UNIVERSAL COMPUTABLY ENUMERABLE EQUIVALENCE RELATIONS

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Abstract. We study computably enumerable equivalence relations (ceers) under the reducibility $R \leq S$ if there exists a computable function f such that, for every x, y, $x \; R \; y$ if and only if $f(x) \; S \; f(y)$. We show that the degrees of ceers under the equivalence relation generated by $\leq$ form a bounded poset that is neither a lower semilattice, nor an upper semilattice, and its first order theory is undecidable. We then study the universal ceers. We show that 1) the uniformly effectively inseparable ceers are universal, but there are effectively inseparable ceers that are not universal; 2) a ceer R is universal if and only if $R' \leq R$, where R' denotes the halting jump operator introduced by Gao and Gerdes (answering an open question of Gao and Gerdes); and 3) both the index set of the universal ceers and the index set of the uniformly effectively inseparable ceers are Σ^0_3-complete (the former answering an open question of Gao and Gerdes).

1. Introduction

We are interested in the following reducibility on equivalence relations: If R, S are equivalence relations on the set ω of the natural numbers, we say that R is *reducible to* S (notation: $R \leq S$) if there exists a computable function f such that, for every x, y,

$$x \; R \; y \Leftrightarrow f(x) \; S \; f(y).$$

This reducibility was introduced by Ershov [9] while considering monomorphisms in the category of numberings. We recall that a *numbering* is a pair $\langle \nu, S \rangle$, where $\nu : \omega \longrightarrow S$ is an onto function. Numberings are the objects in a category Num, called the *category of numberings*: The *morphisms* from a numbering $\langle \nu_1, S_1 \rangle$ to a numbering $\langle \nu_2, S_2 \rangle$ are the functions $\mu : S_1 \longrightarrow S_2$ for which there is a

2010 *Mathematics Subject Classification.* 03D25.

Key words and phrases. computably enumerable equivalence relations, precomplete numberings.

Lempp's research was partially supported by an AMS-Simons Foundation Collaboration Grant 209087. Miller's research was partially supported by NSF grant DMS-1001847. Sorbi's research was partially carried out while he was visiting the Department of Mathematics of the University of Wisconsin-Madison in March 2011. The second and last author would like to thank the Isaac Newton Institute for Mathematical Sciences of Cambridge, England, for its hospitality during the final phase this paper was completed.

computable function f so that the following diagram commutes:

$$\begin{array}{ccc} \omega & \xrightarrow{f} & \omega \\ {\scriptstyle \nu_1}\big\downarrow & & \big\downarrow{\scriptstyle \nu_2} \\ S_1 & \xrightarrow[\mu]{} & S_2; \end{array}$$

we say in this case that the computable function f *induces the morphism* μ.

Numberings are equivalence relations in disguise: If $\langle \nu, S \rangle$ is a numbering then one can consider the equivalence relation $\sim_{\langle \nu, S \rangle}$ on ω,

$$x \sim_{\langle \nu, S \rangle} y \Leftrightarrow \nu(x) = \nu(y),$$

and, vice versa, starting from an equivalence relation R on ω, one can consider the numbering $\langle \nu_R, S_R \rangle$, where

$$S_R = \{[x]_R : x \in \omega\}$$

is the set of equivalence classes, and $\nu_R(x) = [x]_R$. In fact, in the category of numberings, we have an isomorphism $\langle \nu, S \rangle \simeq \langle \nu_{\sim_{\langle \nu, S \rangle}}, S_{\sim_{\langle \nu, S \rangle}} \rangle$.

Viewing equivalence relations as numberings, we are led to introducing the category of equivalence relations Eq, whose objects are the equivalence relations on ω, and the morphisms from R_1 to R_2 are functions $\mu : \omega/_{R_1} \longrightarrow \omega/_{R_2}$ such that there is a computable function f for which

$$\mu([x]_{R_1}) = [f(x)]_{R_2},$$

so that a morphism from R_1 to R_2 is induced by a computable function f satisfying

$$x \; R_1 \; y \Rightarrow f(x) \; R_2 \; f(y).$$

By the previous observations, Eq can therefore be viewed as a full subcategory of Num such that every object of Num is isomorphic to exactly one object of Eq.

Lemma 1.1. *In the category* Eq *the monomorphisms coincide with the* 1-1 *morphisms.*

Proof. Suppose $\mu : R \longrightarrow S$, induced by a computable function f, is a monomorphism, i.e., for every pair of morphisms $\nu_1, \nu_2 : E \longrightarrow R$,

$$\mu \circ \nu_1 = \mu \circ \nu_2 \Rightarrow \nu_1 = \nu_2.$$

Assume for a contradiction that μ is not 1-1, and let $[a_1]_R, [a_2]_R$ be distinct equivalence classes such that $\mu([a_1]_R) = \mu([a_2]_R)$. For $i = 1, 2$, define the computable function $g_i(x) = a_i$. Then, for every equivalence relation E, the functions g_1 and g_2 induce distinct morphisms $\nu_1, \nu_2 : E \longrightarrow R$, such that $\mu \circ \nu_1 = \mu \circ \nu_2$, a contradiction. □

Thus we have the following

Corollary 1.2. *If R and S are equivalence relations on ω, then $R \leq S$ if and only if there is a monomorphism $\mu : R \longrightarrow S$.*

From the point of view of category theory, $R \leq S$ may also be expressed by saying that R is a *subobject* of S, see Mac Lane [13, p. 122] and Ershov [8, 9].

We conclude the section with asking if the statement in Lemma 1.1 can be dualized:

Question 1. *Is it true that in* Eq *every epimorphism (i.e., a morphism μ satisfying: $\nu_1 \circ \mu = \nu_2 \circ \mu \Rightarrow \nu_1 = \nu_2$) is onto?*

By Lemma 3.16, the above holds in the subcategory Eq^P introduced below.

1.1. **The category Eq^P of ceers.** A *computably enumerable equivalence relation* (or, simply, *ceer*) is an equivalence relation R on the set of natural numbers such that the set $\{\langle x, y\rangle : x \; R \; y\}$ is computably enumerable (c.e.). Here, $\langle \_, \_ \rangle$ denotes the Cantor pairing function.

