MAXIMAL EQUILATERAL SETS

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Abstract. A subset of a normed space X is called equilateral if the distance between any two points is the same. Let $m(X)$ be the smallest possible size of an equilateral subset of X maximal with respect to inclusion. We first observe that Petty's construction of a d-dimensional X of any finite dimension $d \geq 4$ with $m(X) = 4$ can be generalised to show that $m(X \oplus_1 \mathbb{R}) = 4$ for any X of dimension at least 2 which has a smooth point on its unit sphere. By a construction involving Hadamard matrices we then show that both $m(\ell_p)$ and $m(\ell_p^d)$ are finite and bounded above by a function of p, for all $1 \leq p < 2$. Also, for all $p \in [1, \infty)$ and $d \in \mathbb{N}$ there exists $c = c(p, d) > 1$ such that $m(X) \leq d + 1$ for all d-dimensional X with Banach-Mazur distance less than c from ℓ_p^d. Using Brouwer's fixed-point theorem we show that $m(X) \leq d + 1$ for all d-dimensional X with Banach-Mazur distance less than $3/2$ from ℓ_∞^d. A graph-theoretical argument furthermore shows that $m(\ell_\infty^d) = d + 1$.

The above results lead us to conjecture that $m(X) \leq 1 + \dim X$.

1. Introduction

Vector spaces in this paper are over the field $\mathbb{R}$ of real numbers. Write $[d] := \{1, 2, \dots, d\}$ for any $d \in \mathbb{N}$ and $\binom{V}{k} := \{A \subseteq V : |A| = k\}$ for any set V and $k \in \mathbb{N}$. Consider d-dimensional vectors to be functions $\boldsymbol{x} : [d] \to \mathbb{R}$ denoted using the superscript notation $\boldsymbol{x} = (\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d)})$. Similarly, write $\boldsymbol{x} = (\boldsymbol{x}^{(n)})_{n \in \Gamma}$ for any scalar-valued function defined on a set Γ. Write $\boldsymbol{o}$ for zero vectors and the zero function. For any $\gamma \in \Gamma$, let $\boldsymbol{e}_\gamma$ denote the indicator function of $\{\gamma\}$, i.e., $\boldsymbol{e}_\gamma(\gamma) = 1$ and $\boldsymbol{e}_\gamma(\delta) = 0$ for all $\delta \in \Gamma \setminus \{\gamma\}$. Given $\boldsymbol{a} = (\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(d)}) \in \mathbb{R}^d$ and $\boldsymbol{b} \in X$ whith X any vector space, define the *Kronecker product* $\boldsymbol{a} \otimes \boldsymbol{b}$ by $(\boldsymbol{a}^{(1)}\boldsymbol{b}, \dots, \boldsymbol{a}^{(d)}\boldsymbol{b}) \in X^d$.

Let X denote a real normed space with norm $\|\cdot\| = \|\cdot\|_X$. Denote the multiplicative Banach-Mazur distance between two isomorphic normed spaces X_1 and X_2 by

$$d(X_1, X_2) := \inf\left\{\|T\| \cdot \|T^{-1}\| : T \text{ is a linear isomorphism from } X_1 \text{ to } X_2\right\}.$$

Here, as usual, the notation $\|T\|$ doubles as the operator norm. Let Γ be any set. For $p \in [1, \infty)$ let $\ell_p(\Gamma)$ denote the Banach space of all functions $\boldsymbol{x} \colon \Gamma \to \mathbb{R}$ such that $\sum_{n \in \Gamma} \left|\boldsymbol{x}^{(n)}\right|^p < \infty$ with norm $\|\boldsymbol{x}\|_p = \left(\sum_{n \in \Gamma} \left|\boldsymbol{x}^{(n)}\right|^p\right)^{1/p}$. Let $\ell_p(\Gamma)$ denote the Banach space of all bounded scalar-valued functions on Γ with norm $\|\boldsymbol{x}\|_\infty := \max_{n \in \Gamma} \left|\boldsymbol{x}^{(n)}\right|$. As usual, write ℓ_p for the sequence spaces $\ell_p(\mathbb{N})$ and ℓ_p^d for $\ell_p([d])$. If X and Y are two normed spaces, their ℓ_p-sum $X \oplus_p Y$ is defined to be the direct sum $X \oplus Y$ with norm $\|(\boldsymbol{x}, \boldsymbol{y})\|_p := \|(\|\boldsymbol{x}\|_X, \|\boldsymbol{y}\|_Y)\|_p$. Also, write c for the subspace of ℓ_∞ of convergent sequences, and c_0 for the subspace of null

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sequences. Denote the *sphere* in X with center $\boldsymbol{c} \in X$ and radius $r > 0$ by

$$S(\boldsymbol{c}, r) = S_X(\boldsymbol{c}, r) := \{\boldsymbol{x} \in X : \|\boldsymbol{x} - \boldsymbol{c}\| = r\}.$$

Definition 1. *A subset $A \subseteq X$ is λ-*equilateral *if $\|\boldsymbol{x} - \boldsymbol{y}\| = \lambda$ for all $\{\boldsymbol{x}, \boldsymbol{y}\} \in \binom{A}{2}$. A set $A \subseteq X$ is* equilateral *if A is λ-equilateral for some $\lambda > 0$. An equilateral set $A \subseteq X$ is* maximal *if there does not exist an equilateral set $A' \subseteq X$ with $A \subsetneqq A'$.*

It is clear that a λ-equilateral set is a maximal equilateral set if and only if it does not lie on a sphere of radius λ.

For a survey on equilateral sets, see [8]. See also [9] for recent results on the existence of large equilateral sets in finite-dimensional spaces. This paper will be exclusively concerned with maximal equilateral sets.

Definition 2. *Let $m(X)$ denote the minimum cardinality of a maximal equilateral set in the normed space X.*

By a simple continuity argument, any set of two points in a normed space of dimension at least 2 can be extended to an equilateral set of size 3. It is also easy to find a maximal equilateral set of size 3 in any 2-dimensional X. It follows that $m(X) = 3$ for all 2-dimensional X.

Using a topological result, Petty [7] showed that if the dimension of X is at least 3, any equilateral set of size 3 can be extended to one of size 4. He also constructed, for each dimension $d \geq 3$, a d-dimensional normed space with a maximal equilateral set of size 4. Below we modify his example to show that ℓ_1^d also has this property. Petty showed furthermore that an equilateral set in a d-dimensional normed space has size at most 2^d, attained by ℓ_∞^d. Thus his results may be summarized as saying that $4 \leq m(X) \leq 2^d$ when $\dim X = d \geq 3$, with equality possible in the first inequality in each dimension.

A simple linear algebra argument shows that $m(\ell_2^d) = d + 1$. Brass [2] and Dekster [3] independently showed that if $d(X, \ell_2^d) < 1 + 1/(d+1)$, then $m(X) = d + 1$. In particular, since $d(\ell_p^d, \ell_2^d) = d^{|1/p - 1/2|}$, it follows that

$$m(\ell_p^d) = d + 1 \quad \text{if} \quad \left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2d \ln d}. \tag{1}$$

Even though ℓ_∞^d has an equilateral set of size 2^d, it has a maximal equilateral set of size $d + 1$. More generally, we show the following:

Theorem 3. *If $d(X, \ell_\infty^d) < 3/2$, then $m(X) \leq d + 1$. In addition, $m(\ell_\infty^d) = d + 1$.*

Theorem 3 will follow from Propositions 8 and 10 below. A similar result holds for the ℓ_p^d spaces.

Theorem 4. *For each $p \in (1, \infty)$ and $d \geq 3$ there exists $c = c(p, d) > 1$ such that $m(X) \leq d + 1$ for any d-dimensional X with $d(X, \ell_p^d) < c$.*

Theorem 4 will be proved in Section 5 below. Our main result is the following surprising property of ℓ_p where $p < 2$.

