HODGE POLYNOMIALS OF $\mathrm{SL}(2,\mathbb{C})$-CHARACTER VARIETIES FOR CURVES OF SMALL GENUS

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ABSTRACT. We compute the E-polynomials of the moduli spaces of representations of the fundamental group of a complex surface into $\mathrm{SL}(2,\mathbb{C})$, for the case of small genus g, and allowing the holonomy around a fixed point to be any matrix of $\mathrm{SL}(2,\mathbb{C})$, that is $\mathrm{Id}\,, -\,\mathrm{Id}\,$, diagonalisable, or of either of the two Jordan types.

For this, we introduce a new geometric technique, based on stratifying the space of representations, and on the analysis of the behaviour of the E-polynomial under fibrations.

1. INTRODUCTION

Let X be a smooth complex projective curve of genus $g \geq 1$, and let G be a complex reductive group. The G-character variety of X is defined as the moduli space of semisimple representations of $\pi_1(X)$ into G. The topology of G-character varieties has been studied extensively in the past ten years. The main purpose was to prove certain conjectures on the mirror symmetry phenomena exhibited in the non-abelian Hodge theory of a curve.

The G-character variety is the space

$$\mathcal{M}(G) = \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \mid \prod_{i=1}^{g}[A_i, B_i] = \mathrm{Id}\,\}//G.$$

For complex linear groups $G = \mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C})$, the representations of $\pi_1(X)$ into G can be understood as G-local systems $E \to X$, hence defining a flat bundle E which has $\deg E = 0$. A natural generalization consists of allowing bundles E of non-zero degree d. The G-local systems on X correspond to representations $\rho : \pi_1(X \setminus \{p_0\}) \to G$, where $p_0 \in X$ is a fixed point, and $\rho(\gamma) = -\frac{d}{n}\,\mathrm{Id}\,$, γ a loop around p_0, giving rise to the moduli space

$$\mathcal{M}^d(G) = \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \mid \prod_{i=1}^{g}[A_i, B_i] = e^{2\pi i d/n}\,\mathrm{Id}\,\}//G.$$

The space $\mathcal{M}^d(G)$ is known in the literature as the Betti moduli space. This space is closely related to two other spaces: the De Rham and Dolbeault moduli spaces. The De

Rham moduli space $\mathcal{D}^d(G)$ is the moduli space parameterizing flat bundles, i.e., (E, ∇), where $E \to X$ is an algebraic bundle of degree d and rank n (and fixed determinant in the case $G = \mathrm{SL}(n, \mathbb{C})$), and ∇ is an algebraic connection on $X \setminus \{p_0\}$ with a logarithmic pole at p_0 with residue $-\frac{d}{n}\,\mathrm{Id}$. The Riemann-Hilbert correspondence [4, 24] gives an analytic isomorphism (but not an algebraic isomorphism) $\mathcal{D}^d(G) \cong \mathcal{M}^d(G)$.

The Dolbeault moduli space $\mathcal{H}^d(G)$ is the moduli space of G-Higgs bundles (E, Φ), consisting of an algebraic bundle $E \to X$ of degree d and rank n (also with fixed determinant in the case $G = \mathrm{SL}(n, \mathbb{C})$), and a homomorphism $\Phi : E \to E \otimes K_X$ (known as the Higgs field - in the case $G = \mathrm{SL}(n, \mathbb{C})$, the Higgs field has trace 0). In this situation the theory of harmonic bundles [3, 23] gives a homeomorphism $\mathcal{M}^d(G) \simeq \mathcal{H}^d(G)$.

When $\gcd(n, d) = 1$ these moduli spaces are smooth and the underlying differential manifold is hyperkähler, but the complex structures do not correspond under the previous isomorphisms. The cohomology of these moduli spaces has been computed in several particular cases, but mostly from the point of view of the Dolbeault moduli space $\mathcal{H}^d(G)$. Poincaré polynomials for $G = \mathrm{SL}(2, \mathbb{C})$ were computed by Hitchin in his seminal paper on Higgs bundles [11] and for $G = \mathrm{SL}(3, \mathbb{C})$ by Gothen in [15]. More recently the techniques involved in the computations by Gothen and Hitchin have been improved to give the Poincaré polynomials for $G = \mathrm{GL}(4, \mathbb{C})$ in [14].

Hausel and Thaddeus [10] gave a new perspective for the topological study of these varieties giving the first non-trivial example of the Strominger-Yau-Zaslow Mirror Symmetry using the so called Hitchin system [12] for the Dolbeault moduli space. They conjectured also (and checked for $G = \mathrm{SL}(2, \mathbb{C})$ and $G = \mathrm{SL}(3, \mathbb{C})$ using the previous results by Hitchin and Gothen) that a version of the topological Mirror Symmetry holds, i.e., some Hodge numbers $h^{p,q}$ of $\mathcal{H}^d(G)$ and $\mathcal{H}^d(G^L)$, for G and Langlands dual G^L, agree. It was noticed that while the mixed Hodge structures of $\mathcal{H}^d(G)$ and $\mathcal{D}^d(G)$ agree [10, Theorem 6.2] and are pure [17, 7] this is not the case for $\mathcal{M}^d(G)$. This fact motivates the study of E-polynomials of the G-character varieties.

Hausel and Rodriguez-Villegas started the computation of the E-polynomials of G-character varieties focusing on $G = \mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C})$ and $\mathrm{PGL}(n, \mathbb{C})$ using arithmetic methods inspired on the Weil conjectures. In [9] they obtained the E-polynomials of $\mathcal{M}^d(G)$ for $G = \mathrm{GL}(n, \mathbb{C})$ in terms of a simple generating function. Following the methods of Hausel and Rodriguez-Villegas, Mereb [16] studied the case of $\mathrm{SL}(n, \mathbb{C})$ giving an explicit formula for the E-polynomial in the case $G = \mathrm{SL}(2, \mathbb{C})$, while for $\mathrm{SL}(n, \mathbb{C})$ these polynomials are given in terms of a generating function. He proved also that the E-polynomials for $\mathrm{SL}(n, \mathbb{C})$ are palindromic and monic (so the $\mathrm{SL}(n, \mathbb{C})$-character varieties are connected).

Another direction of interest is the moduli spaces of parabolic bundles. Let X be a complex curve with s marked points $p_1, \ldots, p_s$. For $i \leq s$, fix conjugacy classes $\mathcal{C}_i \subset G$ given by semisimple elements. The corresponding Betti moduli space of parabolic

representations (or *parabolic character variety*) is

$$PM^{\mathcal{C}_1,\dots,\mathcal{C}_s}(G) \quad := \quad \{(A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_s) \in G^{2g+s} \mid$$
$$\prod_{i=1}^{g}[A_i, B_i] \prod_{j=1}^{s} C_j = \mathrm{Id}, \ C_j \in \mathcal{C}_j, \ j = 1, \dots, s\}//G.$$

In [25] Simpson proved that this space is analytically isomorphic to the moduli space of flat logarithmic G-connections and homeomorphic to a moduli space of Higgs bundles with parabolic structures at $p_1, \dots, p_s$. Very recently, formulae for the E-polynomials of the parabolic character varieties for $G = \mathrm{GL}(n, \mathbb{C})$ and generic semisimple $\mathcal{C}_1, \dots, \mathcal{C}_s$ have been obtained by Hausel, Letellier and Rodriguez-Villegas [8] using arithmetic methods. In addition, the Poincaré polynomials have been computed in the context of parabolic Higgs bundles for $G = \mathrm{SL}(2, \mathbb{C})$ in [1] and for $G = \mathrm{SL}(3, \mathbb{C})$ and $\mathrm{GL}(3, \mathbb{C})$ in [13].

In this paper we consider certain character varieties for the group $G = \mathrm{SL}(2, \mathbb{C})$. For any element ξ belonging to a conjugacy class $\mathcal{C} \subset \mathrm{SL}(2, \mathbb{C})$, we define

$$\mathcal{M}_\xi \quad = \quad \{(A_1, B_1, \dots, A_g, B_g) \in \mathrm{SL}(2, \mathbb{C})^{2g} \mid \prod_{i=1}^{g}[A_i, B_i] = \xi\}// \mathrm{Stab}(\xi)$$
$$= \quad \{(A_1, B_1, \dots, A_g, B_g) \in \mathrm{SL}(2, \mathbb{C})^{2g} \mid \prod_{i=1}^{g}[A_i, B_i] \in \mathcal{C}\}//G.$$

This space has the structure of an algebraic variety independent of the complex structure on X. When $\mathcal{C}$ is semisimple, $\mathcal{M}_\xi$ is a parabolic character variety as defined earlier, but we do not make such restriction on $\mathcal{C}$. Also we do not assume that $\mathcal{C}$ satisfies a genericity condition.

Our basic aim is to compute the E-polynomial (or Hodge-Deligne polynomial) of $\mathcal{M}_\xi$. For this purpose, we propose a geometric technique, which has the advantage of allowing us to deal with the cases in which ξ is of Jordan type. Our method depends on certain basic properties of E-polynomials, in particular the additive property (Proposition 2.3(i)), which allows us to compute these polynomials using stratifications. We also require methods for handling fibrations which are locally trivial in the analytic topology but not in the Zariski topology (Propositions 2.4, 2.6 and 2.10). Our main results are explicit formulae for the E-polynomials for $g = 1$ and $g = 2$ for the five different moduli spaces corresponding to ξ being Id, $-\mathrm{Id}$, diagonal with different eigenvalues $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, and of either of the two Jordan types $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. These results can be summarised as follows:

Theorem 1.1. *Let X be a complex curve of genus $g = 1$. Then the E-polynomials of $\mathcal{M}_\xi$ are as follows:*

$$
\begin{aligned}
e(\mathcal{M}_{\mathrm{Id}}) &= q^2 + 1\,, \\
e(\mathcal{M}_{-\mathrm{Id}}) &= 1\,, \\
e(\mathcal{M}_{J_+}) &= q^2 - 2q - 3\,, \\
e(\mathcal{M}_{J_-}) &= q^2 + 3q\,, \\
e(\mathcal{M}_{\xi_\lambda}) &= q^2 + 4q + 1\,,
\end{aligned}
$$

where $q = uv$, $e(\mathcal{M}_\xi) \in \mathbb{Z}[u,v]$.

Theorem 1.2. *Let X be a complex curve of genus $g = 2$. Then*

$$
\begin{aligned}
e(\mathcal{M}_{\mathrm{Id}}) &= q^6 + 17q^4 - 26q^3 + 67q^2 + 26q - 65\,. \\
e(\mathcal{M}_{-\mathrm{Id}}) &= q^6 - 2q^4 - 30q^3 - 2q^2 + 1\,, \\
e(\mathcal{M}_{J_+}) &= q^8 - 3q^6 - 4q^5 - 39q^4 - 4q^3 - 15q^2\,, \\
e(\mathcal{M}_{J_-}) &= q^8 - 3q^6 + 15q^5 + 6q^4 + 45q^3\,, \\
e(\mathcal{M}_{\xi_\lambda}) &= q^8 + q^7 - 2q^6 + 13q^5 - 26q^4 + 13q^3 - 2q^2 + q + 1\,.
\end{aligned}
$$

The layout of the paper is as follows. Section 2 consists of a review of the basic properties of E-polynomials, including a description of the methods which we use for computing these polynomials. This is followed in section 3 by a computation of the E-polynomials for the groups $\mathrm{GL}(2,\mathbb{C})$, $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PGL}(2,\mathbb{C})$ and also for the stratification of $\mathrm{SL}(2,\mathbb{C})$ by conjugacy class types. In section 4, we stratify $\mathrm{SL}(2,\mathbb{C})^2$ and compute the E-polynomials for each stratum. In section 5, we use these computations to prove Theorem 1.1. It turns out that we can do slightly better and compute all Hodge numbers of $\mathcal{M}_\xi$ (except in the case $\xi = J_-$).

Sections 6 and 7 are key sections in which we introduce the *Hodge monodromy representation*. For this, we define first

$$
\overline{X}_4 := \left\{ (A, B, \lambda) \,|\, AB = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} BA, \text{ for some } \lambda \neq 0, \pm 1 \right\}.
$$

The map $(A, B, \lambda) \mapsto \lambda$ fibres this space over $\mathbb{C} \setminus \{0, \pm 1\}$, but the fibration is only analytically locally trivial. The idea of the Hodge monodromy representation is that it encodes in a convenient form all the information on the E-polynomials of both the fibre and the total space. We compute this for $\overline{X}_4$ in section 6 and extend it to $\overline{X}_4/\mathbb{Z}_2$ in section 7 (here $\mathbb{Z}_2$ acts on $\overline{X}_4$ by interchange of basis vectors). In sections 8–12, we prove Theorem 1.2.

Our arguments can certainly be extended to higher genus, but the computations become rapidly more complicated as g increases. Another advantage of a geometrical method of this type is that it may allow one to compute the motives of the character varieties. This will be pursued in future work.

2. E-POLYNOMIALS

In this section we recall some properties of E-polynomials, which include methods for computing them using stratifications and fibrations. We start by reviewing the Hodge theory of algebraic varieties over $\mathbb{C}$.

A *pure Hodge structure of weight k* consists of a finite dimensional complex vector space H with a real structure, and a decomposition

$$H = \bigoplus_{k=p+q} H^{p,q}$$

such that $H^{q,p} = \overline{H^{p,q}}$, the bar meaning complex conjugation on H. A Hodge structure of weight k gives rise to the so-called *Hodge filtration* which is a descending filtration $F^p = \bigoplus_{s \geq p} H^{s,k-s}$. We define $\mathrm{Gr}_F^p(H) := F^p/F^{p+1} = H^{p,k-p}$.

A *mixed Hodge structure* consists of a finite dimensional complex vector space H with a real structure, an ascending (weight) filtration $\ldots \subset W_{k-1} \subset W_k \subset \ldots \subset H$ (defined over $\mathbb{R}$) and a descending (Hodge) filtration F such that F induces a pure Hodge structure of weight k on each $\mathrm{Gr}_k^W(H) = W_k/W_{k-1}$. We define

$$H^{p,q} := \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W(H)$$

and write $h^{p,q}$ for the *Hodge number*

$$h^{p,q} := \dim H^{p,q}.$$

Let Z be any quasi-projective algebraic variety (maybe non-smooth or non-compact). By [5], the cohomology groups $H^k(Z) := H^k(Z; \mathbb{C})$ and the cohomology groups with compact support $H_c^k(Z)$ are endowed with mixed Hodge structures. Moreover, these Hodge structures are pure of weight k if Z is smooth and projective. We define the *Hodge numbers* by

$$
\begin{aligned}
h^{k,p,q}(Z) &= h^{p,q}(H^k(Z)) = \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^k(Z), \\
h_c^{k,p,q}(Z) &= h^{p,q}(H_c^k(Z)) = \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^k(Z).
\end{aligned}
$$

When Z is smooth of dimension n, Poincaré duality provides an equality $h^{k,p,q} = h_c^{2n-k,n-p,n-q}$.

We consider the Euler characteristic

$$\chi_c^{p,q}(Z) = \sum_k (-1)^k h_c^{k,p,q}(Z)$$

and define the E-polynomial by

$$e(Z) = e(Z)(u,v) := \sum_{p,q} \chi_c^{p,q}(Z) u^p v^q.$$

Hence, when Z is smooth and projective of dimension n, Poincaré duality implies that

$$e(Z)(u,v) = (uv)^n e(Z)\left(\frac{1}{u}, \frac{1}{v}\right).$$

Remark 2.1. Note that, when $h_c^{k,p,q} = 0$ for $p \neq q$, the polynomial $e(Z)$ depends only on the product uv. This will happen in all the cases that we shall investigate here. In this situation, it is conventional to use the variable

$$q := uv.$$

If this happens, we shall say that the variety is *of balanced type*.

Remark 2.2. In the literature $e(Z)(-u, -v)$ is sometimes referred to as the Hodge-Deligne polynomial of Z. The polynomial $H_c(Z)(u, v, t) := \sum_{p,q,k} h_c^{p,q,k} u^p v^q t^k$, which encodes all the Hodge numbers, is known as the (compactly supported) *mixed Hodge polynomial*. (In the case of balanced type, we write also $H_c(Z)(q, t)$.) This generalises both the E-polynomial and the Poincaré polynomial. The E-polynomial is especially amenable to computation precisely because it is defined by an Euler characteristic. The preprint [8] is built around a conjectural formula for $H_c(P\mathcal{M}^{\mathcal{C}_1,\ldots,\mathcal{C}_s}(\mathrm{GL}(n, \mathbb{C})))$ in the case of generic semisimple conjugacy classes.

The key property of E-polynomials that permits their calculation is that they are additive for stratifications of Z. This and some other properties that we shall need are summarised in the following propositions.

Proposition 2.3. *Let Z be an algebraic variety over an algebraically closed field.*

   (i) *If $Z = \bigsqcup_{i=1}^n Z_i$, where all Z_i are locally closed in Z, then $e(Z) = \sum_{i=1}^n e(Z_i)$,*
   (ii) *$e(\mathbb{C}^n) = q^n$,*
   (iii) *$e(\mathbb{P}^{n-1}) = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$,*
   (iv) *$e(\mathbb{C}^n \setminus \{0\}) = q^n - 1$.*

Proof. (i) is [18, Theorem 2.2] (see also [5]).

   (ii) and (iii) follow from the known cohomological structure of $\mathbb{C}^n$ and $\mathbb{P}^{n-1}$.

   (iv) follows from (i) and (ii).   □

Proposition 2.4. *Suppose that B is connected and $\pi : Z \to B$ is an algebraic fibre bundle with fibre F (not necessarily locally trivial in the Zariski topology) and that the action of $\pi_1(B)$ on $H_c^*(F)$ is trivial. Then $e(Z) = e(F)e(B)$.*

Proof. The Leray spectral sequence has E_2-term equal to $H_c^*(B) \otimes H_c^*(F)$ and abuts to $H_c^*(Z)$. This spectral sequence respects mixed Hodge structures [21, Corollary 6.6], so taking Euler characteristics, we have that $e(F)e(B) = e(Z)$.   □

Remark 2.5. The hypotheses of Proposition 2.4 hold in particular in the following cases:

   • B is irreducible and π is locally trivial in the Zariski topology. (The key point here is that there is an open covering $\bigcup U_k$ of B such that the monodromy action of each $\pi_1(U_k)$ is trivial and $\bigcap U_k$ is connected, allowing us to apply van Kampen's theorem.)
   • $\pi : Z \to B$ is a morphism between quasi-projective varieties which is a locally trivial fibre bundle for the usual topology and F is a projective space $\mathbb{P}^N$ [19, Lemma 2.4].

- π is a principal G-bundle with G a connected algebraic group. (In this case, any loop in B based at x_0 lifts to an automorphism of the fibre F_{x_0} induced by the action of an element of G; since G is connected, this isomorphism is homotopic to the identity.)

We will also require a method of calculating $e(Z/\mathbb{Z}_2)$ when $\mathbb{Z}_2$ acts on Z. In these circumstances, we denote by $H_c^*(Z)^\pm$ the $\pm$-invariant part of $H_c^*(Z)$, with a corresponding notation

$$e(Z)^\pm := e(H_c^*(Z)^\pm).$$

Note that

$$
\begin{aligned}
e(Z)^+ &= e(Z/\mathbb{Z}_2), \\
e(Z)^- &= e(Z) - e(Z)^+.
\end{aligned}
$$

For example, if $Z = \mathbb{C}^*$ with the action $\lambda \mapsto \lambda^{-1}$, then $Z/\mathbb{Z}_2 \cong \mathbb{C}$, and

(1) $$e(\mathbb{C}^*)^+ = q, \quad e(\mathbb{C}^*)^- = -1.$$

Proposition 2.6. *Let*

$$
\begin{array}{ccc}
F & \xrightarrow{\;=\;} & F \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Z/\mathbb{Z}_2 \\
\widetilde{\pi} \downarrow & & \downarrow \pi \\
B & \xrightarrow{\;2:1\;} & B/\mathbb{Z}_2
\end{array}
$$

be a diagram of fibrations, where B is irreducible, $\widetilde{\pi}$ and π are smooth morphisms, $\widetilde{\pi}$ is a locally trivial fibration in the usual topology and the monodromy action of $\pi_1(B)$ on $H_c^*(F)$ is trivial. Then

(2) $$e(Z/\mathbb{Z}_2) = e(F)^+ e(B)^+ + e(F)^- e(B)^-,$$

where $e(F)^\pm$ are defined by an action of $\mathbb{Z}_2$ on $H_c^*(F)$, which is compatible with the Hodge structure.

Proof. By hypothesis, the monodromy action of $\pi_1(B/\mathbb{Z}_2)$ on $H_c^*(F)$ factors through $\mathbb{Z}_2$ and therefore induces a splitting $H_c^*(F) = H_c^*(F)^+ \oplus H_c^*(F)^-$. We consider the spectral sequences of the fibrations π and $\widetilde{\pi}$ and the restriction map between their E_2-terms:

$$H_c^i(B/\mathbb{Z}_2, H_c^j(F)) \longrightarrow H_c^i(B, H_c^j(F)) = H_c^i(B) \otimes H_c^j(F).$$

This induces an isomorphism

$$H_c^i(B/\mathbb{Z}_2, H_c^j(F)) \xrightarrow{\cong} (H_c^i(B) \otimes H_c^j(F))^+ = (H_c^i(B)^+ \otimes H_c^j(F)^+) \oplus (H_c^i(B)^- \otimes H_c^j(F)^-).$$

The first spectral sequence abuts to $H_c^*(Z/\mathbb{Z}_2)$ and the isomorphism is compatible with Hodge structures. Since all differentials in the spectral sequence go between two groups for which $i + j$ has opposite parity, it follows that any degeneration in the spectral sequence leads to the cancellation of equal terms in the Hodge polynomial. The result follows. $\square$

Remark 2.7. In the situation of Proposition 2.6, we have an action of $\mathbb{Z}_2$ on Z, and we can write

$$
\begin{aligned}
e(Z)^+ &= e(Z/\mathbb{Z}_2) = e(F)^+ e(B)^+ + e(F)^- e(B)^- \\
e(Z)^- &= e(Z) - e(Z/\mathbb{Z}_2) = e(F)^+ e(B)^- + e(F)^- e(B)^+.
\end{aligned}
$$

For the next proposition, recall that a variety is of balanced type if all Hodge numbers $h_c^{k,p,q}$ with $p \neq q$ are zero.

Proposition 2.8.   (i) *Let $Y = Z \sqcup U$, where $Z \subset Y$ is closed. Then, if two of the spaces Y, Z, U are of balanced type, so is the third.*
  (ii) *If $\pi : Z \to B$ is as in Proposition 2.4 and B and F are of balanced type, so is Z.*
 (iii) *If $\mathbb{Z}_2$ acts on Z and Z is of balanced type, so is $Z/\mathbb{Z}_2$.*

Proof. (i) Consider the long exact sequence of mixed Hodge structures

$$
\ldots \to H_c^k(U) \to H_c^k(Y) \to H_c^k(Z) \to H_c^{k+1}(U) \to \ldots
$$

and take the (p,q)-components. Using the fact that the functor $H \mapsto H^{p,q}$ is exact [5], we get an exact sequence

$$
\ldots \to H_c^{k,p,q}(U) \to H_c^{k,p,q}(Y) \to H_c^{k,p,q}(Z) \to H_c^{k+1,p,q}(U) \to \ldots
$$

The result follows.

(ii) This follows from the fact that the Leray spectral sequence respects mixed Hodge structures (see the proof of Proposition 2.4).

(iii) is obvious since $h_c^{k,p,q}(Z/\mathbb{Z}_2) = h_c^{k,p,q}(Z)^+$. $\square$

Remark 2.9. The effect of this proposition is that all the spaces we consider will be of balanced type.

We shall also deal with fibrations which are locally trivial in the analytic topology, for which we want to compute the E-polynomial of the total space. Consider a fibration

$$
(3) \qquad\qquad F \longrightarrow Z \xrightarrow{\;\pi\;} B = \mathbb{C} \setminus \{p_1, \ldots, p_\ell\},
$$

with basis the complex line minus a collection of points, and with fibre F which is a variety of balanced type. The fibration defines a local system

$$
\mathcal{H}_c^k \to B
$$

whose fibres are the cohomology groups $H_c^k(F_b)$, $b \in B$, $F_b := \pi^{-1}(b)$. These fibres possess mixed Hodge structures, thus defining a variation of Hodge structures. The subspaces $W_t(H_c^k(F_b))$, $b \in B$, define a subbundle. As this is defined over the rational numbers, it is locally constant, i.e., a local subsystem. Therefore, the holonomy preserves W_t, and hence it induces a variation of Hodge structures on the pure Hodge structure $\mathrm{Gr}_t^W H_c^k(F_b)$. Now we use the assumption that F is of balanced type. Hence we must have $t = 2p$ and $\mathrm{Gr}_t^W H_c^k(F_b) = H_c^{k,p,p}(F_b)$.

