curve $C_{\gamma} = \text{Supp}(\gamma)$ and its normalization,*

$$\widetilde{C}_{\gamma} \rightarrow C_{\gamma}.$$

As discussed in [20, Lemma 2.11], the invariant $N_{n,\gamma}$ vanishes unless \widetilde{C}_{γ} is a disjoint union of \mathbb{P}^1 . The idea for the proof is as follows: suppose for simplicity that C_{γ} is a smooth curve of positive genus, $C_{\gamma} \subset U \subset X$ be a sufficiently small analytic neighborhood of C_{γ} in X , and $\text{Pic}^0(U)$ the group of line bundles on U which restricts to degree zero line bundles on C_{γ} . The group $\text{Pic}^0(U)$ acts on the moduli space which defines the invariant $N_{n,\gamma}$. We can also find a subgroup $S^1 \subset \text{Pic}^0(U)$ whose induced action on the above moduli space is free. Hence $N_{n,\gamma} = 0$ follows by the localization argument.

Now suppose that \tilde{C}_γ is a union of \mathbb{P}^1 , C_γ has at worst nodal singularities, and the arithmetic genus of C_γ is positive. In this case, the group $\text{Pic}^0(U)$ contains \mathbb{C}^* , so we may try to localize by this action. In this case, there may be \mathbb{C}^* -fixed sheaves supported on C_γ . However it is not easy to investigate the contribution of $\epsilon_{0,\gamma,n}$ at the \mathbb{C}^* -fixed sheaf, and the above localization argument is not obvious in this case.

In [20], instead of the invariant $N_{n,\gamma}$, we apply the \mathbb{C}^* -localization to the parabolic stable pair invariant $\text{DT}_{n,\gamma}^{\text{par}}$. The moduli space $M_n^{\text{par}}(C_\gamma, \gamma)$ also admits the \mathbb{C}^* -action, (while the moduli space of PT or JS stable pairs do not,) and the definition of $\text{DT}_{n,\gamma}^{\text{par}}$ does not require the technique on Hall algebras. There is no technical difficulty in applying the localization on $M_n^{\text{par}}(X, \gamma)$, and can investigate the contribution of \mathbb{C}^* -fixed parabolic stable pairs to the invariant $\text{DT}_{n,\gamma}^{\text{par}}$. It will turn out that these localization argument is relevant to show Conjecture 4.13 in some cases, and these details will be pursued in [20].

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