Lemma 1.3. *Let R, S be ceers. The following hold:*

- *If $\mu : R \longrightarrow S$ is a morphism (monomorphism, respectively) and S yields a partition into infinite sets then there is a computable* 1-1 *function that induces the morphism (monomorphism, respectively).*
- *If $\mu : R \longrightarrow S$ is an isomorphism, and R, S yield partitions into infinite sets, then there is a computable permutation of ω that induces μ.*

Proof. Assume that h induces a (mono)morphism $h : R \longrightarrow S$, where R and S are ceers, and the equivalence classes of S are infinite. We show how to build from h a 1-1 computable function k still inducing the same (mono)morphism: Define

$$k(n) = \text{first } m \in [h(n)]_S - \{k(i) : i < n\},$$

where "first" refers to some computable enumeration, uniform in n, of the c.e. set $[h(n)]_S - \{k(i) : i < n\}$.

In the second item, the function f inducing the isomorphism can be made a computable permutation of ω by a straightforward back-and-forth argument: At stage $2n$, we make sure that f is defined on n, and at stage $2n + 1$, we make sure that $n \in \mathrm{range}(f)$. Injectivity is preserved as in the previous item. □

Once again, interest in ceers originally arose from the theory of numberings, see, e.g., Malcev [15], where an important role is played by the notion of a *positive numbering*, i.e., a numbering $\langle \nu, S\rangle$ such that $\sim_{\langle \nu, S\rangle}$ is a ceer. (In fact, computably enumerable equivalence relations are also called *positive*, see, e.g., Ershov's paper [8] which contains the first detailed investigation of ceers.)

Definition 1.4. The category Eq^P of ceers is the full subcategory of Eq whose objects are exactly the ceers.

For later reference, we fix an effective numbering of all ceers. One natural way to number ceers is via the following lemma. For every set of numbers X, let X^* denote the equivalence relation on ω generated by X, where of course we view X as a subset of ω^2, via the Cantor pairing function.

Lemma 1.5. *There exists a computable function γ such that, for every e,*

$$W_{\gamma(e)} = W_e^*,$$

and if W_e is already an equivalence relation on ω, then $W_{\gamma(e)} = W_e$.

Proof. Trivial. □

We say that a numbering ν of a family of ceers is *computable* if

$$\{\langle e, x\rangle : x \in \nu(e)\}$$

is a c.e. set.

Theorem 1.6. *Let $R_e = W_{\gamma(e)}$. Then the numbering of all ceers, $\nu(e) = R_e$, is computable. Moreover ν is* universal*, or* principal*, i.e., for every computable numbering ρ of all ceers, there exists a computable function f such that $\rho = \nu \circ f$.*

Proof. The proof is straightforward. □

One could alternatively consider the following numbering, suggested by Ershov [8]: Let

$$x\ S_e\ y \Leftrightarrow (\exists m, n)[\varphi_e^m(x)\downarrow = \varphi_e^n(y)\downarrow],$$

where, given a partial function ψ, $\psi^n(x)$ denotes the n-th iterate of ψ on x, with $\psi^0(x) = x$. The function $\rho(e) = S_e$ yields a computable and principal numbering of all ceers (see Ershov [8]) and by standard methods of the theory of numberings one can easily see that μ and ρ are in fact computably isomorphic, i.e., there exists a computable permutation f of ω such that $\rho = \nu \circ f$.

In the following, we fix the numbering $\{R_e : e \in \omega\}$ of ceers provided by Theorem 1.6.

Lemma 1.7. *There exists a computable sequence $\{R_e^s : e, s \in \omega\}$ of finite sets such that for all s,*

(1) *$R_e^s \subset R_e^{s+1}$, and $R_e = \bigcup_s R_e^s$;*
(2) *R_e^s is an equivalence relation with domain a finite subset of ω;*
(3) *either $R_e^{s+1} - R_e^s = \{\langle a, a\rangle\}$ for some a, or there exists exactly one pair $[x]_{R_e^s}, [y]_{R_e^s}$ of equivalence classes, such that $[x]_{R_e^s} \cap [y]_{R_e^s} = \emptyset$, but $[x]_{R_e^{s+1}} = [y]_{R_e^{s+1}}$.*

Proof. Straightforward. □

The following corollary to the previous lemma will be used often:

Corollary 1.8. *For every ceer R there exists a computable sequence $\{R^s : s \in \omega\}$ of equivalence relations on ω, with $R = \bigcup_s R^s$, such that R^0 is the identity relation; $R^s \subseteq R^{s+1}$; and the equivalence classes of each R^s are finite. If $R - \{\langle x, x\rangle : x \in \omega\}$ is infinite, we may also assume that R^{s+1} is obtained from R^s by the collapse of exactly one pair of equivalence classes of R^s.*

Proof. Straightforward. □

Definition 1.9. A ceer R is *universal* if for every ceer S, $S \leq R$.

The following are examples of universal ceers; more interesting examples will be provided in Section 3 and in Section 4:

(1) The ceer R, where

$$\langle i, x\rangle\ R\ \langle j, y\rangle \Leftrightarrow [i = j \text{ and } x\ R_i\ y];$$

(2) if u is a universal partial computable function (i.e., there exists a computable function $t(x, y)$ such that $u(t(x, y)) = \varphi_x(y)$, for every x, y), then for every index e of u, the ceer S_e is universal, where S_e is as defined in the remark following Theorem 1.6 (see Ershov [8]).

The rest of this paper is organized as follows: In Section 2, we present results due to the last two authors on the degrees of ceers. Section 3 is mainly a review of the existing literature on universal ceers, with the exceptions of Lemma 3.4 and Lemma 3.16, which are due to the last two authors. The remaining sections present further results on ceers due to the first four and the last author.

2. The poset of degrees of ceers

Define $R \equiv S$ if $R \leq S$ and $S \leq R$. Denote by $\deg(R)$ the $\equiv$-equivalence class, or *degree*, of R, and define

$$\deg(R) \leq \deg(S) \Leftrightarrow R \leq S.$$

Let $\mathcal{P} = \langle \mathrm{ob}(\mathrm{Eq}^P)/_{\equiv}, \leq \rangle$ denote the poset of degrees of ceers. (Of course, $\mathrm{ob}(\mathrm{Eq}^P)$ denotes the set of objects of the category Eq^P, i.e., all ceers.) For every $n \geq 1$, let Id_n denote the ceer

$$x \,\mathrm{Id}_n\, y \Leftrightarrow x \equiv y \mod n;$$

moreover, let Id be the identity equivalence relation. The following information about $\mathcal{P}$ is readily available:

