

SINGULAR WEYL–TITCHMARSH–KODAIRA THEORY FOR JACOBI OPERATORS

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ABSTRACT. We develop singular Weyl–Titchmarsh–Kodaira theory for Jacobi operators. In particular, we establish existence of a spectral transformation as well as local Borg–Marchenko and Hochstadt–Liebermann type uniqueness results.

1. INTRODUCTION

Classical Weyl–Titchmarsh–Kodaira theory was originally developed for one-dimensional Schrödinger operators with one regular endpoint. Moreover, it has been shown by Kodaira [12], Kac [11] and more recently by Fulton [7], Gesztesy and Zinchenko [10], Fulton and Langer [8], Kurasov and Luger [18], and Kostenko, Sakhnovich, and Teschl [13], [15], [16], [17], that many aspects of this classical theory still can be established at a singular endpoint. It has recently proven to be a powerful tool for inverse spectral theory for these operators and further refinements were given by us in [3], [5], [6], [13].

Of course the analog of classical Weyl–Titchmarsh–Kodaira theory is also a basic ingredient for inverse spectral theory for Jacobi operators [22]. The purpose of the present paper is to extend singular Weyl–Titchmarsh–Kodaira theory to the case of Jacobi operators. While the overall approach generalizes in a straightforward manner, there are some significant differences in the proofs of our main inverse uniqueness result: Theorem 4.2 and Theorem 6.5. This is related to the fact that in the case of Jacobi operators one needs to determine two coefficients $a(n)$ and $b(n)$ while in the case of one-dimensional Schrödinger operators there is only one coefficient $q(x)$. In fact, it is well known that in the case of general Sturm–Liouville operators with three coefficients $r(x)$, $p(x)$, and $q(x)$ the operator can only be determined up to a Liouville transform (cf. e.g. [1], [4]).

As our main result we first prove a local Borg–Marchenko result (Theorem 4.2) which generalizes the classical result whose local version was first established by Gesztesy, Kiselev, and Makarov [9] (see also Weikard [23]). Moreover, we show that in the case of purely discrete spectra the spectral measure uniquely determines the operator (Theorem 5.2) and use this to establish a general Hochstadt–Liebermann-type uniqueness result (Theorem 5.3). Finally we use the connection with de Branges spaces of entire functions in order to give another general criterion when the spectral measure uniquely determines the operator (Theorem 6.5).

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2. SINGULAR WEYL–TITCHMARSH–KODAIRA THEORY

We will be concerned with operators in $\ell^2(\mathbb{Z})$ associated with the difference expression

$$(2.1) \quad (\tau f)(n) = a(n)f(n+1) + a(n-1)f(n-1) + b(n)f(n), \quad n \in \mathbb{Z},$$

where the sequences $a, b \in \ell(\mathbb{Z})$ satisfy

Hypothesis 2.1. *Suppose*

$$(2.2) \quad a(n) > 0, \quad b(n) \in \mathbb{R}, \quad n \in \mathbb{Z}.$$

If τ is limit point (*l.p.*) at both $\pm\infty$ (cf., e.g., [22]), then τ gives rise to a unique self-adjoint operator H when defined maximally. Otherwise, we need to fix a boundary condition at each endpoint where τ is limit circle (*l.c.*) (cf., e.g., [22]). Throughout this paper we denote by $u_{\pm}(z, \cdot)$, $z \in \mathbb{C}$, nontrivial solutions of $\tau u = zu$ which satisfy the boundary condition at $\pm\infty$ (if any) with $u_{\pm}(z, \cdot) \in \ell_{\pm}^2(\mathbb{Z})$, respectively. Here $\ell_{\pm}^2(\mathbb{Z})$ denotes the sequences in $\ell(\mathbb{Z})$ being ℓ^2 near $\pm\infty$. The solution $u_{\pm}(z, \cdot)$ might not exist for $z \in \mathbb{R}$, but if it exists it is unique up to a constant multiple.

Picking a fixed $z_0 \in \mathbb{C} \setminus \mathbb{R}$ we can characterize H by

$$(2.3) \quad H : \mathfrak{D}(H) \subseteq \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}), \quad f \mapsto \tau f,$$

where the domain of H is explicitly given by

$$(2.4) \quad \mathfrak{D}(H) = \{f \in \ell^2(\mathbb{Z}) \mid \tau f \in \ell^2(\mathbb{Z}), \lim_{n \rightarrow \pm\infty} W(u_{\pm}(z_0), f)(n) = 0\}$$

and

$$(2.5) \quad W(f, g)(n) = a(n) \left(f(n)g(n+1) - f(n+1)g(n) \right), \quad n \in \mathbb{Z}$$

denotes the (modified) Wronskian. The boundary condition at $\pm\infty$ imposes no additional restriction on f if τ is *l.p.* at $\pm\infty$ and can hence be omitted in this case. We will also consider the operators H_{\pm} which are obtained by restricting H to $\pm\mathbb{N}$, respectively with a Dirichlet boundary condition at 0.

For the rest of this section we will closely follow the presentation from [16]. Most proofs can be done literally following the arguments in [16] and hence we will omit them here. In order to define an analogous singular Weyl function at $-\infty$ we first need a system of entire solutions $\theta(z, n)$ and $\phi(z, n)$ such that $\phi(z, n)$ lies in the domain of H near $-\infty$ and such that the Wronskian $W(\theta(z), \phi(z)) = 1$. To this end we start with a hypothesis which will turn out necessary and sufficient for such a system of solutions to exist.

Hypothesis 2.2. *Suppose that the spectrum of H_- is purely discrete.*

Note that this hypothesis is for example satisfied if $a(n) = 1$ and $b(n) \rightarrow +\infty$ as $n \rightarrow -\infty$ or if $a(n) \rightarrow 0$ as $n \rightarrow -\infty$.