Fix a base point $b_0 \in B$, and write $F = F_{b_0}$. Associated to the fibration, there is a *monodromy representation*

$$
\rho : \pi_1(B, b_0) \to \mathrm{GL}(H_c^{k,p,p}(F)) \,.
$$

Now note that ρ factors through the homology $H_1(B)$. For if D is a 2-simplex, the local system is defined over D. Therefore the monodromy over the boundary of ∂D is trivial. Write

$$\Gamma := H_1(B) = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_\ell \rangle \,,$$

where γ_i is a loop around b_i. Then the monodromy representation is given by

$$\rho : H_1(B) \to \mathrm{GL}(H_c^{k,p,p}(F)) \,.$$

Consider the representation ring $R(\Gamma)$. Then the $H_c^{k,p,p}(F)$ are modules over $R(\Gamma)$. In particular, there is a well defined element

$$(4) \qquad R(Z) := \sum (-1)^k H_c^{k,p,p}(F)\, q^p \in R(\Gamma)[q]\,,$$

which we shall call the *Hodge monodromy representation* of π.

We want to recover $e(Z)$ from the information of the Hodge monodromy representation. We introduce the notation $e(F)^{inv}$ to denote the E-polynomial of the invariant part of the monodromy representation.

Proposition 2.10. *The E-polynomial of the total space is given by*

$$e(Z) = (q-1)\, e(F)^{inv} - (\ell - 1)\, e(F)\,.$$

Proof. The Leray spectral sequence of the fibration (3) has E_2-term

$$(5) \qquad\qquad E_2^{l,m} = H_c^l(B, H_c^m(F))$$

and abuts to $H_c^k(Z)$, where $k = l + m$. We have non-zero terms only for $l = 1, 2$.

The mixed Hodge structure of $B = \mathbb{C} \setminus \{p_1, \ldots, p_\ell\}$ has

$$H_c^{2,1,1}(B) = \mathbb{C}, \qquad H_c^{1,0,0}(B) = \mathbb{C}\langle \gamma_1, \ldots, \gamma_\ell \rangle \cong \mathbb{C}^\ell$$

(and $e(B) = q - \ell$).

To compute (5), recall that the cohomology of the local system $H_c^m(F)$ is given by the cohomology of the complex

$$d : \bigoplus_{i=1}^{\ell} \gamma_i\, H_c^m(F) \quad\longrightarrow\quad H_c^m(F)$$

$$\sum \gamma_i x_i \quad\mapsto\quad \sum (\rho(\gamma_i)(x_i) - x_i)\,.$$

We realize, a posteriori, that this map d has degree $(1, 1, 1)$ w.r.t. the mixed Hodge structures.

Decompose $H_c^*(F)$ into irreducible representations, and take one of them $V \subset H_c^*(F)$. Then we have the map

$$d : V^\ell \quad\longrightarrow\quad V$$

$$\sum \gamma_i x_i \quad\mapsto\quad \sum (\rho(\gamma_i)(x_i) - x_i)\,,$$

which is either zero if the representation is trivial, or surjective if it is non-trivial. In the first case, the contribution to the E-polynomial is $q\, e(V) - e(V^\ell) = (q - \ell)\, e(V)$. In the second case, the contribution is $0 - (e(V^\ell) - e(V)) = -(\ell - 1)\, e(V)$.

So the E-polynomial is

$$
\begin{aligned}
e(Z) &= e(E_2^{**}) = (q - \ell)e(H_c^*(F)^{inv}) - (\ell - 1)(e(H_c^*(F)) - e(H_c^*(F)^{inv})) \\
&= (q - 1)\, e(F)^{inv} - (\ell - 1)e(F)\,.
\end{aligned}
$$

$\square$

3. E-POLYNOMIALS OF $GL(2,\mathbb{C})$ AND $SL(2,\mathbb{C})$

We start with the following very simple computation.

Lemma 3.1. *The E-polynomials for the algebraic groups $GL(2,\mathbb{C})$, $SL(2,\mathbb{C})$ and $PGL(2,\mathbb{C})$ are*

(i) $e(GL(2,\mathbb{C})) = q(q+1)(q-1)^2 = q^4 - q^3 - q^2 + q$,
(ii) $e(PGL(2,\mathbb{C})) = q(q+1)(q-1) = q^3 - q$,
(iii) $e(SL(2,\mathbb{C})) = q(q+1)(q-1) = q^3 - q$.

Proof. (i) Consider the following locally trivial fibration in the Zariski topology

$$
\begin{array}{ccc}
\mathbb{C}^2 \setminus \mathbb{C} \longrightarrow GL(2,\mathbb{C}) & \longrightarrow & \mathbb{C}^2 \setminus \{0\} \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & (a,b)
\end{array}
$$

It follows from Propositions 2.3 and 2.4 that

$$
e(GL(2,\mathbb{C})) = e(\mathbb{C}^2 \setminus \mathbb{C})e(\mathbb{C}^2 \setminus \{0\}) = (q^2 - q)(q^2 - 1).
$$

(ii) Notice that

$$
\mathbb{C}^* \longrightarrow GL(2,\mathbb{C}) \longrightarrow PGL(2,\mathbb{C})
$$

satisfies the hypotheses of Proposition 2.4. So

$$
e(PGL(2,\mathbb{C})) = e(GL(2,\mathbb{C}))/e(\mathbb{C}^*) = q(q^2 - 1).
$$

(iii) As in (i), we have a fibration

$$
\begin{array}{ccc}
\mathbb{C} \longrightarrow SL(2,\mathbb{C}) & \longrightarrow & \mathbb{C}^2 \setminus \{0\} \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & (a,b)
\end{array}
$$

which is locally trivial in the Zariski topology. Hence $e(SL(2,\mathbb{C})) = q(q^2 - 1)$. $\square$

3.1. E-polynomial of coset spaces of $GL(2,\mathbb{C})$. Let

$$
\begin{aligned}
D &:= \{\text{diagonal matrices in } GL(2,\mathbb{C})\} \cong \mathbb{C}^* \times \mathbb{C}^*, \\
U &:= \left\{ \begin{pmatrix} \mu & \nu \\ 0 & \mu \end{pmatrix} \in GL(2,\mathbb{C}) \right\} \cong \mathbb{C}^* \times \mathbb{C}.
\end{aligned}
$$

Note that $\mathbb{Z}_2$ acts on $GL(2,\mathbb{C})$ by switching basis vectors, i.e., multiplication by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and that this action descends to $GL(2,\mathbb{C})/D$.

Proposition 3.2. (i) $e(GL(2,\mathbb{C})/D) = q(q+1) = q^2 + q$,

(ii) $e(\mathrm{GL}(2,\mathbb{C})/D)^+ = q^2$ *and* $e(\mathrm{GL}(2,\mathbb{C})/D)^- = q$,
(iii) $e(\mathrm{GL}(2,\mathbb{C})/U) = q^2 - 1$.

Proof. (i) The map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ([a:b],[c:d])$ defines an isomorphism $\mathrm{GL}(2,\mathbb{C})/D \cong$
$(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$, where Δ denotes the diagonal. The E-polynomial is therefore $e(\mathbb{P}^1)^2 - e(\mathbb{P}^1) = q(q+1)$.

(ii) By the above,

$$(\mathrm{GL}(2,\mathbb{C})/D)/\mathbb{Z}_2 \cong ((\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta)/\mathbb{Z}_2 = \mathbb{P}^2 \setminus C,$$

where C is a conic. So

$$e(\mathrm{GL}(2,\mathbb{C})/D)^+ = e((\mathrm{GL}(2,\mathbb{C})/D)/\mathbb{Z}_2) = e(\mathbb{P}^2) - e(C) = e(\mathbb{P}^2) - e(\mathbb{P}^1) = q^2.$$

(iii) We have a fibration

$$\mathbb{C}^* \longrightarrow \mathrm{GL}(2,\mathbb{C})/U \longrightarrow \mathbb{P}^1,$$

given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [c:d]$. This fibration is locally trivial in the Zariski topology, so

$$e(\mathrm{GL}(2,\mathbb{C})/U) = e(\mathbb{C}^*)e(\mathbb{P}^1) = (q-1)(q+1).$$

$\square$

Corollary 3.3. *The action of $\mathbb{Z}_2$ descends to $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{GL}(2,\mathbb{C})/\mathbb{C}^*$. Moreover*
$$e(\mathrm{PGL}(2,\mathbb{C}))^+ = q^3 - q, \qquad e(\mathrm{PGL}(2,\mathbb{C}))^- = 0.$$

Proof. To calculate $\mathrm{PGL}(2,\mathbb{C})/\mathbb{Z}_2$, we use the diagram of fibrations

$$
\begin{array}{ccc}
\mathbb{C}^* & \overset{=}{\longrightarrow} & \mathbb{C}^* \\
\downarrow & & \downarrow \\
\mathrm{GL}(2,\mathbb{C})/\mathbb{C}^* & \longrightarrow & (\mathrm{GL}(2,\mathbb{C})/\mathbb{C}^*)/\mathbb{Z}_2 \\
\downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} \\
\mathrm{GL}(2,\mathbb{C})/D & \overset{2:1}{\longrightarrow} & (\mathrm{GL}(2,\mathbb{C})/D)/\mathbb{Z}_2
\end{array}
$$

and apply Proposition 2.6 with $B = \mathrm{GL}(2,\mathbb{C})/D$ and $F = \mathbb{C}^*$, where the inclusion of the fibre is given by $\mu \mapsto \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$. From Proposition 3.2, we have $e(B)^+ = q^2$, $e(B)^- = q$, while the action in the fibre is given by $\mu \mapsto \mu^{-1}$, since $\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \sim \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. So $e(F)^+ = q$, $e(F)^- = -1$, and

$$e(\mathrm{PGL}(2,\mathbb{C})/\mathbb{Z}_2) = e(F)^+ e(B)^+ + e(F)^- e(B)^- = q^3 - q = e(\mathrm{PGL}(2,\mathbb{C})).$$

The remaining assertion is clear. $\square$

3.2. **E-polynomials for a stratification of** $SL(2, \mathbb{C})$**.** We use the $GL(2, \mathbb{C})$-action on $SL(2, \mathbb{C})$ by conjugation to decompose the space $W = SL(2, \mathbb{C})$ into strata whose E-polynomials we compute. The strata we use are orbit spaces:

- $W_0 :=$ orbit through $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- $W_1 :=$ orbit through $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.
- $W_2 :=$ orbit through $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- $W_3 :=$ orbit through $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.
- $W_4 := \{A \in SL(2, \mathbb{C}) \mid \operatorname{Tr}(A) \neq \pm 2\}$.

The stratum W_4 can also be described as the union of the orbits through $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for $\lambda \in \mathbb{C} \backslash \{0, \pm 1\}$.

3.2.1. W_0 *and* W_1. These strata are single points, so
$$e(W_0) = e(W_1) = 1.$$

3.2.2. W_2 *and* W_3. The stabiliser of $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where $\lambda = \pm 1$, is the subgroup U of Proposition 3.2. So
$$e(W_2) = e(W_3) = q^2 - 1$$
by Proposition 3.2(iii).

3.2.3. W_4. The stabiliser of $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \neq \pm 1$, is the diagonal group D. We therefore have a diagram

$$
\begin{array}{ccc}
GL(2, \mathbb{C})/D & \xrightarrow{=} & GL(2, \mathbb{C})/D \\
\downarrow & & \downarrow \\
\widetilde{W_4} & \longrightarrow & W_4 \\
\downarrow & & \downarrow \\
\mathbb{C} \backslash \{0, \pm 1\} & \xrightarrow{2:1} & \mathbb{C} \backslash \{\pm 2\}.
\end{array}
$$

where $\widetilde{W_4} := \{(\lambda, P) \mid \lambda \in \mathbb{C} \backslash \{0, \pm 1\}, P \in GL(2, \mathbb{C})/D\}$. The middle map is given by $(\lambda, P) \mapsto P \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} P^{-1}$, the bottom map is given by $\lambda \mapsto \lambda + \lambda^{-1}$ and the map $W_4 \to \mathbb{C} \backslash \{\pm 2\}$ is the trace map, Tr. Writing $F = GL(2, \mathbb{C})/D$ and $B = \mathbb{C} \backslash \{0, \pm 1\}$, we have $e(B)^+ = q - 2$, $e(B)^- = -1$ and, from Proposition 3.2, $e(F)^+ = q^2$, $e(F)^- = q$. So, by Proposition 2.6,
$$e(W_4) = e(F)^+ e(B)^+ + e(F)^- e(B)^- = q^2(q - 2) - q = q^3 - 2q^2 - q.$$

One may note also that $e(\widetilde{W_4}) = q(q+1)(q-3) = q^3 - 2q^2 - 3q$.

Remark 3.4. Note that

$$e(W_0) + e(W_1) + e(W_2) + e(W_3) + e(W_4) = q^3 - q = e(\mathrm{SL}(2, \mathbb{C}))\,,$$

as expected.

4. E-POLYNOMIALS OF $\mathrm{SL}(2, \mathbb{C})^2$ USING ORBIT SPACES

Let us consider the map

$$\begin{aligned}
f : \mathrm{SL}(2, \mathbb{C})^2 &\longrightarrow \mathrm{SL}(2, \mathbb{C}) \\
(A, B) &\mapsto [A, B] = ABA^{-1}B^{-1}
\end{aligned}$$

Note that $\mathrm{PGL}(2, \mathbb{C})$ acts on both spaces and f is equivariant. We stratify $X = \mathrm{SL}(2, \mathbb{C})^2$ as follows

- $X_0 := f^{-1}(W_0)$,
- $X_1 := f^{-1}(W_1)$,
- $X_2 := f^{-1}(W_2)$,
- $X_3 := f^{-1}(W_3)$,
- $X_4 := f^{-1}(W_4)$.

Hence $X = \bigsqcup_{i=0}^{4} X_i$, so again we will test that

$$e(X) = e(\mathrm{SL}(2, \mathbb{C}))^2 = \sum_i e(X_i) = q^2(q+1)^2(q-1)^2 = q^6 - 2q^4 + q^2.$$

It will be convenient to study the varieties $f^{-1}(\xi)$ for fixed ξ and we define accordingly

- $\overline{X}_0 := f^{-1}(\mathrm{Id}\,) = X_0$,
- $\overline{X}_1 := f^{-1}(-\mathrm{Id}\,) = X_1$,
- $\overline{X}_2 := f^{-1}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$,
- $\overline{X}_3 := f^{-1}\left(\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right)$,
- $\overline{X}_{4,\lambda} := f^{-1}\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right)$, where $\lambda \neq 0, \pm 1$.

It will also be convenient to define

- $\overline{X}_4 := \left\{ (A, B, \lambda) \,|\, AB = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} BA, \text{ for some } \lambda \neq 0, \pm 1 \right\}.$

4.1. **E-polynomial for X_0.** We decompose

$$X_0 = (\mathrm{SL}(2, \mathbb{C}) \times \{\pm \mathrm{Id}\,\}) \cup (\{\pm \mathrm{Id}\,\} \times \mathrm{SL}(2, \mathbb{C})) \cup X_0'\,,$$

where

$$X_0' = \{(A, B) \in X \mid AB = BA;\ A, B \neq \pm \mathrm{Id}\,\}.$$

Hence

$$e(X_0) = 4\, e(\mathrm{SL}(2, \mathbb{C})) - 4 + e(X_0').$$

We subtract 4 because $(\mathrm{SL}(2,\mathbb{C}) \times \{\pm \mathrm{Id}\}) \cap (\{\pm \mathrm{Id}\} \times \mathrm{SL}(2,\mathbb{C}))$ consists of 4 points, $\{\pm \mathrm{Id}\} \times \{\pm \mathrm{Id}\}$, which we have counted twice in $(\mathrm{SL}(2,\mathbb{C}) \times \{\pm \mathrm{Id}\}) \cup (\{\pm \mathrm{Id}\} \times \mathrm{SL}(2,\mathbb{C}))$.

Furthermore we split X_0' into the following subsets $X_0' = X_0'^a \sqcup X_0'^b$

- $X_0'^a := \{(A,B) \in X_0' | AB = BA; \; A, B \neq \pm \mathrm{Id}, \mathrm{Tr}\, A \neq \pm 2\}$. This set consists of pairs of matrices $(A,B) \sim \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \right)$, where $\lambda, \mu \in \mathbb{C}\backslash\{0, \pm 1\}$.

- $X_0'^b := \{(A,B) \in X_0' | AB = BA; \; A, B \neq \pm \mathrm{Id}, \mathrm{Tr}\, A = \pm 2\}$. This set consists of pairs of matrices $(A,B) \sim \left(\begin{pmatrix} \lambda & b \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & y \\ 0 & \mu^{-1} \end{pmatrix} \right)$, where $\lambda = \pm 1, \mu = \pm 1$ and $y, b \in \mathbb{C}^*$.

We use the following diagram to compute $e(X_0'^a)$:

$$
\begin{array}{ccc}
F = \mathbb{C}\backslash\{0, \pm 1\} & \xrightarrow{\;=\;} & \mathbb{C}\backslash\{0, \pm 1\} \\
\downarrow & & \downarrow \\
\widetilde{X}_0'^a & \longrightarrow & X_0'^a \\
\downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} \\
B = \widetilde{W}_4 & \xrightarrow{\;2:1\;} & W_4
\end{array}
$$

where $W_4, \widetilde{W}_4$ are as in subsection 3.2.3, the map π is given by $(A,B) \mapsto A$ and $\widetilde{X}_0'^a = (\mathbb{C} \backslash \{0, \pm 1\}) \times \widetilde{W}_4$. The bottom map is $(\lambda, P) \mapsto P \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} P^{-1}$ and the middle map is

$$
(\mu, \lambda, P) \mapsto \left(P \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} P^{-1}, P \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} P^{-1} \right).
$$

The maps are $2:1$ and reflect the action of $\mathbb{Z}_2$ consisting of interchanging the eigenvalues and eigenvectors. The E-polynomials of W_4 and $\widetilde{W}_4$ have already been computed in subsection 3.2.3, so we have $e(B)^+ = q^3 - 2q^2 - q$, $e(B)^- = -2q$, while $e(F)^+ = q - 2$, $e(F)^- = -1$ since the action on F is given by $\lambda \mapsto \lambda^{-1}$. So

$$
e(X_0'^a) = (q^3 - 2q^2 - q)(q - 2) + (-2q)(-1) = q^4 - 4q^3 + 3q^2 + 4q.
$$

Now we compute $e(X_0'^b)$. Note that $X_0'^b$ has 4 components, according to the signs of $\mathrm{Tr}\, A$ and $\mathrm{Tr}\, B$. We analyse one of these components, say the one given by $\lambda = \mu = 1$, the others being analogous. Note that the stabiliser of A (which is the same as the stabiliser of B) is the subgroup U of Proposition 3.2. It is easy to check that each orbit contains exactly one element of the form

$$
(A,B) = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right),
$$

so

$$
e(X_0'^b) = 4\, e(\mathrm{GL}(2,\mathbb{C})/U) e(\mathbb{C}^*) = 4(q^2 - 1)(q - 1) = 4q^3 - 4q^2 - 4q + 4\,.
$$

Hence
$$e(X_0') = e(X_0'^a) + e(X_0'^b) = q^4 - q^2 + 4$$
and
$$\begin{aligned} e(X_0) &= 4\,e(\mathrm{SL}(2,\mathbb{C})) - 4 + e(X_0') \\ &= 4q^3 - 4q - 4 + q^4 - q^2 + 4 = q^4 + 4q^3 - q^2 - 4q\,. \end{aligned}$$

4.2. E-polynomial for X_1.
$$X_1 = \{(A, B) \in X \mid AB = -BA\}.$$

For $(A, B) \in X_1$,
$$\mathrm{Tr}\ B = \mathrm{Tr}\,(ABA^{-1}) = \mathrm{Tr}\,(-B) = -\,\mathrm{Tr}\ B$$
by the properties of the trace. Hence $\mathrm{Tr}\ B = 0$ and analogously $\mathrm{Tr}\ A = 0$. The characteristic equations of these matrices are then $x^2 + 1 = 0$, therefore $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ in some basis. The equation $AB = -BA$ implies that, in the same basis, $B = \begin{pmatrix} 0 & y \\ -y^{-1} & 0 \end{pmatrix}$ where $y \neq 0$. Conjugating by a diagonal matrix, we can arrange that $y = 1$. Finally note that $\mathrm{Stab}(A, B) = \mathbb{C}^*$. So
$$e(X_1) = e(\mathrm{PGL}(2,\mathbb{C})) = q(q+1)(q-1) = q^3 - q.$$

4.3. E-polynomials for X_2, $\overline{X}_2$.
$$\begin{aligned} X_2 &= \{(A, B) \in X \mid AB \sim \xi BA\}, \\ \overline{X}_2 &= \{(A, B) \in X \mid AB = \xi BA\}, \end{aligned}$$
where $\xi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Firstly, $AB = \xi BA$ implies that $\mathrm{Tr}\ B = \mathrm{Tr}\,(ABA^{-1}) = \mathrm{Tr}\ \xi B$. Together with $\det B = 1$, this gives us
$$B = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}.$$
Similarly
$$A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

Now the condition $AB = \xi BA$ may be rewritten as
$$\begin{pmatrix} ax & ay + bx^{-1} \\ 0 & a^{-1}x^{-1} \end{pmatrix} = \begin{pmatrix} ax & bx + a^{-1}y + a^{-1}x^{-1} \\ 0 & a^{-1}x^{-1} \end{pmatrix}$$
or
$$(6) \qquad y(a - a^{-1}) = b(x - x^{-1}) + a^{-1}x^{-1},$$
which can be written as
$$y(x(a^2 - 1)) - b(a(x^2 - 1)) = 1\,.$$

So we have a family of lines over the open set $\mathbb{C}^* \times \mathbb{C}^* \setminus \{(\pm 1, \pm 1)\}$ in the plane parameterised by (a, x). The stabiliser of ξ in $\mathrm{PGL}(2, \mathbb{C})$ is $\mathbb{C}$, and it acts on each line by translations:

$$(y, b) \mapsto (y + \lambda x^{-1}(1 - x^2), b + \lambda a^{-1}(1 - a^2)).$$

So

$$e(\overline{X}_2) = q\big((q-1)^2 - 4\big) = q^3 - 2q^2 - 3q$$

and

$$\begin{aligned}
e(X_2) &= e(\mathrm{GL}(2, \mathbb{C})/U)e(\overline{X}_2) \\
&= (q^2 - 1)(q^3 - 2q^2 - 3q) \\
&= q^5 - 2q^4 - 4q^3 + 2q^2 + 3q.
\end{aligned}$$

4.4. E-polynomials for X_3, $\overline{X}_3$.

$$\begin{aligned}
X_3 &= \{(A, B) \in \mathrm{SL}(2, \mathbb{C})^2 \mid AB \sim \xi BA\}, \\
\overline{X}_3 &= \{(A, B) \in \mathrm{SL}(2, \mathbb{C})^2 \mid AB = \xi BA\},
\end{aligned}$$

where $\xi = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.

For $(A, B) \in \overline{X}_3$, write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Then $\mathrm{Tr}\, B = \mathrm{Tr}\, \xi B$ implies

$$\text{(7)} \qquad\qquad\qquad z = 2(x + w),$$

while $\mathrm{Tr}\, A = \mathrm{Tr}\, \xi^{-1} A$ implies

$$\text{(8)} \qquad\qquad\qquad c = -2(a + d).$$

and the relation $AB = \xi BA$ is translated into

$$\text{(9)} \qquad\qquad\qquad cy + 2dw + bz = 0$$

We consider the following cases

- $\overline{X}_3'$, X_3', when $x + w = 0$,
- $\overline{X}_3''$, X_3'', when $x + w \neq 0$.

4.4.1. $\overline{X}_3' := \{(A, B) \in \overline{X}_3 \mid x + w = 0\}$ and $X_3' := \{(A, B) \in X_3 \mid x + w = 0\}$. In this case (7) implies $xw = 1$; together with the equation $w = -x$, this gives $x = \pm i$, $w = \mp i$. The first vector of our basis is an eigenvector for B. We choose the second vector to be an eigenvector of B for the other eigenvalue, so that

$$B = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}.$$

Note that $\mathrm{Stab}(\xi, B) = \mathbb{C}^*$. From equation (9), we get that $d = 0$. So A is of the form

$$A = \begin{pmatrix} a & \frac{1}{2a} \\ -2a & 0 \end{pmatrix}$$

with $a \in \mathbb{C}^*$. These (A, B) form a slice for the action of the stabiliser of ξ in $\mathrm{PGL}(2, \mathbb{C})$ on $\overline{X}'_3$, and this stabiliser is isomorphic to $\mathbb{C}$. So

$$\overline{X}'_3 \cong \mathbb{C} \times \mathbb{C}^* \times \{\pm i\}, \quad X'_3 \cong \mathrm{PGL}(2, \mathbb{C}) \times \mathbb{C}^* \times \{\pm i\}$$

and

$$e(\overline{X}'_3) = 2q(q-1) = 2q^2 - 2q, \quad e(X'_3) = 2q(q^2-1)(q-1) = 2q^4 - 2q^3 - 2q^2 + 2q.$$

4.4.2. $\overline{X}''_3 := \{(A, B) \in \overline{X}_3 \,|\, x + w \neq 0\}$ and $X''_3 := \{(A, B) \in X_3 \,|\, x + w \neq 0\}$. Changing the second vector in the basis corresponds to conjugating by the matrix $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. But

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x + \alpha z & * \\ z & w - \alpha z \end{pmatrix}.$$

So we can arrange to have $w = 0$, and hence by (7),

$$B = \begin{pmatrix} x & -\frac{1}{2x} \\ 2x & 0 \end{pmatrix},$$

where $x \neq 0$. Again these (A, B) form a slice, which we shall denote by S. The equation (9) in this situation is $cy + bz = 0$, i.e., $b = \frac{1}{4x^2}c$. The determinant is $ad - bc = 1$; using (8), we get

$$(10) \qquad\qquad ad - \frac{(a+d)^2}{x^2} = 1.$$

Let us compute the E-polynomial of this variety S. We have the contributions:

- $S^a := \{(A, B) \in S \,|\, a + d = 0\}$. Then $a = -d = \pm i$, $x \neq 0$. So $e(S^a) = 2(q-1)$.
- The case $a + d \neq 0$ has several contributions. If $a + d \neq 0$, take $t = (a+d)/x$. Then $t \in \mathbb{C}^*$, and the equation (10) is $ad = 1 + t^2$. This is a family of conics over $\mathbb{C}^*$. The first contribution is

$$S^b := \{(A, B) \in S \,|\, a + d \neq 0, t = \pm i\}.$$

  This corresponds to degenerate conics. For $t = \pm i$, the conic is $ad = 0$. As $a + d \neq 0$, we have that $(a, d) \neq (0, 0)$. So we get $e(S^b) = 4(q-1)$.