(1) $\mathcal{P}$ is a bounded poset: The least element is given by $\deg(\mathrm{Id}_1)$; the greatest element is given by the degree of universal ceers;
(2) $\mathcal{P}$ has a linearly ordered initial segment of order type $\omega + 1$,

$$\deg(\mathrm{Id}_1) < \deg(\mathrm{Id}_2) < \cdots < \deg(\mathrm{Id}_n) < \cdots < \deg(\mathrm{Id}),$$

with the mapping $n \mapsto \deg(\mathrm{Id}_{n+1})$ providing the order-theoretic isomorphism of ω with $\mathcal{I}_\omega = \{\deg(\mathrm{Id}_n) : n \geq 1\}$.
(3) Every ceer with n equivalence classes lies in $\deg(\mathrm{Id}_n)$, whereas $\deg(\mathrm{Id})$ consists of all decidable ceers with infinitely many equivalence classes.
(4) For every $\mathbf{R} \in \mathcal{P} - \mathcal{I}_\omega$, we have that $\mathcal{I}_\omega < \mathbf{R}$, i.e.,

$$(\forall \mathbf{S} \, [\mathbf{S} \in \mathcal{I}_\omega \Rightarrow \mathbf{S} \leq \mathbf{R}] \, .$$

Definition 2.1. Given a set A, define R_A by

$$x \, R_A \, y \Leftrightarrow x, y \in A \text{ or } x = y.$$

Lemma 2.2. *If A is simple, then* $\mathrm{Id} \not\leq R_A$.

Proof. If $\mathrm{Id} \leq R_A$ via a computable function f, then f is 1-1, so there is at most one number a such that $f(a) \in A$. Then either $f[\omega - \{a\}] \subseteq A^c$, or $f[\omega] \subseteq A^c$ if no such a exists. In either case, A^c contains an infinite c.e. set, a contradiction. □

The following lemma, appearing in San Mauro [21], has been independently proved by Coskey, Hamkins, and Miller in [7].

Lemma 2.3. *If A, B are c.e. sets with B infinite, then*

$$A \leq_1 B \Leftrightarrow R_A \leq R_B.$$

Proof. The left-to-right implication, observed in Gao and Gerdes [10], is immediate: Any computable 1-1 function reducing A to B induces a monomorphism from R_A to R_B.

For the other direction, we first claim that under the assumption $R_A \leq R_B$ there is a computable function g that induces a monomorphism from R_A to R_B, and $g[A] \subseteq B$. Let f witness that $R_A \leq R_B$. Indeed, if $f[A] \subseteq B$, then there is nothing that needs to be done. Otherwise, let $a \notin B$ be such that $f[A] = \{a\}$; moreover, let $b \in B$. Define

$$g(x) = \begin{cases} b & \text{if } f(x) = a \\ a & \text{if } f(x) = b \\ f(x) & \text{otherwise.} \end{cases}$$

The function g is computable, and it is easily seen that it has the desired properties.

At this point, it is only left to show that there is a 1-1 computable function h that induces the same monomorphism as g. This is essentially the same as in the proof of Lemma 1.3. Define

$$h(x) = \begin{cases} g(x) & \text{if } \neg(\exists y < x)[g(y) = g(x)], \\ \text{first } z \in B - \{h(y) : y < x\} & \text{otherwise,} \end{cases}$$

where "first" refers of course to some fixed computable enumeration of B. □

Lemma 2.4. *If* $\mathrm{Id} \leq R \leq R_A$ *then there exists a c.e. set* B *such that* $R \equiv R_B$.

Proof. If $\mathrm{Id} \leq R \leq R_A$, and $R \leq R_A$ via a computable f, then the range of f is an infinite c.e. set, and thus computably isomorphic to ω; let $g : \mathrm{range}(f) \longrightarrow \omega$ be a computable bijection. Finally take $B = g[A \cap \mathrm{range}(f)]$. Then $R_B \leq R$ via h where

$$h(x) = \text{first } y.\, [g(f(y)) = x].$$

On the other hand, $R \leq R_B$ via the computable function $g \circ f$. □

Let $\mathbf{0}_1$ denote the 1-degree of any computable, infinite and coinfinite set, and let $\mathbf{0}_1'$ denote the 1-degree of K.

Corollary 2.5. *We have* $[\deg(\mathrm{Id}), \deg(R_K)] \simeq [\mathbf{0}_1, \mathbf{0}_1']$, *where* $\simeq$ *denotes order isomorphism. Thus* $\mathcal{P}$ *is neither an upper semilattice nor a lower semilattice.*

Proof. The isomorphism between $[\deg(\mathrm{Id}), \deg(R_K)]$ and $[\mathbf{0}_1, \mathbf{0}_1']$ is provided by Lemma 2.3 and Lemma 2.4: Notice that $\mathrm{Id} \equiv R_A$ for every computable, infinite and coinfinite set A. The claim that $\mathcal{P}$ is neither an upper semilattice nor a lower semilattice follows from the fact that the poset of c.e. 1-degrees, $[\mathbf{0}_1, \mathbf{0}_1']$, is neither an upper semilattice nor a lower semilattice, see Young [27]. □

Corollary 2.6. *The first-order theory of* $\mathcal{P}$*, in fact its* Π^0_3*-fragment, is undecidable.*

Proof. The claim follows from Lachlan's result [11] that the topped finite initial segments of $[\mathbf{0}_1, \mathbf{0}_1']$ are exactly the finite distributive lattices (see also Odifreddi [19, p. 584]); thus the same is true of the interval $[\deg(\mathrm{Id}), \deg(R_K)]$ of $\mathcal{P}$. Hence the first-order theory of the finite distributive lattices is Σ_1-elementarily definable with parameters (see Nies [18] for the terminology) in $\mathcal{P}$. On the other hand, the Π^0_3-theory of the finite distributive lattices is hereditarily undecidable (Nies [18, Theorem 4.8]). Hence by the Nies Transfer Lemma, [18], the Π^0_3-theory of $\mathcal{P}$ is undecidable. □

Additional results about the poset of degrees of ceers will be given in Corollaries 4.3 and 5.8. For more information on $\mathcal{P}$, see Gao and Gerdes [10].

3. The world of universal ceers

We now turn our attention to universal ceers. In this section, we (mostly) review the existing literature concerning this topic. The section is divided into three subsections, dedicated to precomplete ceers, uniformly finitely precomplete ceers, and e-complete ceers, respectively.

3.1. Precomplete ceers.

Precomplete equivalence relations were introduced by Malcev [14]. For a detailed study of precomplete numberings, the reader is referred to Ershov's monograph [9].

Definition 3.1. An equivalence relation R is *precomplete* if for every partial computable function φ there exists a total computable function f such that for all n,

$$\varphi(n){\downarrow} \Rightarrow \varphi(n) \; R \; f(n).$$

We say in this case that f *makes* φ *total modulo* R, or f *is an* R*-totalizer of* φ.

