Monodromy evolving deformations and confluent Halphen's systems

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Abstract

We study Halphen's confluent systems corresponding to Whittaker, Bessel, Weber and Airy functions. We show that Halphen's confluent systems are represented by Monodromy evolving deformation found by Chakravarty and Ablowitz.

1 Introduction

In this note, we represent quadratic differential systems of the Halphen type as monodromy evolving deformations (MED). We explain quadratic systems of the Halphen type at first. In 1881, G. Halphen [10] [14] found a quadratic system of differential equations

$$
\begin{aligned}
X' + Y' &= 2XY, \\
Y' + Z' &= 2YZ, \\
Z' + X' &= 2ZX.
\end{aligned}
$$

This equation can be solved by theta constants:

$$
X = 2\frac{d}{d\tau}\log\theta_2(0,\tau), \quad Y = 2\frac{d}{d\tau}\log\theta_3(0,\tau), \quad Z = 2\frac{d}{d\tau}\log\theta_4(0,\tau).
$$

In these thirty years, (1) becomes popular in many mathematical fields. It is a reduction from the Bianchi IX cosmological models or the self-dual Yang-Mills equation [5] [6] and gives a special self-dual Einstein metric [9] [3]. If we set $y = 2(X + Y + Z)$, y satisfies Chazy's equation

$$
y''' = 2yy'' - 3(y')^2. \tag{1}
$$

Chazy's equation appeared in his classification of the third order Painlevé type equation [7], but (1) does not have the Painlevé property because generic solutions has a natural boundary.

In the same year Halphen found another system, called Halphen's second equation [11]

$$
\begin{array}{rcl}
x_1' &=& x_1^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2, \\
x_2' &=& x_2^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2, \\
x_3' &=& x_3^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2,
\end{array} \tag{2}
$$

which is less familiar. Halphen's second equation (2) appears in the study on hypercomplex structure [12]. Halphen's second equation is also studies in [2] as the DH-IX system (the Darboux-Halphen system of ninth order)

$$
\frac{dM}{dt} = {}^t(\operatorname{adj} M) + {}^t MM - (\operatorname{Tr} M)M,
$$

for a 3×3 matrix $M = M(t)$.

In case $a = b = c = -\frac{1}{8}$, (2) is equivalent to (1) by the transform $2X = x_2 + x_3, 2Y = x_3 + x_1, 2Z = x_1 + x_2$. Since solutions of (2) may have a natural boundary or moving branch points, (2) does not have the Painlevé property neither.

We may consider confluent case of the Halphen's equation [15]. The system

$$
\begin{array}{rcl}
x_1' &=& x_1^2 + \left(m^2 - \dfrac{1}{4}\right)(x_2 - x_1)^2 - k(x_2 - x_1)(x_3 - x_1) + \dfrac{1}{4}(x_3 - x_1)^2, \\[2mm]
x_2' &=& x_2^2 + \left(m^2 - \dfrac{1}{4}\right)(x_2 - x_1)^2 - k(x_2 - x_1)(x_3 - x_1) + \dfrac{1}{4}(x_3 - x_1)^2, \\[2mm]
x_3' &=& x_3^2 + \left(m^2 - \dfrac{1}{4}\right)(x_2 - x_1)^2 - k(x_2 - x_1)(x_3 - x_1) - \dfrac{3}{4}(x_3 - x_1)^2,
\end{array} \tag{3}
$$

is solved by the confluent hypergeometric functions $W_{k,m}(z)$. We can construct a system of Halphen's type from any second order linear equation [15]. The Bessel type was studied also in [1] and the Airy type was studied in [8] independently. These confluent systems also do not have the Painlevé property in general.