- $S^c := \{(A, B) \in S \,|\, a + d \neq 0, t \neq 0, \pm i\}$. For $t \in \mathbb{C} \setminus \{0, \pm i\}$, we get a smooth conic $ad = 1 + t^2$. Consider the family of projective conics obtained by completing these affine conics. This gives a conic bundle, i.e., a fibration

$$\mathbb{P}^1 \to S^{c,1} \to \mathbb{C} \setminus \{0, \pm i\}.$$

  Now:
  (1) By Remark 2.5, we get $e(S^{c,1}) = (q+1)(q-3)$.
  (2) $S^{c,2}$ corresponds to the points at infinity of the conics in the conic bundle. These are given by $ad = 0$, where $t \in \mathbb{C} \setminus \{0, \pm i\}$. So $e(S^{c,2}) = 2(q-3)$.
  (3) $S^{c,3}$ corresponds to the points in the conic bundle with $a + d = 0$. So these points are given by $ad = 1 + t^2$, $a + d = 0$, which gives the affine hyperbola $-a^2 = 1 + t^2$, from which we must remove $t = 0, \pm i$. The contribution is $e(S^{c,3}) = q - 5$.

Thus $e(S^c) = e(S^{c,1}) - e(S^{c,2}) - e(S^{c,3}) = (q+1)(q-3) - 2(q-3) - (q-5) = q^2 - 5q + 8$.

Finally we get

$$e(S) = e(S^a) + e(S^b) + e(S^c) = 2(q-1) + 4(q-1) + (q^2 - 5q + 8) = q^2 + q + 2$$

and (using again the fact that the stabiliser of ξ in $\mathrm{PGL}(2,\mathbb{C})$ is isomorphic to $\mathbb{C}$)

$$e(\overline{X}_3'') = q(q^2 + q + 2), \quad e(X_3'') = q(q^2 - 1)(q^2 + q + 2) = q^5 + q^4 + q^3 - q^2 - 2q.$$

Thus

$$e(\overline{X}_3) = e(\overline{X}_3') + e(\overline{X}_3'') = q(q^2 + 3q) = q^3 + 3q^2$$

and

$$e(X_3) = e(X_3') + e(X_3'') = q(q^2 - 1)(q^2 + 3q) = q^5 + 3q^4 - q^3 - 3q^2.$$

4.5. E-polynomials for X_4, $\overline{X}_{4,\lambda}$ and $\overline{X}_4$. Finally consider

$$X_4 = \left\{ (A, B) \mid AB \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} BA, \lambda \neq 0, \pm 1 \right\},$$

$$\overline{X}_{4,\lambda} = \left\{ (A, B) \mid AB = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} BA \right\}, \quad \lambda \in \mathbb{C} \setminus \{0, \pm 1\}$$

and

$$\overline{X}_4 = \left\{ (A, B, \lambda) \mid AB = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} BA, \lambda \neq 0, \pm 1 \right\}.$$

Write $\xi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. For $(A, B) \in \overline{X}_{4,\lambda}$, we have $\mathrm{Tr}\, B = \mathrm{Tr}\, \xi B$ and $\mathrm{Tr}\, A = \mathrm{Tr}\, \xi^{-1} A$, therefore

$$A = \begin{pmatrix} a & b \\ c & \lambda^{-1}a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} x & y \\ z & \lambda x \end{pmatrix}.$$

Now $AB = \xi BA$ means

$$\begin{pmatrix} ax + bz & ay + \lambda bx \\ cx + \lambda^{-1}az & cy + ax \end{pmatrix} = \begin{pmatrix} \lambda(ax + cy) & \lambda bx + ay \\ \lambda^{-1}az + cx & \lambda^{-1}(ax + bz) \end{pmatrix}.$$

Combining this with the equations obtained from the determinants we have the following equations for X_4:

$$\lambda^{-1}a^2 - bc = 1 \tag{11}$$

$$\lambda x^2 - yz = 1 \tag{12}$$

$$ax + bz = \lambda(ax + cy). \tag{13}$$

It cannot happen that $b = 0$, $y = 0$ simultaneously, since in this case $a \neq 0$, $x \neq 0$, but then $\lambda = 1$ by (13), which is a contradiction.

We shall split X_4 into strata X_4^i (with corresponding strata for $\overline{X}_{4,\lambda}$ and $\overline{X}_4$) and construct a slice S^i for each stratum $\overline{X}_4^i$. We will then have (noting that the stabiliser of ξ in $\mathrm{PGL}(2,\mathbb{C})$ is isomorphic to $\mathbb{C}^*$ and writing S_λ^i for $S^i \cap \overline{X}_{4,\lambda}^i$)

$$\overline{X}_{4,\lambda}^i \cong \mathbb{C}^* \times S_\lambda^i, \quad \overline{X}_4^i \cong \mathbb{C}^* \times S^i,$$

with corresponding formulae for the E-polynomials. There is an action of $\mathbb{Z}_2$ on $\overline{X}_4$ given by $(A, B, \lambda) \mapsto (P_0 A P_0^{-1}, P_0 B P_0^{-1}, \lambda^{-1})$, where $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Provided S^i is invariant under this action, we have

$$X_4^i \cong (\mathrm{PGL}(2, \mathbb{C}) \times S^i)/\mathbb{Z}_2.$$

Since $e(\mathrm{PGL}(2, \mathbb{C}))^- = 0$ by Corollary 3.3, it follows that

(14) $$e(X_4^i) = e(\mathrm{PGL}(2, \mathbb{C}))e(S^i/\mathbb{Z}_2) = (q^3 - q)e(S_i)^+ .$$

In describing the strata, we shall always assume that $AB = \xi BA$ and equations (11), (12) and (13) hold.

4.5.1. $X_4^0 := \{(A, B) \in X_4 \mid b = c = 0\}$. Equations (11), (12), (13) specialize to $a^2 = \lambda$, $x = 0$ and $yz = -1$. Rescaling the two basis vectors, we can assume that

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Note that $a \in \mathbb{C} \setminus \{0, \pm 1, \pm i\}$. This gives our slice S^0, and S_λ^0 consists of the two points given by $a^2 = \lambda$. So

$$\begin{aligned} e(\overline{X}_{4,\lambda}^0) &= 2(q - 1) = 2q - 2, \\ e(\overline{X}_4^0) &= (q - 1)(q - 5) = q^2 - 6q + 5 \end{aligned}$$

and, by (14),

$$e(X_4^0) = (q^3 - q)e((\mathbb{C} \setminus \{0, \pm 1, \pm i\})/\mathbb{Z}_2) = (q^3 - q)(q - 3) = q^4 - 3q^3 - q^2 + 3q .$$

4.5.2. $X_4^1 := \{(A, B) \in X_4 \mid y = z = 0\}$. Analogously

$$\begin{aligned} e(\overline{X}_{4,\lambda}^1) &= 2(q - 1), \\ e(\overline{X}_4^1) &= (q - 1)(q - 5), \\ e(X_4^1) &= q^4 - 3q^3 - q^2 + 3q . \end{aligned}$$

4.5.3. $X_4^2 := \{(A, B) \in X_4 \mid b = 0, c \neq 0\} \cup \{(A, B) \in X_4 \mid c = 0, b \neq 0\}$. In the case $b = 0$, rescaling fixes $c = 1$. Equations (11), (12), (13) in this situation are

(15) $$a^2 = \lambda$$

(16) $$a^2 x^2 - yz = 1$$

(17) $$(1 - \lambda)ax = \lambda y.$$

Equation (15) implies that $a \neq 0, \pm 1, \pm i$. Equation (17) allows us to obtain y uniquely. Note that (16) and (17) imply that $x \neq 0$ and $y \neq 0$. Equation (16) uniquely determines z. So we are reduced to considering $x \in \mathbb{C}^*$ and $a \in \mathbb{C} \setminus \{0, \pm 1, \pm i\}$, which defines our slice S^2. Moreover, for S_λ^2, we have again $a^2 = \lambda$, so we get two copies of $\mathbb{C}^*$. Since we have a second component given by $c = 0$, $b \neq 0$, we must double everything and we get

$$\begin{aligned} e(\overline{X}_{4,\lambda}^2) &= 4(q - 1)^2 = 4q^2 - 8q + 4, \\ e(\overline{X}_4^2) &= 2(q - 1)^2(q - 5) = 2q^3 - 14q^2 + 22q - 10. \end{aligned}$$

The $\mathbb{Z}_2$-action interchanges the two components, so we simply consider one of them and get $X_4^2 \cong \mathrm{PGL}(2, \mathbb{C}) \times \mathbb{C}^* \times (\mathbb{C} \setminus \{0, \pm 1, \pm i\})$. Hence

$$e(X_4^2) = (q^3 - q)(q - 1)(q - 5) = q^5 - 6q^4 + 4q^3 + 6q^2 - 5q.$$

4.5.4. $X_4^3 := \{(A, B) \in X_4 \mid y = 0, z \neq 0\} \cup \{(A, B) \in X_4 \mid z = 0, y \neq 0\}$. This runs similarly to X_4^2, so

$$\begin{aligned} e(\overline{X}_{4,\lambda}^3) &= 4(q - 1)^2 = 4q^2 - 8q + 4, \\ e(\overline{X}_4^3) &= 2(q - 1)^2(q - 5) = 2q^3 - 14q^2 + 22q - 10, \\ e(X_4^3) &= q^5 - 6q^4 + 4q^3 + 6q^2 - 5q. \end{aligned}$$

4.5.5. $X_4^4 := \{(A, B) \in X_4 \mid b = z = 0\} \cup \{(A, B) \in X_4 \mid c = y = 0\}$. Note that it is impossible that $b = 0$ and $y = 0$ (or $c = 0$ and $z = 0$) simultaneously. Therefore, this gives exactly the intersection of the previous two strata

$$X_4^4 = X_4^2 \cap X_4^3.$$

For the first component, we scale so that $c = 1$. The equations (11), (12), (13) in this situation give $a^2 = \lambda$, $x^2 = \lambda^{-1}$ and $(1 - \lambda)ax = \lambda y$. Therefore $a \in \mathbb{C} \setminus \{0, \pm 1, \pm i\}$, $x = \pm \frac{1}{a}$ and $y = \pm \frac{1 - a^2}{a^2}$. The slice S^4 is parameterised by a and x, while S_λ^4 consists of 4 points. Doubling up to take into consideration the two components, we get

$$\begin{aligned} e(\overline{X}_{4,\lambda}^4) &= 8(q - 1) = 8q - 8, \\ e(\overline{X}_4^4) &= 4(q - 1)(q - 5) = 4q^2 - 24q + 20. \end{aligned}$$

Moreover

$$e(X_4^4) = 2(q^3 - q)(q - 5) = 2q^4 - 10q^3 - 2q^2 + 10q.$$

Note that

$$e(X_4^2 \cup X_4^3) = e(X_4^2) + e(X_4^3) - e(X_4^4).$$

4.5.6. *The case where b, c, y, z are all $\neq 0$.* There will be several contributions in this case. We begin by scaling so that $b = 1$. The equations (11), (12), (13) are now

$$\begin{aligned} (18) \qquad\qquad\qquad c &= \lambda^{-1}a^2 - 1 \\ (19) \qquad\qquad\qquad 1 &= \lambda x^2 - yz \\ (20) \qquad\qquad\qquad z &= (\lambda - 1)ax + \lambda cy \end{aligned}$$

and the condition $c \neq 0$ translates into $a^2 \neq \lambda$.

Using (18) in (20), we get

$$z = (\lambda - 1)ax + a^2 y - \lambda y$$

and then (19) gives

$$(21) \qquad\qquad \lambda x^2 + a(1 - \lambda)xy + (\lambda - a^2)y^2 = 1.$$

This is a family of conics parameterised by (x, y) over the plane (a, λ), $a^2 \neq \lambda$. We continue stratifying the space X_4 with respect to the possibilities of these conics being degenerate or non-degenerate.

The discriminant is

$$(22) \qquad D := 4\lambda(a^2 - \lambda) + (\lambda - 1)^2 a^2 = ((\lambda + 1)a - 2\lambda)((\lambda + 1)a + 2\lambda).$$

We consider below

- X_4^5 corresponding to $D = 0$ (degenerate conics) and
- X_4^6, X_4^7 and X_4^8 for $D \neq 0$ (non-degenerate conics).

We shall also need to make explicit the $\mathbb{Z}_2$-action. This acts on coordinates as follows:

$$(23) \qquad \begin{aligned} a &\mapsto \lambda^{-1}a \\ c &\mapsto b = 1 \mapsto c \\ b = 1 &\mapsto c \mapsto 1 \\ x &\mapsto \lambda x \\ y &\mapsto z \mapsto \frac{z}{c} = \frac{(\lambda - 1)ax}{c} + \lambda y = \lambda y + \frac{\lambda(\lambda - 1)a}{a^2 - \lambda}x \\ z &\mapsto y \mapsto yc. \end{aligned}$$

Recall that the $\mathbb{Z}_2$-action is given by conjugation with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ together with the slice fixing condition which makes $b = 1$.

4.5.7. $X_4^5 := \{(A, B) \in X_4 \mid c, y, z \neq 0,\ b = 1,\ D = 0\}$. The discriminant of the conics in (21) is zero over two lines given by

$$a = \pm \frac{2\lambda}{\lambda + 1}.$$

Note that $c = 0 \iff a^2 = \lambda \iff (\lambda - 1)^2 = 0$, which does not hold. So $c \neq 0$ automatically.

Suppose

$$(24) \qquad a = \frac{2\lambda}{\lambda + 1};$$

the other situation $a = -\frac{2\lambda}{\lambda+1}$ is similar. The equation of the family of conics in (21) is then

$$\lambda x^2 + \frac{2\lambda}{\lambda + 1}xy(1 - \lambda) + \left(\lambda - \left(\frac{2\lambda}{\lambda + 1}\right)^2\right)y^2 = 1,$$

that is

$$\lambda x^2 - \frac{2\lambda(\lambda - 1)}{\lambda + 1}xy + \lambda\frac{(\lambda - 1)^2}{(\lambda + 1)^2}y^2 = 1.$$

Hence

$$(25) \qquad \left(x - \frac{\lambda - 1}{\lambda + 1}y\right)^2 = \frac{1}{\lambda}.$$

These are two parallel lines. Recall that we have to impose the condition $yz \neq 0$, i.e., $\lambda x^2 \neq 1$. This means that $x \neq \pm\frac{1}{\sqrt{\lambda}}$.

Set $\mu := x - \frac{\lambda-1}{\lambda+1}y$, so that $\mu^2 = \lambda^{-1}$ by (25) and $\mu \in \mathbb{C}\backslash\{0, \pm 1, \pm i\}$. The condition $\lambda x^2 \neq 1$ becomes $x \neq \pm\mu$ and the map $(x, \mu) \mapsto (x, y, \lambda)$ is an isomorphism. This gives our slice S^5 with $e(S^5) = (q-2)(q-5)$, while $e(S_\lambda^5) = 2(q-2)$. So (recalling that we have two components of this type),

$$
\begin{aligned}
e(\overline{X}_{4,\lambda}^5) &= 4(q-1)(q-2) = 4q^2 - 12q + 8, \\
e(\overline{X}_4^5) &= 2(q-1)(q-2)(q-5) = 2q^3 - 16q^2 + 34q - 20.
\end{aligned}
$$

The $\mathbb{Z}_2$-action leaves each of the two components invariant. We can therefore consider the action of $\mathbb{Z}_2$ on the slice parameterised by (x, μ). The action maps $x \mapsto \mu^{-2}x$ and

$$
\begin{aligned}
\mu = x - \frac{\lambda-1}{\lambda+1}y \;\mapsto\; & \lambda x - \frac{\lambda^{-1}-1}{\lambda^{-1}+1}\left(\lambda y + \frac{\lambda(\lambda-1)a}{a^2-\lambda}x\right) \\
&= \lambda x + \lambda\left(\frac{\lambda-1}{\lambda+1}y - 2x\right) \\
&= -\lambda\left(x - \frac{\lambda-1}{\lambda+1}y\right) \\
&= -\lambda\mu^2 = -\mu^{-1},
\end{aligned}
$$

using (24).

We make a change of variable $X = x/\mu$ so that $X \neq \pm 1$ and $\mathbb{Z}_2$ acts as $X \mapsto -X$. Using the fibration

$$
F = \mathbb{C}\backslash\{\pm 1\} \longrightarrow E \longrightarrow \mathbb{C}\backslash\{0, \pm 1, \pm i\} = B,
$$

where $E = F \times B = \{(X, \mu) | X \neq \pm 1, \mu \neq 0, \pm 1, \pm i\}$, we have $e(F)^+ = q - 1$, $e(F)^- = -1$, $e(B)^+ = q - 3$, $e(B)^- = -2$. So, taking into account that we have two components corresponding to the two possibilities for a, and using (14),

$$
\begin{aligned}
e(X_4^5) &= 2(q^3 - q)\big((q-1)(q-3) + 2\big) = 2(q^3 - q)(q^2 - 4q + 5) \\
&= 2q^5 - 8q^4 + 8q^3 + 8q^2 - 10q.
\end{aligned}
$$

4.5.8. X_4^6. Now consider the case $D \neq 0$. From equation (22) this is equivalent to

$$
\text{(26)} \qquad\qquad\qquad a \neq \pm\frac{2\lambda}{\lambda+1}.
$$

Recall the equation (21) defining our family of conics over an open subset of the plane parameterised by (a, λ), $a^2 \neq \lambda$. We complete each conic to a projective conic in $\mathbb{P}^2$ with coordinates $[u : x : y]$,

$$
\lambda x^2 + a(1 - \lambda)xy + (\lambda - a^2)y^2 = u^2.
$$

We define $X_4^6, \overline{X}_4^6$ and $\overline{X}_{4,\lambda}^6$ as the corresponding $\mathbb{P}^1$-fibrations. In particular, we have a fibration

$$
\text{(27)} \qquad \mathbb{P}^1 \to S^6 \to \{(a, \lambda) \mid \lambda \neq 0, \pm 1; \; a^2 \neq \lambda; \; a \neq \pm 2\lambda/(\lambda+1)\}.
$$

The plane (a, λ) with $\lambda \neq 0, \pm 1$ and satisfying (26) has E-polynomial $(q-2)(q-3)$. We remove the subset $a^2 = \lambda$, with $a \neq 0, \pm 1, \pm i$, hence getting

$$
(q-2)(q-3) - (q-5) = q^2 - 6q + 11.
$$

The resulting variety is our slice S^6 and $e(S^6) = (q+1)(q^2 - 6q + 11)$. Moreover S_λ^6 is a conic bundle over $\mathbb{C} \setminus \{4 \text{ points}\}$, so $e(S_\lambda^6) = (q+1)(q-4)$. Thus

$$\begin{aligned} e(\overline{X}_{4,\lambda}^6) &= (q-1)(q+1)(q-4) = q^3 - 4q^2 - q + 4, \\ e(\overline{X}_4^6) &= (q-1)(q+1)(q^2 - 6q + 11) = q^4 - 6q^3 + 10q^2 + 6q - 11. \end{aligned}$$

The action of $\mathbb{Z}_2$ takes $(a, \lambda) \mapsto (\lambda^{-1}a, \lambda^{-1})$, and takes each fibre of the conic bundle to another fibre, so defining a conic bundle over the plane quotiented by $\mathbb{Z}_2$. To compute the latter, set $\mu^2 := \lambda$ and $A := \mu^{-1}a$, so the action by $\mathbb{Z}_2$ sends $\mu \mapsto \mu^{-1}$ and A remains invariant. The equation in (26) is then

$$A \neq \pm \frac{2}{\mu + \mu^{-1}} \, .$$

Also $a^2 \neq \lambda$ means $A \neq \pm 1$. We now change variables, writing $s = \mu + \mu^{-1}$, to take account of the $\mathbb{Z}_2$-action. We still have to quotient by $(A, s) \mapsto (-A, -s)$, due to the sign indeterminacy of μ. Let $B = As$ and $S = s^2$, so that the variables (B, S) define the required quotient, with the conditions

$$A \neq \pm \frac{2}{s}, \pm 1 \iff B \neq \pm 2, B^2 \neq S$$

and

$$\lambda \neq 0, \pm 1 \iff \mu \neq 0, \pm 1, \pm i \iff s \neq 0, \pm 2 \iff S \neq 0, 4.$$

Thus $S^6/\mathbb{Z}_2$ consists of a family of conics over $(\mathbb{C} \setminus \{\pm 2\}) \times (\mathbb{C} \setminus \{0, 4\})$ minus the set $S = B^2$, which is isomorphic to $\mathbb{C} \setminus \{0, \pm 2\}$. The E-polynomial for the base is $(q-2)^2 - (q-3) = q^2 - 5q + 7$. Finally the E-polynomial is

$$e(X_4^6) = (q^3 - q)(q+1)(q^2 - 5q + 7) = q^6 - 4q^5 + q^4 + 11q^3 - 2q^2 - 7q \, .$$

4.5.9. X_4^7. In X_4^6, we have added points when considering the projective conics which should not be included in the computation of $e(X_4)$. We now remove these points. Consider the sets X_4^7, $\overline{X}_4^7$, $\overline{X}_{4,\lambda}^7$, S^7 corresponding to the points in each of the conics over the plane (27) satisfying the condition $yz = 0 \Leftrightarrow \lambda x^2 = 1$. This means the slice S^7 is parameterised by (a, λ, x, y) subject to $\lambda \neq 0, \pm 1$, $a \neq \pm \frac{2\lambda}{1+\lambda}$, $a^2 \neq \lambda$, $\lambda x^2 = 1$ and $a(1-\lambda)xy + (\lambda - a^2)y^2 = 0$. Therefore $\lambda = x^{-2}$, $x \neq 0, \pm 1, \pm i$, and $y = 0$ or $y = \frac{a(1-\lambda)}{a^2 - \lambda}x$.

We have two contributions:

- $a = 0$, $x \neq 0, \pm 1, \pm i$, $y = 0$.
- $a \neq 0$. Then the two values of y are different. Also

$$a \neq \pm \frac{2x^{-1}}{x + x^{-1}}, a \neq \pm x^{-1}.$$

For fixed λ, the case $a = 0$ yields 2 points, while $a \neq 0$ gives 2 values for x, each of which gives two values of y and there are 5 excluded values for a. So $e(S_\lambda^7) = 2 + 4(q-5) = 2(2q - 9)$.

For variable λ, when $a = 0$, we can take x as a parameter, giving a contribution of $q - 5$. When $a \neq 0$, we can take a and x as parameters and we have 2 values of y for each (a, x). So $e(S^7) = 2(q-5)^2 + q - 5 = (q-5)(2q - 9)$.