We observe that uniformity holds: If R is precomplete then there exists a computable function $f(e,x)$ such that $f(e,\_)$ is a totalizer of φ_e, using the existence of a universal partial computable function u.

The following is a useful characterization of precomplete equivalence relations.

Theorem 3.2 (Ershov [9])**.** *An equivalence relation R is precomplete if and only if there is a computable function* fix *such that, for every n,*

$$\varphi_n(\mathrm{fix}(n)){\downarrow} \Rightarrow \varphi_n(\mathrm{fix}(n)) \; R \; \mathrm{fix}(n).$$

Notice that Id_1, the trivial equivalence relation having only one equivalence class, is precomplete. Henceforth, to avoid this singular case, we will always assume that we are dealing with nontrivial equivalence relations.

Theorem 3.3 (Bernardi and Sorbi [4])**.** *Every precomplete ceer is universal.*

Proof. See Bernardi and Sorbi [4]. The proof follows also from Lemma 3.4 below. □

Although implicit in Bernardi and Sorbi [4], the following lemma perhaps deserves some attention, in that it yields a slightly stronger result than the one in Bernardi and Sorbi [4], and most of all because it can be easily used to prove Lemma 3.16 below.

If X is a set of natural numbers and R is an equivalence relation on ω, then let

$$[X]_R = \{y : (\exists x \in X)[y \; R \; x]\},$$

and say that X is *R-closed* if $X = [X]_R$.

Lemma 3.4. *Let R, S be ceers, with S precomplete; let A be a nonempty R-closed c.e. set, and let φ be a partial computable function, with* domain$(\varphi) = A$, *that induces a non-onto partial monomorphism ν from R to S, i.e., for every $x, y \in A$,*

$$x \mathrel{R} y \Leftrightarrow \varphi(x) \mathrel{S} \varphi(y).$$

Then ν can be extended to a total monomorphism.

Proof. Let R, S, A, φ be as in the statement of the lemma, and let $\{A^s\}_{s\in\omega}$ and $\{R^s\}_{s\in\omega}$ be computable approximations to A and R, respectively.

We recall from Visser [26] that every precomplete ceer yields a partition of ω into pairs of effectively inseparable (or simply, e.i.) sets: For the definition, see Section 4 below; an alternative proof of Visser's result can be derived from Theorem 4.8, since we will see that every precomplete ceer is u.f.p. and hence weakly u.f.p. We also need the following result from Visser [26], called the Anti Diagonal Normalization Theorem (ADN-Theorem). First of all, a partial computable function Δ is called a *diagonal* function for an equivalence relation E if for every x such that $\Delta(x)\downarrow$, we have that $\Delta(x) \not\mathrel{E} x$. The ADN-Theorem reads: If E is a precomplete equivalence relation and Δ is a diagonal function for E, then for every partial computable function ψ, there exists (uniformly from indices of ψ and Δ) a total computable function g such that, for every x,

- $\psi(x)\downarrow \Rightarrow \psi(x) \mathrel{E} g(x)$;
- $\psi(x)\uparrow \Rightarrow g(x) \notin \text{domain}(\Delta)$.

We now define by induction a computable function f which induces a monomorphism extending the partial monomorphism induced by φ. Since ν is not onto, we can pick a number b such that, for every $x \in \varphi[A]$, $x \not\mathrel{S} b$. Let $a \in \varphi[A]$. Since S yields a partition into effectively inseparable sets, let χ be a productive function for the pair $([a]_S, [b]_S)$.

Stage 0. Let

$$\psi_0(x) = \begin{cases} \varphi(x), & \text{if } x \in A, \\ \uparrow, & \text{otherwise,} \end{cases}$$

and

$$\Delta_0(x) = \begin{cases} \chi(u_0, v_0), & \text{if } x \in [\varphi[A]]_S \cup [b]_S, \\ \uparrow & \text{otherwise,} \end{cases}$$

where u_0, v_0 are indices of $[\varphi[A]]_S$ and $[b]_S$, respectively. Hence, Δ_0 is diagonal for S. Finally define $f(0) = g_0(0)$, where g_0 is a computable function provided by the ADN-Theorem for the partial computable functions ψ_0, Δ_0.

Stage $n+1$. Given c.e. sets X and Y, with computable approximations $\{X_s\}_{s\in\omega}$ and $\{Y_s\}_{s\in\omega}$, define $x \in X \preceq Y$ and $x \in X \prec Y$, respectively, by

$$(\exists s)[x \in X_s \,\&\, (\forall t < s)[x \notin Y_t]]$$

and

$$(\exists s)[x \in X_s \,\&\, (\forall t \leq s)[x \notin Y_t]].$$

Let

$$\psi_{n+1}(x) = \begin{cases} \varphi(x), & \text{if } x \in A \preceq \bigcup_{i\leq n}[i]_R, \\ f(i) & \text{if } x \in \bigcup_{i\leq n}[i]_R \prec A, \text{ and } i \text{ first such that } x \text{ appears in } [i]_R, \\ \uparrow, & \text{otherwise,} \end{cases}$$

and

$$\Delta_{n+1}(x) = \begin{cases} \chi(u_{n+1}, v_{n+1}) & \text{if } x \in [\varphi[A]]_S \cup \bigcup_{i\leq n}[f(i)]_S \cup [b]_S, \\ \uparrow & \text{otherwise,} \end{cases}$$

where u_{n+1} and v_{n+1} are indices of $[\varphi[A]]_S \cup \bigcup_{i\leq n}[f(i)]_S$ and $[b]_S$, respectively. Hence the partial computable function Δ_{n+1} is diagonal for S. Finally define $f(n+1) = g_{n+1}(n+1)$, where g_{n+1} is a computable function provided by the ADN-Theorem for the partial computable functions ψ_{n+1}, Δ_{n+1}.

This completes the construction. A simple inductive argument shows that the sequences $\{\psi_n\}$, $\{\Delta_n\}$, $\{g_n\}$, $\{u_n\}$, $\{v_n\}$ are computable; for every n, the partial function Δ_n is diagonal for S, and $\left([\varphi[A]]_S \cup \bigcup_{i\leq n}[f(i)]_S\right) \cap [b]_R = \emptyset$. Then it follows that f induces a monomorphism μ extending ν. Notice that $[b]_S \notin \text{range}(\mu)$. □

Notice that Theorem 3.3 follows from Lemma 3.4; indeed, by taking, e.g., A to be any R-equivalence class, and φ constant on A, one can find a monomorphism f (extending φ) from R into S.