Theorem 5. *For each $p \in [1, 2)$ there exist $C = C(p) \in \mathbb{N}$ and $d_0 = d_0(p) \in \mathbb{N}$ such that for any normed space X, any $d \geq d_0$, and any $q \in [1, \infty)$, $m(\ell_p^d \oplus_q X) \leq C$. For p close to 1, upper bounds are given in Table 1. When $p \to 2$, $C(p) = O(1/(2 - p))$ and $d_0(p) = O(1/(2 - p))$.*

Note that the bound on $C(p)$ in the above theorem for p close to 2 is close to optimal, as (1) implies that

$$C(p) = \Omega\left(\frac{1}{(2 - p) \ln(2 - p)^{-1}}\right).$$

Range of p	$C(p)$	$d_0(p)$	Reason
$1 \leq p < \dfrac{\log 5/2}{\log 2} \approx 1.32$	5	4	Prop. 17
$\dfrac{\log 5/2}{\log 2} \leq p < \dfrac{\log 3}{\log 2} \approx 1.58$	8	6	Prop. 21 with $(k_1, k_2) = (2, 2)$
$\dfrac{\log 3}{\log 2} \leq p < \dfrac{\log 13/4}{\log 2} \approx 1.70$	12	10	Prop. 21 with $(k_1, k_2) = (2, 4)$
$\dfrac{\log 13/4}{\log 2} \leq p < \dfrac{\log 7/2}{\log 2} \approx 1.81$	16	14	Prop. 21 with $(k_1, k_2) = (4, 4)$
$\dfrac{\log 7/2}{\log 2} \leq p < \dfrac{\log 29/8}{\log 2} \approx 1.86$	24	22	Prop. 21 with $(k_1, k_2) = (4, 8)$
$\dfrac{\log 29/8}{\log 2} \leq p < \dfrac{\log 15/4}{\log 2} \approx 1.907$	32	30	Prop. 21 with $(k_1, k_2) = (8, 8)$
$\dfrac{\log 15/4}{\log 2} \leq p < \dfrac{\log 91/24}{\log 2} \approx 1.923$	40	38	Prop. 21 with $(k_1, k_2 = (8, 12)$
$\dfrac{\log 91/24}{\log 2} \leq p < \dfrac{\log 23/4}{\log 2} \approx 1.939$	48	46	Prop. 21 with $(k_1, k_2) = (12, 12)$

TABLE 1. Values of $C(p)$ and $d_0(p)$ in Theorem 5

Theorem 5 will be proved in Section 6 below.

We do not know of any d-dimensional space X for which $m(X) > d + 1$. The above theorems give some evidence for the following conjecture:

Conjecture 6. *For any d-dimensional normed space X, $m(X) \leq d + 1$.*

2. Generalising Petty's example

Petty [7] showed that $m(\ell_2^d \oplus_1 \mathbb{R}) = 4$ for all $d \geq 2$. In his argument ℓ_2^d can in fact be replaced by any, not necessarily finite-dimensional, normed space which has a smooth point on its unit sphere. By a theorem of Mazur [6] any separable normed space enjoys this property.

Proposition 7. *Let X be a normed space of dimension at least 2 with a norm that is Gâteaux differentiable at some point. Then $m(X \oplus_1 \mathbb{R}) = 4$.*

Proof. Since $X \oplus \mathbb{R}$ is at least 3-dimensional, $m(X) \geq 4$, as mentioned in Section 1. For the upper bound, let $\boldsymbol{u} \in X$ be a unit vector such that the norm of X is Gâteaux differentiable at $\boldsymbol{u}$. Let $A := \{(\boldsymbol{o}, 1), (\boldsymbol{o}, -1), (\boldsymbol{u}, 0), (-\boldsymbol{u}, 0)\}$. Then A is 2-equilateral. If there exist $(\boldsymbol{x}, r) \in X \oplus_1 \mathbb{R}$ at distance 2 to each point in A, then it easily follows that $r = 0$, $\|\boldsymbol{x}\| = 1$ and $\|\boldsymbol{x} \pm \boldsymbol{u}\| = 2$. Then $\pm\frac{1}{2}\boldsymbol{x} \pm \frac{1}{2}\boldsymbol{u}$ are all unit vectors, which implies that the unit ball of the subspace generated by $\boldsymbol{u}$ and $\boldsymbol{x}$ is the parallelogram with vertices $\pm\boldsymbol{u}, \pm\boldsymbol{x}$. In particular, $\boldsymbol{u}$ is not a point of Gâteaux differentiability of the norm. □

As special cases, $m(\ell_1) = m(\ell_1^d) = 4$ for $d \geq 3$. However, if Γ is an uncountable set, then the norm of $\ell_1(\Gamma)$ is nowhere Gâteaux differentiable. It will follow from the results in Section 6 that $m(\ell_1(\Gamma)) \leq 5$.

3. Using Brouwer's fixed point theorem

Proposition 8. *If $d(X, \ell_\infty^d) < 3/2$, then there exists a maximal equilateral set with $d+1$ elements. As a consequence, $m(X) \leq d+1$.*

Proof. As preparation for the proof, we first exhibit a 2-equilateral set A of $d+1$ points in ℓ_∞ such that $S(\boldsymbol{o}, 1)$ is the unique sphere (of any radius) that passes through A. For $i \in [d+1]$ and $n \in [d]$, let

$$\boldsymbol{p}_i^{(n)} := \begin{cases} -1 & \text{if } n = i, \\ 0 & \text{if } n > i, \\ 1 & \text{if } n < i, \end{cases}$$

and set $A = \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_{d+1}\}$. Suppose that $A \subset S(\boldsymbol{x}, r)$ for some $\boldsymbol{x} \in X$ and $r > 0$. Then for each $n \in [d]$, $\left|x^{(n)} \pm 1\right| \leq r$, hence $\left|x^{(n)}\right| \leq r - 1$ and $r \geq 1$. If we can show that $r = 1$, we would also get $\boldsymbol{x} = \boldsymbol{o}$. Suppose for the sake of contradiction that $r > 1$.

We first show that $\boldsymbol{x} = (r-1, r-1, \dots, r-1)$. If not, let m be the smallest index such that $\boldsymbol{x}^{(m)} \neq r - 1$. Then for all $n < m$, $\left|\boldsymbol{x}^{(n)} - \boldsymbol{p}_m^{(n)}\right| = |r - 1 - 1| < r$, and for $n > m$, $\left|\boldsymbol{x}^{(n)} - \boldsymbol{p}_m^{(n)}\right| = \left|x^{(n)}\right| \leq r - 1$. It follows that $r = \|\boldsymbol{x} - \boldsymbol{p}_m\|_\infty = \left|\boldsymbol{x}^{(m)} + 1\right|$. Thus $\boldsymbol{x}^{(m)} = -1 \pm r$, which contradicts $\left|x^{(n)}\right| \leq r - 1$ and the choice of m. Therefore, $\boldsymbol{x} = (r-1, r-1, \dots, r-1)$.

Since $r = \|\boldsymbol{x} - \boldsymbol{p}_{d+1}\|_\infty = |r - 1 - 1| < r$, we have obtained a contradiction. Therefore, A lies on a unique sphere. Since this sphere has radius 1, A is maximal equilateral. This shows that $m(\ell_\infty^d) \leq d+1$.

We now prove the general result. Let $D := d(X, \ell_\infty^d) < 3/2$, and assume without loss of generality that $X = (\mathbb{R}^d, \|\cdot\|)$ such that

$$\|\boldsymbol{x}\| \leq \|\boldsymbol{x}\|_\infty \leq D\|\boldsymbol{x}\| \text{ for all } \boldsymbol{x} \in \mathbb{R}^d. \tag{2}$$

We will prove that $m(X) \leq d+1$ by finding a perturbation of the above set A that will be maximal equilateral in X. We use Brouwer's theorem as in [2] and [9]. Consider the space $\mathbb{R}^{\binom{[d+1]}{2}}$ of vectors indexed by unordered pairs of elements from $[d+1]$. Write $\boldsymbol{z}^{\{i,j\}}$ for the coordinate of $\boldsymbol{z} \in \mathbb{R}^{\binom{[d+1]}{2}}$ indexed by $\{i, j\}$. For $\boldsymbol{z} \in I := [0,1]^{\binom{[d+1]}{2}} \subset \mathbb{R}^{\binom{[d+1]}{2}}$, define $\boldsymbol{p}_1(\boldsymbol{z}), \dots, \boldsymbol{p}_{d+1}(\boldsymbol{z}) \in \mathbb{R}^d$ as follows. For $i \in [d+1]$ and $n \in [d]$, let

$$\boldsymbol{p}_i^{(n)}(\boldsymbol{z}) := \begin{cases} -1 & \text{if } n = i, \\ 0 & \text{if } n > i, \\ 1 + \boldsymbol{z}^{\{n,i\}} & \text{if } n < i. \end{cases} \tag{3}$$