This means that

$$
\begin{aligned}
e(\overline{X}_{4,\lambda}^7) &= 2(q-1)(2q-9) = 4q^2 - 22q + 18, \\
e(\overline{X}_4^7) &= (q-1)(q-5)(2q-9) = 2q^3 - 21q^2 + 64q - 45.
\end{aligned}
$$

The action of $\mathbb{Z}_2$ on S^7 sends $x \mapsto \lambda x = x^{-1}$, $a \mapsto x^2 a$ (so $A = ax$ is invariant) and $y \mapsto \frac{z}{c} = \lambda y + \frac{\lambda(\lambda-1)a}{a^2-\lambda}x$. So

$$
y = 0 \mapsto \frac{\lambda(\lambda-1)a}{a^2-\lambda}x = \frac{\lambda^{-1}a(1-\lambda^{-1})}{\lambda^{-2}a^2 - \lambda^{-1}}x^{-1}.
$$

For $a = 0$, we get simply $x \mapsto x^{-1}$, giving a contribution of $q - 3$ to the quotient. For $a \neq 0$, the two lines corresponding to the two values $y \in \{0, \frac{a(1-\lambda)}{a^2-\lambda}x\}$ are interchanged. Thus we just have to compute the E-polynomial of the variety with $y = 0$. This is parameterised by (x, A), subject to the conditions $A \neq \pm\frac{2}{x+x^{-1}}$, $A \neq 0, \pm 1$. Then $x \neq 0, \pm 1, \pm i$, $A \neq 0, \pm 1, \pm\frac{2}{x+x^{-1}}$. As before, the contribution is $(q-5)^2$. So

$$
\begin{aligned}
e(X_4^7) &= (q^3 - q)\big((q-5)^2 + (q-3)\big) = (q^3 - q)(q^2 - 9q + 22) \\
&= q^5 - 9q^4 + 21q^3 + 9q^2 - 22q.
\end{aligned}
$$

4.5.10. X_4^8. It now just remains to take out from X_4^6 the points at infinity in each conic. We consider the sets X_4^8, $\overline{X}_4^8$, $\overline{X}_{4,\lambda}^8$, S^8 corresponding to the points at infinity in each of the conics in the projective conic bundle. For each value (a, λ), these points are given by

$$
(\lambda - a^2)y^2 + a(1-\lambda)xy + \lambda x^2 = 0,
$$

in projective coordinates $[x : y]$. Put $Y = y/x$, so the equation for S^8 is

$$
(\lambda - a^2)Y^2 + a(1-\lambda)Y + \lambda = 0,
$$

where the variables are (a, λ, Y), $\lambda \neq 0, \pm 1$, $a \neq \pm\frac{2\lambda}{1+\lambda}$, $a^2 \neq \lambda$.

Note that we know that $a^2 \neq \lambda$, so we may rewrite the eqation as

$$
Y^2 + \frac{a(1-\lambda)}{\lambda - a^2}Y + \frac{\lambda}{\lambda - a^2} = 0.
$$

Equivalently,

$$
(28) \qquad \left(Y + \frac{1}{2}\frac{a(1-\lambda)}{\lambda - a^2}\right)^2 = \frac{D}{4(\lambda - a^2)^2},
$$

where $D = (a(1+\lambda) + 2\lambda)(a(1+\lambda) - 2\lambda)$. Write $e = \frac{1}{2}\frac{a(1-\lambda)}{\lambda - a^2}$. By the action of $\mathbb{Z}_2$, $e \mapsto -e$. Now

$$
Y = \frac{y}{x} \mapsto \frac{z/c}{\lambda x} = \frac{y}{x} + \frac{(\lambda-1)a}{a^2 - \lambda} = Y + 2e.
$$

Introduce a new variable $\alpha = Y + e$. Then α remains invariant by the action of $\mathbb{Z}_2$. The equation is now

$$
\alpha^2 = \frac{D}{4(\lambda - a^2)^2}.
$$

Consider the change of variables $(a, \alpha) \mapsto (A, \Omega)$ given by

$$A \quad := \quad \frac{(1 + \lambda)a}{\lambda}$$

$$\Omega \quad := \quad \frac{2\alpha(\lambda - a^2)}{\lambda},$$

possible because $a^2 \neq \lambda$. These are invariant under the action of $\mathbb{Z}_2$. The equation (28) is now

(29) $$\Omega^2 = A^2 - 4,$$

and the conditions are $\lambda \neq 0, \pm 1$, $A \neq \pm 2$ and $A^2 \neq \frac{(1+\lambda)^2}{\lambda}$. This is our final equation for S^8. For fixed λ, we have a conic, from which we must remove the two points at infinity and 6 further points corresponding to the excluded values of A; so $e(S^8_\lambda) = q - 7$, and

$$e(\overline{X}^8_{4,\lambda}) = (q - 1)(q - 7) = q^2 - 8q + 7.$$

To compute $e(S^8)$, we take the contribution of (29) with $\lambda \neq 0, \pm 1$, $A \neq \pm 2$, which gives $(q - 3)^2$, and subtract the contribution of $A^2 = \frac{(1+\lambda)^2}{\lambda}$. Note that then $\Omega^2 = \frac{(1-\lambda)^2}{\lambda}$, so $A = \pm \frac{1+\lambda}{1-\lambda}\Omega$. Writing $B = \frac{1+\lambda}{A}$, we have the equation $\lambda = B^2$, with $B \neq 0, \pm 1, \pm i$. Thus, $e(S^8) = (q - 3)^2 - 2(q - 5) = q^2 - 8q + 19$ and

$$e(\overline{X}^8_4) = (q - 1)(q^2 - 8q + 19) = q^3 - 9q^2 + 27q - 19.$$

Finally consider the action of $\mathbb{Z}_2$. As already observed, we find that this action leaves A, Ω invariant and sends $\lambda \mapsto \lambda^{-1}$. We can therefore take $s = \lambda + \lambda^{-1}$ as a parameter on the quotient. Ignoring the final condition $A^2 \neq \frac{(1+\lambda)^2}{\lambda}$ for the moment, we have a conic with 4 points deleted and s has 2 excluded values, so we obtain a contribution of $(q - 3)(q - 2)$. From this we have to delete the subvariety defined by $A^2 = s + 2$. This subvariety is a conic $\Omega^2 = A^2 - 4$ with the 2 points at infinity and the 4 points $(A, \Omega) = (0, \pm 2i), (\pm 2, 0)$ removed; this contributes $q - 5$. Thus

$$
\begin{aligned}
e(X^8_4) &= (q^3 - q)\big((q - 3)(q - 2) - (q - 5)\big) \\
&= (q^3 - q)(q^2 - 6q + 11) \\
&= q^5 - 6q^4 + 10q^3 + 6q^2 - 11q.
\end{aligned}
$$

We obtain finally

$$
\begin{aligned}
e(X_4) &= e(X^0_4) + e(X^1_4) + e(X^2_4) + e(X^3_4) - e(X^4_4) + e(X^5_4) + e(X^6_4) - e(X^7_4) - e(X^8_4) \\
\text{(30)} \quad &= (q^3 - q)(q^3 - 2q^2 - 3q - 2) = q^6 - 2q^5 - 4q^4 + 3q^2 + 2q,
\end{aligned}
$$

and similarly

(31) $$e(\overline{X}_{4,\lambda}) = (q - 1)(q^2 + 4q + 1) = q^3 + 3q^2 - 3q - 1$$

and

(32) $$e(\overline{X}_4) = q^4 - 3q^3 - 6q^2 + 5q + 3.$$

Note also that

$$e(X) = e(X_0) + e(X_1) + e(X_2) + e(X_3) + e(X_4) = q^6 - 2q^4 + q^2 = (q^3 - q)^2,$$

which agrees with the known formula for $e(\mathrm{SL}(2,\mathbb{C})^2)$.

5. $\mathrm{SL}(2,\mathbb{C})$-CHARACTER VARIETIES FOR $g = 1$

We now use the previous work to calculate the E-polynomials of the character varieties

$$
\begin{aligned}
\mathcal{M}_\xi &= \{(A,B) \in SL(2,\mathbb{C}) \,|\, [A,B] = \xi\}//\mathrm{Stab}(\xi) \\
&= \{(A,B) \in SL(2,\mathbb{C}) \,|\, [A,B] \in \mathcal{C}\}//\mathrm{PGL}(2,\mathbb{C})\,,
\end{aligned}
$$

where $\mathcal{C}$ is the conjugacy class of $\xi \in \mathrm{SL}(2,\mathbb{C})$.

Corresponding to the five strata X_i of section 4, we have five character varieties, which we shall denote more logically by $\mathcal{M}_{\mathrm{Id}}$, $\mathcal{M}_{-\mathrm{Id}}$, $\mathcal{M}_{J_+}$, $\mathcal{M}_{J_-}$ and $\mathcal{M}_\lambda$.

5.1. Hodge polynomials.

- $\mathcal{M}_{\mathrm{Id}} = X_0//\mathrm{PGL}(2,\mathbb{C})$. This character variety contains reducibles (in fact, all elements are reducible), so we have to make a GIT quotient, identifying S-equivalence classes. The non-diagonalisable orbits have limits in the diagonalisable ones, so every S-equivalence class contains an element of the form

$$
(A,B) = \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \right),
$$

  unique up to $(\lambda,\mu) \mapsto (\lambda^{-1},\mu^{-1})$. We have the fibration $F \to E = F \times B \to B$, where λ parametrizes $F = \mathbb{C}\backslash\{0\}$ and μ parametrizes $B = \mathbb{C}\backslash\{0\}$. So $e(F)^+ = q$, $e(F)^- = -1$, $e(B)^+ = q$, $e(B)^- = -1$, and Proposition 2.6 gives

$$
e(\mathcal{M}_{\mathrm{Id}}) = q^2 + 1\,.
$$

- $\mathcal{M}_{-\mathrm{Id}} = X_1//\mathrm{PGL}(2,\mathbb{C})$. We have already seen that the character variety is just one point, so

$$
e(\mathcal{M}_{-\mathrm{Id}}) = 1\,.
$$

- $\mathcal{M}_{J_+} = X_2//\mathrm{PGL}(2,\mathbb{C})$. Here we put $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The computations in subsection 4.3 show that

$$
\begin{aligned}
\mathcal{M}_{J_+} &\cong \mathbb{C}^* \times \mathbb{C}^* \setminus \{(\pm 1, \pm 1)\}, \\
e(\mathcal{M}_{J_+}) &= (q-1)^2 - 4 = q^2 - 2q - 3\,.
\end{aligned}
$$

Remark 5.1. Given the shape of the matrices A, B in subsection 4.3, we see that all representations in X_2 are reducible. Therefore there is a map $\mathcal{M}_{J_+} \to \mathcal{M}_{\mathrm{Id}}$ which consists of quotienting by the involution $(a,x) \mapsto (a^{-1}, x^{-1})$.

- $\mathcal{M}_{J_-} = X_3//\mathrm{PGL}(2,\mathbb{C})$. Here we put $J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. We can stratify as in subsection 4.4 to show that

$$
e(\mathcal{M}_{J_-}) = 2(q-1) + (q^2 + q + 2) = q^2 + 3q\,.
$$

Remark 5.2. Note that there are no reducibles in X_3. If A, B form a reducible representation, then there is a common eigenvector $v \in \mathbb{C}^2$. Then $[A, B](v) = v$, so the only eigenvalue of $[A, B]$ is 1. Therefore either $(A, B) \in X_0$, or $(A, B) \in X_2$.

- $\mathcal{M}_\lambda = \mathcal{M}_\xi = \overline{X}_{4,\lambda}//\mathbb{C}^*$. Here we have set $\xi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \neq 0, \pm 1$. The slices S_λ^i constructed in subsection 4.5 give a stratification of $\mathcal{M}_\lambda$. It follows at once that

$$e(\mathcal{M}_\lambda) = e(\overline{X}_{4,\lambda})/(q-1) = q^2 + 4q + 1.$$

Remark 5.3. Note the equality $e(\mathcal{M}_{J_-}) + (q+1)e(\mathcal{M}_{-\mathrm{Id}}) = e(\mathcal{M}_\lambda)$ holds. This has been suggested to us by T. Hausel, and it is predicted to hold for arbitrary genus.

5.2. **Poincaré polynomials and Hodge numbers.**

The information obtained above for the E-polynomials is not sufficient to compute Poincaré polynomials or the $h_c^{k,p,q}$ of the character varieties, since there can be cancellations in the computation of the E-polynomial. However, using the explicit description and, in one case, the known Poincaré polynomials, we can compute the Hodge numbers (except in the case of $\mathcal{M}_{J_-}$).

- For $\mathcal{M}_{\mathrm{Id}}$, we have an explicit description. We know that $h_c^{k,p,q}(\mathbb{C}^*) = 1$ for $(k, p, q) = (2, 1, 1)$ and $(k, p, q) = (1, 0, 0)$ and is otherwise non-zero. So, for $\mathbb{C}^* \times \mathbb{C}^*$, we have non-zero Hodge numbers as follows: $h_c^{4,2,2} = 1, h_c^{3,1,1} = 2, h_c^{2,0,0} = 1$. It is easily checked that, for the action of $\mathbb{Z}_2$ on the cohomology of $\mathbb{C}^* \times \mathbb{C}^*$, we have $(H_c^*)^+ = H_c^2 \oplus H_c^4$ and $(H_c^*)^- = H_c^3$. It follows that the non-zero Hodge numbers of $\mathcal{M}_{\mathrm{Id}}$ are

$$h_c^{4,2,2}(\mathcal{M}_{\mathrm{Id}}) = h_c^{2,0,0}(\mathcal{M}_{\mathrm{Id}}) = 1.$$

It follows also that the Poincaré polynomial P_t^c for compact cohomology is given by $P_t^c(\mathcal{M}_{Id}) = t^4 + t^2$. A similar computation using ordinary cohomology gives $P_t(\mathcal{M}_{\mathrm{Id}}) = 1 + t^2$ (or we can use Poincaré duality because $\mathcal{M}_{\mathrm{Id}}$ is smooth). We can summarise these results in the statement

$$H_c(\mathcal{M}_{\mathrm{Id}})(q, t) = q^2 t^4 + t^2.$$

- For $\mathcal{M}_{-\mathrm{Id}}$, there is nothing to prove.
- For $\mathcal{M}_{J_+}$, we again have an explicit description, from which it follows that the non-zero Hodge numbers are

$$h_c^{4,2,2}(\mathcal{M}_{J_+}) = 1, \quad h_c^{3,1,1}(\mathcal{M}_{J_+}) = 2, \quad h_c^{2,0,0}(\mathcal{M}_{J_+}) = 1, \quad h_c^{1,0,0}(\mathcal{M}_{J_+}) = 4.$$

The Poincaré polynomials are

$$P_t^c(\mathcal{M}_{J_+}) = t^4 + 2t^3 + t^2 + 4t, \quad P_t(\mathcal{M}_{J_+}) = 4t^3 + t^2 + 2t + 1.$$

Note that this space is smooth, so Poincaré duality applies to relate the two polynomials. Any one of our formulae implies that the Euler characteristic is -4. We can again summarise our results as

$$H_c(\mathcal{M}_{J_+})(q, t) = q^2 t^4 + 2q t^3 + t^2 + 4t.$$

- To deal with the case $\mathcal{M}_{J_-}$, let us use the computations in subsection 4.4 to describe explicitly $\mathcal{M}_{J_-}$. First, we have maps $\pi_1 : \mathcal{M}_{J_-} \to \mathbb{C}$, $\pi_1(A, B) = \mathrm{Tr}\, A$, and $\pi_2 : \mathcal{M}_{J_-} \to \mathbb{C}$, $\pi_2(A, B) = \mathrm{Tr}\, B$. This produces a map $\pi = \pi_1 \times \pi_2 : \mathcal{M}_{J_-} \to \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$. Note that $\overline{X}_3''$ is the part corresponding to $\pi_2^{-1}(\mathbb{C} \setminus \{0\})$. So this latter set is described by the equation (10), which is a smooth surface. By symmetry, $\pi_1^{-1}(\mathbb{C} \setminus \{0\})$ is also smooth, so $\mathcal{M}_{J_-}$ is smooth, being covered by $\pi_1^{-1}(\mathbb{C} \setminus \{0\})$ and $\pi_2^{-1}(\mathbb{C} \setminus \{0\})$. It follows that $h_c^{k,p,q} = 0$ whenever $p + q > k$.

  The map $\pi_2 : \mathcal{M}_{J_-} \to \mathbb{C}$ is a fibration over $\mathbb{C}$ with some singular fibres. The generic fibre is a hyperbola, isomorphic to $\mathbb{C}^*$; this conic (10) degenerates into two (parallel) lines when $x = \pm 2$; the fibre over 0 consists of two copies of $\mathbb{C}^*$, since it corresponds to $\overline{X}_3'$. Thus the manifold $\mathcal{M}_{J_-}$ is constructed as follows: take the surface $\mathbb{C} \times \mathbb{P}^1$, blow it up at three points in the fibres over $0, 2, -2$, obtaining thus three nodal fibres, with three nodes, which we call p_{-2}, p_0, p_2. Now we remove a bi-section of this non-minimal ruled surface, which passes through p_{-2}, p_2, but not through p_0. This bisection cannot be reducible (otherwise, we would not get the correct E-polynomial), so it has to be a double cover of $\mathbb{C}$ ramified exactly at p_{-2}, p_2. Hence it is isomorphic to $\mathbb{C}^*$. Finally, we also remove the point p_0. The result is $\mathcal{M}_{J_-}$.

  From this it follows that $b_0 = 1$, $b_1 = 1$, $b_2 = 5$ and $b_3 = 1$. The Euler characteristic is 4. The Poincaré polynomials are

  $$P_t^c(\mathcal{M}_{J_-}) = t^4 + t^3 + 5t^2 + t, \quad P_t(\mathcal{M}_{J_-}) = t^3 + 5t^2 + t + 1\,.$$

  Although the combination of $e(\mathcal{M}_{J_-})$ and $P_t^c(\mathcal{M}_{J_-})$ gives considerable information about the values of the Hodge numbers, this is not sufficient to determine them completely. In fact $h_c^{4,2,2}(\mathcal{M}_{J_-}) = 1$ and $h_c^{1,0,0}(\mathcal{M}_{J_-}) = 1$, and the other non-zero Hodge numbers are either

  $$h_c^{3,1,1}(\mathcal{M}_{J_-}) = 1, \quad h_c^{2,1,1}(\mathcal{M}_{J_-}) = 4, \quad h_c^{2,0,0}(\mathcal{M}_{J_-}) = 1$$

  or

  $$h_c^{3,0,0}(\mathcal{M}_{J_-}) = 1, \quad h_c^{2,1,1}(\mathcal{M}_{J_-}) = 3, \quad h_c^{2,0,0}(\mathcal{M}_{J_-}) = 2.$$

- Finally, let us consider $\mathcal{M}_\lambda$. Since this is smooth, $h_c^{k,p,q} = 0$ whenever $p + q > k$. Moreover $h_c^4 = 1$ since $\mathcal{M}_\lambda$ is connected and $h_c^0 = 0$ since $\mathcal{M}_\lambda$ is not compact. The moduli space $\mathcal{M}_\lambda$ is homeomorphic to the moduli space of parabolic $\mathrm{SL}(2, \mathbb{C})$-Higgs bundles $\mathcal{H}_\lambda$ on an elliptic curve C. This is the space $\mathcal{N}_\alpha^0$ considered in [1] (with $g = n = 1$), for which the Poincaré polynomials are

$$P_t(\mathcal{N}_\alpha^0) = \frac{(1 + t^3)^2 + t^3(1 + t)(t - 2)}{(1 - t^2)^2} + 3t^2 = 5t^2 + 1, \quad P_t^c(\mathcal{N}_\alpha^0) = t^4 + 5t^2.$$

Comparing this with the formula for $e(\mathcal{M}_\lambda)$ and noting that all Hodge numbers of this space must be of type (p, p) (see Proposition 2.8), we find that the non-zero Hodge numbers are

$$h_c^{4,2,2}(\mathcal{M}_\lambda) = 1, \quad h_c^{2,1,1}(\mathcal{M}_\lambda) = 4, \quad h_c^{2,0,0}(\mathcal{M}_\lambda) = 1.$$

Hence

$$H_c(\mathcal{M}_\lambda) = q^2 t^4 + q t^2 + t^2.$$

Remark 5.4. The space $\mathcal{H}_{\mathrm{Id}}$ is the moduli space of $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles on C, which is known to be isomorphic to $C \times \mathbb{C}/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is given by $(\lambda, a) \mapsto (\lambda^{-1}, -a)$. The non-zero Hodge numbers of $C \times \mathbb{C}$ are $h_c^{4,2,2} = h_c^{3,2,1} = h_c^{3,1,2} = h_c^{2,1,1} = 1$. Now $\mathbb{Z}_2$ acts on H^4 and H^2 by $+1$ and on H^3 by -1. So the non-zero Hodge numbers of the moduli space are $h_c^{4,2,2} = h_c^{2,1,1} = 1$ and the E-polynomial is

$$e(\mathcal{H}_{\mathrm{Id}}) = q^2 + q$$

(different from that of $\mathcal{M}_{\mathrm{Id}}$). This result could also be obtained by noting that the moduli space maps to $C/\mathbb{Z}_2 \cong \mathbb{P}^1$ and all the fibres are isomorphic to $\mathbb{C}$ (although it is not a locally trivial fibration).

5.3. **Failure of Mirror Symmetry for $\mathcal{M}_{J_\pm}$.**

Mirror Symmetry predicts an equality of the E-polynomials of the G-character varieties and G^L-character varieties, where G^L is the Langlands dual of G, for *semisimple holonomy*.

Mirror Symmetry was proved for the moduli spaces of G-Higgs bundles of ranks 2 and 3 by Hausel and Thaddeus [10], showing that the mirror partners have equal Hodge numbers. It was conjectured and proved by Hausel and Rodriguez-Villegas [9] that the same happens for the twisted G-character variety.

For $G = \mathrm{SL}(2,\mathbb{C})$, we have $G^L = \mathrm{PGL}(2,\mathbb{C})$. We want to show now that the analogous statement to the conjecture of [9] does not hold for G-character varieties with non-semisimple holonomy $\xi = J_\pm$. We check it for curves of genus $g = 1$. For this, let us compute the E-polynomial of

$$\mathcal{M}_\xi^L := \mathcal{M}_\xi(\mathrm{PGL}(2,\mathbb{C})),$$

for $\xi = \mathrm{Id}, -\mathrm{Id}, J_+, J_-$ and $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, with $g = 1$.

Note that $\mathcal{M}_\xi^L$ appears as the quotient of $\mathcal{M}_\xi$ with the action of $Z = \mathbb{Z}_2 \times \mathbb{Z}_2$ given by $A \mapsto -A$, $B \mapsto -B$, for $(A, B) \in \mathcal{M}_\xi$.

- We have $\mathcal{M}_{\mathrm{Id}} = (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as $(\lambda, \mu) \mapsto (\lambda^{-1}, \mu^{-1})$. We quotient now by $\lambda \mapsto -\lambda$ and $\mu \mapsto -\mu$. As $\mathbb{C}^*/\pm 1 \cong \mathbb{C}^*$, we get that $\mathcal{M}_{\mathrm{Id}}^L = \mathcal{M}_{\mathrm{Id}}/Z \cong (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2$. Therefore $e(\mathcal{M}_{\mathrm{Id}}^L) = q^2 + 1$, as expected.
- $\mathcal{M}_{-\mathrm{Id}}$ is one point, so $\mathcal{M}_{-\mathrm{Id}}^L$ is one point also, and $e(\mathcal{M}_{-\mathrm{Id}}^L) = 1$.
- The discussion in subsection 4.3 produced a description $\mathcal{M}_{J_+} = \mathbb{C}^* \times \mathbb{C}^* \setminus \{(\pm 1, \pm 1)\}$. The action of Z on $(a, x) \in \mathcal{M}_{J_+}$ is given by

$$(33) \qquad\qquad a \mapsto -a, \quad x \mapsto -x.$$

  So $\mathcal{M}_{J_+}^L = (\mathbb{C}^* \times \mathbb{C}^* \setminus \{(\pm 1, \pm 1)\})/Z \cong \mathbb{C}^* \times \mathbb{C}^* \setminus \{(1, 1)\}$. Therefore $e(\mathcal{M}_{J_+}^L) = q^2 - 2q \neq e(\mathcal{M}_{J_+})$.
- We have to work a little bit for $\mathcal{M}_{J_-}^L$. We decompose $\mathcal{M}_{J_-}$ in several pieces, following the discussion in subsection 4.4. The first piece, corresponding to $\overline{X}_3'$, yields a space $\mathbb{C}^* \times \{\pm i\}$, and Z acts on $(a, x) \in \mathbb{C}^* \times \{\pm i\}$ by (33). This produces $(\mathbb{C}^* \times \{\pm i\})/Z \cong \mathbb{C}^*$ with E-polynomial $q - 1$.

  The second piece, corresponding to S^a, is described by $a = -d = \pm i$, $x \in \mathbb{C}^*$, with the action (33). This contributes $q - 1$ to the E-polynomial.

Finally, the remainder is given by equation (10), where (a, d, x) satisfies $a+d \neq 0$, $x \neq 0$, and subject to the quotient by (33). We do the change of variables $t = (a + d)/x$, where $t \sim -t$. Now we introduce the variable $\alpha = t^2$, and have the equation $ad - \alpha = 1$, modulo $(a, d) \sim (-a, -d)$. Here $\alpha = ad - 1$ is uniquely determined, so we can forget about it, only recalling that $ad - 1 \neq 0$. The resulting space is $\{(a, d) \mid a + d \neq 0, ad \neq 1\}/\mathbb{Z}_2$, whose E-polynomial is easily computed to be $q^2 - 2q + 2$. Hence

$$e(\mathcal{M}_{J_-}^L) = q^2 - 2q + 2 + 2(q - 1) = q^2 \neq e(\mathcal{M}_{J_-}).$$

- The E-polynomial of $\mathcal{M}_\lambda^L$ has not been computed yet. The conjecture in [9] predicts that $e(\mathcal{M}_\lambda^L) = q^2 + 4q + 1$. We shall undertake a full study of the case of $\mathrm{PGL}(2, \mathbb{C})$-character varieties in future work, including this case and many others.