In order to study such non-Painlevé type equations, we may use monodromy evolving deformations, which do not preserve monodromy data. In 1996, Chakravarty and Ablowitz [4] showed that a fifth-order equation

$$
\begin{array}{rcl}
\omega_1' &=& \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3) + \phi^2, \\
\omega_2' &=& \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1) + \theta^2, \\
\omega_3' &=& \omega_1\omega_2 - \omega_3(\omega_1 + \omega_2) - \phi\theta, \\
\phi' &=& \omega_1(\theta - \phi) - \omega_3(\theta + \phi), \\
\theta' &=& -\omega_2(\theta - \phi) - \omega_3(\theta + \phi),
\end{array} \tag{4}
$$

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can be represented by MED. The above system, called the Darboux-Halphen fifth equation (DH-V), arises in complex Bianchi IX cosmological models. The DH-V has two first integrals and is reduced to the third order equation, which is a special case of Halphen's second equation.

The author has generalized the result of [4] and has shown that all of Halphen's second equation can be represented by MED. In this paper, we show that Halphen's confluent systems are also represented by MED.

2   Quadratic systems and non-associative algebras

We take a homogeneous quadratic system

$$\frac{dX_i}{dt} = \sum_{j,k=1}^{n} a_{jk}^{i} X_j X_k, \tag{5}$$

where $i = 1, 2, ..., n$ and $a_{jk} = a_{kj}$. For (5), we define a commutative and non-associative algebra generated by $x_1, x_2, ..., x_n$ with a multiplication table

$$x_j \cdot x_k = \sum_{i=1}^{n} a_{jk}^{i} x_i.$$

Theorem 1 *[16] We take a quadratic system with rank three. If and only if the associated algebra has a unit and its automorphism group is finite, the quadratic system is corresponding to one of hypergeometric type $HG(a, b, c)$, Whittaker type $W(k, m)$, Bessel type $B(\nu)$, Hermite-Weber type $HW(n)$ or Airy type Ai.*

If the associated algebra has a unit and its automorphism group is infinite, the algebra is isomorphic to J (Jordan algebra), $X(t)$, Y_1, Y_2, Y_3 or Y_4 defined below. For these cases the corresponding quadratic systems can be solved by elementary functions.

$HG(0, 0, 0)$, $X(0)$, Y_1 and Y_2 are associative. J and Y_3 are not associative but power-associative. $X(t)$, Y_1, Y_2, Y_3 and Y_4 has a non-trivial ideal.

We list up three-dimensional commutative algebras with a unit e. We assume that the algebra is $\mathbb{C}e + \mathbb{C}f + \mathbb{C}g$ as a $\mathbb{C}$-vector space. We may take any base

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field, whose characteristic is not two.

$$HG(a,b,c): \qquad\qquad\qquad e = x + y + z \quad \text{and}$$
$$x \cdot x = x + (b+c)(x+y+z), \quad x \cdot y = -c(x+y+z),$$
$$y \cdot y = y + (c+a)(x+y+z), \quad y \cdot z = -a(x+y+z),$$
$$z \cdot z = z + (a+b)(x+y+z), \quad z \cdot x = -b(x+y+z).$$

$$W(k,m): \qquad\qquad g \cdot g = m^2 e, \; f \cdot f = e, \; f \cdot g = -\frac{f}{2} - ke$$

$$B(\nu): \qquad\qquad g \cdot g = \frac{\nu^2}{4} e, \; f \cdot f = 0, \; f \cdot g = -\frac{f}{2} + e$$

$$HW(n): \qquad\qquad g \cdot g = \frac{n^2}{4} e, \; f \cdot f = 0, \; f \cdot g = -\frac{f}{2} + e$$

$$\text{Ai}: \qquad\qquad w \cdot w = v, \; v \cdot v = 0, \; w \cdot v = e$$

$$J: \qquad\qquad f \cdot f = e, \; g \cdot g = e, \; f \cdot g = 0$$

$$X(t): \qquad\qquad f \cdot f = 0, \; g \cdot g = te, \; f \cdot g = f$$

$$Y_j \; (j = 1,2,3,4): \qquad f \cdot f = 0, \; f \cdot g = 0, \; g \cdot g = 0, f, e \text{ or } f + e$$

Here, $a, b, c; k, m; \nu; n; t$ are complex parameters.