Define the mapping $\varphi : I \to I$ by

$$\varphi^{\{i,j\}}(\boldsymbol{z}) := \|\boldsymbol{p}_i(\boldsymbol{z}) - \boldsymbol{p}_j(\boldsymbol{z})\|_\infty - \|\boldsymbol{p}_i(\boldsymbol{z}) - \boldsymbol{p}_j(\boldsymbol{z})\| = 2 + z^{\{i,j\}} - \|\boldsymbol{p}_i(\boldsymbol{z}) - \boldsymbol{p}_j(\boldsymbol{z})\|$$

for each $\{i, j\} \in \binom{[d+1]}{2}$. Then by (2), $\varphi^{\{i,j\}}(\boldsymbol{z}) \geq 0$ and

$$\begin{aligned} \varphi^{\{i,j\}}(\boldsymbol{z}) &\leq \|\boldsymbol{p}_i(\boldsymbol{z}) - \boldsymbol{p}_j(\boldsymbol{z})\|_\infty - \frac{1}{D}\|\boldsymbol{p}_i(\boldsymbol{z}) - \boldsymbol{p}_j(\boldsymbol{z})\|_\infty \\ &= \left(1 - \frac{1}{D}\right)(2 + \boldsymbol{z}^{\{i,j\}}) \\ &< \left(1 - \frac{2}{3}\right)(2 + 1) = 1. \end{aligned}$$

Thus φ is well-defined. It is clearly continuous, and so has a fixed point z_0 by Brouwer's theorem:

$$2 + z_0^{\{i,j\}} - \|\boldsymbol{p}_i(z_0) - \boldsymbol{p}_j(z_0)\| = z_0^{\{i,j\}} \quad \text{for all } \{i,j\} \in \binom{[d+1]}{2}.$$

Therefore, $\{\boldsymbol{p}_1(z_0), \dots, \boldsymbol{p}_{d+1}(z_0)\}$ is 2-equilateral in X.

From now on, write $\boldsymbol{p}_i$ for $\boldsymbol{p}_i(z_0)$. We have to show that $\{\boldsymbol{p}_1, \dots \boldsymbol{p}_{d+1}\}$ is maximal equilateral. Suppose for the sake of contradiction that $\boldsymbol{x} \in X$ satisfies $\|\boldsymbol{x} - \boldsymbol{p}_i\| = 2$ for each $i \in [d+1]$. We first show that $\left|\boldsymbol{x}^{(n)}\right| < 2$ for all $n \in [d]$, and then obtain a contradiction.

By (2),

$$2 \le \|\boldsymbol{x} - \boldsymbol{p}_i\|_\infty \le 2D \quad \text{for each } i \in [d+1].$$

In particular, $\left|\boldsymbol{x}^{(n)} - \boldsymbol{p}_n^{(n)}\right| = \left|\boldsymbol{x}^{(n)} + 1\right| \le 2D$, which gives $\boldsymbol{x}^{(n)} \le 2D - 1 < 2$ for all $n \in [d]$. Also, $\left|\boldsymbol{x}^{(n)} - \boldsymbol{p}_{n+1}^{(n)}\right| \le 2D$, i.e., $\left|\boldsymbol{x}^{(n)} - 1 - z^{\{n,n+1\}}\right| \le 2D$, which gives $\boldsymbol{x}^{(n)} \ge 1 + z^{\{n,n+1\}} - 2D > -2$. It follows that $\left|\boldsymbol{x}^{(n)}\right| < 2$ for all $n \in [d]$.

Since $\|\boldsymbol{x} - \boldsymbol{p}_i\|_\infty \ge 2$ for each $i \in [d+1]$, by the pigeon-hole principle there exist a coordinate $n \in [d]$ and two points $\boldsymbol{p}_i, \boldsymbol{p}_j$, $\{i,j\} \in \binom{[d+1]}{2}$, such that $\left|\boldsymbol{x}^{(n)} - \boldsymbol{p}_i^{(n)}\right|, \left|\boldsymbol{x}^{(n)} - \boldsymbol{p}_j^{(n)}\right| \ge 2$. Without loss of generality, $i \ne n$. Then $\boldsymbol{p}_i^{(n)} \ge 0$ from (3), and it follows that $\left|\boldsymbol{x}^{(n)} - \boldsymbol{p}_i^{(n)}\right| < 2$, a contradiction.

We have shown that $\{\boldsymbol{p}_1, \dots \boldsymbol{p}_{d+1}\}$ is maximal equilateral. □

4. Using graph theory

In their studies of neighborly axis-parallel boxes, Zaks [10] and Alon [1] considered coverings of complete graphs by complete bipartite subgraphs. We will also use graphs in the proof that an arbitrary equilateral set of at most d points in ℓ_∞^d can be extended to a larger equilateral set. Our proof shows in fact that any collection of at most d pairwise touching, axis-parallel boxes in $\mathbb{R}^d$ can be extended to a pairwise touching collection of $d+1$ axis-parallel boxes.

As usual, the edges of a graph are considered to be unordered pairs. Let K_k denote the complete graph with vertex set $[k]$ and edge set $\binom{[k]}{2}$. For $A, B \subseteq [k]$ such that $A \cap B = \emptyset$, $A \cup B \ne \emptyset$, define their *unordered product* to be $A \bowtie B := \{\{a,b\} : a \in A, b \in B\}$. Thus $A \bowtie B$ is the set of edges of a complete bipartite subgraph of K_k, where we allow one, but not more than one, of A or B to be empty. As the definition implies, $A \bowtie B = B \bowtie A$.

Lemma 9. *Let $d \ge k \ge 1$ be integers. Suppose that the edges of the complete graph K_k are covered by d (not necessarily distinct) unordered products $A_n^0 \bowtie A_n^1$, $n \in [d]$, where for each n, $A_n^0, A_n^1 \subseteq [k]$, $A_n^0 \cap A_n^1 = \emptyset$, and $A_n^0 \cup A_n^1 \ne \emptyset$. Then there exist $\sigma_1, \dots, \sigma_d \in \{0,1\}$ such that $A_1^{\sigma_1} \cup \dots \cup A_d^{\sigma_d} = [k]$.*

Proof. We use induction on $k \in \mathbb{N}$. The case $k = 1$ is trivial, so we assume that $k \ge 2$ and that the theorem holds for K_{k-1}. If for each $j \in [k]$, some $A_n^0 \bowtie A_n^1 = \emptyset \bowtie \{j\}$, take σ_n such that $A_n^{\sigma_n} = \{j\}$ for each of these n. Then choose all remaining σ_n arbitrarily to obtain the required covering of $[k]$.

Thus assume without loss of generality that $\emptyset \bowtie \{k\}$ does not occur as a $A_n^0 \bowtie A_n^1$. The edge $\{1,k\}$ is covered by some $A_n^0 \bowtie A_n^1$ (note $k \ge 2$). Without loss of generality, $n = d$, i.e., $k \in A_d^{\sigma_d}$ for some $\sigma_d \in \{0,1\}$. Set $B_n^0 := A_n^0 \setminus \{k\}$ and $B_n^1 := A_n^1 \setminus \{k\}$ for each $n \in [d]$. Then $B_n^0 \bowtie B_n^1$, $n \in [d-1]$, cover the edges of K_{k-1}. Since all $A_n^0 \bowtie A_n^1 \ne \emptyset \bowtie \{k\}$, we still have $B_n^0 \cup B_n^1 \ne \emptyset$, so we may apply the induction hypothesis to obtain $B_n^{\sigma_n}$, $n \in [d-1]$, that cover $[k-1]$. Together with $A_d^{\sigma_d}$ we have obtained the required covering of $[k]$. □

Proposition 10. $m(\ell_\infty^d) \geq d+1$.

Proof. We show that any 1-equilateral set $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_k\} \subset \ell_\infty^d$ of size at most $k \leq d$ can be extended. Without loss of generality, $k \geq 1$.

Since $\left|\boldsymbol{p}_i^{(n)} - \boldsymbol{p}_j^{(n)}\right| \leq 1$ for all $\{i,j\} \in \binom{[k]}{2}$ and $n \in [d]$, we may assume after a suitable translation that all $\boldsymbol{p}_i \in [0,1]^d$. For each $n \in [d]$, define $A_n^0 := \left\{i : p_i^{(n)} = 0\right\}$ and $A_n^1 := \left\{i : p_i^{(n)} = 1\right\}$. Again by making a suitable translation we may assume that each $A_n^0 \cup A_n^1 \neq \emptyset$.