Lemma 2.3. *The following properties are equivalent:*

- (i) *The spectrum of H_- is purely discrete.*
- (ii) *There is a real entire solution $\phi(z, n)$, which is non-trivial and lies in the domain of H near $-\infty$ for each $z \in \mathbb{C}$.*
- (iii) *There are real entire solutions $\theta(z, n)$, $\phi(z, n)$ with $W(\theta, \phi) = 1$, such that $\phi(z, n)$ is non-trivial and lies in the domain of H near $-\infty$ for each $z \in \mathbb{C}$.*

Given a system of real entire solutions $\theta(z, n)$ and $\phi(z, n)$ as in the above lemma we can define the singular Weyl function

$$(2.6) \quad M(z) = -\frac{W(\theta(z), u_+(z))}{W(\phi(z), u_+(z))}$$

such that the solution which lies in the domain of H near $+\infty$ is given by

$$(2.7) \quad u_+(z, n) = \alpha(z)(\theta(z, n) + M(z)\phi(z, n)), \quad n \in \mathbb{Z},$$

where $\alpha(z) = -W(\phi(z), u_+(z))$. By construction we obtain that the singular Weyl function $M(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ and satisfies $M(z) = M(z^*)^*$. Rather than $u_+(z, n)$ we will use

$$(2.8) \quad \psi(z, n) = \theta(z, n) + M(z)\phi(z, n), \quad n \in \mathbb{Z}.$$

Recall also from [16, Lem. 3.2] that associated with $M(z)$ is a corresponding spectral measure by virtue of the Stieltjes–Livšić inversion formula

$$(2.9) \quad \frac{1}{2}(\rho((x_0, x_1)) + \rho([x_0, x_1])) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{x_0}^{x_1} \text{Im}(M(x + i\varepsilon)) dx.$$

Theorem 2.4. *Define*

$$(2.10) \quad \hat{f}(\lambda) = \lim_{n \rightarrow \infty} \sum_{m=-\infty}^n \phi(\lambda, m) f(m),$$

where the right-hand side is to be understood as a limit in $L^2(\mathbb{R}, d\rho)$. Then the map

$$(2.11) \quad U : \ell^2(\mathbb{Z}) \rightarrow L^2(\mathbb{R}, d\rho), \quad f \mapsto \hat{f},$$

is unitary and its inverse is given by

$$(2.12) \quad f(n) = \lim_{r \rightarrow \infty} \int_{-r}^r \phi(\lambda, n) \hat{f}(\lambda) d\rho(\lambda),$$

where again the right-hand side is to be understood as a limit in $\ell^2(\mathbb{Z})$. Moreover, U maps H to multiplication with the independent variable in $L^2(\mathbb{R}, d\rho)$.

Corollary 2.5. *The following sets*

$$(2.13) \quad \begin{aligned} \Sigma_{ac} &= \{\lambda \in \mathbb{R} \mid 0 < \limsup_{\varepsilon \downarrow 0} \text{Im}(M(\lambda + i\varepsilon)) < \infty\}, \\ \Sigma_s &= \{\lambda \in \mathbb{R} \mid \limsup_{\varepsilon \downarrow 0} \text{Im}(M(\lambda + i\varepsilon)) = \infty\}, \\ \Sigma_p &= \{\lambda \in \mathbb{R} \mid \lim_{\varepsilon \downarrow 0} \varepsilon \text{Im}(M(\lambda + i\varepsilon)) > 0\}, \end{aligned}$$

$$(2.14) \quad \Sigma = \Sigma_{ac} \cup \Sigma_s = \{\lambda \in \mathbb{R} \mid 0 < \limsup_{\varepsilon \downarrow 0} \text{Im}(M(\lambda + i\varepsilon))\}$$

are minimal supports for ρ_{ac} , ρ_s , ρ_{pp} , and ρ , respectively. In fact, we could even restrict ourselves to values of λ , where the \limsup is a \lim (finite or infinite).

Moreover, the spectrum of H is given by the closure of Σ ,

$$(2.15) \quad \sigma(H) = \overline{\Sigma},$$

the set of eigenvalues is given by

$$(2.16) \quad \sigma_p(H) = \Sigma_p,$$

and the absolutely continuous spectrum of H is given by the essential closure of Σ_{ac} ,

$$(2.17) \quad \sigma(H_{ac}) = \overline{\Sigma_{ac}^{ess}}.$$

Hereby recall $\overline{\Omega}^{ess} = \{\lambda \in \mathbb{R} \mid |(\lambda - \varepsilon, \lambda + \varepsilon) \cap \Omega| > 0 \text{ for all } \varepsilon > 0\}$, where $|\Omega|$ denotes the Lebesgue measure of a Borel set Ω .

Lemma 2.6. *We have*

$$(2.18) \quad (U\partial_z^k G(z, n, \cdot))(\lambda) = \frac{k!\phi(\lambda, n)}{(\lambda - z)^{k+1}}$$

for every $k \in \mathbb{N}_0$, and every $z \in \mathbb{C} \setminus \sigma(H)$, where $G(z, n, m) = \langle \delta_n, (H - z)^{-1} \delta_m \rangle$ is the Green's function of H .

Remark 2.7. *It is important to point out that a fundamental system $\theta(z, n)$, $\phi(z, n)$ of solutions is not unique and any other such system is given by*

$$\tilde{\theta}(z, n) = e^{-g(z)}\theta(z, n) - f(z)\phi(z, n), \quad \tilde{\phi}(z, n) = e^{g(z)}\phi(z, n),$$

where $g(z)$, $f(z)$ are entire functions with $f(z)$ real and $g(z)$ real modulo $i\pi$. The singular Weyl functions are related via

$$\tilde{M}(z) = e^{-2g(z)}M(z) + e^{-g(z)}f(z)$$

and the corresponding spectral measure is given by

$$d\tilde{\rho}(\lambda) = e^{-2g(\lambda)}d\rho(\lambda).$$

Hence the two measures are mutually absolutely continuous and the associated spectral transformations just differ by a simple rescaling with the positive function $e^{-2g(\lambda)}$.

Finally, $M(z)$ can be reconstructed from ρ up to an entire function via the following integral representation.

Theorem 2.8 ([16]). *Let $M(z)$ be a singular Weyl function and ρ its associated spectral measure. Then there exists an entire function $g(z)$ such that $g(\lambda) \geq 0$ for $\lambda \in \mathbb{R}$ and $e^{-g(\lambda)} \in L^2(\mathbb{R}, d\rho)$.*

Moreover, for any entire function $\hat{g}(z)$ such that $\hat{g}(\lambda) > 0$ for $\lambda \in \mathbb{R}$ and $(1 + \lambda^2)^{-1}\hat{g}(\lambda)^{-1} \in L^1(\mathbb{R}, d\rho)$ (e.g. $\hat{g}(z) = e^{2g(z)}$) we have the integral representation

$$(2.19) \quad M(z) = E(z) + \hat{g}(z) \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) \frac{d\rho(\lambda)}{\hat{g}(\lambda)}, \quad z \in \mathbb{C} \setminus \sigma(H),$$

where $E(z)$ is a real entire function.