6. The Hodge monodromy representation of X_4

Recall that

$$\overline{X}_4 = \left\{ (A, B, \lambda) \mid [A, B] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \ \lambda \neq 0, \pm 1 \right\}.$$

In this section we want to study in detail the fibration

$$\pi : \overline{X}_4 \longrightarrow \mathbb{C} \setminus \{0, \pm 1\},$$

where $(A, B, \lambda) \mapsto \lambda$. The fibres of π are the spaces $\overline{X}_{4,\lambda}$, all of them diffeomorphic and of balanced type. In particular, all fibres have the same E-polynomial.

Consider loops γ_0, γ_1, γ_{-1} surrounding the points $0, 1, -1$ respectively. These are free generators of $\Gamma = H_1(\mathbb{C} \setminus \{0, \pm 1\})$. We are interested in computing the Hodge monodromy representation of π,

$$R(\overline{X}_4) \in R(\Gamma)[q].$$

Theorem 6.1. *The actions of $\gamma_{\pm 1}$ are trivial, and the action of γ_0 is an involution. Consider the associated quotient $\Gamma \twoheadrightarrow \mathbb{Z}_2$. Then*

$$R(\overline{X}_4) = (q^3 - 1) T + (3q^2 - 3q) N \in R(\mathbb{Z}_2)[q],$$

where T is the trivial representation, and N is the non-trivial one.

Proof. According to subsection 4.5, we have a stratification $\overline{X}_4 = \bigsqcup \overline{X}_4^i$, which is compatible with the fibration. Therefore

$$R(\overline{X}_4) = \sum R(\overline{X}_4^i).$$

Note that we have isomorphisms $\overline{X}_{4,\lambda}^i \cong S_\lambda^i \times \mathbb{C}^*$, so $e(\overline{X}_{4,\lambda}^i) = e(S_\lambda^i)(q - 1)$. Therefore $R(\overline{X}_4^i) = (q-1)R(S^i)$. Let us compute each $R(S^i)$ individually. We follow the notations of subsection 4.5.

- S_λ^0. The equations are $a^2 = \lambda$, where λ is fixed. So S_λ^0 consists of two points. The actions of γ_1 and γ_{-1} are trivial. The action of γ_0 is an involution on the fibre, as it swaps the two points. Therefore $R(S^0) = T + N$.

- S_λ^1. Exactly as in the previous case, $R(S^1) = T + N$.
- S_λ^2. Recall that there are two different (isomorphic) strata, given by either $b \neq 0, c = 0$ or $b = 0, c \neq 0$. The equations of the first stratum are $a^2 = \lambda$, $y = (1 - \lambda)ax/\lambda$, and $z = (\lambda x^2 - 1)/y$. Hence a admits two values, and $x \neq 0$. The elements $\gamma_{\pm 1}$ act trivially, and γ_0 swaps the two values of a. Doubling the contribution, we get $R(S^2) = 2(q - 1)T + 2(q - 1)N$.
- S_λ^3. Exactly as in the previous case, $R(S^3) = 2(q - 1)T + 2(q - 1)N$.
- S_λ^4. Again we have two different (isomorphic) strata. The equations determining one of them are $a^2 = \lambda$, $x = \pm\frac{1}{a}$, $y = \pm\frac{1-a^2}{a^2}$, so it consists of 4 points. The elements $\gamma_{\pm 1}$ act trivially, and γ_0 swaps the two values of a. Doubling the contribution, we get $R(S^4) = 4T + 4N$.
- S_λ^5. Now $a = \pm\frac{2\lambda}{\lambda+1}$ has two values, giving two (isomorphic) strata. For each value, we have the equation:

$$\left(x - \frac{\lambda - 1}{\lambda + 1}y\right)^2 = \frac{1}{\lambda},$$

with the conditions $x \neq \pm\frac{1}{\sqrt{\lambda}}$. Hence we get two parallel lines with two points removed in each of them. The actions of γ_1 and γ_{-1} are trivial, and the action of γ_0 interchanges the two lines. Doubling the contribution, we get

$$R(S^5) = 2(q - 2)T + 2(q - 2)N.$$

- S_λ^6. For a fixed λ, S_λ^6 is a surface which consists of a family of projective conics

$$\lambda^2 x + a(1 - \lambda)xy + (\lambda - a^2)y^2 = u^2,$$

parameterised by $a \in B = \mathbb{C} \setminus \{\pm\sqrt{\lambda}, \pm\frac{2\lambda}{\lambda+1}\}$. The actions of γ_1 and γ_{-1} in S_λ^6 are again trivial. The action of γ_0 interchanges two of the points in the base B of the fibration. So $e(B)^{inv} = q - 3$, and $e(B) - e(B)^{inv} = -1$. As $S_\lambda^6 \to B$ is a fibration whose fibres are conics, we have

$$R(S^6) = (q + 1)\big((q - 3)T - N\big) = (q^2 - 2q - 3)T - (q + 1)N.$$

- S_λ^7. From S_λ^6 we have to remove some points. Firstly, from each fibre (conic) we consider the points $x = \pm\frac{1}{\sqrt{\lambda}}$, $y = 0$; and $x = \pm\frac{1}{\sqrt{\lambda}}$, $y = -a(1 - \lambda)x/(\lambda - a^2)$, where $a \in \mathbb{C} \setminus \{\pm\sqrt{\lambda}, \pm\frac{2\lambda}{\lambda+1}\}$. Geometrically, this produces two disjoint components, each of which consists of 2 punctured lines (punctured at 4 points), intersecting at one point (corresponding to the value $a = 0$). This object has E-polynomial $2q - 9$, and there are two copies of it.

  The action of $\gamma_{\pm 1}$ is trivial, and the action of γ_0 interchanges the components. So $R(S^7) = (2q - 9)T + (2q - 9)N$.
- S_λ^8. We also remove the points of the hyperbola $\Omega^2 = A^2 - 4$ minus $(A, \Omega) = (\pm 2, 0), (\pm\frac{1+\lambda}{\sqrt{\lambda}}, \pm\frac{1-\lambda}{\sqrt{\lambda}})$ The action of $\gamma_{\pm 1}$ is again trivial, and the action of γ_0 leaves the first two points fixed and interchanges the other four in two pairs. So $R(S^8) = (q - 5)T - 2N$.

Altogether, we get

$$
\begin{aligned}
R(\overline{X}_4) &= R(\overline{X}_4^0) + R(\overline{X}_4^1) + R(\overline{X}_4^2) + \\
&\quad + R(\overline{X}_4^3) - R(\overline{X}_4^4) + R(\overline{X}_4^5) + R(\overline{X}_4^6) - R(\overline{X}_4^7) - R(\overline{X}_4^8) \\
&= (q-1)((q^2 + q + 1)T + 3qN) \\
&= (q^3 - 1)T + (3q^2 - 3q)N \,.
\end{aligned}
$$

$\square$

A lot of information can be extracted from Theorem 6.1. For instance, the E-polynomial of the fibre is obtained by

$$
aT + bN \mapsto a + b
$$

(i.e., evaluating the characters $\chi \mapsto \chi(1)$). So

$$
e(\overline{X}_{4,\lambda}) = (q^3 - 1) + (3q^2 - 3q) = q^3 + 3q^2 - 3q - 1 \,.
$$

The E-polynomial of the total space is evaluated using Proposition 2.10,

$$
aT + bN \mapsto (q-1)a - 2(a+b) = (q-3)a - 2b \,.
$$

Using Theorem 6.1, we get

$$
e(\overline{X}_4) = (q-3)(q^3 - 1) - 2(3q^2 - 3q) = q^4 - 3q^3 - 6q^2 + 5q + 3 \,,
$$

which agrees with (32).

7. The quotient of $\overline{X}_4$ by $\mathbb{Z}_2$

Now we recall that there is an action of $\mathbb{Z}_2$ on $\overline{X}_4$, given by

$$
(A, B, \lambda) \mapsto (P_0 A P_0^{-1}, P_0 B P_0^{-1}, \lambda^{-1}),
$$

where $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then there is a diagram

$$
\begin{array}{ccc}
\overline{X}_4 & \longrightarrow & \overline{X}_4/\mathbb{Z}_2 \\
\downarrow & & \downarrow \\
\mathbb{C} \setminus \{0, \pm 1\} & \longrightarrow & \mathbb{C} \setminus \{\pm 2\},
\end{array}
$$

where the bottom map is $\lambda \mapsto s = \lambda + \lambda^{-1}$. The fibration $\overline{X}_4/\mathbb{Z}_2 \to \mathbb{C} \setminus \{\pm 2\}$ is given by $(A, B) \mapsto \mathrm{Tr}\,[A, B] = s = \lambda + \lambda^{-1}$.

7.1. **The Hodge monodromy representation of $\overline{X}_4/\mathbb{Z}_2$.** We want to calculate the Hodge monodromy representation

$$R(\overline{X}_4/\mathbb{Z}_2) \in R(\Gamma')[q] \,,$$

where $\Gamma' = H_1(\mathbb{C} \setminus \{\pm 2\})$.

Write $B = \mathbb{C} \setminus \{0, \pm 1\}$ and $B' = \mathbb{C} \setminus \{\pm 2\}$, where $B' = B/\mathbb{Z}_2$. Let η_2, η_{-2} denote small loops in B' around 2 and -2, respectively. The group Γ' is freely generated by η_2, η_{-2}. There is an exact sequence

$$\Gamma \to \Gamma' \to \mathbb{Z}_2 \to 0 \,,$$

where $\gamma_1 \mapsto 2\eta_2$, $\gamma_{-1} \mapsto 2\eta_{-2}$ and $\gamma_0 \mapsto \eta_{-2} + \eta_2$. By Theorem 6.1, the action of $\gamma_{\pm 1}$ on $\overline{X}_4$ is trivial, and the action of $2\gamma_0$ is also trivial. This implies that the actions of $2\eta_2$ and $2\eta_{-2}$ are trivial. Hence, the Hodge monodromy representation of $\overline{X}_4/\mathbb{Z}_2$ descends to a representation of the quotient group

$$\Gamma' = H_1(B') \twoheadrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \,.$$

The conclusion is that

$$R(\overline{X}_4/\mathbb{Z}_2) \in R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q] \,.$$

We shall denote by T, S_2, S_{-2}, S_0 the four irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The trivial representation is T, S_2 is characterised by $\rho(\eta_2) = \mathrm{Id}$, S_{-2} is characterised by $\rho(\eta_{-2}) = \mathrm{Id}$, and S_0 is characterised by $\rho(\gamma_0) = \mathrm{Id}$, where $\gamma_0 = \eta_2 + \eta_{-2}$.

Theorem 7.1. *For $\overline{X}_4/\mathbb{Z}_2$, we have*

$$R(\overline{X}_4/\mathbb{Z}_2) = q^3 \, T - 3q \, S_2 + 3q^2 \, S_{-2} - S_0 \in R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q] \,.$$

Proof. As in the proof of Theorem 6.1, we shall follow the stratification described in subsection 4.5. Note that $\mathbb{Z}_2$ acts on each of the strata $\overline{X}_4^i$.

We need to check the monodromy when we describe small loops around -2 and 2 in the s-plane $\mathbb{C} \setminus \{\pm 2\}$. The composition of the two loops is homotopic to a big loop encircling both ± 2. To understand the monodromy action, we have to lift the loops to paths in $B = \mathbb{C} \setminus \{0, \pm 1\}$. Due to the monodromy around 0, we cannot trivialize the local system on B. So we cut B along the axis $L = \{-y\, i \mid y > 0\}$, and take a determination of $\sqrt{\lambda}$ there. In this way, we can identify all fibres over $B \setminus L$. The image of L under $h : \lambda \mapsto s = \lambda + \lambda^{-1}$ is the whole imaginary axis. We shall fix the fibre over $b_0 = 1 + \epsilon$, where $\epsilon > 0$ is a small real number, and identify all fibres to it.

(1) Going around η_2 consists (in the λ-plane) of a path from $1 + \epsilon$ to $1/(1 + \epsilon)$, followed by the $\mathbb{Z}_2$-action.

(2) Going around η_{-2} consists of a path from $-1 + \epsilon$ to $1/(-1 + \epsilon)$ followed by the $\mathbb{Z}_2$-action. Here we have to identify the fiber over $-1 + \epsilon$ with the fiber over $b_0 = 1 + \epsilon$ via a path σ in the λ-plane. Returning from $-1 + \epsilon$ to b_0 is the $\mathbb{Z}_2$-transform of going from $1/(-1 + \epsilon)$ to $1/(1 + \epsilon)$ via the transform of σ^{-1}. We can take σ not to cross L, and then this new path does cross L once.

(3) Going around $\gamma_0 = \eta_2 + \eta_{-2}$ consists of doing both loops and joining them. This crosses the vertical axis. Taking the preimage under h, it gives a loop that encircles 0 and crosses L once.

Note that the action of γ_0 is the composition of the actions of η_2 and η_{-2}. We shall use below this fact in the form: the action of η_{-2} is the composition of the actions of η_2 and γ_0. The action of γ_0 is basically known thanks to the computations in the proof of Theorem 6.1.

To abbreviate, we shall write $F^i = \overline{X}_{4,\lambda}^i \cong S_\lambda^i \times \mathbb{C}^*$ for the fiber of $\overline{X}_{4,\lambda}^i \longrightarrow B'$. For a loop $\alpha = \gamma_0, \eta_2, \eta_{-2}$, we shall use the notation $e(F^i)^\alpha$ to mean the E-polynomial of $H_c^*(F^i)^\alpha$, i.e., the invariant part of the cohomology $H_c^*(F^i)$ under the action of α. Also $e(F)^{inv}$ will stand for the E-polynomial of $H_c^*(F)^{\Gamma'}$, as usual. We have the following:

- $\overline{X}_4^0/\mathbb{Z}_2$. The fibre consists of 2 copies of $\mathbb{C}^*$. We already know that γ_0 acts by permuting the two copies of $\mathbb{C}^*$. Note that the parameter for $\mathbb{C}^*$ is $y \neq 0$, and that the $\mathbb{Z}_2$-action sends $y \mapsto z = -1/y$. The path corresponding to η_2 does not permute the copies of $\mathbb{C}^*$, and turns $\mathbb{C}^*$ inside out (sending 0 to ∞ and vice-versa). Therefore, η_{-2} swaps the two copies of $\mathbb{C}^*$ (and sends 0 to ∞ and vice-versa). Thus

$$
\begin{aligned}
e(F^0) &= 2(q-1), \\
e(F^0)^{inv} &= q, \\
e(F^0)^{\eta_2} &= 2q, \\
e(F^0)^{\eta_{-2}} &= q-1, \\
e(F^0)^{\gamma_0} &= q-1.
\end{aligned}
$$

The conclusion is that $R(\overline{X}_4^0/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$, where $a = e(F)^{inv}$, $a+b = e(F)^{\eta_2}$, $a+c = e(F)^{\eta_{-2}}$ and $a+d = e(F)^{\gamma_0}$. Thus

$$
R(\overline{X}_4^0/\mathbb{Z}_2) = qT + qS_2 - S_{-2} - S_0 \,.
$$

Note that $e(F^0) = a+b+c+d = 2q-2$.

- $\overline{X}_4^1/\mathbb{Z}_2$. Exactly as above, $R(\overline{X}_4^1/\mathbb{Z}_2) = qT + qS_2 - S_{-2} - S_0$.
- $\overline{X}_4^2/\mathbb{Z}_2$. By our previous analysis, we have a $\mathbb{C}^*$-bundle (the slice fixing condition) over a collection of two sets (given by either $b \neq 0, c = 0$ or $b = 0, c \neq 0$), each of which consists of two copies of $\mathbb{C}^*$ (given by the two solutions of $a^2 = \lambda$). The involution γ_0 interchanges the two values of a. On the other hand, the $\mathbb{Z}_2$-action interchanges the two sets. So

$$
\begin{aligned}
e(F^2) &= 4(q-1)^2, \\
e(F^2)^{inv} &= (q-1)^2, \\
e(F^2)^{\eta_2} &= 2(q-1)^2, \\
e(F^2)^{\eta_{-2}} &= 2(q-1)^2, \\
e(F^2)^{\gamma_0} &= 2(q-1)^2.
\end{aligned}
$$

Thus

$$
R(\overline{X}_4^2/\mathbb{Z}_2) = (q-1)^2 T + (q-1)^2 S_2 + (q-1)^2 S_{-2} + (q-1)^2 S_0 \,.
$$

- $\overline{X}_4^3/\mathbb{Z}_2$. As before, $R(\overline{X}_4^3/\mathbb{Z}_2) = (q-1)^2 T + (q-1)^2 S_2 + (q-1)^2 S_{-2} + (q-1)^2 S_0$.

- $\overline{X}_4^4/\mathbb{Z}_2$. Now we have 8 copies of $\mathbb{C}^*$ (2 solutions to $a^2 = \lambda$; a choice of sign in $x = \pm\frac{1}{a}$; and the choice of whether $b = z = 0$ or $c = y = 0$). The action of γ_0 interchanges the copies of $\mathbb{C}^*$ in pairs (corresponding to the solutions to $a^2 = \lambda$). The action of η_2 (or η_{-2}) interchanges the strata $b = z = 0$ and $c = y = 0$. So finally,

$$\begin{aligned}
e(F^4) &= 8(q-1), \\
e(F^4)^{inv} &= 2(q-1), \\
e(F^4)^{\eta_2} &= 4(q-1), \\
e(F^4)^{\eta_{-2}} &= 4(q-1), \\
e(F^4)^{\gamma_0} &= 4(q-1).
\end{aligned}$$

Thus

$$R(\overline{X}_4^4/\mathbb{Z}_2) = 2(q-1)T + 2(q-1)S_2 + 2(q-1)S_{-2} + 2(q-1)S_0\,.$$

- $\overline{X}_4^5/\mathbb{Z}_2$. Now there are two possibilites $a = \pm\frac{2\lambda}{\lambda+1}$. Consider one of the possibilities, say $a = \frac{2\lambda}{\lambda+1}$ (the other one is analogous). Then the fibre F^5 is a $\mathbb{C}^*$-bundle over S_λ^5, which consists of two lines (given by equation (25)), parameterised by a square root of λ, each with two points removed. The action of γ_0 interchanges the two lines.

   To find the monodromy of η_2, we have to consider how the $\mathbb{Z}_2$-action behaves in a small neighbourhood of $\lambda = 1$. There, we have a well-defined square root $\sqrt{\lambda}$. Hence the two lines have equations

(34)
$$\sqrt{\lambda}\,x - \sqrt{\lambda}\frac{\lambda - 1}{\lambda + 1}y = \pm 1\,.$$

   The $\mathbb{Z}_2$-action sends $\sqrt{\lambda} \mapsto \sqrt{\lambda}^{-1}$, $\sqrt{\lambda}\,x \mapsto \sqrt{\lambda}\,x$, and

$$\begin{aligned}
\sqrt{\lambda}\frac{\lambda - 1}{\lambda + 1}y \;\mapsto\;& \sqrt{\lambda}^{-1}\frac{\lambda^{-1} - 1}{\lambda^{-1} + 1}\left(\lambda y + \frac{\lambda(\lambda - 1)a}{a^2 - \lambda}x\right) \\
=\;& -\sqrt{\lambda}\frac{\lambda - 1}{\lambda + 1}y + 2\sqrt{\lambda}\,x\,.
\end{aligned}$$

   Therefore the two lines (34) get interchanged.

   By composition, the action of η_{-2} preserves each line. With the notation of subsection 4.5.7, each line may be described as $F = \mathbb{C} \setminus \{\pm 1\}$, and the $\mathbb{Z}_2$-action swaps ± 1. As we have a $\mathbb{C}^*$-bundle over F, we compute using $e(\mathbb{C}^*)^+ = q$, $e(\mathbb{C}^*)^- = -1$, $e(F)^+ = q - 1$, $e(F)^- = -1$, to obtain the Hogde-Deligne polynomial $q(q-1) + 1$. Doubling all polynomials to account for the other value $a = -\frac{2\lambda}{\lambda+1}$, we summarise this as

$$\begin{aligned}
e(F^5) &= 4(q-1)(q-2), \\
e(F^5)^{inv} &= 2(q^2 - q + 1), \\
e(F^5)^{\eta_2} &= 2(q-1)(q-2), \\
e(F^5)^{\eta_{-2}} &= 4(q^2 - q + 1), \\
e(F^5)^{\gamma_0} &= 2(q-1)(q-2).
\end{aligned}$$

Thus
$$R(\overline{X}_4^5/\mathbb{Z}_2) = 2(q^2 - q + 1)T + 2(-2q + 1)S_2 + 2(q^2 - q + 1)S_{-2} + 2(-2q + 1)S_0\,.$$

- $\overline{X}_4^6/\mathbb{Z}_2$. We have a $\mathbb{C}^*$-bundle over a family of complete smooth conics over the space of $a \in \mathbb{C} \setminus \{\pm\sqrt{\lambda}, \pm\frac{2\lambda}{\lambda+1}\}$. The action of γ_0 interchanges the first two of the points in the line parameterised by a. The action of η_{-2} does the same, but turns $\mathbb{C}^*$ inside-out. The action of η_2 fixes the line and turns $\mathbb{C}^*$ inside-out. Multiplying all polynomials by $(q + 1)$ since we are dealing with conic bundles, we obtain

$$\begin{aligned}
e(F^6) &= (q - 1)(q - 4)(q + 1), \\
e(F^6)^{inv} &= q(q - 3)(q + 1), \\
e(F^6)^{\eta_2} &= q(q - 4)(q + 1), \\
e(F^6)^{\eta_{-2}} &= (q(q - 3) + 1)(q + 1), \\
e(F^6)^{\gamma_0} &= (q - 1)(q - 3)(q + 1).
\end{aligned}$$

Thus
$$R(\overline{X}_4^6/\mathbb{Z}_2) = (q + 1)\big((q^2 - 3q)T - qS_2 + S_{-2} - (q - 3)S_0\big).$$

- $\overline{X}_4^7/\mathbb{Z}_2$. By our previous discussion, the fibre consists of a $\mathbb{C}^*$-bundle over two copies of the variety consisting of the nodal curve N which is two lines $\mathbb{C} \setminus \{4 \text{ points}\}$ joined by the origin. Note that $e(N) = 2q - 9$. The action of γ_0 interchanges the two nodal curves in pairs. Clearly, η_{-2} does the same, and η_2 fixes each nodal curve. The nodal curve is parameterised by $y \in \{0, a(1 - \lambda)x/(a^2 - \lambda)\}$, where $a \in \mathbb{C} \setminus \{\pm\sqrt{\lambda}, \pm\frac{2\lambda}{\lambda+1}\}$. We know that the $\mathbb{Z}_2$-action sends $y = 0 \mapsto y = a(1 - \lambda)x/(a^2 - \lambda)$, so the quotient $N/\mathbb{Z}_2$ is a punctured line $\mathbb{C} \setminus \{4 \text{ points}\}$, hence $e(N)^+ = q - 4$ and $e(N)^- = q - 5$. As $e(\mathbb{C}^*)^+ = q$, $e(\mathbb{C}^*)^- = -1$, we get the polynomial $(q - 4)q - (q - 5) = q^2 - 5q + 5$. Hence

$$\begin{aligned}
e(F^7) &= 2(q - 1)(2q - 9), \\
e(F^7)^{inv} &= q^2 - 5q + 5, \\
e(F^7)^{\eta_2} &= 2(q^2 - 5q + 5), \\
e(F^7)^{\eta_{-2}} &= (q - 1)(2q - 9), \\
e(F^7)^{\gamma_0} &= (q - 1)(2q - 9).
\end{aligned}$$

Thus
$$R(\overline{X}_4^7/\mathbb{Z}_2) = (q^2 - 5q + 5)T + (q^2 - 5q + 5)S_2 + (q^2 - 6q + 4)S_{-2} + (q^2 - 6q + 4)S_0\,.$$

- $\overline{X}_4^8/\mathbb{Z}_2$. We follow the notations of subsection 4.5.10. The fiber F^8 is a $\mathbb{C}^*$-bundle over the hyperbola $\Omega^2 = A^2 - 4$ minus six points: $(A, \Omega) = (\pm 2, 0)$ and the four points $(\pm\frac{1+\lambda}{\sqrt{\lambda}}, \pm\frac{1-\lambda}{\sqrt{\lambda}})$ (all signs allowed).