Since $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_k\}$ is 1-equilateral, the edges of K_k are covered by $A_n^0 \bowtie A_n^1$, $n \in [d]$. Indeed, since for any edge $\{i,j\}$ of K_k, $\|\boldsymbol{p}_i - \boldsymbol{p}_j\|_\infty = 1$, there exists an $n \in [d]$ with $\left|\boldsymbol{p}_i^{(n)} - \boldsymbol{p}_j^{(n)}\right| = 1$. Since $0 \leq \boldsymbol{p}_i^{(n)}, \boldsymbol{p}_j^{(n)} \leq 1$, it follows that $\boldsymbol{p}_i^{(n)}, \boldsymbol{p}_j^{(n)} = \{0,1\}$, which gives $\{i,j\} \in A_n^0 \bowtie A_n^1$.

By Lemma 9 we may choose $A_n^{\sigma_n}$, $\sigma_n \in \{0,1\}$, such that $A_1^{\sigma_1} \cup \dots \cup A_d^{\sigma_d} = [k]$. Define $\boldsymbol{q} = (1,1,\dots,1) - (\sigma_1, \dots, \sigma_d)$. We show that for each $i \in [k]$, $\|\boldsymbol{p}_i - \boldsymbol{q}\|_\infty = 1$. Since $\boldsymbol{q} \in [0,1]^d$, $\|\boldsymbol{p}_i - \boldsymbol{q}\|_\infty \leq 1$. There exists $n \in [d]$ such that $i \in A_n^{\sigma_n}$, i.e., $\boldsymbol{p}_i^{(n)} = \sigma_n$. It follows that $\left|\boldsymbol{p}_i^{(n)} - \boldsymbol{q}^{(n)}\right| = 1$, which gives $\|\boldsymbol{p}_i - \boldsymbol{q}\|_\infty = 1$. □

5. A calculation

We omit the simple proof of the following lemma.

Lemma 11. *For any $p \geq 1$ and $\lambda > 0$ the function $f(x) = |x+\lambda|^p - |x|^p$, $x \in \mathbb{R}$, is increasing, and strictly increasing if $p > 1$.*

Proposition 12. *For any $p \geq 1$, $m(\ell_p^d) \leq d+1$.*

Proof. We have already observed above that $m(X) = 3$ for any two-dimensional X, so we may assume that $d \geq 3$. We have also observed that $m(\ell_1^d) \leq 4$ for all $d \geq 3$, so we may assume that $p > 1$.

The set of standard unit basis vectors $S = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_d\}$ in $\mathbb{R}^d$ is $2^{1/p}$-equilateral in ℓ_p^d. We show that S can be extended, and if S is extended in two ways $S \cup \{\boldsymbol{p}\}$ and $S \cup \{\boldsymbol{q}\}$, then the distance $\|\boldsymbol{p} - \boldsymbol{q}\|_p > 2^{1/p}$. Thus both $S \cup \{\boldsymbol{p}\}$ and $S \cup \{\boldsymbol{q}\}$ will be maximal equilateral. (In fact S has exactly two extensions, but we don't need this for the proof.)

Let $\boldsymbol{p}$ be equidistant to all points of S, say $\|\boldsymbol{p}_i - \boldsymbol{e}_i\|_p = c$ for all $i \in [d]$ where $c > 0$ is fixed. It then follows that $\left|\boldsymbol{p}^{(i)} - 1\right|^p - \left|\boldsymbol{p}^{(i)}\right|^p = c^p - \|\boldsymbol{p}\|_p^p$ for all i. By Lemma 11, $\boldsymbol{p}^{(1)} = \dots = \boldsymbol{p}^{(d)}$, i.e., $\boldsymbol{p}$ is a multiple of $\boldsymbol{j} = (1,1,\dots,1) \in \mathbb{R}^d$.

Suppose now $\boldsymbol{p} = x\boldsymbol{j}$ satisfies $\|\boldsymbol{p} - \boldsymbol{e}_i\|_p = 2^{1/p}$ for all $i \in [d]$. It follows that

$$|x-1|^p + (d-1)\,|x|^p = 2. \tag{4}$$

Consider the function $f(x) = |x-1|^p + (d-1)\,|x|^p$. It is clearly strictly decreasing on $(-\infty, 0]$, and since $f(0) = 1$ and $f(-1) > 2$, equation (4) has a unique negative solution $-\mu$, say, in the interval $(-1,0)$. Let λ be any other solution to (4). Then $\lambda > 0$ (there is in fact a unique positive solution to (4), but we don't need to show this), and we have to show that $\|-\mu\boldsymbol{j} - \lambda\boldsymbol{j}\|_p > 2^{1/p}$, i.e., $\lambda + \mu > (2/d)^{1/p}$. Since λ is a solution to (4), it follows that $2 = (1-\lambda)^p + (d-1)\lambda^p < 1 + d\lambda^p$, hence $\lambda > (1/d)^{1/p}$. It remains to show that $\mu \geq (2^{1/p} - 1)/d^{1/p}$. Suppose then that

$$\mu < \frac{2^{1/p} - 1}{d^{1/p}}. \tag{5}$$

Since $x = -\mu$ is a solution of (4),

$$\begin{aligned} 2 &= (1+\mu)^p + (d-1)\mu^p \\ &\le (1+\mu)^p - \mu^p + (2^{1/p}-1)^p \quad \text{by (5),} \end{aligned}$$

hence

$$(1+2^{1/p}-1)^p - (2^{1/p}-1)^p \le (1+\mu)^p - \mu^p.$$

By Lemma 11, $2^{1/p} - 1 \le \mu$, which contradicts (5). □

Proposition 13. *Let $1 < p < \infty$, $d \ge 3$, $0 < \varepsilon \le (d-2)^{-1/(p-1)}$, and $R = (1 + \frac{p-1}{2}\varepsilon)^{1/p}$. Suppose that $X = (\mathbb{R}^d, \|\cdot\|)$ is given such that*

$$\|x\| \le \|x\|_p \le R\|x\| \quad \textit{for all } \boldsymbol{x} \in \mathbb{R}^d.$$

Then X has a λ-equilateral set $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_d\}$, where $\lambda = (2 + (d-2)\varepsilon^p)^{1/p}$, such that $\boldsymbol{p}_i^{(i)} = 1$ for all $i \in [d]$, $-\varepsilon < \boldsymbol{p}_i^{(j)} < 0$ for all $i, j \in [d]$ with $j < i$, and $\boldsymbol{p}_i^{(j)} = 0$ for all $i, j \in [d]$ with $j > i$.

Proof. Let $R > 1$ and $\beta, \gamma > 0$ be arbitrary (to be fixed later). For $i \in [d]$ define $\boldsymbol{p}_i \colon \mathbb{R}^{\binom{[d]}{2}} \to \mathbb{R}^d$ by setting for each $n \in [d]$,

$$\boldsymbol{p}_i^{(n)}(\boldsymbol{z}) = \begin{cases} z^{\{i,j\}} & \text{if } n < i, \\ -\gamma & \text{if } n = i, \\ 0 & \text{if } n > i. \end{cases}$$

That is,

$$\boldsymbol{p}_i(\boldsymbol{z}) = (z^{\{1,i\}}, \dots, z^{\{i-1,i\}}, -\gamma, 0, \dots, 0).$$

Let $I = [0, \beta]^{\binom{[d]}{2}}$ and define $\varphi \colon I \to I$ by

$$\varphi^{\{i,j\}}(\boldsymbol{z}) = 1 + z^{\{i,j\}} - \|\boldsymbol{p}_i(\boldsymbol{z}) - \boldsymbol{p}_j(\boldsymbol{z})\| \quad \text{for each } \{i,j\} \in \binom{[d]}{2}.$$

It is clear that φ is continuous. We next show that φ is well defined if R, β, and γ are chosen appropriately. Let $\boldsymbol{z} \in I$. Then $0 \le z^{\{i,j\}} \le \beta$ for all $\{i,j\} \in \binom{[d]}{2}$. We first bound $\|\boldsymbol{p}_i(\boldsymbol{z}) - \boldsymbol{p}_j(\boldsymbol{z})\|_p$. Without loss of generality, $i < j$. Then

$$\begin{aligned} \|\boldsymbol{p}_i(\boldsymbol{z}) - \boldsymbol{p}_j(\boldsymbol{z})\|_p^p &= \sum_{k=1}^{i-1} \left| z^{\{k,i\}} - z^{\{k,j\}} \right|^p + \left| \gamma + z^{\{i,j\}} \right|^p \\ &\qquad + \sum_{k=i+1}^{j-1} \left| z^{\{k,j\}} \right|^p + \gamma^p \\ &\le (i-1)\beta^p + (\gamma + z^{\{i,j\}})^p + (j-1-i)\beta^p + \gamma^p \\ &= (j-2)\beta^p + \gamma^p + (\gamma + z^{\{i,j\}})^p \end{aligned} \tag{6}$$

and

$$\|\boldsymbol{p}_i(\boldsymbol{z}) - \boldsymbol{p}_j(\boldsymbol{z})\|_p^p \ge \gamma^p + (\gamma + z^{\{i,j\}})^p. \tag{7}$$