Remark 2.9. *Choosing a real entire function $g(z)$ such that $\exp(-2g(\lambda)) \in L^1(\mathbb{R}, d\rho)$ we see that*

$$(2.20) \quad M(z) = e^{2g(z)} \int_{\mathbb{R}} \frac{1}{\lambda - z} e^{-2g(\lambda)} d\rho(\lambda) - E(z)$$

for some real entire function $E(z)$. Hence if we choose $f(z) = \exp(-g(z))E(z)$ and switch to a new system of solutions as in Remark 2.7, we see that the new singular Weyl function is a Herglotz–Nevanlinna function

$$(2.21) \quad \tilde{M}(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} e^{-2g(\lambda)} d\rho(\lambda).$$

As a final ingredient we will need the following simple lemma on high energy asymptotics of our real entire solution $\phi(z, n)$.

Lemma 2.10. *If $\phi(z, n)$ is a real entire solution which lies in the domain of H near $-\infty$, then for every $n, \tilde{n} \in \mathbb{Z}$*

$$(2.22) \quad \phi(z, n) = \phi(z, \tilde{n}) \begin{cases} z^{n-\tilde{n}} \prod_{m=\tilde{n}}^{n-1} a(m)^{-1} (1 + O(z^{-1})), & n \geq \tilde{n} \\ z^{n-\tilde{n}} \prod_{m=n}^{\tilde{n}-1} a(m) (1 + O(z^{-1})), & n < \tilde{n} \end{cases}$$

as $|z| \rightarrow \infty$ along any non-real ray.

Proof. This follows from induction using [22, Lemma 6.6],

$$m_-(z, n) = -\frac{\phi(z, n-1)}{a(n-1)\phi(z, n)} = -\frac{1}{z} + O(z^{-2})$$

as $|z| \rightarrow \infty$ along non-real rays. \square

3. EXPONENTIAL GROWTH RATES

While a real entire fundamental system $\theta(z, n)$, $\phi(z, n)$ as in Section 2 is good enough to define a singular Weyl function and an associated spectral measure, it does not suffice for the proofs of our inverse uniqueness results. For them we will need information on the exponential growth rate of the functions $\theta(\cdot, n)$ and $\phi(\cdot, n)$. Our presentation in this section will closely follow [6, Section 3].

We will say a real entire solution $\phi(z, n)$ is of exponential growth order at most $s \geq 0$ if $\phi(\cdot, n)$ and $\phi(\cdot, n+1)$ are of growth order at most s for one (and hence for all) $n \in \mathbb{Z}$.

Our first aim is to extend Lemma 2.3 and to show how the growth order of $\phi(z, n)$ is connected with the convergence exponent of the spectrum. To this end we begin by recalling some basic notation. We refer to the classical book by Levin [19] for proofs and further background.

Given some discrete set $S \subseteq \mathbb{C}$, the number

$$(3.1) \quad \inf \left\{ s \geq 0 \mid \sum_{\mu \in S} \frac{1}{1 + |\mu|^s} < \infty \right\} \in [0, \infty],$$

is called the convergence exponent of S . Moreover, the smallest integer $p \in \mathbb{N}_0$ for which

$$(3.2) \quad \sum_{\mu \in S} \frac{1}{1 + |\mu|^{p+1}} < \infty$$

will be called the genus of S . Introducing the elementary factors

$$(3.3) \quad E_p(\zeta, z) = \left(1 - \frac{z}{\zeta}\right) \exp\left(\sum_{k=1}^p \frac{1}{k} \frac{z^k}{\zeta^k}\right), \quad z \in \mathbb{C},$$

if $\zeta \neq 0$ and $E_p(z, 0) = z$, we recall that the product $\prod_{\mu \in S} E_p(\mu, z)$ converges uniformly on compact sets to an entire function of growth order s , where s and p are the convergence exponent and genus of S , respectively.

Theorem 3.1. *For each $s \geq 0$ the following properties are equivalent:*

- (i) *The spectrum of H_- is purely discrete and has convergence exponent at most s .*
- (ii) *There is a real entire solution $\phi(z, n)$ of exponential growth order at most s which is non-trivial and lies in the domain of H near $-\infty$ for each $z \in \mathbb{C}$.*

Proof. Suppose the spectrum of H_- is purely discrete and has convergence exponent at most s . The same then holds true for the spectrum of the operator H'_- which is obtained by restricting H to $-\mathbb{N}_0$ with a Dirichlet boundary condition at 1. The spectra of these operators will be denoted with

$$(3.4) \quad \sigma(H_-) = \{\mu_j\}_{j \in N} \quad \text{and} \quad \sigma(H'_-) = \{\nu_{j-1}\}_{j \in N},$$

where the index set N is either \mathbb{N} or \mathbb{Z} . Note that the eigenvalues $\mu_j, \nu_{j-1}, j \in N$ are precisely the zeros of $\phi(\cdot, 0)$ and $\phi(\cdot, 1)$, respectively. Also recall that both spectra are interlacing

$$(3.5) \quad \nu_{j-1} < \mu_j < \nu_j, \quad j \in N,$$

and that Krein's theorem [19, Thm. 27.2.1] states

$$(3.6) \quad m_-(z, 1) = C \prod_{j \in N} \frac{E_0(\mu_j, z)}{E_0(\nu_{j-1}, z)}$$

for some real constant $C \neq 0$. Now consider the real entire functions

$$\alpha(z) = \prod_{j \in N} E_p(\nu_{j-1}, z) \quad \text{and} \quad \tilde{\beta}(z) = \prod_{j \in N} E_p(\mu_j, z),$$

where $p \in \mathbb{N}_0$ is the genus of these sequences. Then $\alpha(z)$ and $\tilde{\beta}(z)$ are of exponential growth order at most s by Borel's theorem (see [19, Thm. 4.3.3]). Next note that

$$m_-(z, 1) = e^{h(z)} \frac{\tilde{\beta}(z)}{\alpha(z)}$$

for some entire function $h(z)$ since the right-hand side has the same poles and zeros as $m_-(z, 1)$. Comparing this with Krein's formula (3.6) we obtain that $h(z)$ is in fact a polynomial of degree at most p :

$$h(z) = \sum_{k=1}^p \frac{z^k}{k} \sum_{j \in N} \left(\frac{1}{\nu_{j-1}^k} - \frac{1}{\mu_j^k} \right) + \log(C), \quad z \in \mathbb{C},$$

where the sums converge absolutely by our interlacing assumption. In particular,

$$(3.7) \quad \beta(z) = -a(0)m_-(z, 1)\alpha(z) = -a(0)e^{h(z)}\tilde{\beta}(z)$$

is of exponential growth order at most s as well. Hence the solutions $\phi(z, n)$ with

$$\phi(z, 1) = \alpha(z) \quad \text{and} \quad \phi(z, 0) = \beta(z), \quad z \in \mathbb{C}$$

lie in the domain of H near $-\infty$ and are of growth order at most s .