Example 3.5. We list some examples of precomplete ceers:

- S_e, where $u = \varphi_e$ is a universal partial computable function, Malcev [15];
- (Visser [26]) Consider any consistent c.e. theory T (e.g., $T = PA$) extending Robinson's Arithmetic such that for every $n \geq 1$, T has a Σ_n-truth predicate $T_n(v)$, i.e., a Σ_n-formula such that for all Σ_n-sentences σ

$$T \vdash \sigma \longleftrightarrow T_n(\overline{\ulcorner\sigma\urcorner})$$

where $\ulcorner\_\urcorner$ is a suitable Gödel numbering for all sentences in the language of T, and $\overline{m}$ denotes the numeral term for the number m. (For unexplained proof-theoretic notions, the reader is referred to [23].) Define $\sim_n$ on pairs σ, τ of Σ_n-sentences by

$$\ulcorner\sigma\urcorner_n \sim_n \ulcorner\tau\urcorner_n \Leftrightarrow T \vdash \sigma \longleftrightarrow \tau$$

where $\ulcorner\_\urcorner_n$ is a suitable Gödel numbering identifying Σ_n sentences with numbers: Notice that we use here $\ulcorner\_\urcorner_n$ instead of $\ulcorner\_\urcorner$, as otherwise the domain of $\sim_n$ would be a proper subset of ω. Then $\sim_n$ is a precomplete ceer. Given the relevance, throughout the paper, of this example, we sketch the proof of why $\sim_n$ is precomplete, limiting ourselves to the case $n = 1$. Given a partial computable function φ, let F be a representing Σ_1 formula for the partial computable function ψ, where

$$\psi(\ulcorner\sigma\urcorner_1) = \begin{cases} \ulcorner\tau\urcorner, & \text{if } \varphi(\ulcorner\sigma\urcorner_1)\downarrow = \ulcorner\tau\urcorner_1, \\ \uparrow, & \text{if } \varphi(\ulcorner\sigma\urcorner_1)\uparrow. \end{cases}$$

In particular we assume that F satisfies:

– $\psi(m)\downarrow= n \Leftrightarrow T \vdash F(\overline{m}, \overline{n})$;
– for every m, $T \vdash F(\overline{m}, u) \wedge F(\overline{m}, v) \longrightarrow u = v$.

Let $T_1(v)$ be a Σ_1-truth predicate, and define

$$f(m) = \ulcorner(\exists v)[F(\overline{m}, v) \wedge T_1(v)]\urcorner_1.$$

Assume now that $\varphi(\ulcorner\sigma\urcorner_1)\downarrow= \ulcorner\tau\urcorner_1$, where σ and τ are Σ_1-sentences. Then

$$\begin{aligned} T \vdash (\exists v)[F(\overline{\ulcorner\sigma\urcorner_1}, v) \wedge T_1(v)] &\longleftrightarrow F(\overline{\ulcorner\sigma\urcorner_1}, \overline{\ulcorner\tau\urcorner}) \wedge T_1(\overline{\ulcorner\tau\urcorner}) \\ &\longleftrightarrow T_1(\overline{\ulcorner\tau\urcorner}) \\ &\longleftrightarrow \tau, \end{aligned}$$

which implies that $\varphi(\ulcorner\sigma\urcorner_1) \sim_1 f(\ulcorner\sigma\urcorner_1)$.

- (Visser [26]) Provable equality $\sim_{\lambda\beta}$ of terms of $\lambda\beta$-calculus:

$$\ulcorner M\urcorner \sim_{\lambda\beta} \ulcorner N\urcorner \Leftrightarrow M =^\beta N,$$

where $\ulcorner\_\urcorner$ is a suitable Gödel numbering for all terms, and $M =^\beta N$ denotes that M reduces to N by finitely many applications of the β-rule.

Theorem 3.6 (Lachlan [12]). *All precomplete ceers are isomorphic.*

Proof. See Lachlan [12]. Notice that since precomplete ceers partition ω into infinite c.e. sets, by Lemma 1.3 being isomorphic here is equivalent to saying that there is a computable permutation f of ω such that

$$x\ R\ y \Leftrightarrow f(x)\ S\ f(y). \qquad \square$$

Let T be any theory as in Example 3.5, and let $\sim_T$ be the ceer defined by

$$\ulcorner\sigma\urcorner \sim_T \ulcorner\tau\urcorner \Leftrightarrow T \vdash \sigma \longleftrightarrow \tau.$$

We have

Theorem 3.7 (Bernardi and Sorbi [4]). *The ceer $\sim_T$ is not precomplete, but $\sim_T$ is universal.*

Proof. The computable function induced by the negation connective $\neg$ has no fixed point by the consistency of T. Thus $\sim_T$ is not precomplete by Theorem 3.2. On the other hand, for every $n \geq 1$, $\sim_n \leq \sim_T$, and since $\sim_n$ is universal so is $\sim_T$. $\square$

Despite the fact that $\sim_T$ and all precomplete ceers are universal, it follows that $\sim_T$ is not isomorphic to any precomplete ceer, thus showing the failure of the Myhill Isomorphism Theorem for universal ceers!

3.2. **Uniformly finitely precomplete ceers.** Notice that, although not precomplete, $\sim_T$ is "locally" precomplete, i.e., every partial computable function with finite range can be totalized modulo $\sim_T$ since there is some $n \geq 1$ such that all sentences in the range of φ are Σ_n, and thus we can totalize modulo $\sim_n$. This leads to the following definition (for which we immediately give the uniform version)

Definition 3.8. (Montagna [16]) R is *uniformly finitely precomplete* (or *u.f.p.* for short) if there exists a total computable function $f(D, e, x)$ such that for every finite set D and every e, x,

$$\varphi_e(x)\downarrow \in [D]_R \Rightarrow \varphi_e(x)\ R\ f(D, e, x).$$

$f(\_, e, \_)$ is called a *totalizer of* φ_e.