We call quadratic systems with the rank three associated with the algebras $W(k,m), B(\nu), HW(n)$ and Ai *Halphen's confluent system*. A quadratic system associated with $HG(a,b,c)$ is Halphen's second equation (2):

$$
\begin{aligned}
x_1' &= x_1^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2, \\
x_2' &= x_2^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2, \\
x_3' &= x_3^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2.
\end{aligned}
$$

Halphen's confluent system of type $W(k,m)$ is (3):

$$
\begin{aligned}
x_1' &= x_1^2 + \left(m^2 - \frac{1}{4}\right)(x_2 - x_1)^2 - k(x_2 - x_1)(x_3 - x_1) + \frac{1}{4}(x_3 - x_1)^2, \\
x_2' &= x_2^2 + \left(m^2 - \frac{1}{4}\right)(x_2 - x_1)^2 - k(x_2 - x_1)(x_3 - x_1) + \frac{1}{4}(x_3 - x_1)^2, \\
x_3' &= x_3^2 + \left(m^2 - \frac{1}{4}\right)(x_2 - x_1)^2 - k(x_2 - x_1)(x_3 - x_1) - \frac{3}{4}(x_3 - x_1)^2.
\end{aligned}
$$

Halphen's confluent system of type $B(\nu)$:

$$
\begin{aligned}
x_1' &= x_1^2 + (\nu^2 - 1)(x_2 - x_1)^2 - 2(x_2 - x_1)(x_3 - x_1), \\
x_2' &= x_2^2 + (\nu^2 - 1)(x_2 - x_1)^2 - 2(x_2 - x_1)(x_3 - x_1), \\
x_3' &= x_3^2 + (\nu^2 - 1)(x_2 - x_1)^2 - 2(x_2 - x_1)(x_3 - x_1) - (x_3 - x_1)^2,
\end{aligned}
$$

which is solved by the Bessel function $J_\nu(2\sqrt{2x})$.

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Halphen's confluent system of type $W(n)$:

$$
\begin{aligned}
x_1' &= x_1^2 + (n^2 - 1)(x_2 - x_1)^2 - \frac{1}{4}(x_2 - x_1)(x_3 - x_1), \\
x_2' &= x_2^2 + (n^2 - 1)(x_2 - x_1)^2 - \frac{1}{4}(x_2 - x_1)(x_3 - x_1) - (x_2 - x_1)^2, \\
x_3' &= x_3^2 + (n^2 + 1)(x_2 - x_1)^2 - \frac{1}{4}(x_2 - x_1)(x_3 - x_1) - (x_3 - x_1)^2,
\end{aligned}
$$

which is solved by the parabolic cylinder function $D_n(x)$.

Halphen's confluent system of type Ai:

$$
\begin{aligned}
x_1' &= x_1^2 + (x_2 - x_1)(x_3 - x_1), \\
x_2' &= x_2^2 + (x_2 - x_1)(x_3 - x_2), \\
x_3' &= x_3^2 + (x_2 - x_3)(x_3 - x_1) + (x_2 - x_1)^2,
\end{aligned} \tag{6}
$$

which is solved by the Airy function $\mathrm{Ai}(x)$. The system (6) is written as a single equation

$$
x_1''' - 12x_1 x_1'' + 48x_1{}^2 x_1' - 6(x_1')^2 - 24x_1{}^4 = 0,
$$

which is known by Clarskon and Olver [8]. The author does not know a condition when a Halphen-type system can be written as a single equation.