Conversely let $\phi(z, n)$ be a real entire solution of exponential growth order at most s which lies in the domain of H near $-\infty$. Then since $m_-(z, 0) = -\frac{\phi(z, -1)}{a(-1)\phi(z, 0)}$, the spectrum of H_- is purely discrete and coincides with the zeros of $\phi(\cdot, 0)$. Now since $\phi(\cdot, 0)$ is of growth order at most s , its zeros are of convergence exponent at most s . \square

Unfortunately, given a real entire solution $\phi(z, n)$ of exponential growth order $s \geq 0$ we are not able to prove the existence of a second solution of the same growth order. However, at least under some additional assumptions we get a second solution $\theta(z, n)$ of growth order arbitrarily close to s .

Lemma 3.2 ([6]). *Suppose $\phi(z, n)$ is a real entire solution of exponential growth order $s \geq 0$ and let $\varepsilon > 0$, $\tilde{n} \in \mathbb{Z}$. Then there is a real entire second solution $\theta(z, n)$ with*

$$(3.8) \quad |\theta(z, \tilde{n})| + |\theta(z, \tilde{n} + 1)| \leq B e^{A|z|^{s+\varepsilon}}, \quad z \in \mathbb{C}$$

for some constants $A, B > 0$ and $W(\theta, \phi) = 1$ if and only if

$$(3.9) \quad |\phi(z, \tilde{n})| + |\phi(z, \tilde{n} + 1)| \geq b e^{-a|z|^{s+\varepsilon}}, \quad z \in \mathbb{C}$$

for some constants $a, b > 0$.

We are also able to provide a sufficient condition for a second solution of exponential order to exist, in terms of the interlacing zeros of $\phi(\cdot, \tilde{n})$ and $\phi(\cdot, \tilde{n} + 1)$ which we denote by $\{\mu_j\}_{j \in N}$ and $\{\nu_{j-1}\}_{j \in N}$ respectively, where N is either \mathbb{N} or \mathbb{Z} .

Lemma 3.3 ([6]). *Suppose $\phi(z, n)$ is a real entire solution of growth order $s \geq 0$ and that for some $r > 0$ all but finitely many of the circles given by*

$$(3.10) \quad |z - \mu_j| < |\mu_j|^{-r} \quad \text{and} \quad |z - \nu_{j-1}| < |\nu_{j-1}|^{-r}, \quad j \in N,$$

are disjoint. Then for every $\varepsilon > 0$ there is a real entire second solution $\theta(z, n)$ with exponential growth at most $s + \varepsilon$ and $W(\theta, \phi) = 1$.

Remark 3.4. *By the Hadamard product theorem [19, Thm. 4.2.1], a solution $\phi(z, n)$ of exponential growth order $s \geq 0$ is unique up to a factor $e^{g(z)}$, for some real polynomial $g(z)$ of degree at most p , where $p \in \mathbb{N}_0$ is the genus of the eigenvalues of H_- . A solution $\theta(z, n)$ of growth order at most s is unique only up to $f(z)\phi(z, n)$, where $f(z)$ is an entire function of growth order at most s .*

Finally, note that under the assumptions in this section one can use $\hat{g}(z) = \exp(z^{2\lceil(p+1)/2\rceil})$ in Theorem 2.8. If in addition H is bounded from below, then one can also use $\hat{g}(z) = \exp(z^{p+1})$.

4. A LOCAL BORG–MARCHENKO UNIQUENESS RESULT

The purpose of the present section is again to extend the results from [6] to the case of Jacobi operators.

Lemma 4.1 ([16]). *The singular Weyl function $M(z)$ and the Weyl solution $\psi(z, n)$ defined in (2.8) have the following asymptotics*

$$(4.1) \quad M(z) = -\frac{\theta(z, n)}{\phi(z, n)} + \mathcal{O}\left(\frac{1}{z\phi(z, n)^2}\right),$$

$$(4.2) \quad \psi(z, n) = \frac{-1}{z\phi(z, n)} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right)$$

as $|z| \rightarrow \infty$ in any sector $|\operatorname{Im}(z)| \geq \delta |\operatorname{Re}(z)|$.

Proof. This follows by solving the well-known asymptotical formula ([22, Thm. 6.2]) for the diagonal of Green's function

$$G(z, n, n) = \phi(z, n)\psi(z, n) = -\frac{1}{z} + O(z^{-2})$$

for $\psi(z, n)$ and $M(z)$. \square

In particular, (4.1) shows that asymptotics of $M(z)$ immediately follow once one has corresponding asymptotics for the solutions $\theta(z, n)$ and $\phi(z, n)$. Moreover, the leading asymptotics depend only on the values of the sequences a, b near the endpoint $-\infty$ (and on the choice of $\theta(z, n)$ and $\phi(z, n)$). The following Borg–Marchenko type uniqueness result shows that the converse is also true.

In order to state this theorem let $\{a_0, b_0\}$ and $\{a_1, b_1\}$ be two sets of coefficients on \mathbb{Z} satisfying Hypothesis 2.1. By H_0 and H_1 we denote some corresponding self-adjoint operators with separate boundary conditions. Furthermore, for $j = 0, 1$ let $\theta_j(z, n), \phi_j(z, n)$ be some real entire fundamental system of solutions with $W_j(\theta_j, \phi_j) = 1$ such that $\phi_j(z, n)$ lie in $\ell^2(\mathbb{Z})$ near $-\infty$ and satisfy the boundary condition of H_j there (if any). The associated singular Weyl functions are denoted by $M_0(z)$ and $M_1(z)$. We will also use the common short-hand notation $\phi_0(z, n) \sim \phi_1(z, n)$ to abbreviate the asymptotic relation $\phi_0(z, n) = \phi_1(z, n)(1 + o(1))$ as $|z| \rightarrow \infty$ in some specified manner.

Theorem 4.2. *Let $\tilde{n} \in \mathbb{Z}$, suppose $\theta_0(z, n), \theta_1(z, n), \phi_0(z, n), \phi_1(z, n)$ are of growth order at most s for some $s > 0$ and $\phi_0(z, \tilde{n}) \sim \phi_1(z, \tilde{n})$ as $|z| \rightarrow \infty$ along some non-real rays dissecting the complex plane into sectors of length less than π/s . Then the following properties are equivalent:*

- (i) *We have $a_0(n) = a_1(n)$ for $n < \tilde{n}$, $b_0(n) = b_1(n)$ for $n \leq \tilde{n}$ and*

$$\lim_{n \rightarrow -\infty} a_0(n) \left(\phi_0(z, n)\phi_1(z, n+1) - \phi_0(z, n+1)\phi_1(z, n) \right) = 0.$$

- (ii) *For each $\delta > 0$ there is an entire function $f(z)$ of growth order at most s such that*

$$M_1(z) - M_0(z) = f(z) + o\left(\frac{1}{\phi_0(z, \tilde{n} + 1)^2}\right),$$

as $|z| \rightarrow \infty$ in the sector $|\operatorname{Im}(z)| \geq \delta |\operatorname{Re}(z)|$.