Clearly

Corollary 3.9. *Every precomplete equivalence relation is u.f.p. Moreover, the relation* $\sim_T$ *is u.f.p.*

Proof. Immediate: As explained at the beginning of this section, in order to prove that $\sim_T$ is u.f.p. use the fact that, given φ and a finite D, all sentences in D fall into some finite level Σ_n. □

The u.f.p. ceers provide further examples of universal ceers, as shown in Theorem 3.11 below. In the proof of the theorem, and in other proofs of results contained in later sections, we make many appeals to the Recursion Theorem in the construction of a certain computable sequence $\{e_i\}_{i\in\omega}$ of fixed points. Since a computable sequence of indices can be viewed as the range of a computable function f, a formal justification to our argument is provided by the Case Functional Recursion Theorem:

Lemma 3.10 (Case Functional Recursion Theorem, Case [5])**.** *Given a partial computable functional F, there is a* 1-1 *total computable function f such that, for every e, x,*

$$F(f, e, x) = \varphi_{f(e)}(x).$$

Proof. See also Odifreddi [19, p. 135], where the statement of the theorem does not make any explicit reference to f being 1-1, but this clearly follows from the fact that f is provided by the *s-m-n* Theorem. □

Theorem 3.11 (Montagna [16])**.** *Every u.f.p. ceer is universal.*

Proof. See Montagna [16]. The claim also follows from Theorem 4.8, as we will see that every u.f.p. ceer is weakly u.f.p. We include an outline of a proof here (different from the original proof given by Montagna [16]) to motivate the proof of Lemma 4.10. Let R be u.f.p. via the computable function f. As usual, we are assuming that R is nontrivial, and thus fix a and b with $a \not\mathrel{R} b$. In order to show that R is universal, we fix a universal ceer E with $0 \not\mathrel{E} 1$ and demonstrate that $E \le R$. By the Recursion Theorem (or more precisely, by the Case Functional Recursion Theorem), we assume that we control φ_{e_i} for a computable sequence $\{e_i\}_{i\in\omega}$ of indices. Define the computable sequence y_i by $y_0 = a$, $y_1 = b$ and $y_i = f(\{y_j \mid j < i\}, e_i, i)$ for each $i \ge 2$. By our choice of whether to make $\varphi_{e_i}(i)$ converge, we can control whether y_i and y_j are R-equivalent. We show that $E \le R$ via the function $i \mapsto y_i$.

By threat of forcing a contradiction via the Recursion Theorem, we will ensure that at no stage will it happen that $y_i\ R\ y_j$ but $i \not\mathrel{E} j$. Suppose y_i and y_j are the least y's in their R-equivalence classes at stage s, and those classes become

R-equivalent at stage $s+1$. We will ensure in the construction that if y_i is the least y in its R-equivalence class at stage s, then $\varphi_{e_i,s}(i)\uparrow$ and thus similarly $\varphi_{e_j,s}(j)\uparrow$. We then will cause $\varphi_{e_i,s+1}(i)\downarrow= a$ and $\varphi_{e_j,s+1}(j)\downarrow= b$, thus forcing that $a \mathrel{R} y_i \mathrel{R} y_j \mathrel{R} b$ contradicting that $a \not\mathrel{R} b$. Simply the threat of this action ensures that at no stage will it happen that $y_i \mathrel{R} y_j$ but $i \not\mathrel{E} j$.

When we witness at stage s that $i \mathrel{E} j$ for $i \neq j$ with y_i, and y_j being least in their respective R-equivalence classes, and, say $y_i < y_j$ (the case $y_j < y_i$ is treated similarly), then we cause $\varphi_{e_j,s}(j)\downarrow= y_i$ and wait for R to agree that $y_i \mathrel{R} y_j$. As $f(\_, e_j, \_)$ is a totalizer of φ_{e_j}, it must occur that y_j becomes R-equivalent to y_i. Notice that y_i becomes the least y in the combined R-equivalence class and, as promised, that we have not yet caused $\varphi_{e_i}(i)$ to converge. This completes the construction. □

3.3. **e-complete ceers.** The ceer $\sim_T$ has an interesting additional property which is captured by the following definition, due to Montagna [16], and later independently rediscovered by Lachlan [12]: The equivalence relations described by this definition were called *uniformly finitely m-complete* by Montagna [16], and *extension complete* (or, simply, *e-complete*) by Lachlan [12]. We adopt here Lachlan's terminology, although a terminology more appropriate to the rest of the paper would perhaps be *uniformly universal*.

Definition 3.12. (Montagna [16], Lachlan [12]) An equivalence relation S is *e-complete* if for every ceer R and every pair of m-tuples $(a_1,\dots,a_m)$, $(b_1,\dots,b_m)$ such that the assignment $a_i \mapsto b_i$ induces a partial monomorphism from R to S, one can extend the assignment (uniformly from the two tuples and an index for R) to a computable function inducing a monomorphism. (Notice that uniformity extends also to the case in which the assignment does not provide a partial monomorphism.)

Example 3.13. The ceer $\sim_T$ is e-complete, as follows from the following

Theorem 3.14 (Montagna [16], Bernardi and Montagna [3])**.** *A ceer R is e-complete if and only if R is u.f.p. and R has a total diagonal function, i.e., a computable function d such that for all x, $d(x) \not\mathrel{R} x$.*

Proof. Montagna [16] shows that, for every ceer R, R is e-complete if and only if R is u.f.p. and R has an *extended diagonal function*, i.e., a computable function d, such that for every finite set D, we have that $x \not\mathrel{R} d(D)$, for every $x \in D$. It is then observed in Bernardi and Montagna [3] that every u.f.p. ceer with a total diagonal function has also an extended diagonal function. □

The following theorem was first proved by Montagna [16], and later independently rediscovered by Lachlan [12].

Theorem 3.15 (Montagna [16], Lachlan [12])**.** *e-complete ceers are universal. Moreover, the e-complete ceers are all isomorphic with each other.*

Proof. Both properties easily follows from the uniform extension property provided by e-completeness: To show isomorphism, one uses a straightforward back-and-forth argument. Notice that universality is also a consequence of the fact that, by Theorem 3.14, every e-complete ceer is u.f.p. □

3.4. **Miscellanea.** Although some of it is not directly relevant to the scope of this paper, we collect here some additional useful information about the classes of universal ceers dealt with in this section.