Thus

$$\varphi^{\{i,j\}} \ge 1 + z^{\{i,j\}} - \Big((j-2)\beta^p + \gamma^p + (\gamma + z^{\{i,j\}})^p \Big)^{1/p}.$$

Let $f(x) = 1 + x - ((j-2)\beta^p + \gamma^p + (\gamma+x)^p)^{1/p}$, $0 \le x \le \beta$. Then

$$\begin{aligned} f'(x) &= 1 - \frac{1}{p}\left((j-1)\beta^p + \gamma^p + (\gamma+x)^p\right)^{1/p} p(\gamma+x)^{p-1} \\ &= 1 - \left(\frac{(j-1)\beta^p + \gamma^p + (\gamma+x)^p}{(\gamma+x)^p}\right)^{\frac{1}{p}-1} \\ &> 1 - 1 = 0 \quad \text{since } \frac{1}{p} - 1 < 0. \end{aligned}$$

It follows that f is strictly increasing, which gives that

$$\begin{aligned} \varphi^{\{i,j\}} &\ge f\left(z^{\{i,j\}}\right) \ge f(0) = 1 - ((j-2)\beta^p + 2\gamma^p)^{1/p} \\ &\ge 1 - ((d-2)\beta^p + 2\gamma^p)^{1/p}. \end{aligned}$$

If we require that

$$(d-2)\beta^p + 2\gamma^p = 1 \tag{8}$$

then $\varphi^{\{i,j\}} \ge 0$ for all $\boldsymbol{z} \in I$. Also,

$$\begin{aligned} \varphi^{\{i,j\}}(\boldsymbol{z}) &\le 1 + z^{\{i,j\}} - \frac{1}{R}\|\boldsymbol{p}_i(\boldsymbol{z}) - \boldsymbol{p}_j(\boldsymbol{z})\|_p \\ &\le 1 + z^{\{i,j\}} - \frac{1}{R}\left(\gamma^p + (\gamma + z^{\{i,j\}})^p\right)^{1/p}. \end{aligned}$$

Let $g(x) = 1 + x - \frac{1}{R}\left(\gamma^p + (\gamma+x)^p\right)^{1/p}$, $0 \le x \le \beta$. Then

$$\begin{aligned} g'(x) &= 1 - \frac{1}{R}\left(\gamma^p + (\gamma+x)^p\right)^{\frac{1}{p}-1} p(\gamma+x)^{p-1} \\ &= 1 - \frac{1}{R}\left(\frac{\gamma^p + (\gamma+x)^p}{(\gamma+x)^p}\right)^{\frac{1}{p}-1} \\ &> 1 - \frac{1}{R} > 0. \end{aligned}$$

Therefore, g is strictly increasing, which gives that

$$\varphi^{\{i,j\}}(\boldsymbol{z}) \le g(z^{\{i,j\}} \le g(\beta) = 1 + \beta - \frac{1}{R}\left(\gamma^p + (\gamma+\beta)^p\right)^{1/p}.$$

To derive $\varphi^{\{i,j\}}(\boldsymbol{z}) \le \beta$, it is sufficient to require that

$$\gamma^p + (\gamma+\beta)^p \ge R^p. \tag{9}$$

If we can find $\beta, \gamma > 0$ and $R > 1$ such that (8) and (9) are satisfied, then φ is well defined, and by Brouwer's fixed point theorem φ has a fixed point, that is, for some $\boldsymbol{z}_0 \in I$, $\varphi(\boldsymbol{z}_0) = \boldsymbol{z}_0$, which implies that $\{\boldsymbol{p}_i(\boldsymbol{z}_0) : i \in [d]\}$ is 1-equilateral. Since $\boldsymbol{p}_i^{(i)} = \boldsymbol{p}_i^{(i)}(\boldsymbol{z}_0) = -\gamma$, we have to divide each vector in this set by $-\gamma$. This means we have to set $\gamma = 1/\lambda$ and $\beta/\gamma = \varepsilon$. We can then rewrite (8) as

$$(d-2)\varepsilon^p + 2 = \lambda^p$$

and (9) as

$$\frac{1 + (1+\varepsilon)^p}{2 + (d-2)\varepsilon^p} \ge R^p.$$

Now assume that $\varepsilon \le (d-2)^{-1/(p-1)}$ and $R^p = 1 + \frac{p-1}{2}\varepsilon$. Since $p > 1$, $(1+\varepsilon)^p \ge 1 + p\varepsilon + \frac{p}{p-1}2\varepsilon^2$ for all $\varepsilon \ge 0$, and it is thus sufficient to show that

$$\frac{2 + p\varepsilon + \frac{p}{p-1}2\varepsilon^2}{2 + (d-2)\varepsilon^p} \ge 1 + \frac{p-1}{2}\varepsilon.$$

However,

$$\begin{aligned}
&2 + p\varepsilon + \frac{p}{p-1}2\varepsilon^2 - (2+(d-2)\varepsilon^p)\left(1+\frac{p-1}{2}\varepsilon\right) \\
&= -(d-2)\varepsilon^p + \varepsilon - \frac{1}{2}(d-2)(p-1)\varepsilon^{p+1} + \frac{p(p-1)}{2}\varepsilon^2 \\
&= \left(1-(d-2)\varepsilon^{p-1}\right)\varepsilon + \frac{1}{2}(p-1)\left(1-(d-2)\varepsilon^{p-1}\right)\varepsilon^2 + \frac{1}{2}(p-1)^2\varepsilon^2 \\
&> 0 \quad \text{since } 1-(d-2)\varepsilon^{p-1} \geq 0 \text{ and } p > 1.
\end{aligned}$$

We have shown that if we choose $\gamma = 1/\lambda = (2+(d-2)\varepsilon^p)^{-1/p}$, $\beta = \varepsilon\gamma$, and $R = \left(1+\frac{p-1}{2}\varepsilon\right)^{1/p}$, then (8) and (9) are satisfied, which finishes the proof. □

Proof of Theorem 4. Suppose that the theorem is false. Then for some $p \in (0,\infty)$ and $d \geq 3$ and for all $c > 1$, there exists a d-dimensional X such that $d(X, \ell_p^d) < c$ and $m(X) \geq d+2$. Choose a sequence $X_n = (\mathbb{R}^d, \|\cdot\|_{(n)})$ such that $m(X_n) \geq d+2$ and

$$\|\boldsymbol{x}\|_{(n)} \leq \|\boldsymbol{x}\|_p \leq \left(1+\frac{1}{n}\right)^{1/p} \|\boldsymbol{x}\|_{(n)} \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^d.$$

If n is sufficiently large, in particular,

$$n > \frac{2(d-2)^{1/(p-1)}}{p-1},$$

and if we choose $\varepsilon = \frac{2}{n(p-1)}$, then $\frac{1}{n} = \frac{p-1}{2}\varepsilon$ and $\varepsilon < (d-2)^{-1/(p-1)}$, and we may apply Proposition 13 to obtain an equilateral set $\{\boldsymbol{p}_i(n) : i \in [d]\}$ in X_n such that $\boldsymbol{p}_i^{(i)}(n) = 1$ for all $i \in [d]$ and $-\varepsilon < \boldsymbol{p}_i^{(j)}(n) \leq 0$ for all $i,j \in [d]$, $i \neq j$. Since $m(X_n) \geq d+2$, there exist points $\boldsymbol{p}_{d+1}(n), \boldsymbol{p}_{d+2}(n) \in X_n$ such that $\{\boldsymbol{p}_i(n) : i \in [d+2]\}$ is equilateral. By passing to a subsequence we may assume without loss of generality that $\boldsymbol{p}_{d+1}(n) \to \boldsymbol{p}$ and $\boldsymbol{p}_{d+2}(n) \to \boldsymbol{q}$ as $n \to \infty$. Since $\boldsymbol{p}_i(n) \to \boldsymbol{e}_i$ and $d(\|\cdot\|_{(n)}, \|\cdot\|_p) \to 0$ as $n \to \infty$, it follows that $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_d, \boldsymbol{p}, \boldsymbol{q}\}$ is equilateral in ℓ_p^d. However, in the proof of Proposition 12 we have shown this to be impossible. □

6. Using Hadamard matrices

Before introducing the properties of Hadamard matrices that will be needed, we first do a special case to illustrate the construction.