  The monodromies around γ_0, $\eta_{\pm 2}$ fix the points at infinity and the first two points. Let us see what happens with the remaining four points: they are parameterised by $\Omega^2 = A^2 - 4$ and $A^2 = s + 2$, where $s = \lambda + \lambda^{-1}$. Clearly, γ_0 changes both signs simultaneously $(r_1, r_2) \mapsto (-r_1, -r_2)$, $(r_1, r_2) \in \{(\pm\frac{1+\lambda}{\sqrt{\lambda}}, \pm\frac{1-\lambda}{\sqrt{\lambda}})\}$. The map $(A, \Omega) \mapsto s = A^2 - 2$ ramifies at $A = 0$, $s = -2$, so η_{-2} has the effect

$(r_1, r_2) \mapsto (-r_1, r_2)$. Similarly, $(A, \Omega) \mapsto s = \Omega^2 + 2$ ramifies at $\Omega = 0$, $s = 2$, so η_2 has the effect $(r_1, r_2) \mapsto (r_1, -r_2)$. Recalling that $\eta_{\pm 2}$ also turns $\mathbb{C}^*$ inside-out, we obtain

$$
\begin{aligned}
e(F^8) &= (q-1)(q-7), \\
e(F^8)^{inv} &= a, \\
e(F^8)^{\eta_2} &= q(q-5) + 2, \\
e(F^8)^{\eta_{-2}} &= q(q-5) + 2, \\
e(F^8)^{\gamma_0} &= (q-1)(q-5),
\end{aligned}
$$

for some number a. Let us determine the value of a :

$$R(\overline{X}_4^8/\mathbb{Z}_2) = aT + (q^2 - 5q + 2 - a)S_2 + (q^2 - 5q + 2 - a)S_{-2} + (q^2 - 6q + 5 - a)S_0.$$

Considering the E-polynomial of the fibre, we get $3q^2 - 16q + 9 - 2a = q^2 - 8q + 7$, and hence $a = q^2 - 4q + 1$. This gives

$$R(\overline{X}_4^8/\mathbb{Z}_2) = (q^2 - 4q + 1)T + (-q + 1)S_2 + (-q + 1)S_{-2} + (-2q + 4)S_0.$$

Adding up all contributions together, we get

$$R(\overline{X}_4/\mathbb{Z}_2) = q^3 T - 3q S_2 + 3q^2 S_{-2} - S_0.$$

$\square$

Remark 7.2. In the proof of Theorem 7.1, we have chosen a base point b_0. This produces an asymmetry in some of the calculations. If we chose a different base point (or a different cut-locus for the plane B), some of the intermediate calculations would change, but the final answer would be the same. The asymmetry between the coefficients of S_2 and S_{-2} in $R(\overline{X}_4/\mathbb{Z}_2)$ is inherent to the geometry of the situation.

It is clear that if

$$R(\overline{X}_4/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$$

then

$$R(\overline{X}_4) = (a + d)T + (b + c)N = (q^3 - 1)T + (3q^2 - 3q)N,$$

agreeing with Theorem 6.1. For the fibre,

$$e(\overline{X}_{4,\lambda}) = a + b + c + d = q^3 + 3q^2 - 3q - 1,$$

agreeing with (31).

We also recover the E-polynomial of the total space $e(\overline{X}_4/\mathbb{Z}_2)$ using Proposition 2.10,

$$
\begin{aligned}
e(\overline{X}_4/\mathbb{Z}_2) &= (q-1)a - (a + b + c + d) \\
&= (q-2)a - (b + c + d) \\
&= (q-2)q^3 - (3q^2 - 3q - 1) \\
&= q^4 - 2q^3 - 3q^2 + 3q + 1.
\end{aligned}
$$

(35)

Remark 7.3. More generally, suppose that we have a fibration

$$\overline{X} \longrightarrow B = \mathbb{C} \setminus \{0, \pm 1\},$$

with a $\mathbb{Z}_2$-action compatible with the action $\lambda \mapsto \lambda^{-1}$ in the base. Then we have a fibration

$$\overline{X}/\mathbb{Z}_2 \longrightarrow B' = \mathbb{C} \setminus \{\pm 2\}.$$

Suppose that the Hodge monodromy representation is of the form

$$R(\overline{X}/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0,$$

then we have

$$
\begin{aligned}
e(\overline{X}) &= (q-3)(a+d) - 2(b+c), \\
e(\overline{X})^+ &= (q-2)a - (b+c+d), \\
e(\overline{X})^- &= (q-2)d - (a+b+c).
\end{aligned}
$$

7.2. Application: Recovering the E-polynomial of X_4.

Consider the set

$$\widetilde{X}_4 = \{(A, B, \ell) \mid \xi = [A, B], \operatorname{Tr}(\xi) = \lambda + \lambda^{-1}, \lambda \neq 0, \pm 1, \ell \text{ eigenspace of } \xi\}.$$

There are fibrations

(36)
$$
\begin{array}{ccccc}
\mathbb{C}^* & \longrightarrow & \mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_4 & \longrightarrow & \widetilde{X}_4 \\
\| & & \downarrow & & \downarrow{\scriptstyle 2:1} \\
\mathbb{C}^* & \longrightarrow & (\mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_4)/\mathbb{Z}_2 & \longrightarrow & X_4
\end{array}
$$

where the top map is $(P, A, B, \lambda) \mapsto \left(PAP^{-1}, PBP^{-1}, P\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, which is a principal bundle with fibre $D/\mathbb{C}^* \cong \mathbb{C}^*$, given by $\mu \sim \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$. The action by $\mathbb{Z}_2$ in the middle is given by $(P, A, B, \lambda) \mapsto (PP_0^{-1}, P_0AP_0^{-1}, P_0BP_0^{-1}, \lambda^{-1})$, $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. On the right hand column, we have the $2:1$-map $(A, B, \ell) \mapsto (A, B)$. Note that the $\mathbb{Z}_2$-action on the fibre is

$$\mu \sim \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \sim \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 1 \end{pmatrix} \sim \mu^{-1}.$$

From (36) and using Remark 2.5, we get first that

$$e(\widetilde{X}_4) = \frac{e(\mathrm{PGL}(2,\mathbb{C}))e(\overline{X}_4)}{e(\mathbb{C}^*)} = q(q+1)e(\overline{X}_4).$$

Now Proposition 2.6 and equation (1) yield

$$
\begin{aligned}
e((\mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_4)/\mathbb{Z}_2) &= e(\widetilde{X}_4)^+ e(\mathbb{C}^*)^+ + e(\widetilde{X}_4)^- e(\mathbb{C}^*)^- \\
&= e(X_4)q - (e(\widetilde{X}_4) - e(X_4)) \\
&= (q+1)e(X_4) - q(q+1)e(\overline{X}_4).
\end{aligned}
$$

On the other hand, using Corollary 3.3,

$$
\begin{aligned}
e((\mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_4)/\mathbb{Z}_2) &= e((\mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_4)^+ \\
&= e(\mathrm{PGL}(2,\mathbb{C}))^+ e(\overline{X}_4)^+ + e(\mathrm{PGL}(2,\mathbb{C}))^- e(\overline{X}_4)^- \\
&= e(\mathrm{PGL}(2,\mathbb{C})) e(\overline{X}_4/\mathbb{Z}_2) \\
&= q(q^2 - 1) e(\overline{X}_4/\mathbb{Z}_2) \,.
\end{aligned}
$$

Equating these two formulae, we get

$$
q(q^2 - 1) e(\overline{X}_4/\mathbb{Z}_2) = (q+1) e(X_4) - q(q+1) e(\overline{X}_4)
$$

and hence

$$
e(X_4) = (q^2 - q) e(\overline{X}_4/\mathbb{Z}_2) + q\, e(\overline{X}_4).
$$

Now use (35) and (32) to get

$$
\begin{aligned}
e(X_4) &= (q^2 - q)(q^4 - 2q^3 - 3q^2 + 3q + 1) + q(q^4 - 3q^3 - 6q^2 + 5q + 3) \\
&= q^6 - 2q^5 - 4q^4 + 3q^2 + 2q \,,
\end{aligned}
$$

agreeing with our previous formula (30).

Remark 7.4. Suppose that we are in the situation of Remark 7.3, and that we have a diagram

$$
\begin{array}{ccccc}
\mathbb{C}^* & \longrightarrow & \mathrm{PGL}(2,\mathbb{C}) \times \overline{X} & \longrightarrow & \widetilde{X} \\
\| & & \downarrow & & \downarrow{\scriptstyle 2:1} \\
\mathbb{C}^* & \longrightarrow & (\mathrm{PGL}(2,\mathbb{C}) \times \overline{X})/\mathbb{Z}_2 & \longrightarrow & X
\end{array}
$$

in which the action of $\mathbb{Z}_2$ on the fiber $\mathbb{C}^*$ takes $\mu \mapsto \mu^{-1}$. Then

$$
\begin{aligned}
e(X) &= (q^2 - q) e(\overline{X})^+ + q\, e(\overline{X}) \\
(37) \qquad &= q(q-1)\big((q-2)a - (b+c+d)\big) + q\big((q-3)(a+d) - 2(b+c)\big) \\
&= q(q^2 - 2q - 1)a - q(q+1)(b+c) - 2qd \,,
\end{aligned}
$$

where $R(\overline{X}/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$.

8. E-POLYNOMIAL OF THE $SL(2,\mathbb{C})$-CHARACTER VARIETY FOR GENUS 2

We consider now the map

$$
\begin{aligned}
F : \mathrm{SL}(2,\mathbb{C})^4 &\longrightarrow \mathrm{SL}(2,\mathbb{C}), \\
(A,B,C,D) &\mapsto [A,B][C,D],
\end{aligned}
$$

where $\mathrm{PGL}(2,\mathbb{C})$ acts on both spaces by conjugation and F is equivariant. The character variety to consider is then

$$
\mathcal{M}_{\mathrm{Id}} = Y // \mathrm{PGL}(2,\mathbb{C}),
$$

where

$$
Y = F^{-1}(\mathrm{Id}) = \{(A,B,C,D) \in \mathrm{SL}(2,\mathbb{C})^4 \,|\, [A,B] = [D,C]\}.
$$

(Here we have to take the GIT quotient because there are reducibles.)

In order to compute its E-polynomial, we stratify Y as follows.

- $Y_0 := \{(A, B, C, D)| \, [A, B] = [D, C] = \mathrm{Id}\,\}$,
- $Y_1 := \{(A, B, C, D)| \, [A, B] = [D, C] = -\,\mathrm{Id}\,\}$,
- $Y_2 := \left\{(A, B, C, D)| \, [A, B] = [D, C] \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$,
- $Y_3 := \left\{(A, B, C, D)| \, [A, B] = [D, C] \sim \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right\}$,
- $Y_4 := \left\{(A, B, C, D)| \, [A, B] = [D, C] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \; \lambda \neq 0, \pm 1\right\}$.

Therefore

$$(38) \qquad\qquad Y = \bigsqcup_{i=0}^{4} Y_i.$$

We also introduce the sets

- $\overline{Y}_0 := \{(A, B, C, D)| \, [A, B] = [D, C] = \mathrm{Id}\,\} = Y_0$,
- $\overline{Y}_1 := \{(A, B, C, D)| \, [A, B] = [D, C] = -\,\mathrm{Id}\,\} = Y_1$,
- $\overline{Y}_2 := \left\{(A, B, C, D)| \, [A, B] = [D, C] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$,
- $\overline{Y}_3 := \left\{(A, B, C, D)| \, [A, B] = [D, C] = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right\}$,
- $\overline{Y}_4 := \left\{(A, B, C, D)| \, [A, B] = [D, C] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \; \lambda \neq 0, \pm 1\right\}$,
- $\overline{Y}_{4,\lambda} := \left\{(A, B, C, D)| \, [A, B] = [D, C] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\}$, for $\lambda \neq 0, \pm 1$.

8.1. **E-polynomial of Y.** We use the stratification (38).

- E-polynomial of Y_0. Clearly $Y_0 = X_0 \times X_0$, hence

$$e(Y_0) = e(X_0)^2 = (q^4 + 4q^3 - q^2 - 4q)^2 = q^8 + 8q^7 + 14q^6 - 16q^5 - 31q^4 + 8q^3 + 16q^2\,.$$

- E-polynomial of Y_1. Similarly, $Y_1 = X_1 \times X_1$, so

$$e(Y_1) = e(X_1)^2 = (q^3 - q)^2 = q^6 - 2q^4 + q^2\,.$$

- E-polynomial of Y_2. Clearly,

$$\overline{Y}_2 = \overline{X}_2 \times \overline{X}_2\,,$$

so $e(\overline{Y}_2) = e(\overline{X}_2)^2$. The stabiliser of $\xi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is $U = \mathbb{C}^* \times \mathbb{C}$. So we have a fibration

$$U \longrightarrow \mathrm{GL}(2, \mathbb{C}) \times \overline{Y}_2 \longrightarrow Y_2\,.$$

Hence

$$\begin{aligned} e(Y_2) &= e(\mathrm{GL}(2, \mathbb{C})/U)e(\overline{X}_2)^2 = (q^2 - 1)(q^3 - 2q^2 - 3q)^2 \\ &= q^8 - 4q^7 - 3q^6 + 16q^5 + 11q^4 - 12q^3 - 9q^2. \end{aligned}$$

- E-polynomial of Y_3. Again as above, $\overline{Y}_3 = \overline{X}_3 \times \overline{X}_3$, and

$$\begin{aligned} e(Y_3) &= e(\mathrm{GL}(2,\mathbb{C})/U)e(\overline{X}_3)^2 = (q^2-1)(q^3+3q^2)^2 \\ &= q^8 + 6q^7 + 8q^6 - 6q^5 - 9q^4. \end{aligned}$$

- E-polynomial of Y_4. It is clear that

(39)
$$\overline{Y}_{4,\lambda} = \overline{X}_{4,\lambda} \times \overline{X}_{4,\lambda}.$$

There is a fibration

$$\overline{Y}_4 \longrightarrow B = \mathbb{C} \setminus \{0, \pm 1\},$$

with a $\mathbb{Z}_2$-action that covers the action $\lambda \mapsto \lambda^{-1}$ on the base. The Hodge monodromy representation of

$$\overline{Y}_4/\mathbb{Z}_2 \longrightarrow B' = \mathbb{C} \setminus \{\pm 2\}$$

is given by

$$R(\overline{Y}_4/\mathbb{Z}_2) = R(\overline{X}_4/\mathbb{Z}_2) \otimes R(\overline{X}_4/\mathbb{Z}_2).$$

Write $R(\overline{X}_4/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$, where

$$a = q^3, \ b = -3q, \ c = 3q^2, \ d = -1.$$

Now, using the fact that $T \otimes S_i = S_i$, $i = 0, \pm 2$, $S_0 \otimes S_2 = S_{-2}$, $S_0 \otimes S_{-2} = S_2$ and $S_2 \otimes S_{-2} = S_0$, we have

$$\begin{aligned} R(\overline{Y}_4/\mathbb{Z}_2) &= (a^2+b^2+c^2+d^2)T + 2(ab+cd)S_2 + 2(ac+bd)S_{-2} + 2(ad+bc)S_0 \\ &= (1+5q^2-8q^3+4q^4+q^6)T + (4q-2q^2+2q^4+4q^5)S_2 \\ &\quad +(2q-4q^2-4q^4)S_{-2} + (-12q^3+q^4)S_0. \end{aligned}$$

(40)

We introduce the set

$$\widetilde{Y}_4 = \{(A,B,C,D,\ell) | [A,B]=[D,C]=\xi, \ \mathrm{Tr}\,\xi = \lambda+\lambda^{-1}, \ \lambda \neq 0, \pm 1, \ \ell \text{ eigenspace of } \xi\}.$$

Then we have a diagram

$$\begin{array}{ccccc} \mathbb{C}^* & \longrightarrow & \mathrm{PGL}(2,\mathbb{C}) \times \overline{Y}_4 & \longrightarrow & \widetilde{Y}_4 \\ \| & & \downarrow & & \downarrow{\scriptstyle 2:1} \\ \mathbb{C}^* & \longrightarrow & (\mathrm{PGL}(2,\mathbb{C}) \times \overline{Y}_4)/\mathbb{Z}_2 & \longrightarrow & Y_4 \end{array}$$

Using Remark 7.4, we have

$$\begin{aligned} e(Y_4) &= q(q-1)e(\overline{Y}_4/\mathbb{Z}_2) + q\,e(\overline{Y}_4) \\ &= q(q^2-2q-1)a' - q(q+1)(b'+c') - 2qd' \\ &= q^9 - 2q^8 + 2q^7 - 12q^6 + 8q^5 + 16q^4 - 12q^3 - 2q^2 + q, \end{aligned}$$

where $R(\overline{Y}_4/\mathbb{Z}_2) = a'T + b'S_2 + c'S_{-2} + d'S_0$ is given by (40).

Adding up all contributions above, we get

$$e(Y) = q^9 + q^8 + 12q^7 + 8q^6 + 2q^5 - 15q^4 - 16q^3 + 6q^2 + q.$$

8.2. **The E-polynomial of $\mathcal{M}_{\mathrm{Id}}$.** We want to calculate the E-polynomial of

$$\mathcal{M}_{\mathrm{Id}} = Y /\!/ \mathrm{PGL}(2,\mathbb{C}) \,.$$

It is clear that there are no reducibles in Y_1 and Y_4. Also there are no reducibles in Y_3 by Remark 5.2. All elements in Y_2 are reducible by Remark 5.1. Finally, Y_0 contains both reducibles and irreducibles.

Contribution from reducibles. A reducible (A, B, C, D) is S-equivalent to a direct sum of two representations in $\mathbb{C}^*$, given as

$$\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_3^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_4 & 0 \\ 0 & \lambda_4^{-1} \end{pmatrix} \right) .$$

Therefore the space of S-equivalence classes of reducible representations is $\mathcal{R} = (\mathbb{C}^*)^4 / \mathbb{Z}_2$, where the $\mathbb{Z}_2$-action is

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}, \lambda_4^{-1}).$$

Using the fact that $e(\mathbb{C}^*)^+ = q$ and $e(\mathbb{C}^*) = -1$, the above variety has E-polynomial

$$\begin{aligned} e(\mathcal{R}) &= (e(\mathbb{C}^*)^+)^4 + \binom{4}{2}(e(\mathbb{C}^*)^+)^2(e(\mathbb{C}^*)^-)^2 + (e(\mathbb{C}^*)^-)^4 \\ &= q^4 + 6q^2 + 1 \,. \end{aligned}$$

Contribution from irreducibles in Y_0. We consider the space

$$\mathcal{I} := \{\text{irreducible elements } (A, B, C, D) | AB = BA, CD = DC\} / \mathrm{PGL}(2,\mathbb{C}).$$

The irreducibility means that there the four matrices have no common eigenvector.

Consider the trace map

$$(A, B, C, D) \mapsto (t_1, t_2, t_3, t_4) = (\mathrm{Tr}\, A, \mathrm{Tr}\, B, \mathrm{Tr}\, C, \mathrm{Tr}\, D) \,.$$

We stratify $\mathcal{I}$ according to various cases:

- $\mathcal{I}_1$ given by the conditions $(t_1, t_2) \neq (\pm 2, \pm 2)$ and $(t_3, t_4) \neq (\pm 2, \pm 2)$. This space is described with the techniques of [20].

  The matrices A, B are diagonalizable and have two well-defined common eigenvectors e_1, e_2 (up to multiplication by scalars). Let λ_1, λ_2 be the eigenvalues of the eigenvector e_1 for A, B, respectively. Also C, D have two eigenvectors f_1, f_2, and let λ_3, λ_4 be the eigenvalues of f_1 for C, D, respectively. The four elements $[e_1], [e_2], [f_1], [f_2] \in \mathbb{P}^1$ are different, so they have a well-defined cross-ratio

  $$r = ([e_1], [e_2], [f_1], [f_2]) \in \mathbb{C} \setminus \{0, 1\}.$$

  There is an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by $e_1, e_2 \mapsto e_2, e_1$ and $f_1, f_2 \mapsto f_2, f_1$. This yields

  $$\begin{aligned} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, r) &\sim (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3, \lambda_4, 1 - r) \\ &\sim (\lambda_1, \lambda_2, \lambda_3^{-1}, \lambda_4^{-1}, 1 - r) \sim (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}, \lambda_4^{-1}, r) \,. \end{aligned}$$

  We write

  $$U_1 := (\mathbb{C}^* \times \mathbb{C}^*) \setminus \{(\pm 1, \pm 1)\}, \ U_2 := (\mathbb{C}^* \times \mathbb{C}^*) \setminus \{(\pm 1, \pm 1)\}, \ U_3 = \mathbb{C} \setminus \{0, 1\}$$

and define

$$\sigma_1 : U_1 \to U_1, \quad \sigma_1(\lambda_1, \lambda_2) = (\lambda_1^{-1}, \lambda_2^{-1}),$$
$$\sigma_2 : U_2 \to U_2, \quad \sigma_2(\lambda_3, \lambda_4) = (\lambda_3^{-1}, \lambda_4^{-1}),$$
$$\tau : U_3 \to U_3, \quad \tau(r) = 1 - r.$$

Then

$$\mathcal{I}_1 = (U_1 \times U_2 \times U_3)/\langle \sigma_1 \times \tau, \sigma_2 \times \tau \rangle.$$

Thus $e(U_1)^+ = e(U_2)^+ = q^2 - 3$, $e(U_1)^- = e(U_2)^- = -2q$, $e(U_3)^+ = q - 1$ and $e(U_3)^- = -1$. We have

$$
\begin{aligned}
e(\mathcal{I}_1) &= e\left((U_1 \times U_2 \times U_3)^{\sigma_1 \times \tau, \sigma_2 \times \tau}\right) \\
&= e(U_1)^+ e(U_2)^+ e(U_3)^+ + e(U_1)^- e(U_2)^- e(U_3)^- \\
&= (q^2 - 3)^2(q - 1) - 4q^2 = q^5 - q^4 - 6q^3 + 2q^2 + 9q - 9.
\end{aligned}
$$

- $\mathcal{I}_2'$ corresponding to $(t_1, t_2) \neq (\pm 2, \pm 2)$, $(t_3, t_4) = (\pm 2, \pm 2)$ and $C = \pm \operatorname{Id}$, or $D = \pm \operatorname{Id}$. Note that we cannot have simultaneously $C, D = \pm \operatorname{Id}$, since in this case the element (A, B, C, D) is reducible.

  We suppose $C = \operatorname{Id}$ and $D \sim J_+$ (the other cases are similar, and they account for 8 cases). The matrices A, B define two eigenvectors e_1, e_2. The matrix D has a unique eigenvector f, and $f \neq e_1, e_2$. Use the basis $\{f, e_2\}$. By multiplying the vectors by suitable scalars, we can suppose $D = J_+$. Then

$$
A = \begin{pmatrix} \lambda_1 & 0 \\ x_1 & \lambda_1^{-1} \end{pmatrix}, \ B = \begin{pmatrix} \lambda_2 & 0 \\ x_2 & \lambda_2^{-1} \end{pmatrix}.
$$

  Note that $(x_1, x_2) \neq (0, 0)$, since $e_1 \neq f$. The commutation relation $AB = BA$ translates to $x_1(\lambda_2 - \lambda_2^{-1}) = x_2(\lambda_1 - \lambda_1^{-1})$. Therefore there is one parameter $x_1 \in \mathbb{C}^*$ or $x_2 \in \mathbb{C}^*$ (depending on the case). Note that $x_1 = \operatorname{Tr}(AD) - \operatorname{Tr} A$, $x_2 = \operatorname{Tr}(BD) - \operatorname{Tr} B$.

  We have made a choice of basis. If we choose $\{f, e_1\}$ instead, then we get the change: $(\lambda_1, \lambda_2, x_1) \sim (\lambda_1^{-1}, \lambda_2^{-1}, x_1)$. We deduce that

$$
\begin{aligned}
e(\mathcal{I}_2') &= 8(q - 1)e(((\mathbb{C}^* \times \mathbb{C}^*) \setminus \{(\pm 1, \pm 1)\})/\mathbb{Z}_2) \\
&= 8(q - 1)(q^2 - 3) = 8q^3 - 8q^2 - 24q + 24.
\end{aligned}
$$

- $\mathcal{I}_2''$ corresponding to $(t_1, t_2) \neq (\pm 2, \pm 2)$, $(t_3, t_4) = (\pm 2, \pm 2)$ and C, D of Jordan form. Suppose that $C, D \sim J_+$ (the other cases are similar). As C, D commute, they share an eigenvector f. Also A, B have two common eigenvectors e_1, e_2 (and $f \neq e_1, e_2$). Choose the basis $\{f, e_2\}$, and arrange that $C = J_+$. Then

$$
A = \begin{pmatrix} \lambda_1 & 0 \\ x_1 & \lambda_1^{-1} \end{pmatrix}, \ B = \begin{pmatrix} \lambda_2 & 0 \\ x_2 & \lambda_2^{-1} \end{pmatrix}, \ C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 1 & x_4 \\ 0 & 1 \end{pmatrix},
$$

  where $x_1, x_2, x_4 \neq 0$. Again $AB = BA$ fixes a relation between x_1 and x_2. Now $x_1 = \operatorname{Tr}(AC) - \operatorname{Tr} A$ and $x_1 x_4 = \operatorname{Tr}(AD) - \operatorname{Tr} A$, so x_1, x_4 are well-determined. However, $(\lambda_1, \lambda_2, x_1, x_4) \sim (\lambda_1^{-1}, \lambda_2^{-1}, x_1, x_4)$. We get

$$
e(\mathcal{I}_2'') = 4(q - 1)^2(q^2 - 3) = 4q^4 - 8q^3 - 8q^2 + 24q - 12.
$$

- $\mathcal{I}_3 = \mathcal{I}_3' \cup \mathcal{I}_3''$ corresponding to $(t_1, t_2) = (\pm 2, \pm 2)$, $(t_3, t_4) \neq (\pm 2, \pm 2)$. This is similar to the previous two cases, $\mathcal{I}_2 = \mathcal{I}_2' \cup \mathcal{I}_2''$,

$$e(\mathcal{I}_3) = e(\mathcal{I}_2) = e(\mathcal{I}_2') + e(\mathcal{I}_2'') = 4q^4 - 16q^2 + 12.$$

- $\mathcal{I}_4$ given by $(t_1, t_2) = (\pm 2, \pm 2)$ and $(t_3, t_4) = (\pm 2, \pm 2)$, where one of A, B is $\pm \operatorname{Id}$ and the other is of Jordan form, and one of C, D is $\pm \operatorname{Id}$ and the other is of Jordan form (there are 64 cases).