- The e-complete ceers are all isomorphic to $\sim_T$ (by Example 3.13 and Theorem 3.15);
- the precomplete ceers are all isomorphic to $\sim_1$ (by Example 3.5 and Theorem 3.6);
- As a consequence of Theorem 3.14, all u.f.p. ceers not computably isomorphic to $\sim_T$ are weakly precomplete in the sense of Badaev [1]: An equivalence relation R is *weakly precomplete* if there exists a partial computable function fix such that for all n, if φ_n is total then $\text{fix}(n)\downarrow$ and
$$\varphi_n(\text{fix}(n))\ R\ \text{fix}(n);$$
- (Shavrukov [22]) There exists a u.f.p. ceer that is neither precomplete nor e-complete;
- a formula $F(v)$ of the theory T as in Example 3.5 is *extensional* if for every x, y,
$$x \sim_T y \Rightarrow T \vdash F(\overline{x}) \longleftrightarrow F(\overline{y});$$
let us call $\sim_F$ the ceer
$$x \sim_F y \Leftrightarrow T \vdash F(\overline{x}) \longleftrightarrow F(\overline{y});$$
Bernardi and Montagna [3] have shown that a ceer R is u.f.p. if and only if R is isomorphic with a ceer $\sim_F$ for some extensional formula F. (Again, the isomorphism is induced by a computable permutation of ω). The proof is based, among other things. on the following fact (Bernardi and Sorbi [4]): For every ceer R one can find a Σ_1-formula $F(v)$ in the language of T such that
$$x\ R\ y \Leftrightarrow\ \vdash_T F(\overline{x}) \longleftrightarrow F(\overline{y}).$$
- (Bernardi and Montagna [3]) The ceer $\sim_{\text{Pr}_T(v)}$ is precomplete, where $\text{Pr}_T(v)$ is the provability predicate of T.

Given equivalence relations R, S, we say that R is a *quotient* of S if there is an epimorphism $\mu : S \longrightarrow R$.

Lemma 3.16. *In the category* Eq^P, *the epimorphisms coincide with the onto morphisms.*

Proof. Suppose that $\mu : R \longrightarrow S$ is not onto, and let h be a computable function inducing the morphism. Let $A = [\text{range}(h)]_S$, and let a be such that $a \notin A$. Consider any precomplete ceer E, and by Theorem 3.3, let k be a computable function inducing a monomorphism ρ from S to E. Since $\rho \circ \mu$ is not onto, we can choose $b_1, b_2 \notin [\text{range}(k \circ h)]_E$ such that $b_1 \not\!E\ b_2$. (Indeed, since each distinct pair of E-equivalence classes is effectively inseparable by Visser [26], the complement of $[\text{range}(k \circ h)]_E$ cannot be a decidable set, and thus consists of infinitely many distinct classes: See also Ershov [8].) Let ψ_i be the partial computable function

$$\psi_i(x) = \begin{cases} k(x), & \text{if } x \in A, \\ b_i, & \text{if } x \mathrel{S} a, \\ \uparrow, & \text{otherwise.} \end{cases}$$

Applying Lemma 3.4 to $A \cup [a]_S$ and the partial monomorphisms induced by ψ_i, we obtain two distinct (mono)morphisms ν_1, ν_2, with ν_i induced by ψ_i, which agree on the range of μ, thus showing that μ is not an epimorphism. □

Hence R is a quotient of S if and only if there is an onto morphism $\mu : S \longrightarrow R$. Bernardi and Montagna use the notion of a quotient object to characterize u.f.p. ceers and precomplete ceers:

Theorem 3.17 (Bernardi and Montagna [3]). *The following hold:*

(1) *A ceer R is u.f.p. if and only if R is a quotient of $\sim_T$.*
(2) *A ceer R is precomplete if and only if R is a quotient of every universal ceer.*

Proof. See [3]. □

4. Universal ceers and partitions of the natural numbers into effectively inseparable sets

Effective inseparability plays a crucial role in the theory of universal ceers. We recall the following

Definition 4.1. Two disjoint c.e. sets A and B are *effectively inseparable* if there is a computable function p (called a *productive function*) such that, for all pairs u, v,

$$A \subseteq W_u \text{ and } B \subseteq W_v \text{ and } W_u \cap W_v = \emptyset \Rightarrow p(u, v) \notin W_u \cup W_v.$$

Since every ceer yields a partition of ω into c.e. sets, the previous definition suggests the following

Definition 4.2. A nontrivial ceer R is

- *effectively inseparable* (or *e.i.* for short) if it yields a partition of ω into effectively inseparable sets;
- *uniformly effectively inseparable* (or *u.e.i.* for short) if it is e.i. and there is a uniform productive function, i.e., a computable function $p(a, b, u, v)$ such that if $[a]_R \cap [b]_R = \emptyset$ then $p(a, b, \_, \_)$ is a productive function for the pair $\big([a]_R, [b]_R\big)$.

Visser [26] shows that every precomplete ceer is e.i., thus there exist e.i. universal ceers: This observation can be used to provide a new proof of a result that follows from Ershov [8, Lemma 12]:

Corollary 4.3. *The greatest element of the poset $\mathcal{P}$ of the degrees of ceers is join-irreducible.*

Proof. Suppose that R and S are ceers such that their degrees join to the greatest element. Consider the ceer $R \oplus S$, defined by

$$x \; R \oplus S \; y \Leftrightarrow \begin{cases} u \; R \; v & \text{if } x = 2u \text{ and } y = 2v, \\ u \; S \; v & \text{if } x = 2u+1 \text{ and } y = 2v+1. \end{cases}$$

Being above both R and S we have that $R \oplus S$ is universal. Let now E be any e.i. universal ceer, and let f be a computable function that induces a monomorphism from E to $R \oplus S$. Clearly the range of f cannot contain both even numbers and odd numbers: If say $f(x) = a$ with a even, and $f(y) = b$ with b odd, then $[x]_E \cap [y]_E = \emptyset$, and since f would m-reduce the pair $([x]_E, [y]_E)$ to the pair $([a]_{R \oplus S}, [b]_{R \oplus S})$, it would follow that the latter pair is e.i., contradicting the fact that the sets in the pair are separated by the decidable set of the even numbers. If the range of f is contained in the even numbers, then from f one can easily construct a monomorphism reducing E to R, giving that R is universal; similarly if the range of f is contained in the odd numbers, then from f one can easily construct a monomorphism reducing E to S, giving that S is universal. This shows that the greatest element of $\mathcal{P}$ is join-irreducible. □

Ceers yielding partitions into effectively inseparable sets had been previously studied also by Bernardi [2]. The main result of this section shows that every u.e.i. ceer is universal; on the other hand, there exist e.i. ceers that are not universal. The proof that u.e.i. ceers are universal proceeds by showing that the u.e.i. ceers coincide with yet two new classes of ceers, introduced in the next two definitions.