3   Monodromy evolving deformations

We give a basic theory of monodromy evolving deformation (MED). In [4], Chakravarty and Ablowitz have shown DH-V is given by MED. The author has shown that Halphen's second equation is given by MED [17]. As the same as monodromy preserving deformation, we consider

$$
\frac{\partial Y}{\partial x} = A(x,t)Y, \qquad \frac{\partial Y}{\partial t} = B(x,t)Y \tag{7}
$$

where $A(x,t), B(x,t)$ are $m \times m$ matrices and $A(x,t)$ is a rational function on x. When $B(x,t)$ is also rational on x, the monodromy data of the first equation in (7) is invariant for any t. When $B(x,t)$ is not rational, the monodromy data may be changed.

Assume that $A(x,t)$ has a singularity at $x = a$. Then Y can be developed as

$$
Y \sim Y_0(x) \exp\left[\sum_{k=1}^{r} T_k (x - a)^{-k}\right] (x - a)^L.
$$

Here $Y_0(x)$ is a (formal) power series around $x = a$, r is the Poincaré rank at $x = a$ (when $x = a$ is regular singular, $r = 0$), T_k $(k = 1, 2, ..., r)$ and L are commutative. We consider the following monodromy evolving deformation:

$$
\frac{\partial L}{\partial t} = fI,
$$

5

where I is a unit matrix. This deformation changes only the trace part of local exponent matrix and preserves projective monodromy [13]. In this case the deformation equation is given by

$$\frac{\partial Y}{\partial t} = [B_0(x, t) + f \log(x - a)] Y.$$

We review the result in [17]. We set $Q(x) = x^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2$, $P(x) = (x - x_1)(x - x_2)(x - x_3)$ and $R(x) = -(x + x_1 + x_2 + x_3)$. Halphen's second equation is nothing but $x_j' = Q(x_j)\,(j = 1, 2, 3)$. We take the following 2×2 linear system

$$\frac{\partial Y}{\partial x} = \left(\frac{\mu}{P} + \sum_{j=1}^{3} \frac{c_j S}{x - x_j} \right) Y, \qquad \frac{\partial Y}{\partial t} = \left(\nu + \sum_{j=1}^{3} c_j x_j S \right) Y - Q(x) \frac{\partial Y}{\partial x}. \quad (8)$$

Here μ and c_j's are constants with $c_1 + c_2 + c_3 = 0$, and S is any traceless constant matrix. We assume

$$\frac{\partial \nu}{\partial x} = \frac{R}{P} \mu.$$

Theorem 2 *[17] The compatibility condition of* (8) *gives the Halphen's second equation.*

This theorem contains the result in [4].

We assume the local monodromy of $Y_j(x)$ around $x = x_j$ is $e^{2\pi i L_j}$. This deformation does not preserve monodromy data. The local exponent L_j at $x = x_j$ evolves as

$$\frac{dL_j}{dt} = -\frac{R(x_j)}{\prod_{m \neq j}(x_j - x_m)} \mu.$$

We can eliminate the variables μ and ν in (8) by the rescaling $Y = fZ$ for a scalar function $f = f(x, t)$. f satisfies the linear equations

$$\frac{\partial f}{\partial x} = \frac{\mu}{P(x)} f, \quad \frac{\partial f}{\partial t} = \nu f - Q(x) \frac{\partial f}{\partial x}, \quad \frac{\partial \nu}{\partial x} = \frac{R(x)}{P(x)} \mu. \quad (9)$$

Here $R(x) = -(x + x_1 + x_2 + x_3)$.

The integrability condition for f is

$$\frac{\partial P}{\partial t} + Q \frac{\partial P}{\partial x} - P \frac{\partial Q}{\partial x} + RP = 0, \quad (10)$$

which gives the Halphen's equation. Therefore MED is essentially deformations of a scalar equation (10). Our problem is *to find P, Q and R in* (9) *which gives Halphen's confluent equation.*

4 Halphen's confluent systems and monodromy evolving deformations

We list up P, Q and R in (9) which gives Halphen's confluent systems:

1) The Whittaker type

$$P(x) = (x - x_1)^2(x - x_2), \qquad R(x) = -x - 2x_1 - x_2,$$

$$Q(x) = x^2 + \left(m^2 - \frac{1}{4}\right)(x_1 - x_2)^2 - \frac{1}{4}(x_3 - x_1)^2 - k(x_2 - x_1)(x_3 - x_1).$$