Lemma 14. *Let $1 \leq p \leq 2$. For each $\lambda \in [2^{1-1/p}, 2^{1/p}]$ there exist unit vectors $\boldsymbol{u}, \boldsymbol{v} \in \ell_p^2$ such that $\|\boldsymbol{u}+\boldsymbol{v}\|_p = \|\boldsymbol{u}-\boldsymbol{v}\|_p = \lambda$.*

Proof. Let $\boldsymbol{u} = (\alpha, \beta)$ and $\boldsymbol{v} = (-\beta, \alpha)$ where $\alpha, \beta \geq 0$ and $\alpha^p + \beta^p = 1$. Then $\|\boldsymbol{u} \pm \boldsymbol{v}\|_p^p = |\alpha+\beta|^p - |\alpha-\beta|^p$, which ranges from 2 when $\alpha = 0$ and $\beta = 1$, to 2^{p-1} when $\alpha = \beta = 2^{1/p}$. □

Lemma 15 (Monotonicity lemma)**.** *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be linearly independent unit vectors in a strictly convex 2-dimensional normed space. Let $\boldsymbol{p} \neq \boldsymbol{o}$ be any point such that $\boldsymbol{u}$ is between $\frac{1}{\|\boldsymbol{p}\|}\boldsymbol{p}$ and $\boldsymbol{v}$ on the boundary of the unit ball. Then $\|\boldsymbol{p}-\boldsymbol{u}\| < \|\boldsymbol{p}-\boldsymbol{v}\|$.*

For a proof of the above lemma, see [5, Proposition 31]. For non-strictly convex norms the above lemma still holds with a non-strict inequality. On the other hand, the following corollary of the monotonicity lemma is false when the norm is not strictly convex, as easy examples show.

Lemma 16. *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be linearly independent unit vectors in a strictly convex 2-dimensional normed space. Suppose that $\boldsymbol{x}$ is such that $\|\boldsymbol{x}-\boldsymbol{u}\| = \|\boldsymbol{x}+\boldsymbol{u}\|$ and $\|\boldsymbol{x}-\boldsymbol{v}\| = \|\boldsymbol{x}+\boldsymbol{v}\|$. Then $\boldsymbol{x}=\boldsymbol{o}$.*

Proof. Without loss of generality, $\boldsymbol{x} = \alpha\boldsymbol{u} + \beta\boldsymbol{v}$ with $\alpha, \beta \geq 0$. If $\boldsymbol{x} \neq \boldsymbol{o}$, then by Lemma 15,

$$\|\boldsymbol{x}-\boldsymbol{v}\| < \|\boldsymbol{x}+\boldsymbol{u}\| = \|\boldsymbol{x}-\boldsymbol{u}\| < \|\boldsymbol{x}+\boldsymbol{v}\|,$$

a contradiction. □

Proposition 17. *Let X be any normed space, $q \in [1,\infty)$, and $1 \leq p < \frac{\log 5/2}{\log 2}$. Then $m(\ell_p^4 \oplus_q X) \leq 5$. If $p = \frac{\log 5/2}{\log 2}$, then $m(\ell_p^4 \oplus_q X) \leq 6$.*

Proof. Consider the following subset of $\ell_p^4 \oplus_q X$:

$$\begin{aligned} S = \{ & (\ \ 1, \ \ 1, \ \ 1, \ 0, \ \boldsymbol{o}\ ),\\ & (\ \ 1, -1, -1, \ 0, \ \boldsymbol{o}\ ),\\ & (-1, \ \ 1, -1, \ 0, \ \boldsymbol{o}\ ),\\ & (-1, -1, \ \ 1, \ 0, \ \boldsymbol{o}\ ),\\ & (\ \ 0, \ \ 0, \ \ 0, \ \lambda, \ \boldsymbol{o}\ )\ \}. \end{aligned}$$

By setting $\lambda = (2^{p+1}-3)^{1/p}$, S becomes a $2^{1+1/p}$-equilateral set. We show that S is maximal equilateral. Suppose that $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \boldsymbol{x})$ has distance $2^{1+1/p}$ to each point in S.

Then $(\alpha_1, \alpha_2, \alpha_3)$ has the same distance in ℓ_p^3 to the points

$$(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1).$$

Then

$$\|(\alpha_1,\alpha_2)-(1,1)\|_p = \|(\alpha_1,\alpha_2)-(-1,-1)\|_p$$

and

$$\|(\alpha_1,\alpha_2)-(1,-1)\|_p = \|(\alpha_1,\alpha_2)-(-1,1)\|_p.$$

It follows (from Lemma 16 if $p > 1$) that $(\alpha_1, \alpha_2) = (0,0)$. Thus $|\alpha_3 - 1| = |\alpha_3 + 1|$, which gives $\alpha_3 = 0$.

It follows that $3 + |\alpha_4|^p = |\alpha_4 - \lambda|^p$. By Lemma 11, the function $f(x) = 3 + |x|^p - |x-\lambda|^p$ is increasing (strictly increasing if $p > 1$). Since $f(\alpha_4) = 0$ and $f(-\lambda) = 2^{p+1}(\frac{5}{2} - 2^p) \geq 0$ (> 0 if $p = 1$), it follows that $\alpha_4 \leq -\lambda$. Then by assumption,

$$\begin{aligned} 2^{1+1/p} &= \|(0,0,0,\alpha_4,\boldsymbol{x}) - (1,1,1,0,\boldsymbol{o})\|_q \\ &= \left((3+|\alpha_4|^p)^{q/p} + \|\boldsymbol{x}\|^q\right)^{1/q} \\ &\geq (3+\lambda^p)^{1/p} = 2^{1+1/p}, \end{aligned}$$

and equality holds throughout, which implies that $p = \frac{\log 5/2}{\log 2}$, $\alpha_4 = -\lambda$ and $\boldsymbol{x} = \boldsymbol{o}$. Therefore, S is maximal equilateral unless $p = \frac{\log 5/2}{\log 2}$, in which case $S \cup \{(0,0,0,-\lambda,\boldsymbol{o})\}$ is maximal equilateral. □

An $n \times n$ matrix H is called a *Hadamard matrix* of order n if each entry equals ± 1 and $HH^{\mathsf{T}} = nI$. It is easy to see that if a Hadamard matrix of order n exists, then $n = 1$, $n = 2$ or n is divisible by 4. It is conjectured that there exist Hadamard matrices of all orders divisible by 4. This is known for all multiples of for 4 up to 664 [4]. The next lemma summarises the only (well-known) results on the existence of Hadamard matrices that we will use.

Lemma 18. *There exist Hadamard matrices of orders* 1, 2, 4, 8, 12.

Let $x \geq 1$. The interval $(x, 2x)$ contains the order of some Hadamard matrix iff $x \notin \{1,2,4\}$.

Let $H(x)$ be the largest order n of a Hadamard matrix with $n < x$. Then $\lim_{x\to\infty} H(x)/x = 1$.

Proof. Given Hadamard matrices H_1 of order n_1 and H_2 of order n_2, the Kronecker product $H_1 \otimes H_2$ will be a Hadamard matrix of order $n_1 n_2$. Starting with the unique Hadamard matrices of orders 2 and 12, we obtain Hadamard matrices of orders 2^k and $12 \cdot 2^k$, $k \in \mathbb{N}$. This is sufficient to cover every interval $(x, 2x)$ except for $(1,2)$, $(2,4)$ and $(4,8)$.

The Paley construction gives a Hadamard matrix of order $q+1$ for any prime power $q \cong 3 \pmod 4$. The prime number theorem for arithmetic progressions states in particular that the number of primes less than x that are congruent to 3 modulo 4 is $(1+o(1))x/(2\ln x)$. This implies that the largest such prime less than x is $\geq (1+o(1))x$, which gives $H(x)/x \to 1$ as $x \to \infty$. □

A Hadamard matrix is *normalised* if its first column are all $+1$s. If

$$H = \begin{bmatrix} 1 & \boldsymbol{h}_1 \\ 1 & \boldsymbol{h}_2 \\ \vdots & \vdots \\ 1 & \boldsymbol{h}_n \end{bmatrix}$$

is a normalised Hadamard matrix we say that $\{\boldsymbol{h}_1, \dots, \boldsymbol{h}_n\} \subset \mathbb{R}^{n-1}$ is a *Hadamard simplex*. Note that a Hadamard simplex is equilateral in ℓ_p^{n-1} for any value of p and lies on a sphere with centre $\boldsymbol{o}$. Note that the next lemma shows in particular that a Hadamard simplex cannot lie on any other sphere of ℓ_p^{n-1} if $p \in [1, \infty)$.