Proof. If (i) holds, then by Remark 3.4 and our assumptions

$$(4.3) \quad \phi_1(z, n) = \phi_0(z, n) \quad \text{and} \quad \theta_1(z, n) = \theta_0(z, n) - f(z)\phi_1(z, n), \quad z \in \mathbb{C}$$

for all $n \leq \tilde{n}$, where $f(z)$ is some real entire function of growth order at most s . In particular we obtain

$$\frac{\theta_1(z, \tilde{n} + 1)}{a_0(\tilde{n})} = \frac{\theta_0(z, \tilde{n} + 1) - f(z)\phi_0(z, \tilde{n} + 1)}{a_1(\tilde{n})}, \quad \frac{\phi_1(z, \tilde{n} + 1)}{a_0(\tilde{n})} = \frac{\phi_0(z, \tilde{n} + 1)}{a_1(\tilde{n})}$$

and the asymptotics in Lemma 4.1 show that

$$\begin{aligned} M_1(z) - M_0(z) &= \frac{\theta_0(z, \tilde{n} + 1)}{\phi_0(z, \tilde{n} + 1)} - \frac{\theta_1(z, \tilde{n} + 1)}{\phi_1(z, \tilde{n} + 1)} + \mathcal{O}\left(\frac{1}{z\phi_0(z, \tilde{n} + 1)^2}\right) \\ &= f(z) + \mathcal{O}\left(\frac{1}{z\phi_0(z, \tilde{n} + 1)^2}\right), \end{aligned}$$

as $|z| \rightarrow \infty$ in any sector $|\operatorname{Im}(z)| \geq \delta |\operatorname{Re}(z)|$.

Now suppose property (ii) holds and for each fixed $n \leq \tilde{n} + 1$ consider the entire function

$$(4.4) \quad G_n(z) = \phi_1(z, n)\theta_0(z, n) - \phi_0(z, n)\theta_1(z, n) - f(z)\phi_0(z, n)\phi_1(z, n), \quad z \in \mathbb{C}.$$

Since away from the real axis this function may be written as

$$\begin{aligned} G_n(z) &= \phi_1(z, n)\psi_0(z, n) - \phi_0(z, n)\psi_1(z, n) \\ &\quad + (M_1(z) - M_0(z) - f(z))\phi_0(z, n)\phi_1(z, n), \quad z \in \mathbb{C} \setminus \mathbb{R}, \end{aligned}$$

it vanishes as $|z| \rightarrow \infty$ along our non-real rays. For the first two terms this follows from (4.2) together with our hypothesis that $\phi_0(\cdot, n)$ and $\phi_1(\cdot, n)$ have the same order of magnitude (in view of Lemma 2.10). The last term tends to zero because of our assumption on the difference of the Weyl functions. Moreover, by our hypothesis $G_n(z)$ is of growth order at most s and thus we can apply the Phragmén–Lindelöf theorem (e.g., [19, Sect. 6.1]) in the angles bounded by our rays. This shows that $G_n(z)$ is bounded on all of \mathbb{C} . By Liouville’s theorem it must be constant and since it vanishes along a ray, it must be zero; that is,

$$\phi_1(z, n)\theta_0(z, n) - \phi_0(z, n)\theta_1(z, n) = f(z)\phi_0(z, n)\phi_1(z, n), \quad n \leq \tilde{n} + 1, \quad z \in \mathbb{C}.$$

Dividing both sides by $\phi_0(z, n)\phi_1(z, n)$ and taking differences shows

$$\frac{\theta_0(z, n-1)}{\phi_0(z, n-1)} - \frac{\theta_0(z, n)}{\phi_0(z, n)} = \frac{\theta_1(z, n-1)}{\phi_1(z, n-1)} - \frac{\theta_1(z, n)}{\phi_1(z, n)}, \quad n \leq \tilde{n} + 1, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

and, using $W_0(\theta_0, \phi_0) = W_1(\theta_1, \phi_1) = 1$,

$$a_0(n-1)\phi_0(z, n-1)\phi_0(z, n) = a_1(n-1)\phi_1(z, n-1)\phi_1(z, n), \quad n \leq \tilde{n} + 1, \quad z \in \mathbb{C}.$$

Hence the Dirichlet and Neumann eigenvalues are equal (we know which is which by interlacing and the high energy asymptotics of $m_-(z, n)$) and the result follows from [21, Thm. 4.6]. \square

Note that the implication (ii) \Rightarrow (i) could also be proved under somewhat weaker conditions. First of all the assumption on the growth of the entire functions $f(z)$ is only due to the use of the Phragmén–Lindelöf principle. Hence it would also suffice that for each $\varepsilon > 0$ we have

$$(4.5) \quad \sup_{|z|=r_n} |f(z)| \leq B e^{A r_n^{s+\varepsilon}}, \quad n \in \mathbb{N}$$

for some increasing sequence of positive reals $r_n \uparrow \infty$ and constants $A, B > 0$. Furthermore, for this implication to hold it would also suffice that $\phi_0(z, n) \asymp \phi_1(z, n)$ for one (and hence all) $n \in \mathbb{Z}$ as $|z| \rightarrow \infty$ along our non-real rays instead of $\phi_0(z, \tilde{n}) \sim \phi_1(z, \tilde{n})$. Here, the notation $\phi_0(z, n) \asymp \phi_1(z, n)$ is short hand for both, $\phi_0(z, n)\phi_1(z, n)^{-1} = \mathcal{O}(1)$ and $\phi_1(z, n)\phi_0(z, n)^{-1} = \mathcal{O}(1)$ to hold.

While at first sight it might look like the condition on the asymptotics of the solutions $\phi_j(z, n)$ requires knowledge about them, this is not the case, since the high energy asymptotics will only involve some qualitative information on behavior of the coefficients as $n \rightarrow -\infty$. Next, the appearance of the additional freedom of the function $f(z)$ just reflects the fact that we only ensure the same normalization for the solutions $\phi_0(z, n)$ and $\phi_1(z, n)$ but not for $\theta_0(z, n)$ and $\theta_1(z, n)$ (cf. Remark 3.4).