2) The Bessel type

$$P(x) = (x - x_1)^2(x - x_2)^2, \qquad R(x) = -2(x + x_1 + x_2),$$

$$Q(x) = x^2 + \left(\frac{\nu^2 - 1}{4}\right)(x - x_2)^2 - 2(x_3 - x_1)(x_2 - x_1).$$

3) The Weber type

$$P(x) = (x - x_1)^3, \qquad R(x) = -x - 3x_1,$$

$$Q(x) = x^2 - \left(n + \frac{1}{2}\right)(x_2 - x_1)^2 + \frac{1}{4}(x_3 - x_1)^2.$$

4) The Airy type

$$P(x) = (x - x_1)^3, \qquad R(x) = -x - 3x_1,$$

$$Q(x) = x^2 + (x_3 - x_1)(x_2 - x_1).$$

References

[1] M. J. Ablowitz, S. Chakravarty and R. Halburd, The generalized Chazy equation from the self-duality equations. *Stud. Appl. Math.* **103** (1999), 75–88.

[2] M. J. Ablowitz, S. Chakravarty and R. Halburd, The Darboux-Halphen system and the singularity structure of its solutions, in "Mathematical and numerical aspects of wave propagation", 408–412, SIAM, 1998.

[3] M. Atiyah, N. J. Hitchin, Low-energy scattering on non-Abelian magnetic monopoles, *Phil. Trans. R. Soc. Lond.* A **315** (1985), 459–469.

[4] S. Chakravarty and M. J. Ablowitz, Integrability, Monodromy Evolving Deformations and Self-Dual Bianchi IX Systems, *Phys. Rev. Lett.* **76** (1996), 857–860.

[5] S. Chakravarty, M. J. Ablowitz, P. A. Clarkson, Reductions of Self-Dual Yang-Mills Fields and Classical Systems, *Phys. Rev. Let.* **65** (1990), 1085–1087.

[6] S. Chakravarty, M. J. Ablowitz, L. A. Takhtajan, Self-dual Yang-Mills equation and new special functions in integrable systems, *Nonlinear evolution equations and dynamical systems*, 3–11, World Sci. Publ., 1992.

[7] J. Chazy, Sur les équations différentielles du trousième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, *Acta Math.* **34** (1911), 317–385.

[8] P. A. Clarkson and P. J. Olver, Symmetry and the Chazy equation, *J. Diff. Eq.* **124** (1996), 225–246.

[9] G. W. Gibbons and C. N. Pope, The Positive Action Conjecture and Asymptotically Euclidean Metrics in Quantum Gravity, *Commun. Math. Phys.* **66**, 267–290 (1979).

[10] G. Halphen, Sur une systém d'équations différentielles, *C. R. Acad. Sci., Paris* **92** (1881), 1101–1103.

[11] G. Halphen, Sur certains système d'équations différentielles, *C. R. Acad. Sci., Paris* **92** (1881), 1404–1406.

[12] N. Hitchin, Hypercomplex manifolds and the space of framings, *The Geometric Universe* (Oxford, 1996), Oxford (1998), 9–30.

[13] C. Mitschi, M. F. Singer; Projective Isomonodromy and Galois Groups, `arXiv:1002.2005`

[14] Y. Ohyama, Differential Relations of Theta Functions, *Osaka J.Math.*, **32** (1995), 431–450.

[15] Y. Ohyama, Systems of nonlinear differential equations related to second order linear equations, *Osaka J. Math.* **33** (1997) 927–949.

[16] Y. Ohyama, Hypergeometric functions and non-associative algebras, *CRM Proceedings and Lecture Notes* **30** (2001), 173–184.

[17] Y. Ohyama, Monodromy evolving deformations and Halphen's equation *CRM Proceedings and Lecture Notes*, **47** (2009) 343–348.