Lemma 19. *Let $\boldsymbol{h}_1, \dots, \boldsymbol{h}_n$ be a Hadamard simplex. Let X be a normed space and let $\boldsymbol{u} \in X$. Suppose that*

$$\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}) \in \underbrace{X \oplus_p \cdots \oplus_p X}_{n-1 \text{ summands}}$$

has the same distance in the p-norm to each $\boldsymbol{h}_i \otimes \boldsymbol{u}$, $i \in [n]$. Then $\|\boldsymbol{x}_i - \boldsymbol{u}\| = \|\boldsymbol{x}_i + \boldsymbol{u}\|$ for all $i \in [n]$.

Proof. Let $\boldsymbol{h}_i = [h_{i,1}, h_{i,2}, \dots, h_{i,n-1}]$ for $i \in [n]$. We may assume without loss of generality that $\boldsymbol{h}_1 = [-1, -1, \dots, -1]$. Since $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{n-1})$ is equidistant to all $\boldsymbol{h}_i \otimes \boldsymbol{u}$, there exists $D \geq 0$ such that $\sum_{j=1}^{n-1} \|\boldsymbol{x}_j - h_{i,j}\boldsymbol{u}\|^p = D^p$ for each $i \in [n]$. Subtract the first of these equations from the others to obtain the system

$$\begin{bmatrix} \boldsymbol{h}_2 - \boldsymbol{h}_1 \\ \boldsymbol{h}_3 - \boldsymbol{h}_1 \\ \vdots \\ \boldsymbol{h}_{n-1} - \boldsymbol{h}_1 \end{bmatrix} \begin{bmatrix} \|\boldsymbol{x}_1 - \boldsymbol{u}\|^p - \|\boldsymbol{x}_1 + \boldsymbol{u}\|^p \\ \|\boldsymbol{x}_2 - \boldsymbol{u}\|^p - \|\boldsymbol{x}_2 + \boldsymbol{u}\|^p \\ \vdots \\ \|\boldsymbol{x}_{k-1} - \boldsymbol{u}\|^p - \|\boldsymbol{x}_{k-1} + \boldsymbol{u}\|^p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10}$$

The Hadamard matrix H is invertible. If we subtract the first row from all the other rows, the resulting matrix

$$\begin{bmatrix} 1 & \boldsymbol{o} \\ 0 & \boldsymbol{h}_2 - \boldsymbol{h}_1 \\ 0 & \vdots \\ 0 & \boldsymbol{h}_{n-1} - \boldsymbol{h}_1 \end{bmatrix}$$

is still invertible. It follows that (10) has the unique solution

$$\|\boldsymbol{x}_j - \boldsymbol{u}\|^p - \|\boldsymbol{x}_j + \boldsymbol{u}\|^p = 0 \quad \text{for all } j \in [n-1].$$

□

Lemma 20. *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be linearly independent unit vectors in a strictly convex 2-dimensional normed space X. Let $\boldsymbol{h}_1, \dots, \boldsymbol{h}_n$ be a Hadamard simplex. Suppose that*

$$\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}) \in \underbrace{X \oplus_p \cdots \oplus_p X}_{n-1 \text{ summands}}$$

has the same distance in the p-norm to each $\boldsymbol{h}_i \otimes \boldsymbol{u}$, $i \in [n]$, *and the same distance to each* $\boldsymbol{h}_i \otimes \boldsymbol{v}$, $i \in [n]$. *Then* $\boldsymbol{x} = \boldsymbol{o}$.

Proof. Combine Lemmas 16 and 19. □

Proposition 21. *Let* $p \in (1,2)$, $q \in [1,\infty)$, *and* X *any normed space. Let* $k_1, k_2 \in \mathbb{N}$ *be such that there exist Hadamard matrices of orders* k_1 *and* k_2 *and such that*

$$2 - 2^{p-1} \leq \frac{1}{k_1} + \frac{1}{k_2} < 4 - 2^p, \tag{11}$$

$$\frac{5}{2} - 2^{p-1} - 2^{1-p} \leq (1 - 2^{1-p})\frac{1}{k_1} + \frac{1}{k_2}, \tag{12}$$

$$\frac{5}{2} - 2^{p-1} - 2^{1-p} \leq \frac{1}{k_1} + (1 - 2^{1-p})\frac{1}{k_2}, \tag{13}$$

$$\text{and if } k_1 = k_2, \text{ then } 2 - 2^{p-1} < \frac{1}{k_1} + \frac{1}{k_2}. \tag{14}$$

Then $m(\ell_p^{2(k_1+k_2-1)} \oplus_q X) \leq 2(k_1 + k_2)$.

Proof. It is sufficient to construct an equilateral set S of cardinality $2(k_1 + k_2)$ in $\ell_p^{2(k_1+k_2-1)}$ that does not lie on any sphere. Then $S \oplus \{\boldsymbol{o}\}$ will be maximal equilateral in $\ell_p^{2(k_1+k_2-1)} \oplus_q X$ for any $q \in [1,\infty)$.

Let $\alpha_1, \alpha_2, \lambda_1, \lambda_2 \in \mathbb{R}$ to be fixed later such that

$$\alpha_1, \alpha_2 \geq 0 \quad \text{and} \quad 2^{1-1/p} \leq \lambda_1, \lambda_2 \leq 2^{1/p}. \tag{15}$$

By Lemma 14 there exist $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \ell_p^2$ such that $\|\boldsymbol{u}_i \pm \boldsymbol{v}_i\|_p = \lambda_i$, $i = 1,2$. Consider the following subset of $\ell_p^{2(k_1+k_2-1)} = \mathbb{R} \oplus_p \ell_p^{2(k_1-1)} \oplus_p \mathbb{R} \oplus_p \ell_p^{2(k_2-1)}$:

$$\begin{aligned}
S_1^- &= \{\, (-\alpha_1, k_1^{-1/p}\boldsymbol{g}_i \otimes \boldsymbol{u}_1, \quad 0, \qquad \boldsymbol{o}) : i \in [k] \}, \\
S_1^+ &= \{\, (\ \ \alpha_1, k_1^{-1/p}\boldsymbol{g}_i \otimes \boldsymbol{v}_1, \quad 0, \qquad \boldsymbol{o}) : i \in [k] \}, \\
S_2^- &= \{\, (\quad 0, \qquad \boldsymbol{o}, -\alpha_2, \boldsymbol{h}_i \otimes \boldsymbol{u}_2) : i \in [k] \}, \\
S_2^+ &= \{\, (\quad 0, \qquad \boldsymbol{o}, \ \ \alpha_2, \boldsymbol{h}_i \otimes \boldsymbol{v}_2) : i \in [k] \}.
\end{aligned}$$

We would like to choose $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ so as to make $S = S_1^- \cup S_1^+ \cup S_2^- \cup S_2^+$ equilateral and non-spherical. Note that $|S| = 2(k_1 + k_2)$.