Corollary 4.3. *Suppose $\theta_0(z, n)$, $\theta_1(z, n)$, $\phi_0(z, n)$, $\phi_1(z, n)$ are of growth order at most s for some $s > 0$ and $\phi_0(z, n) \asymp \phi_1(z, n)$ for one $n \in \mathbb{Z}$ as $|z| \rightarrow \infty$ along*

some non-real rays dissecting the complex plane into sectors of length less than π/s .
If

$$(4.6) \quad M_1(z) - M_0(z) = f(z), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

for some entire function $f(z)$ of growth order at most s , then $H_0 = H_1$.

Proof. By (the remark after the proof of) Theorem 4.2 it remains to show that H_0 and H_1 have the same boundary condition near $+\infty$. If there are one at all, then H_0 and H_1 have a common eigenvalue λ and the claim follows since $\phi_0(\lambda, \cdot)$ and $\phi_1(\lambda, \cdot)$ are linearly dependent and hence satisfy the boundary conditions of both H_0 and H_1 . \square

5. UNIQUENESS RESULTS FOR OPERATORS WITH DISCRETE SPECTRA

Now we are finally able to investigate when the spectral measure determines the potential for operators with purely discrete spectrum. In this respect observe that the uniqueness results for the singular Weyl function from the previous sections do not immediately yield such results. In fact, if $\rho_0 = \rho_1$ then the difference of the corresponding singular Weyl functions is an entire function by Theorem 2.8. However, in order to apply Corollary 4.3 we would need some bound on the exponential growth of this function. Fortunately, in the case of purely discrete spectrum with finite convergence exponent, a refinement of the arguments in the proof of Theorem 4.2 shows that the growth condition is not necessary. This can be shown literally as in [6, Cor. 5.1].

Corollary 5.1. *Suppose $\phi_0(z, n)$, $\phi_1(z, n)$ are of growth order at most s for some $s > 0$ and $\phi_0(z, n) \asymp \phi_1(z, n)$ for one $n \in \mathbb{Z}$ as $|z| \rightarrow \infty$ along some non-real rays dissecting the complex plane into sectors of length less than π/s . Furthermore, assume that H_0 and H_1 have purely discrete spectrum with convergence exponent at most s . If*

$$(5.1) \quad M_1(z) - M_0(z) = f(z), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

for some entire function $f(z)$, then $H_0 = H_1$.

Now the lack of a growth restriction in Corollary 5.1 implies that it immediately translates into a corresponding uniqueness result for the spectral measure. Again this follows literally as in [6, Thm. 5.2].

Theorem 5.2. *Suppose that $\phi_0(z, n)$, $\phi_1(z, n)$ are of growth order at most s for some $s > 0$ and $\phi_0(z, n) \asymp \phi_1(z, n)$ for one $n \in \mathbb{Z}$ as $|z| \rightarrow \infty$ along some non-real rays dissecting the complex plane into sectors of length less than π/s . Furthermore, assume that H_0 and H_1 have purely discrete spectrum with convergence exponent at most s . If the corresponding spectral measures ρ_0 and ρ_1 are equal, then we have $H_0 = H_1$.*

Note that in the case of discrete spectra, the spectral measure is uniquely determined by the eigenvalues $\lambda \in \sigma(H)$ and the corresponding norming constants

$$(5.2) \quad \gamma_\lambda^2 = \sum_{m \in \mathbb{Z}} \phi(\lambda, m)^2, \quad \lambda \in \sigma(H)$$

since in this case we have

$$(5.3) \quad \rho = \sum_{\lambda \in \sigma(H)} \gamma_\lambda^{-2} \delta_\lambda,$$

where δ_λ is the Dirac measure in the point λ .

As another application we are also able to prove a generalization of Hochstadt–Liebermann-type uniqueness results. To this end let us consider an operator H whose spectrum is purely discrete and has convergence exponent (at most) s . Since the operator with an additional Dirichlet boundary condition at zero is a rank one perturbation of H we conclude that the convergence exponents of the spectra of H_- and H_+ are at most s as well. Hence by Theorem 3.1 there are real entire solutions $\phi(z, n)$ and $\chi(z, n)$ of growth order at most s which are in the domain of H near $-\infty$ and $+\infty$, respectively.

Theorem 5.3. *Suppose H_0 is an operator with purely discrete spectrum of finite convergence exponent $s > 0$. Let $\phi_0(z, n)$ and $\chi_0(z, n)$ be entire solutions of growth order at most s which lie in the domain of H_0 near $-\infty$ and $+\infty$, respectively, and suppose there is an $\tilde{n} \in \mathbb{Z}$ such that*

$$(5.4) \quad \frac{\chi_0(z, \tilde{n})}{\phi_0(z, \tilde{n})} = o(1)$$

as $|z| \rightarrow \infty$ along some non-real rays dissecting the complex plane into sectors of length less than π/s .

Then every other isospectral operator H_1 with $a_1(n) = a_0(n)$ for $n < \tilde{n} - 1$, $b_1(n) = b_0(n)$ for $n < \tilde{n}$ and which is associated with the same boundary condition at $-\infty$ (if any) is equal to H_0 .

Proof. Start with some solutions $\phi_j(z, n)$, $\chi_j(z, n)$ of growth order at most s and note that we can choose $\phi_1(z, n) = \phi_0(z, n)$ for $n < \tilde{n}$ since H_1 and H_0 are associated with the same boundary condition at $-\infty$ (if any). Moreover, by Lemma 2.22 we have $\phi_0(z, n) \asymp \phi_1(z, n)$ as $|z| \rightarrow \infty$ along every non-real ray, even for $n \geq \tilde{n}$. Next note that the zeros of the Wronskians $W_j(\phi_j, \chi_j)$ are precisely the eigenvalues of H_j and thus, by assumption, are equal. Hence by the Hadamard factorization theorem $W_1(\phi_1, \chi_1) = e^g W_0(\phi_0, \chi_0)$ for some polynomial g of degree at most s . Since we can absorb this factor in $\chi_1(z, n)$, we can assume $g = 0$ without loss of generality. Hence we have

$$\begin{aligned} 1 &= \frac{W_0(\phi_0, \chi_0)}{W_1(\phi_1, \chi_1)} = \frac{a_0(\tilde{n} - 1)(\phi_0(z, \tilde{n} - 1)\chi_0(z, \tilde{n}) - \phi_0(z, \tilde{n})\chi_0(z, \tilde{n} - 1))}{a_1(\tilde{n} - 1)(\phi_1(z, \tilde{n} - 1)\chi_1(z, \tilde{n}) - \phi_1(z, \tilde{n})\chi_1(z, \tilde{n} - 1))} \\ &= \frac{\phi_0(z, \tilde{n} - 1)\chi_0(z, \tilde{n} - 1)}{\phi_1(z, \tilde{n} - 1)\chi_1(z, \tilde{n} - 1)} \left(\frac{a_0(\tilde{n} - 1)\chi_0(z, \tilde{n})}{\chi_0(z, \tilde{n} - 1)} - \frac{a_0(\tilde{n} - 1)\phi_0(z, \tilde{n})}{\phi_0(z, \tilde{n} - 1)} \right) \times \\ &\quad \times \left(\frac{a_1(\tilde{n} - 1)\chi_1(z, \tilde{n})}{\chi_1(z, \tilde{n} - 1)} - \frac{a_1(\tilde{n} - 1)\phi_1(z, \tilde{n})}{\phi_1(z, \tilde{n} - 1)} \right)^{-1} \end{aligned}$$