  Suppose $A = \operatorname{Id}$, $B \sim J_+$, $C = \operatorname{Id}$, $D \sim J_+$. We take the eigenvector e of B and the eigenvector f of D. We arrange that $D = J_+$. So $B = \begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix}$, and $x_2 \neq 0$. Thus $e(\mathcal{I}_4) = 64(q - 1) = 64q - 64$.

- $\mathcal{I}_5$ given by $(t_1, t_2) = (\pm 2, \pm 2)$ and $(t_3, t_4) = (\pm 2, \pm 2)$, where both A, B are of Jordan form, and one of either C, D is $\pm \operatorname{Id}$ and the other is of Jordan form (or both C, D are of Jordan form, and one of either A, B is $\pm \operatorname{Id}$ and the other is of Jordan form). There are 64 cases.

  Suppose $A, B \sim J_+$, $C = \operatorname{Id}$, $D \sim J_+$. We can arrange that $D = J_+$ and

$$A = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix},$$

  with $x_1, x_2 \neq 0$. So $e(\mathcal{I}_5) = 64(q - 1)^2 = 64q^2 - 128q + 64$.

- $\mathcal{I}_6$ given by $(t_1, t_2) = (\pm 2, \pm 2)$ and $(t_3, t_4) = (\pm 2, \pm 2)$, where all A, B, C, D are of Jordan form. We can assume $A, B, C, D \sim J_+$ (there are 16 cases). Working as before, we can arrange that

$$A = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 1 & x_4 \\ 0 & 1 \end{pmatrix},$$

  where $x_1, x_2, x_4 \neq 0$. Thus $e(\mathcal{I}_6) = 16(q - 1)^3 = 16q^3 - 48q^2 + 48q - 16$.

Therefore

$$e(\mathcal{I}) = \sum_{j=1}^{6} e(\mathcal{I}_j) = q^5 + 7q^4 + 10q^3 - 14q^2 - 7q - 1.$$

Other contributions. The remaining contributions are

$$e(\mathcal{J}) = (e(Y_1) + e(Y_3) + e(Y_4))/e(\operatorname{PGL}(2, \mathbb{C})) = q^6 - q^5 + 9q^4 - 10q^3 + 9q^2 + 7q + 1.$$

Adding up the previous E-polynomials, we get

$$e(\mathcal{M}_{\operatorname{Id}}) = e(\mathcal{R}) + e(\mathcal{I}) + e(\mathcal{J}) = q^6 + 17q^4 + q^2 + 1.$$

9. E-POLYNOMIAL OF THE TWISTED $\operatorname{SL}(2, \mathbb{C})$-CHARACTER VARIETY FOR GENUS $g = 2$

In this section, we consider the *twisted* $\operatorname{SL}(2, \mathbb{C})$-character variety, i.e.,

$$\mathcal{M}_{-\operatorname{Id}} = W/\operatorname{PGL}(2, \mathbb{C}),$$

where $W = F^{-1}(-\operatorname{Id})$, with the definition of F in Section 8. Note that now the GIT quotient is a geometric quotient, since there are no reducibles.

We have
$$W := \{(A, B, C, D) \in \operatorname{SL}(2, \mathbb{C})^4 \,|\, [A, B] = -[D, C]\}$$
We stratify

- $W_0 := \{(A, B, C, D) \,|\, [A, B] = -[D, C] = \xi = \operatorname{Id}\}$,
- $W_1 := \{(A, B, C, D) \,|\, [A, B] = -[D, C] = \xi = -\operatorname{Id}\}$,
- $W_2 := \left\{(A, B, C, D) \,|\, [A, B] = -[D, C] = \xi \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$,
- $W_3 := \left\{(A, B, C, D) \,|\, [A, B] = -[D, C] = \xi \sim \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right\}$,
- $W_4 := \left\{(A, B, C, D) \,|\, [A, B] = -[D, C] = \xi \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1\right\}$.

We also introduce the sets

- $\overline{W}_2 := \left\{(A, B, C, D) \,|\, [A, B] = -[D, C] = \xi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$,
- $\overline{W}_3 := \left\{(A, B, C, D) \,|\, [A, B] = -[D, C] = \xi = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right\}$,
- $\overline{W}_4 := \left\{(A, B, C, D) \,|\, [A, B] = -[D, C] = \xi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1\right\}$,
- $\overline{W}_{4,\lambda} := \left\{(A, B, C, D) \,|\, [A, B] = -[D, C] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\}$, for $\lambda \neq 0, \pm 1$.

It is clear that $W_0 = X_0 \times X_1$, $W_1 = X_1 \times X_0$, so
$$e(W_0) = e(W_1) = e(X_0)e(X_1) = q^7 + 4q^6 - 2q^5 - 8q^4 + q^3 + 4q^2 \,.$$
Also $\overline{W}_2 = \overline{X}_2 \times \overline{X}_3$ and $\overline{W}_3 = \overline{X}_3 \times \overline{X}_2$. So
$$e(W_2) = e(W_3) = e(\operatorname{GL}(2, \mathbb{C})/U)e(\overline{X}_2)e(\overline{X}_3) = q^8 + q^7 - 10q^6 - 10q^5 + 9q^4 + 9q^3 \,.$$

For the last stratum, note that

(41) $$\overline{W}_{4,\lambda} = \overline{X}_{4,\lambda} \times \overline{X}_{4,-\lambda} \,.$$

Consider the involution $\tau : B \to B$, $\tau(\lambda) = -\lambda$. Then we have the fibration $\tau^* \overline{X}_4 \to B$. Since τ commutes with the $\mathbb{Z}_2$-action, we have a fibration
$$\tau^* \overline{X}_4 / \mathbb{Z}_2 \to B' = \mathbb{C} \setminus \{\pm 2\} \,,$$
whose fibre over λ is $\overline{X}_{4,-\lambda}$. Writing $R(\overline{X}_4/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$, where
$$a = q^3, \ b = -3q, \ c = 3q^2, \ d = -1 \,,$$
we have that
$$R(\tau^* \overline{X}_4 / \mathbb{Z}_2) = \tau^* R(\overline{X}_4 / \mathbb{Z}_2) = aT + cS_2 + bS_{-2} + dS_0 \,,$$
since $\tau^* T = T$, $\tau^* S_i = S_{-i}$, $i = 0, \pm 2$.

Now

$$
\begin{aligned}
R(\overline{W}_4/\mathbb{Z}_2) &= R(\overline{X}_4/\mathbb{Z}_2) \otimes R(\tau^*\overline{X}_4/\mathbb{Z}_2) \\
&= (a^2 + 2bc + d^2)T + (ac + ab + bd + cd)S_2 + (ab + ac + bd + cd)S_{-2} \\
&\quad + (2ad + b^2 + c^2)S_0 \\
&= (q^6 - 18q^3 + 1)T + (3q^5 - 3q^4 - 3q^2 + 3q)S_2 \\
&\quad + (3q^5 - 3q^4 - 3q^2 + 3q)S_{-2} + (9q^4 - 2q^3 + 9q^2)S_0 \,.
\end{aligned}
$$

(42)

Using Remark 7.4, we get

$$
e(W_4) = q^9 - 2q^8 - 7q^7 - 18q^6 + 24q^5 + 28q^4 - 17q^3 - 8q^2 - q.
$$

Adding up all contributions,

$$
e(W) = q^9 - 3q^7 - 30q^6 + 30q^4 + 3q^3 - q.
$$

Also

$$
e(\mathcal{M}_{-\mathrm{Id}}) = e(W)/e(\mathrm{PGL}(2,\mathbb{C})) = q^6 - 2q^4 - 30q^3 - 2q^2 + 1.
$$

This agrees with the result of Mereb [16, Theorem 4.6].

The moduli space $\mathcal{M}_{-\mathrm{Id}}$ is homeomorphic to the moduli space of rank 2, fixed determinant and odd degree Higgs bundles on a curve C of genus $g = 2$. By [11, p. 99], the Poincaré polynomial of this space is

$$
P_t(\mathcal{M}_{-\mathrm{Id}}) = 2t^6 + 34t^5 + 2t^4 + 4t^3 + t^2 + 1\,.
$$

This space is smooth and of complex dimension 6 (it is homotopy equivalent to the nilpotent cone, which is Lagrangian and of half the dimension). By Poincaré duality, we have

$$
P_t^c(\mathcal{M}_{-\mathrm{Id}}) = t^{12} + t^{10} + 4t^9 + 2t^8 + 34t^7 + 2t^6\,.
$$

The Euler-Poincaré characteristic of $\mathcal{M}_{-\mathrm{Id}}$ is thus -32. The variety $\mathcal{M}_{-\mathrm{Id}}$ is smooth and of balanced type. So $h_c^{k,p,q} = 0$ if $p \neq q$ or $p + q > k$. However, this is not enough information to get the Hodge numbers of $\mathcal{M}_{-\mathrm{Id}}$.

The moduli space of rank 2 odd degree Higgs bundle with fixed determinant, $\mathcal{H}_{-\mathrm{Id}}$, over a curve of genus 2 has E-polynomial

$$
e(\mathcal{H}_{-\mathrm{Id}}) = u^6v^6 + u^5v^5 - 2u^4v^5 - 2u^5v^4 + 2u^4v^4 - 17u^3v^4 - 17u^4v^3 + 2u^3v^3\,,
$$

which is not of balanced type. This polynomial basically follows from the arguments in [11].

10. E-polynomial of $\mathrm{SL}(2,\mathbb{C})$-character varieties for genus $g = 2$ and diagonalizable holonomy

Let $\xi_0 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix}$, where $\lambda_0 \neq 0, \pm 1$, and consider

$$
\mathcal{M}_{\lambda_0} = Z/\mathbb{C}^*,
$$

where $Z = F^{-1}(\xi_0)$. Thus

$$
Z := \{(A, B, C, D) \in \mathrm{SL}(2,\mathbb{C})^4 \,|\, [A, B] = \xi_0\,[D, C]\}\,.
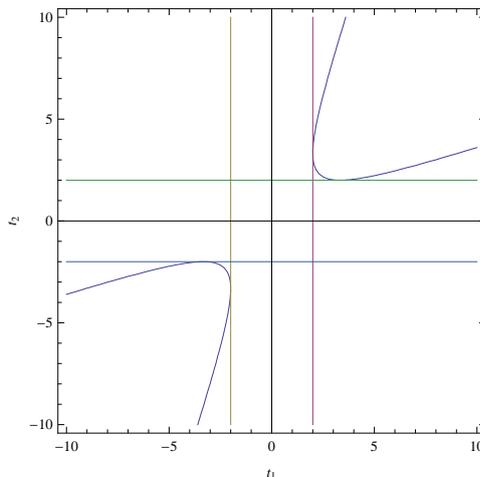$$

FIGURE 1. Distinguished lines in the base of the fibration for Z when $\lambda_0 = 1/3$.

Let
$$\eta = [D, C], \quad \delta = [A, B],$$
and write $t_1 := \operatorname{Tr} \eta$, $t_2 := \operatorname{Tr} \delta$. We put
$$\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
and
$$\delta = \xi_0 \eta = \begin{pmatrix} \lambda_0 a & \lambda_0 b \\ \lambda_0^{-1} c & \lambda_0^{-1} d \end{pmatrix}.$$

We have the equations:

(43)
$$\begin{cases} t_1 = a + d \\ t_2 = \lambda_0 a + \lambda_0^{-1} d \end{cases} \implies \begin{cases} a = \dfrac{-\lambda_0^{-1} t_1 + t_2}{\lambda_0 - \lambda_0^{-1}} \\ d = \dfrac{\lambda_0 t_1 - t_2}{\lambda_0 - \lambda_0^{-1}} \end{cases}$$

We shall pay special attention to the equation $ad = 1$. This becomes
$$(-\lambda_0^{-1} t_1 + t_2)(\lambda_0 t_1 - t_2) = (\lambda_0 - \lambda_0^{-1})^2,$$
or equivalently:

(44)
$$-t_1^2 - t_2^2 + (\lambda_0 + \lambda_0^{-1}) t_1 t_2 = (\lambda_0 - \lambda_0^{-1})^2.$$

This is a hyperbola, that we shall denote by H (see Figure 1).

We stratify Z according to the preimages of different sets over the plane (t_1, t_2), where we consider the map $Z \to \mathbb{C}^2$, $(A, B, C, D) \mapsto (\operatorname{Tr}[D, C], \operatorname{Tr}[A, B])$. We distinguish the lines $t_1 = \pm 2$, $t_2 = \pm 2$ and the curve $ad = 1$, and the points of intersection of these curves.

We consider the following sets:

- Z_1, corresponding to the cases $t_1 = \pm 2$, $t_2 = \pm 2$. First observe that (43) gives us the values of a, d. In this situation $ad \neq 1$ (if $ad = 1$, then $\pm 2 = a + d$ would imply that $a = d = \pm 1$, and $\pm 2 = \lambda_0 a + \lambda_0^{-1} d$ would imply that $\lambda_0 a = \lambda_0^{-1} d = \pm 1$, giving $\lambda_0 = \pm 1$, which is not the case).

  The condition $ad - bc = 1$ then gives $bc \neq 0$, therefore $b \in \mathbb{C}^*$ and c is uniquely determined. Moreover, for each choice of b, we have that the matrices η and δ are fixed, and both are of Jordan type $\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$. Taking into account the four possibilities, this yields the polynomial

$$
\begin{aligned}
e(Z_1) &= (q-1)e(\overline{X}_2)^2 + (q-1)e(\overline{X}_3)^2 + 2(q-1)e(\overline{X}_2)e(\overline{X}_3) \\
&= (q-1)(e(\overline{X}_2) + e(\overline{X}_3))^2 = 4q^7 - 15q^5 + 5q^4 + 15q^3 - 9q^2\,.
\end{aligned}
$$

- Z_2, corresponding to the cases $t_1 = 2$, $t_2 = \lambda_0 + \lambda_0^{-1}$, and $t_2 = 2$, $t_1 = \lambda_0 + \lambda_0^{-1}$. We shall do the first case (the second is analogous), and then multiply the result by 2. So consider the point $t_1 = 2$, $t_2 = \lambda_0 + \lambda_0^{-1}$, which is the intersection of the line $t_1 = 2$ with $ad = 1$. Thus $bc = 0$. There are three possibilities, giving rise to three sub-strata:
  (1) Z_2^1, given by $b = c = 0$. Then $\eta = \mathrm{Id}$ and $\delta = \xi_0$. So $(D, C) \in X_0$, $(A, B) \in \overline{X}_{4,\lambda_0}$. Therefore $Z_2^1 = X_0 \times \overline{X}_{4,\lambda_0}$, and

$$
e(Z_2^1) = e(X_0)e(\overline{X}_{4,\lambda_0}).
$$

  (2) Z_2^2, given by $b \neq 0$, $c = 0$. Then $b \in \mathbb{C}^*$, and η is of Jordan type J_+, and δ is diagonalizable conjugate to ξ_0. So

$$
e(Z_2^2) = (q-1)e(\overline{X}_2)e(\overline{X}_{4,\lambda_0}).
$$

  (3) Z_2^3, given by $b = 0$, $c \neq 0$. It gives a similar contribution $e(Z_2^3) = e(Z_2^2)$.
  Altogether, and doubling the contribution as we mentioned above, we obtain

$$
\begin{aligned}
e(Z_2) &= 2(e(X_0) + 2(q-1)e(\overline{X}_2))e(\overline{X}_{4,\lambda}) \\
&= 6q^7 + 14q^6 - 36q^5 - 8q^4 + 34q^3 - 6q^2 - 4q\,.
\end{aligned}
$$

- Z_3, corresponding to the cases $t_1 = -2$, $t_2 = -\lambda_0 - \lambda_0^{-1}$, and $t_2 = -2$, $t_1 = -\lambda_0 - \lambda_0^{-1}$. Again consider the first case, $t_1 = -2$, $t_2 = -\lambda_0 - \lambda_0^{-1}$. The analysis is analogous to that for Z_2, just changing the sign of the diagonal entries of η to -1. So this produces

$$
\begin{aligned}
e(Z_3) &= 2(e(X_1) + 2(q-1)e(\overline{X}_3))e(\overline{X}_{4,\lambda}) \\
&= 4q^7 + 22q^6 + 6q^5 - 72q^4 + 20q^3 + 18q^2 + 2q\,.
\end{aligned}
$$

- Z_4, corresponding to the cases $t_1 = 2$, $t_2 \neq \pm 2, \lambda_0 + \lambda_0^{-1}$, and $t_2 = 2$, $t_1 \neq \pm 2, \lambda_0 + \lambda_0^{-1}$. We focus on the first case, the second one being analogous. So (t_1, t_2) move in a (three-punctured) line $L = \{(t_1, t_2) \mid t_1 = 2, t_2 \neq \pm 2, \lambda_0 + \lambda_0^{-1}\}$.

  First, we cannot have $\eta = \mathrm{Id}$, since in that case $\delta = \xi_0$ and $t_2 = \lambda_0 + \lambda_0^{-1}$. So η is of Jordan form J_+. Also $ad \neq 1$, since $(t_1, t_2) \notin H$. So the numbers a, b, c, d are determined, up to a choice $b \in \mathbb{C}^*$.

We fix a slice by setting $b = 1$. Then $\eta = \begin{pmatrix} a & 1 \\ ad - 1 & d \end{pmatrix}$, where

$$a = \frac{-2\lambda_0^{-1} + t_2}{\lambda_0 - \lambda_0^{-1}}, \; d = \frac{2\lambda_0 - t_2}{\lambda_0 - \lambda_0^{-1}}.$$

The matrix η has eigenvector $(1, 1 - a)$. Conjugating by $\begin{pmatrix} 0 & 1 \\ 1 & 1 - a \end{pmatrix}$, we transform η into J_+, and the set of matrices $\{(C, D) \mid [D, C] = \eta\}$ into $\overline{X}_2$. This gives a trivial family over the line L. On the other hand,

$$\delta = \begin{pmatrix} \frac{-2 + \lambda_0 t_2}{\lambda_0 - \lambda_0^{-1}} & \lambda_0 \\ * & \frac{2 - \lambda_0^{-1} t_2}{\lambda_0 - \lambda_0^{-1}} \end{pmatrix}.$$

To understand the family of matrices (A, B) with $[A, B] = \delta$ for $t_2 \in L$, we need to take a double cover given by $t_2 = \mu + \mu^{-1}$, $\mu \in \mathbb{C} \setminus \{0, \pm 1, \lambda_0, \lambda_0^{-1}\}$. Then we can conjugate the matrix δ by

$$P_\mu := \begin{pmatrix} \lambda_0 & \lambda_0 \\ \mu + \frac{2 - \lambda_0 t_2}{\lambda_0 - \lambda_0^{-1}} & \mu^{-1} + \frac{2 - \lambda_0 t_2}{\lambda_0 - \lambda_0^{-1}} \end{pmatrix}$$

to take it into diagonal form $\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$. This makes the space isomorphic to $\overline{X}_4$. To recover the original space, we have to quotient by $\mu \mapsto \mu^{-1}$. Note that $P_{\mu^{-1}} = P_\mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so the corresponding action on $\overline{X}_4$ is conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. To summarize, the resulting space is $(\overline{X}_4/\mathbb{Z}_2) \setminus \overline{X}_{4,\lambda_0}$. This gives the polynomial

$$\begin{aligned} e(Z_4) &= 2(q - 1)e(\overline{X}_2)(e(\overline{X}_4/\mathbb{Z}_2) - e(\overline{X}_{4,\lambda_0})) \\ &= 2q^8 - 12q^7 + 4q^6 + 60q^5 - 38q^4 - 60q^3 + 32q^2 + 12q \,. \end{aligned}$$

- Z_5, corresponding to the cases $t_1 = -2$, $t_2 \neq \pm 2, -\lambda_0 - \lambda_0^{-1}$, and $t_2 = -2$, $t_1 \neq \pm 2, -\lambda_0 - \lambda_0^{-1}$. Analogously to the previous case, we obtain

$$\begin{aligned} e(Z_5) &= 2(q - 1)e(\overline{X}_3)(e(\overline{X}_4/\mathbb{Z}_2) - e(\overline{X}_{4,-\lambda_0})) \\ &= 2q^8 - 2q^7 - 30q^6 + 6q^5 + 64q^4 - 28q^3 - 12q^2 \,. \end{aligned}$$

- Z_6, corresponding to the hyperbola H, i.e., those (t_1, t_2) satisfying (44), but $t_1, t_2 \neq \pm 2$. This set is parameterized by $\mathbb{C}^* \setminus \{\pm 1, \pm \lambda_0^{-1}\}$ under

$$\mu \mapsto (\mu + \mu^{-1}, \lambda_0 \mu + \lambda_0^{-1} \mu^{-1}).$$

The elements on H satisfy $ad = 1$, so $bc = 0$. For each value of μ, there are three possibilities:
(1) Z_6^1, for $b = c = 0$;
(2) Z_6^2, for $b \in \mathbb{C}^*$, $c = 0$;
(3) Z_6^3, for $b = 0$, $c \in \mathbb{C}^*$.

This gives a factor $2q - 1$, that multiplies the E-polynomial of the space which is a fibration over $B = \mathbb{C}^* \setminus \{\pm 1, \pm \lambda_0^{-1}\}$, and whose fibre over μ is $\overline{X}_{4,\mu} \times \overline{X}_{4,\lambda_0\mu}$. Call this fibration $\overline{Z}_6 \to B$.

Consider the map $\tau : \mathbb{C}^* \to \mathbb{C}^*$, $\tau(\mu) = \lambda_0\mu$. Then $\overline{Z}_6 = \overline{X}_4 \times_B \tau^* \overline{X}_4$. Using Theorem 6.1, this fibration has monodromy representation:

$$
\begin{aligned}
R(\overline{Z}_6) &= R(\overline{X}_4) \otimes \tau^* R(\overline{X}_4) = R(\overline{X}_4) \otimes R(\overline{X}_4) \\
&= ((q^3 - 1)T + (3q^2 - 3q)N)^2 \\
&= ((q^3 - 1)^2 + (3q^2 - 3q)^2)T + 2(q^3 - 1)(3q^2 - 3q)N \\
&= (q^6 + 9q^4 - 20q^3 + 9q^2 + 1)T + (6q^5 - 6q^4 - 6q^2 + 6q)N \, .
\end{aligned}
$$

Now we use Proposition 2.10 to calculate the E-polynomial. Write $R(\overline{Z}_6) = aT + bN$. Then

$$
\begin{aligned}
e(Z_6) &= (2q - 1)e(\overline{Z}_6) \\
&= (2q - 1)((q - 5)a - 4b) \\
&= (2q - 1)(q^7 - 5q^6 - 15q^5 - 41q^4 + 109q^3 - 21q^2 - 23q - 5) \\
&= 2q^8 - 11q^7 - 25q^6 - 67q^5 + 259q^4 - 151q^3 - 25q^2 + 13q + 5 \, .
\end{aligned}
$$

- Z_7, corresponding to the open part of the plane (t_1, t_2), that is, $(t_1, t_2) \notin H$ and $t_1, t_2 \neq \pm 2$. The equations (43) determine a, b, c, d up to $b \in \mathbb{C}^*$. This gives a space whose fibre over (t_1, t_2) is isomorphic to

$$
\tag{45} \overline{X}_{4,\mu_1} \times \overline{X}_{4,\mu_2},
$$

where $t_i = \mu_i + \mu_i^{-1}$, $i = 1, 2$. Ignoring the condition $ad = 1$, this contributes $e(\overline{X}_4/\mathbb{Z}_2)^2$. The term to be subtracted is the E-polynomial of the fibration with fibres (45) over $\{(t_1, t_2) \in H \mid t_1 \neq \pm 2, t_2 \neq \pm 2\}$. This is the same as $e(\overline{Z}_6)$ above. So

$$
\begin{aligned}
e(Z_7) &= (q - 1)\big(e(\overline{X}_4/\mathbb{Z}_2)^2 - e(\overline{Z}_6)\big) \\
&= q^9 - 6q^8 + 8q^7 + 30q^6 + 7q^5 - 171q^4 + 155q^3 - 5q^2 - 23q - 6 \, .
\end{aligned}
$$

Adding up all contributions, we get

$$
e(Z) = q^9 - 3q^7 + 15q^6 - 39q^5 + 39q^4 - 15q^3 + 3q^2 - 1 \, .
$$

Dividing by the stabilizer, $\mathrm{Stab}(\xi) = \mathbb{C}^* \subset \mathrm{PGL}(2, \mathbb{C})$, we get

$$
e(\mathcal{M}_{\lambda_0}) = q^8 + q^7 - 2q^6 + 13q^5 - 26q^4 + 13q^3 - 2q^2 + q + 1 \, .
$$

Note that this polynomial is palindromic, as suggested to us by T. Hausel.