The p^{th} power of the distance between points

- in the same set $S_1^\pm$ is $\frac{k_1}{2}\frac{1}{k_1}2_p^p = 2^{p-1}$,
- in the same set $S_2^\pm$ is $\frac{k_2}{2}\frac{1}{k_2}2_p^p = 2^{p-1}$,
- in S_1^- and S_1^+ is

$$(2\alpha_1)^p + (k_1 - 1)\frac{1}{k_1}\|\boldsymbol{u}_1 \pm \boldsymbol{v}_1\|_p^p = (2\alpha_1)^p + (1 - \frac{1}{k_1})\lambda_1^p,$$

- in S_2^- and S_2^+ is similarly $(2\alpha_2)^p + (1 - \frac{1}{k_2})\lambda_2^p$,
- in $S_1^- \cup S_1^+$ and $S_2^- \cup S_2^+$ is

$$\alpha_1^p + \alpha_2^p + \frac{k_1 - 1}{k_1} + \frac{k_2 - 1}{k_2} = \alpha_1^p + \alpha_2^p + 2 - \left(\frac{1}{k_1} + \frac{1}{k_2}\right).$$

For S to be equilateral, we need

$$(2\alpha_1)^p + \left(1 - \frac{1}{k_1}\right)\lambda_1^p = 2^{p-1}, \quad (2\alpha_2)^p + \left(1 - \frac{1}{k_2}\right)\lambda_2^p = 2^{p-1} \tag{16}$$

$$\alpha_1^p + \alpha_2^p + 2 - \left(\frac{1}{k_1} + \frac{1}{k_2}\right) = 2^{p-1}. \tag{17}$$

The set S will lie on some sphere iff some $(\beta, \boldsymbol{x}, \gamma, \boldsymbol{y})$ is equidistant to S. This implies that $\boldsymbol{x}$ is equidistant to all $k_1^{-1/p}\boldsymbol{g}_i \otimes \boldsymbol{u}_1$ and also equidistant to all $k_1^{-1/p}\boldsymbol{g}_i \otimes \boldsymbol{v}_1$. By Lemma 20, $\boldsymbol{x} = \boldsymbol{o}$. Similarly, $\boldsymbol{y} = \boldsymbol{o}$. Then $|-\alpha_1 - \beta| = |\alpha_1 - \beta|$, which gives $\beta = 0$. Similarly, $\gamma = 0$. Thus S can only lie on a sphere with centre $\boldsymbol{o}$. It follows that S lies on a sphere iff $\alpha_1 = \alpha_2$. Therefore, for S not to lie on a sphere, we need

$$\alpha_1 \neq \alpha_2. \tag{18}$$

It turns out that the three simultaneous equations (16) and (17) have a solution in $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ given the constraints (15) and (18), iff the hypotheses (11), (12), (13), (14) are satisfied. This can be seen as follows. First use (16) to eliminate α_1 and α_2 from (16), (17) and (18), and set $x_1 = \lambda_1^p$ and $x_2 = \lambda_2^p$ to obtain that the condition is equivalent to the existence of $x_1, x_2 \in \mathbb{R}$ such that

$$2^{p-1} \leq x_i \leq \min\left\{2, 2^{p-1}\left(1 - \frac{1}{k_i}\right)^{-1}\right\}, \quad i = 1, 2 \tag{19}$$

$$\left(1 - \frac{1}{k_1}\right)x_1 + \left(1 - \frac{1}{k_2}\right)x_2 = 2^p\left(3 - 2^{p-1} - \left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right) \tag{20}$$

$$x_1 \neq x_2 \tag{21}$$

This means that the line in the x_1x_2 plane described by (20) should intersect the axis-aligned rectangle with bottom-left corner $(2^{p-1}, 2^{p-1})$ and top-right corner

$$\left(\min\{2, 2^{p-1}(1 - \frac{1}{k_1})^{-1}\}, \min\{2, 2^{p-1}(1 - \frac{1}{k_2})^{-1}\}\right),$$

and if this line intersects the rectangle in a single point (x_1, x_2) which is then necessarily either the bottom-left or top-right corner, then $x_1 \neq x_2$. Define the linear functional $f(x_1, x_2) = \left(1 - \frac{1}{k_1}\right)x_1 + \left(1 - \frac{1}{k_2}\right)x_2$. That the line intersects the rectangle is equivalent to

$$\begin{aligned} f(2^{p-1}, 2^{p-1}) &\leq 2^p\left(3 - 2^{p-1} - \left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right) \\ &\leq \min\Big\{f(2,2), f\left(2, 2^{p-1}(1 - \tfrac{1}{k_2})^{-1}\right), f\left(2^{p-1}(1 - \tfrac{1}{k_1})^{-1}, 2\right), \\ &\qquad f\left(2^{p-1}(1 - \tfrac{1}{k_1})^{-1}, 2^{p-1}(1 - \tfrac{1}{k_2})^{-1}\right)\Big\}, \end{aligned}$$

which is easily seen to be equivalent to (11) (with weak right-hand side inequality), (12), (13). If there is only solution (x_1, x_2) to (19), (20), and it fails to satisfy (21), it follows that $x_1 = x_2$ and (x_1, x_2) is either the bottom-left corner or the top-right corner of the rectangle. In the first case, $x_1 = x_2 = 2^{p-1}$, and $f(2^{p-1}, 2^{p-1}) = 2^p\left(3 - 2^{p-1} - \left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right)$, which implies $\frac{1}{k_1} + \frac{1}{k_2} = 4 - 2^p$, contrary to assumption. In the second case, one of the following four possibilities occurs:

First:

$$2^p\left(3 - 2^{p-1} - \left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right) = f(2,2) \tag{22}$$

and

$$2 \leq 2^{p-1}\left(1-\frac{1}{k_1}\right)^{-1}, 2^{p-1}\left(1-\frac{1}{k_2}\right)^{-1}. \tag{23}$$

The equation (22) implies that $2-2^{p-1}=\frac{1}{k_1}+\frac{1}{k_2}$. Then (23) implies that $1-2^{p-2} \leq \frac{1}{k_1}, \frac{1}{k_2}$, which shows that equality has to hold in both inequalities of (23), hence $k_1 = k_2$, contrary to assumption.

Second:

$$2^p\left(3-2^{p-1}-\left(\frac{1}{k_1}+\frac{1}{k_2}\right)\right) = f\left(2, 2^{p-1}\left(1-\frac{1}{k_2}\right)^{-1}\right),$$
$$2 \leq 2^{p-1}\left(1-\frac{1}{k_1}\right)^{-1}, \quad 2^{p-1}\left(1-\frac{1}{k_2}\right)^{-1} \leq 2,$$
$$x_1 = 2 \quad \text{and} \quad x_2 = 2^{p-1}\left(1-\frac{1}{k_2}\right)^{-1}.$$

Again equality holds in both inequalities of (6), which again gives that $2-2^{p-1}=\frac{1}{k_1}+\frac{1}{k_2}$ and $k_1 = k_2$, contrary to assumption.

Third:

$$2^p\left(3-2^{p-1}-\left(\frac{1}{k_1}+\frac{1}{k_2}\right)\right) = f\left(2^{p-1}\left(1-\frac{1}{k_1}\right)^{-1}, 2\right),$$
$$2^{p-1}\left(1-\frac{1}{k_1}\right)^{-1} \leq 2 \leq 2^{p-1}\left(1-\frac{1}{k_2}\right)^{-1},$$
$$x_1 = 2^{p-1}\left(1-\frac{1}{k_1}\right)^{-1} \quad \text{and} \quad x_2 = 2.$$

This gives a contradiction as before.

Fourth:

$$2^p\left(3-2^{p-1}-\left(\frac{1}{k_1}+\frac{1}{k_2}\right)\right) = f\left(2^{p-1}\left(1-\frac{1}{k_1}\right)^{-1}, 2^{p-1}\left(1-\frac{1}{k_2}\right)^{-1}\right),$$
$$2^{p-1}\left(1-\frac{1}{k_1}\right)^{-1}, 2^{p-1}\left(1-\frac{1}{k_2}\right)^{-1} \leq 2,$$
$$x_1 = 2^{p-1}\left(1-\frac{1}{k_1}\right)^{-1} \quad \text{and} \quad x_2 = 2^{p-1}\left(1-\frac{1}{k_2}\right)^{-1}.$$

This gives a contradiction as before. □

Proof of Theorem 5. The last column of Table 1 indicates how each line in that table is obtained: Proposition 17 covers the case $1 \leq p < \frac{\log 5/2}{\log 2}$, and in the remaining cases Proposition 21 is applied with Hadamard matrices of various orders k_1 and k_2. To derive the asymptotic upper bound of $O(1/(4-2^p))$ as $p \to \infty$, we may assume without loss of generality that p is close to 2. Let $k_1 = k_2 = k$ be the largest order of a Hadamard matrix with $k < 4-2^p$. This ensures that $2/k < 4-2^p$. By Lemma 18 there is a Hadamard matrix of some order in the interval $(2/(4-2^p), 4/(4-2^p))$ if p is sufficiently large. It follows by maximality that $2/(4-2^p) < k$, giving that (11) and (14) are satisfied. The equivalent conditions (12) and (13) are equivalent to $k \leq 4/(4-2^p)$, so they are also satisfied. Proposition 21 gives the upper bound

$$2(k_1+k_2) = 4k \sim \frac{8}{4-2^p} \sim \frac{2}{(2-p)\ln 2}. \qquad \square$$

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