and by virtue of the well-known asymptotics (see [22, Lemma 6.6]) for $j = 0, 1$

$$\frac{a_j(\tilde{n} - 1)\chi_j(z, \tilde{n})}{\chi_j(z, \tilde{n} - 1)} = \mathcal{O}(z^{-1}) \quad \text{and} \quad \frac{a_j(\tilde{n} - 1)\phi_j(z, \tilde{n})}{\phi_j(z, \tilde{n} - 1)} = z - b(\tilde{n} - 1) + \mathcal{O}(z^{-1})$$

as $|z| \rightarrow \infty$ along any non-real ray we conclude

$$\chi_1(z, \tilde{n} - 1) = \chi_0(z, \tilde{n} - 1)(1 + \mathcal{O}(z^{-2}))$$

as $|z| \rightarrow \infty$ along non-real rays. Equality of the Wronskians also implies

$$\chi_1(z, n) = \chi_0(z, n) + F(z)\phi_0(z, n), \quad n < \tilde{n}$$

for some entire function $F(z)$ of growth order at most s . Moreover, our assumption (5.4) implies that

$$F(z) = \frac{\chi_1(z, \tilde{n} - 1) - \chi_0(z, \tilde{n} - 1)}{\phi_0(z, \tilde{n} - 1)} = \frac{\chi_0(z, \tilde{n} - 1)}{\phi_0(z, \tilde{n} - 1)} \left(\frac{\chi_1(z, \tilde{n} - 1)}{\chi_0(z, \tilde{n} - 1)} - 1 \right)$$

vanishes along our rays and thus it must be identically zero by the Phragmén–Lindelöf theorem. Finally, choosing $\theta_j(z, n)$ such that $\theta_1(z, n) = \theta_0(z, n)$ for $n < \tilde{n}$, this implies that the associated singular Weyl functions are equal and the claim follows from Corollary 5.1. \square

Note that by (2.22) the growth of $\phi_0(\cdot, \tilde{n})$ will increase as \tilde{n} increases while (by reflection) the growth of $\chi_0(\cdot, \tilde{n})$ will decrease. In particular, if (5.4) holds for \tilde{n} it will hold for any $n_1 > \tilde{n}$ as well. Moreover, one may change knowledge of the coefficient $b_1(\tilde{n}) = b_0(\tilde{n})$ for the stronger asymptotics $o(z^{-1})$ of the quotient in (5.4).

6. JACOBI OPERATORS AND DE BRANGES SPACES

For each $n \in \mathbb{Z}$ we consider the entire function

$$(6.1) \quad E(z, n) = \phi(z, n) + ia(n)\phi(z, n+1), \quad z \in \mathbb{C}$$

which satisfies

$$\frac{E(z, n)E^\#(\zeta^*, n) - E(\zeta^*, n)E^\#(z, n)}{2i(\zeta^* - z)} = \sum_{m=-\infty}^n \phi(\zeta, m)^* \phi(z, m), \quad \zeta, z \in \mathbb{C}^+,$$

where $F^\#(z) = F(z^*)^*$, $z \in \mathbb{C}$. In particular, taking $\zeta = z$ this shows that $E(\cdot, n)$ is a de Branges function. Moreover, note that $E(\cdot, n)$ does not have any real zero λ , since otherwise both, $\phi(\lambda, n)$ and $\phi(\lambda, n+1)$ would vanish. With $B(n)$ we denote the de Branges space associated with the de Branges function $E(\cdot, n)$ endowed with the inner product

$$[F, G]_{B(n)} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\lambda)^* G(\lambda)}{|E(\lambda, n)|^2} d\lambda = \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\lambda)^* G(\lambda)}{\phi(\lambda, n)^2 + a(n)^2 \phi(\lambda, n+1)^2} d\lambda$$

for $F, G \in B(n)$. The reproducing kernel $K(\cdot, \cdot, n)$ of this space is given by

$$(6.2) \quad K(\zeta, z, n) = \sum_{m=-\infty}^n \phi(\zeta, m)^* \phi(z, m), \quad \zeta, z \in \mathbb{C}.$$

Theorem 6.1. *For every $n \in \mathbb{Z}$ the transformation $f \mapsto \hat{f}$ is unitary from $\ell^2(-\infty, n]$ onto $B(n)$. In particular,*

$$(6.3) \quad B(n) = \{ \hat{f} \mid f \in \ell^2(-\infty, n] \} = \overline{\text{span}\{\phi(\cdot, m) \mid m \leq n\}}.$$

Here we identify $\ell^2(-\infty, n]$ with the subspace of sequences in $\ell^2(\mathbb{Z})$ which vanish on $m > n$.

Proof. For each $\lambda \in \mathbb{R}$ consider the function

$$f_\lambda(m) = \begin{cases} \phi(\lambda, m), & m \leq n, \\ 0, & m > n. \end{cases}$$

The transforms of these functions are given by

$$\hat{f}_\lambda(z) = \sum_{m=-\infty}^n \phi(\lambda, m)\phi(z, m) = K(\lambda, z, n), \quad z \in \mathbb{C}.$$

In particular, this shows that the transforms of the functions f_λ , $\lambda \in \mathbb{R}$, lie in $B(n)$. Moreover, we have for all $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\begin{aligned} \langle f_{\lambda_1}, f_{\lambda_2} \rangle &= \sum_{m=-\infty}^n \phi(\lambda_1, m)\phi(\lambda_2, m) = K(\lambda_1, \lambda_2, n) \\ &= [K(\lambda_1, \cdot, n), K(\lambda_2, \cdot, n)]_{B(n)}. \end{aligned}$$

Hence our transform is an isometry on the linear span D of all functions f_λ , $\lambda \in \mathbb{R}$. But this span is dense in $\ell^2(-\infty, n]$ since it contains the eigenfunctions of the operator H restricted to $m \leq n$ with a Dirichlet boundary condition at $n+1$. Moreover, the linear span of all transforms $K(\lambda, \cdot, n)$, $\lambda \in \mathbb{R}$, is dense in $B(n)$. Indeed, each $F \in B(n)$ such that

$$0 = [F, K(\lambda, \cdot, n)]_{B(n)} = F(\lambda), \quad \lambda \in \mathbb{R}$$

vanishes identically. Thus our transformation restricted to D uniquely extends to a unitary map V from $\ell^2(-\infty, n]$ onto $B(n)$. Finally, since $f \mapsto \hat{f}(z)$ as well as $f \mapsto Vf(z)$ are continuous on $\ell^2(-\infty, n]$ for each $z \in \mathbb{C}$, we infer that this extension coincides with our transformation. \square

As an immediate consequence of Theorem 6.1 and the fact that our transformation from (2.10) extends to a unitary map from $\ell^2(\mathbb{Z})$ onto $L^2(\mathbb{R}, d\rho)$, we get the following corollary.