The space $\mathcal{M}_{\lambda_0}$ is homeomorphic to the moduli space of parabolic Higgs bundles $\mathcal{H}_{\lambda_0}$. By [1], the Poincaré polynomial is

$$
\begin{aligned}
P_t(\mathcal{M}_{\lambda_0}) &= \frac{(1 + t^3)^4 + t^7(1 + t)^4(3t - 4)}{(1 - t^2)^2} + 15t^6(1 + t)^2 \\
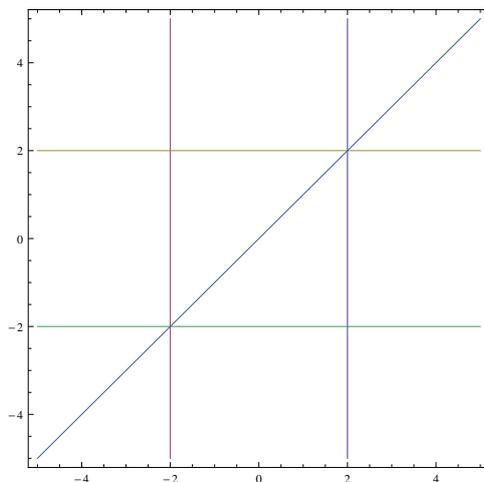&= 19t^8 + 38t^7 + 25t^6 + 8t^5 + 3t^4 + 4t^3 + 2t^2 + 1 \, .
\end{aligned}
$$

FIGURE 2. Distinguished lines in the base of the fibration for Z.

As $\mathcal{M}_{\lambda_0}$ is smooth and of complex dimension 8, Poincaré duality gives
$$P_t^c(\mathcal{M}_{\lambda_0}) = t^{16} + 2t^{14} + 4t^{13} + 3t^{12} + 8t^{11} + 25t^{10} + 38t^9 + 19t^8 \,.$$
Note that the Euler-Poincaré characteristic of $\mathcal{M}_{\lambda_0}$ is 0.

11. E-POLYNOMIAL OF $\mathrm{SL}(2, \mathbb{C})$-CHARACTER VARIETIES FOR GENUS $g = 2$ AND JORDAN FORM J_+

Let $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Recall that $U \subset \mathrm{GL}(2, \mathbb{C})$ is the stabilizer of J_+, so that
$$\mathcal{M}_{J_+} = Z/(U/\mathbb{C}^*)\,,$$
where
$$Z := F^{-1}(J_+) = \{(A, B, C, D) \in \mathrm{SL}(2, \mathbb{C})^4 \mid [A, B] = J_+\,[D, C]\}\,.$$
(The notation Z should not be confused with the one used in section 10.)

Let
$$\eta = [D, C] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,,$$
and
$$\delta = [A, B] = J_+\eta = \begin{pmatrix} a + c & b + d \\ c & d \end{pmatrix}\,.$$

Set $t_1 = \mathrm{Tr}\ \eta = a + d$, and $t_2 = \mathrm{Tr}\ \delta = a + c + d$, so $c = t_2 - t_1$.

We stratify Z into the following sets corresponding to the various possibilities for (t_1, t_2) (see Figure 2).

- Z_1 is the subset of Z lying over the point $(t_1, t_2) = (2, 2)$. Observe that in this case $c = 0$. So $a = d = 1$. There is still a free parameter $b \in \mathbb{C}$. For $b \neq 0, -1$, we have that both η, δ are of Jordan form J_+. This contributes a summand to the E-polynomial equal to $(q - 2)e(\overline{X}_2)^2$. For $b = 0, -1$, one of η, δ is of Jordan form

and the other is diagonal (i.e., equal to Id), thus contributing $2e(\overline{X}_2)e(\overline{X}_0)$. All together, we have

$$
\begin{aligned}
e(Z_1) &= (q-2)e(\overline{X}_2)^2 + 2e(\overline{X}_2)e(\overline{X}_0) \\
&= 3q^7 - 2q^6 - 18q^5 - 12q^4 + 7q^3 + 6q^2 \,.
\end{aligned}
$$

- Z_2, the stratum corresponding to $t_1 = t_2 = -2$. This is completely analogous to the previous case, giving the E-polynomial

$$
\begin{aligned}
e(Z_2) &= (q-2)e(\overline{X}_3)^2 + 2e(\overline{X}_3)e(\overline{X}_1) \\
&= q^7 + 6q^6 + 3q^5 - 20q^4 - 6q^3 \,.
\end{aligned}
$$

- Z_3, corresponding to $t_1 = -t_2 = \pm 2$. These are two points, so we consider the first case $(t_1, t_2) = (2, -2)$, and double the final contribution. Recall that $c = t_2 - t_1$, so $c \neq 0$. Then conjugating by a matrix $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, with $\lambda = d/c$, we can arrange that $d = 0$. Therefore $b = -1/c$, and $a = t_1$. This fixes a, b, c, d. Note now that η is of Jordan type J_+ and δ is of Jordan type J_-. So

$$
e(Z_3) = 2q\,e(\overline{X}_2)e(\overline{X}_3) = 2q^7 + 2q^6 - 18q^5 - 18q^4 \,.
$$

Here the factor q is given by the parameter $\lambda \in \mathbb{C}$.

- Z_4, corresponding to four strata given by the cases $t_1 = 2$, $t_2 \neq \pm 2$; $t_1 = -2$, $t_2 \neq \pm 2$; $t_2 = 2$, $t_1 \neq \pm 2$; and $t_2 = -2$, $t_1 \neq \pm 2$. Let us focus on the line $L = \{(t_1, t_2) \mid t_1 = 2, t_2 \neq \pm 2\}$. Then $c = t_2 - 2 \neq 0$, so η is of Jordan form J_+. Conjugating as before, we can assume $d = 0$, $b = -1/c$ and $a = t_1 = 2$. Therefore

$$
\eta = \begin{pmatrix} 2 & \frac{1}{2-t_2} \\ t_2 - 2 & 0 \end{pmatrix}, \; \delta = \begin{pmatrix} t_2 & \frac{1}{2-t_2} \\ t_2 - 2 & 0 \end{pmatrix}.
$$

Now, conjugating by $\begin{pmatrix} 1 & 1 \\ t_2 - 2 & 0 \end{pmatrix}$, we put η into Jordan form J_+. Hence the fibration given by the matrices (C, D) over L has fibre $\overline{X}_2$ and is trivial.

To study the fibration given by the matrices (A, B) over L, we introduce the variable $t_2 = \mu + \mu^{-1}$, $\mu \in \mathbb{C} \setminus \{0, \pm 1\}$. This gives a double cover of the space we are interested in. The fibre over μ is $\overline{X}_{4,\mu}$, and consists of pairs of matrices (A, B) satisfying

$$
[A, B] = \delta = \begin{pmatrix} \mu + \mu^{-1} & \frac{1}{2-\mu-\mu^{-1}} \\ \mu + \mu^{-1} - 2 & 0 \end{pmatrix}.
$$

We conjugate by the matrix $P_\mu := \begin{pmatrix} \mu + \mu^{-1} - 2 & \mu + \mu^{-1} - 2 \\ -\mu^{-1} & -\mu \end{pmatrix}$, to put δ into standard form $\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$. The quotient by $\mu \mapsto \mu^{-1}$ corresponds to conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, i.e., to the standard $\mathbb{Z}_2$-action on $\overline{X}_4$.

So, the substratum of Z_4 that we are studying is isomorphic to $\mathbb{C} \times \overline{X}_2 \times (\overline{X}_4/\mathbb{Z}_2)$. Analogously, the substratum corresponding to $t_1 = -2$, $t_2 \neq \pm 2$ is

isomorphic $\mathbb{C} \times \overline{X}_3 \times (\overline{X}_4/\mathbb{Z}_2)$. The remaining two substrata are copies of the previous two. Finally, we get

$$
\begin{aligned}
e(Z_4) &= 2q \left(e(\overline{X}_2) + e(\overline{X}_3) \right) e(\overline{X}_4/\mathbb{Z}_2) \\
&= 4q^8 - 6q^7 - 22q^6 + 18q^5 + 28q^4 - 16q^3 - 6q^2 \,.
\end{aligned}
$$

- Z_5, corresponding to $t_1 = t_2 \neq \pm 2$. First, we have $c = t_2 - t_1 = 0$, so $d = a^{-1}$ and $b \in \mathbb{C}$. Thus

$$
\eta = \left(\begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right), \quad \delta = \left(\begin{array}{cc} a & b + a^{-1} \\ 0 & a^{-1} \end{array} \right),
$$

where $t_1 = t_2 = a + a^{-1}$. The fibre over (t_1, t_1) consists of two disjoint sets, one is isomorphic to $\overline{X}_{4,a} \times \overline{X}_{4,a}$, the other isomorphic to $\overline{X}_{4,a^{-1}} \times \overline{X}_{4,a^{-1}}$. Taking the double cover $a \mapsto t_1 = a + a^{-1}$, we get a fibration over the line $B = \mathbb{C} \setminus \{0, \pm 1\}$ whose fibres are $\overline{X}_{4,a} \times \overline{X}_{4,a}$. We call $\overline{Z}_5 \to B$ this fibration. Its monodromy representation is

$$
\begin{aligned}
R(\overline{Z}_5) &= R(\overline{X}_4) \otimes R(\overline{X}_4) \\
&= (aT + bN) \otimes (aT + bN) = (a^2 + b^2)T + 2abN \\
&= (q^6 + 9q^4 - 20q^3 + 9q^2 + 1)T + (6q^5 - 6q^4 - 6q^2 + 6q)N \,,
\end{aligned}
$$

where $R(\overline{X}_4) = aT + bN$, $a = q^3 - 1$, $b = 3q^2 - 3q$ (by Theorem 6.1). Therefore, Proposition 2.10 says that the E-polynomial is

$$
\begin{aligned}
e(\overline{Z}_5) &= (q-3)(q^6 + 9q^4 - 20q^3 + 9q^2 + 1) - 2(6q^5 - 6q^4 - 6q^2 + 6q) \\
&= q^7 - 3q^6 - 3q^5 - 35q^4 + 69q^3 - 15q^2 - 11q - 3 \,.
\end{aligned}
$$

and $e(Z_5) = q \, e(\overline{Z}_5) = q^8 - 3q^7 - 3q^6 - 35q^5 + 69q^4 - 15q^3 - 11q^2 - 3q$.

- Z_6, corresponding to the open stratum, characterised by the equations $t_1, t_2 \neq \pm 2$, and $t_1 \neq t_2$. Then $c = t_2 - t_1 \neq 0$. Conjugating by a matrix $\left(\begin{array}{cc} 1 & \lambda \\ 0 & 1 \end{array} \right)$, with $\lambda = d/c$, we can arrange that $d = 0$. Therefore $b = -1/c$, and $a = t_1$. So the fibre over (t_1, t_2) has

$$
\eta = \left(\begin{array}{cc} t_1 & -1/(t_2 - t_1) \\ t_2 - t_1 & 0 \end{array} \right), \quad \delta = \left(\begin{array}{cc} t_2 & -1/(t_2 - t_1) \\ t_2 - t_1 & 0 \end{array} \right).
$$

Each fibre is isomorphic to $\mathbb{C} \times \overline{X}_{4,\lambda_1} \times \overline{X}_{4,\lambda_2}$, where $t_i = \lambda_i + \lambda_i^{-1}$. Working as before, we see that, ignoring the condition $t_1 \neq t_2$, the total space is isomorphic to $\mathbb{C} \times (\overline{X}_4/\mathbb{Z}_2) \times (\overline{X}_4/\mathbb{Z}_2)$. We have to subtract the contribution corresponding to the space Z_6' parameterized by (t_1, t_1), $t_1 \neq \pm 2$, and with fibers $\mathbb{C} \times \overline{X}_{4,\lambda_1} \times \overline{X}_{4,\lambda_1}$. So $Z_6' = \mathbb{C} \times (\overline{Y}_4/\mathbb{Z}_2)$, where $\overline{Y}_4$ is described in section 8 (see equation (39)). The Hodge monodromy representation $R(\overline{Y}_4/\mathbb{Z}_2)$ is given by equation (40). Using Proposition 2.10, we have that

$$
e(\overline{Y}_4/\mathbb{Z}_2) = q^7 - 2q^6 + 3q^5 - 12q^4 + 29q^3 - 12q^2 - 5q - 2 \,,
$$

and then

$$
\begin{aligned}
e(Z_6) &= q(e(\overline{X}_4/\mathbb{Z}_2)^2 - e(\overline{Y}_4/\mathbb{Z}_2)) \\
&= q^9 - 5q^8 + 15q^6 + 11q^5 - 51q^4 + 15q^3 + 11q^2 + 3q \,.
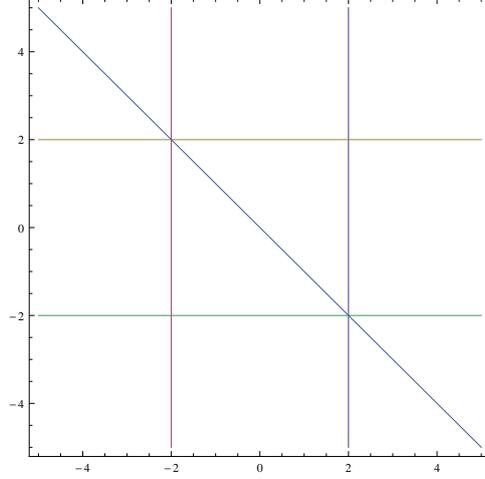\end{aligned}
$$

FIGURE 3. Distinguished lines in the base of the fibration for Z.

Adding all contributions, we get

$$e(Z) = q^9 - 3q^7 - 4q^6 - 39q^5 - 4q^4 - 15q^3 \,,$$

and dividing by the stabilizer, $\mathrm{Stab}(J_+) = U/\mathbb{C}^* \subset \mathrm{PGL}(2,\mathbb{C})$,

$$e(\mathcal{M}_{J_+}) = q^8 - 3q^6 - 4q^5 - 39q^4 - 4q^3 - 15q^2 \,.$$

Remark 11.1. Note that $U/\mathbb{C}^*$ acts freely on Z. Certainly, if $(A,B,C,D) \in \mathrm{SL}(2,\mathbb{C})^4$ is acted on trivially by U, then all matrices $A,B,C,D \in U$, and hence $[A,B] = [C,D] = \mathrm{Id}$.

12. E-POLYNOMIAL OF $\mathrm{SL}(2,\mathbb{C})$-CHARACTER VARIETIES FOR GENUS $g=2$ AND JORDAN FORM J_-

Let $J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, and consider

$$\mathcal{M}_{J_-} = Z/(U/\mathbb{C}^*),$$

where

$$Z := F^{-1}(J_-) = \{(A,B,C,D) \in \mathrm{SL}(2,\mathbb{C})^4 \mid [A,B] = J_- \, [D,C]\} \,.$$

(The notation Z should not be confused with the ones used in section 10 or section 11.)

Let

$$\eta = [D,C] = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and

$$\delta = [A,B] = J_- \, \eta = \begin{pmatrix} -a+c & -b+d \\ -c & -d \end{pmatrix}.$$

Then $t_1 = \mathrm{Tr}\, \eta = a+d$ and $t_2 = \mathrm{Tr}\, \delta = -a+c-d$, so $c = t_2 + t_1$.

We stratify Z in the following sets corresponding to the various possibilities for (t_1, t_2) (see Figure 3).

- Z_1 is the stratum corresponding to $(t_1, t_2) = (2, -2)$. Then $c = t_1 + t_2 = 0$. So $a = d = 1$. There is a parameter $b \in \mathbb{C}$. For $b \neq 0, 1$, we have that both η, δ are of Jordan form (of types J_+ and J_-, resp.). So this contributes $(q-2)e(\overline{X}_2)e(\overline{X}_3)$. For $b = 0, 1$, one of η, δ is of Jordan form and the other is diagonal, contributing $e(\overline{X}_2)e(\overline{X}_1) + e(\overline{X}_3)e(\overline{X}_0)$. Altogether, we have

$$
\begin{aligned}
e(Z_1) &= (q-2)e(\overline{X}_2)e(\overline{X}_3) + e(\overline{X}_2)e(\overline{X}_1) + e(\overline{X}_3)e(\overline{X}_0) \\
&= 2q^7 + 7q^6 - 2q^5 - 2q^4 + 8q^3 + 3q^2 .
\end{aligned}
$$

- Z_2, corresponding to $(t_1, t_2) = (-2, 2)$. It is analogous to the first stratum. We get

$$
\begin{aligned}
e(Z_2) &= (q-2)e(\overline{X}_2)e(\overline{X}_3) + e(\overline{X}_2)e(\overline{X}_1) + e(\overline{X}_3)e(\overline{X}_0) \\
&= 2q^7 + 7q^6 - 2q^5 - 2q^4 + 8q^3 + 3q^2 .
\end{aligned}
$$

- Z_3, corresponding to $t_1 = t_2 = \pm 2$. Consider first the case $(t_1, t_2) = (2, 2)$. We have that $c = t_1 + t_2 = 4$. Conjugating by the matrix $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, with $\lambda = d/c$, we can arrange that $d = 0$. Therefore $b = -1/c$, and $a = t_1$. The matrices η, δ are not diagonal, so they are both of Jordan type J_+. Therefore the contribution is $q\,e(\overline{X}_2)^2$. Analogously, the case $(t_1, t_2) = (-2, -2)$ contributes $q\,e(\overline{X}_3)^2$. Hence

$$
\begin{aligned}
e(Z_3) &= q(e(\overline{X}_2)^2 + e(\overline{X}_3)^2) \\
&= 2q^7 + 2q^6 + 7q^5 + 12q^4 + 9q^3 .
\end{aligned}
$$

- Z_4, corresponding to four strata given by the cases $t_1 = 2$, $t_2 \neq \pm 2$; $t_1 = -2$, $t_2 \neq \pm 2$; $t_2 = 2$, $t_1 \neq \pm 2$; and $t_2 = -2$, $t_1 \neq \pm 2$. Let us focus on the line $L = \{(t_1, t_2) \mid t_1 = 2, t_2 \neq \pm 2\}$.

  Then $c = t_2 + 2 \neq 0$, so η is of Jordan form J_+. Conjugating as before, we can assume $d = 0$, $b = -1/c$ and $a = t_1 = 2$. Therefore

$$
\eta = \begin{pmatrix} 2 & -\frac{1}{t_2+2} \\ t_2 + 2 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} t_2 & \frac{1}{2+t_2} \\ -t_2 - 2 & 0 \end{pmatrix}.
$$

  Working as in the case of the stratum Z_4 of section 11, we can put the matrix η into Jordan form J_+ for all t_2 simultaneously, so that the family parametrizing (C, D) is a trivial family with fiber $\overline{X}_2$ over L. Also, we can make the change of variable $t_2 = \mu + \mu^{-1}$, $\mu \in \mathbb{C} \setminus \{0, \pm 1\}$, to put δ into diagonal form $\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, so that we see that the family parametrizing (A, B) over L is isomorphic to $\overline{X}_4/\mathbb{Z}_2 \to \mathbb{C} \setminus \{\pm 2\}$. So, the substratum of Z_4 that we are dealing with is isomorphic to $\mathbb{C} \times \overline{X}_2 \times (\overline{X}_4/\mathbb{Z}_2)$. The substratum corresponding to $t_1 = -2$, $t_2 \neq \pm 2$ is isomorphic $\mathbb{C} \times \overline{X}_3 \times (\overline{X}_4/\mathbb{Z}_2)$. The remaining two substrata are copies of the previous two. Finally, we get

$$
\begin{aligned}
e(Z_4) &= 2q\,(e(\overline{X}_2) + e(\overline{X}_3))e(\overline{X}_4/\mathbb{Z}_2) \\
&= 4q^8 - 6q^7 - 22q^6 + 18q^5 + 28q^4 - 16q^3 - 6q^2 .
\end{aligned}
$$

- Z_5, corresponding to $t_1 = -t_2 \neq \pm 2$. First, we have $c = t_2 + t_1 = 0$, so $d = a^{-1}$ and $b \in \mathbb{C}$. So

$$\eta = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \ \delta = \begin{pmatrix} -a & -b + a^{-1} \\ 0 & -a^{-1} \end{pmatrix},$$

where $t_1 = -t_2 = a + a^{-1}$. The fibre over $(t_1, -t_1)$ consists of two disjoint sets, one is isomorphic to $\overline{X}_{4,a} \times \overline{X}_{4,-a}$, the other isomorphic to $\overline{X}_{4,a^{-1}} \times \overline{X}_{4,-a^{-1}}$. Taking the double cover $a \mapsto t_1 = a + a^{-1}$, we get a fibration over the plane $B = \mathbb{C} \setminus \{0, \pm 1\}$ whose fibers are $\overline{X}_{4,a} \times \overline{X}_{4,-a}$. We call this fibration $\overline{Z}_5 \to B$. Consider $\tau : B \to B$, $\tau(a) = -a$. The monodromy representation of $\overline{Z}_5$ is

$$\begin{aligned} R(\overline{Z}_5) &= R(\overline{X}_4) \otimes \tau^* R(\overline{X}_4) = R(\overline{X}_4) \otimes R(\overline{X}_4) \\ &= (aT + bN) \otimes (aT + bN) = (a^2 + b^2)T + 2abN \\ &= (q^6 + 9q^4 - 20q^3 + 9q^2 + 1)T + (6q^5 - 6q^4 - 6q^2 + 6q)N, \end{aligned}$$

where $R(\overline{X}_4) = aT + bN$, $a = q^3 - 1$, $b = 3q^2 - 3q$ (by Theorem 6.1). Therefore, Proposition 2.10 says that the E-polynomial is

$$\begin{aligned} e(Z_5) &= q\, e(\overline{Z}_5) \\ &= q^8 - 3q^7 - 3q^6 - 35q^5 + 69q^4 - 15q^3 - 11q^2 - 3q. \end{aligned}$$

- Z_6, corresponding to the open stratum, characterised by the equations $t_1, t_2 \neq \pm 2$, and $t_1 + t_2 \neq 0$. Then $c \neq 0$. Conjugating by a matrix $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, with $\lambda = d/c$, we can arrange that $d = 0$. Therefore $b = -1/c$, and $a = t_1$. So the fibre over (t_1, t_2) has

$$\eta = \begin{pmatrix} t_1 & -1/(t_1 + t_2) \\ t_1 + t_2 & 0 \end{pmatrix}, \ \delta = \begin{pmatrix} t_2 & 1/(t_1 + t_2) \\ -t_1 - t_2 & 0 \end{pmatrix}.$$

So each fibre is isomorphic to $\mathbb{C} \times \overline{X}_{4,\lambda_1} \times \overline{X}_{4,\lambda_2}$, where $t_i = \lambda_i + \lambda_i^{-1}$, $i = 1, 2$. Working as before, we see that, ignoring the condition $t_1 + t_2 \neq 0$, the total space is isomorphic to $\mathbb{C} \times (\overline{X}_4/\mathbb{Z}_2) \times (\overline{X}_4/\mathbb{Z}_2)$, so contributing $q\, e(\overline{X}_4/\mathbb{Z}_2)^2$.

We have to subtract the contribution corresponding to the space Z_6' parametrized by $(t_1, -t_1)$, $t_1 \neq \pm 2$, and with fibres $\mathbb{C} \times \overline{X}_{4,\lambda_1} \times \overline{X}_{4,-\lambda_1}$. Therefore $Z_6' = \mathbb{C} \times (\overline{W}_4/\mathbb{Z}_2)$, where $\overline{W}_4$ is the space of Section 9 (see equation (41)). Therefore the Hodge monodromy representation $R(\overline{W}_4/\mathbb{Z}_2)$ is given by (42), and

$$\begin{aligned} e(\overline{W}_4/\mathbb{Z}_2) &= (q - 2)(q^6 - 18q^3 + 1) - (6q^5 + 3q^4 - 2q^3 + 3q^2 + 6q) \\ &= q^7 - 2q^6 - 6q^5 - 21q^4 + 38q^3 - 3q^2 - 5q - 2. \end{aligned}$$

So

$$\begin{aligned} e(Z_6) &= q\, (e(\overline{X}_4/\mathbb{Z}_2)^2 - e(\overline{W}_4/\mathbb{Z}_2)) \\ &= q^9 - 5q^8 + 24q^6 + 20q^5 - 60q^4 + 6q^3 + 11q^2 + 3q. \end{aligned}$$

Adding up all contributions,

$$e(Z) = q^9 - 3q^7 + 15q^6 + 6q^5 + 45q^4.$$

Dividing by the stabilizer, $\mathrm{Stab}(J_-) = U/\mathbb{C}^* \subset \mathrm{PGL}(2,\mathbb{C})$,

$$e(\mathcal{M}_{J_-}) = q^8 - 3q^6 + 15q^5 + 6q^4 + 45q^3\,.$$

Remark 12.1. As in Remark 5.3, we again see that

$$e(\mathcal{M}_{J_-}) + (q+1)e(\mathcal{M}_{-\mathrm{Id}}) = q^8 + q^7 - 2q^6 + 13q^5 - 26q^4 + 13q^3 - 2q^2 + q + 1 = e(\mathcal{M}_\lambda)$$

for genus $g = 2$. Such equality is predicted to hold for arbitrary genus $g \geq 1$. This amusing fact deserves an explanation.

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