Corollary 6.2. *For each $n \in \mathbb{Z}$ the de Branges space $B(n)$ is isometrically embedded in $L^2(\mathbb{R}, d\rho)$, that is*

$$(6.4) \quad \int_{\mathbb{R}} |F(\lambda)|^2 d\rho(\lambda) = \|F\|_{B(n)}^2, \quad F \in B(n).$$

Moreover, the union of the spaces $B(n)$, $n \in \mathbb{Z}$, is dense in $L^2(\mathbb{R}, d\rho)$, i.e.

$$(6.5) \quad \overline{\bigcup_{n \in \mathbb{Z}} B(n)} = L^2(\mathbb{R}, d\rho).$$

Clearly the de Branges spaces $B(n)$, $n \in \mathbb{Z}$, are totally ordered and strictly increasing in the sense that $B(n) \subsetneq B(\tilde{n})$ for $n < \tilde{n}$. More precisely, Theorem 6.1 shows that $B(n)$ actually has codimension one in $B(n+1)$.

For the proof of our main result we will need the ordering theorem due to de Branges. In order to state it let E_0, E_1 be two de Branges functions and B_0, B_1 be the corresponding de Branges spaces.

Theorem 6.3 ([2], Theorem 35). *Suppose B_0, B_1 are isometrically embedded in $L^2(\mathbb{R}, d\rho)$, for some Borel measure ρ on \mathbb{R} . If E_0/E_1 is of bounded type in the upper complex half-plane and has no real zeros or singularities, then B_0 contains B_1 or B_1 contains B_0 .*

Moreover, one has the following simple converse statement.

Lemma 6.4 ([3], Lemma 2.2). *If B_0 contains B_1 or B_1 contains B_0 , then E_0/E_1 is of bounded type in the upper complex half-plane.*

Now let H_0, H_1 be two Jacobi operators with separate boundary conditions. Suppose there are corresponding nontrivial real entire solutions $\phi_0(z, n), \phi_1(z, n)$ which are square integrable near the left endpoint and satisfy the boundary condition there, if any. In obvious notation we denote all quantities corresponding to H_0 and H_1 with an additional subscript. We say H_0 and H_1 are equal up to a shift if there is some $k \in \mathbb{Z}$ such that

$$H_0 = S^{-k} H_1 S^k,$$

where $S : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}), f(m) \mapsto f(m+1)$, is the usual shift operator.

Theorem 6.5. *Suppose there is some real entire function g such that*

$$(6.6) \quad e^{g(z)} \frac{E_0(z, n_0)}{E_1(z, n_1)}, \quad z \in \mathbb{C}^+$$

is of bounded type for some $n_0, n_1 \in \mathbb{Z}$. If $d\rho_0(\lambda) = e^{-2g(\lambda)} d\rho_1(\lambda)$, then H_0 and H_1 are equal up to a shift.

Proof. First of all note that without loss of generality we may assume that g vanishes identically, since otherwise we replace $\phi_0(z, n)$ with $e^{g(z)}\phi_0(z, n)$. Moreover, because of Lemma 6.4 the function in (6.6) is of bounded type for all points $n_0, n_1 \in \mathbb{Z}$. Hence Corollary 6.2 and Theorem 6.3 (note that (6.6) has no real zeros or singularities because $E_0(\cdot, n_0)$ and $E_1(\cdot, n_1)$ do not have real zeros) imply that either $B_0(n_0)$ is contained in $B_1(n_1)$ or $B_1(n_1)$ is contained in $B_0(n_0)$. Without loss of generality we may assume $B_0(n_0) \subseteq B_1(n_1)$. Moreover, by (6.3) and (6.5) we can further increase n_0 such that $B_0(n_0) \subseteq B_1(n_1) \subsetneq B_0(n_0+1)$. Then clearly $B_0(n_0) = B_1(n_1)$ and we set $k = n_1 - n_0$. Moreover, since $B_j(n)$ has codimension one in $B_j(n+1)$, we obtain by induction $B_0(n) = B_1(n+k)$ for all $n \in \mathbb{Z}$. Now (6.2) shows $\phi_0(z, n) = \sigma_n \phi_1(z, n+k)$ for some sequence $\sigma_n \in \{\pm 1\}$, which is actually independent of n in view of Lemma 2.10. Similarly as in the proof of Theorem 4.2, this ensures that the coefficient sequences and the boundary conditions at $-\infty$ are the same up to a shift by k . Finally, the fact that the boundary conditions at ∞ are the same follows as in the proof of Corollary 4.3. \square

Note that the quotient (6.6) in Theorem 6.5 is of bounded type if and only if

$$e^{g(z)} \frac{\phi_0(z, n_0)}{\phi_1(z, n_1)}, \quad z \in \mathbb{C}^+$$

is of bounded type. We conclude this section by showing that this condition holds if the solutions $\phi_0(z, n), \phi_1(z, n)$ satisfy some growth condition. Therefore recall that an entire function F belongs to the Cartwright class \mathcal{C} if it is of finite exponential type and the logarithmic integral

$$\int_{\mathbb{R}} \frac{\ln^+ |F(\lambda)|}{1 + \lambda^2} d\lambda < \infty,$$

where \ln^+ is the positive part of the natural logarithm. In particular note that the class \mathcal{C} contains all entire functions of exponential order less than one. Now a theorem of Krein [20, Theorem 6.17], [19, Section 16.1] states that the class \mathcal{C} consists of all entire functions which are of bounded type in the upper and in the lower complex half-plane. Since the quotient of two functions of bounded type is of bounded type itself, this immediately yields the following uniqueness result.

Corollary 6.6. *Suppose that $\phi_0(\cdot, n_0)$ and $\phi_1(\cdot, n_1)$ belong to the Cartwright class \mathcal{C} for some $n_0, n_1 \in \mathbb{Z}$. If $\rho_0 = \rho_1$, then H_0 and H_1 are equal up to a shift.*

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