Definition 4.4. We say that a ceer R is *weakly u.f.p.* if there exists a total computable function $f(D, e, x)$ such that for every finite set D where $[i]_R \neq [j]_R$ for every $i, j \in D$, and every e, x,

$$\varphi_e(x){\downarrow} \in [D]_R \;\Rightarrow \varphi_e(x) \; R \; f(D, e, x).$$

Note that the definition differs from that of a u.f.p. ceer in that f need only satisfy the condition when $[i]_R \neq [j]_R$ for every $i, j \in D$. Clearly

Corollary 4.5. *Every u.f.p. ceer is weakly u.f.p.*

Proof. Immediate. □

The following definition is a strengthening of the definition of a uniformly m-complete ceer given in Bernardi and Sorbi [4], namely, a ceer R is *uniformly m-complete* (abbreviated as *u.m.c.*) if for every ceer S and every quadruple a_0, a_1, b_0, b_1 of numbers such that $a_0 \not\mathrel{S} a_1$ and $b_0 \not\mathrel{R} b_1$, there exists a monomorphism from S to R that maps $[a_0]_S$ to $[b_0]_R$, and $[a_1]_S$ to $[b_1]_R$.

Definition 4.6. We say that a ceer R is *strongly u.m.c.* if for every ceer S and for every pair of numbers a_0, a_1, we have that every partial monomorphism $\pi : S \longrightarrow R$ defined on $\{[a_0]_S, [a_1]_S\}$ can be extended uniformly (in a_0, a_1 and an index of a partial computable function inducing π) to a total monomorphism μ, provided that $[a_0]_S \neq [a_1]_S$. (Note that the uniformity extends to the case when $[a_0]_S = [a_1]_S$; however, then no claim is made as to μ being a monomorphism.)

We call a ceer *weakly n-u.f.p.* if Definition 4.4 for weakly u.f.p. holds, but we replace "finite set D" with "finite set D where $|D| \leq n$".

Lemma 4.7. *Each weakly* 2*-u.f.p. ceer is weakly u.f.p.*

Proof. Let f_i be a computable function witnessing that R is weakly i-u.f.p., for $2 \leq i \leq n$. We describe how to effectively get a function f_{n+1} witnessing that R is weakly $n+1$-u.f.p. Let D, e be given. If $|D| \neq n+1$ then $f_{n+1}(D, e, x)$ outputs 0 for every x. We assume $D = \{d_0, \dots, d_n\}$. By the Double Recursion Theorem (see, e.g., Rogers [20, Theorem X(a)]) assume that we build φ_a and φ_b for some a, b. Let $E_x = \{f_n(D - \{d_n\}, a, x), d_n\}$, and $f_{n+1}(D, e, x) = f_2(E_x, b, x)$.

We now specify, for an x, how to compute $\varphi_a(x)$ and $\varphi_b(x)$. We initially start off with both values undefined. We see which event happens first: If we find that $\varphi_e(x)\downarrow R\ d_n$, we define $\varphi_b(x) = d_n$. If we find that $\varphi_e(x)\downarrow R\ d_i$ for some $i < n$, we define $\varphi_b(x) = f_n(D - \{d_n\}, a, x)$ and $\varphi_a(x) = \varphi_e(x)$. Finally if we discover that $f_n(D - \{d_n\}, a, x)\ R\ d_n$, then we define $\varphi_a(x) = d_0$.

Clearly f_{n+1} is a total recursive function, whose index can be found effectively in the indices for $f_2, \dots, f_n$, using the fact that the fixed points in the Double Recursion Theorem can be found effectively from the parameters.

Now we verify that f_{n+1} witnesses that R is weakly $n+1$-u.f.p. Fix D, e, x such that $D = \{d_0, \dots, d_n\}$ where $d_i \not R\ d_j$ for every pair $i \neq j$, and $\varphi_e(x)\downarrow R\ d_i$ for some $i \leq n$. First we claim that $f_n(D - \{d_n\}, a, x) \not R\ d_n$. Suppose otherwise: Then by construction we would set $\varphi_a(x) = d_0$ unless it has previously been defined (to be $\varphi_e(x)\ R\ d_i$, for some $i < n$). In either case we have $\varphi_a(x)\ R\ d_i$ for some $i < n$, which implies that $d_n\ R\ f_n(D - \{d_n\}, a, x)\ R\ d_i$, a contradiction. We have thus that E_x consists of two elements that are not R-equivalent. Since $\varphi_b(x)$ is defined only when $\varphi_e(x)$ converges, it is straightforward to see that $f_{n+1}(D, e, x)\ R\ \varphi_e(x)$. ☐

Theorem 4.8. *The following properties are equivalent for ceers:*

(i) *u.e.i.*
(ii) *weakly u.f.p.*
(iii) *strongly u.m.c.*

We prove Theorem 4.8 via Lemma 4.9, Lemma 4.10, and Lemma 4.11.

Lemma 4.9. *Each u.e.i. ceer is weakly u.f.p.*

Proof. Assume that R is u.e.i. via the uniform productive function $p(a, b, u, v)$ as in Definition 4.2, where for simplicity we denote $p(a, b, \_, \_)$ by $p_{a,b}$. We argue that R is weakly 2-u.f.p. Given any $a \neq b$, and e, we uniformly build a function $f(x) = f(\{a, b\}, e, x)$ witnessing that R is 2-u.f.p. Note that if $a = b$ then we can let f be the constant function with output a. We write p for $p_{a,b}$. Again by the Double Recursion Theorem with parameters we build W_{a_x}, W_{b_x} for computable sequences of indices $\{a_x\}_{x\in\omega}, \{b_x\}_{x\in\omega}$, where the sequence is known to us during the construction.

Let $f(x) = p(a_x, b_x)$, which is a total computable function. Fix x, and let

$$W_{a_x} = \begin{cases} [a]_R, & \text{if } \varphi_e(x) \not R\ b \\ [a]_R \cup \{p(a_x, b_x)\}, & \text{if } \varphi_e(x)\ R\ b, \end{cases}$$

$$W_{b_x} = \begin{cases} [b]_R, & \text{if } \varphi_e(x) \not R\ a \\ [b]_R \cup \{p(a_x, b_x)\}, & \text{if } \varphi_e(x)\ R\ a. \end{cases}$$

Now assume that $a \not R\ b$, and fix e, x such that $\varphi_e(x) \downarrow \in [a]_R \cup [b]_R$. Without loss of generality suppose $\varphi_e(x)\ R\ a$. If $f(x) \not R\ a$ then $W_{a_x} \cap W_{b_x} = \emptyset$ and $p(a_x, b_x) \in W_{a_x} \cup W_{b_x}$, which contradicts p being a productive function. □

Lemma 4.10. *Each weakly u.f.p. ceer is strongly u.m.c.*

Proof. Let R be weakly u.f.p. via the computable function f. This proof will roughly mimic that of Theorem 3.11. The major obstruction is that we cannot define y_i as $f(\{y_j \mid j < i\}, e_i, i)$, since if ever we cause $y_j\ R\ y_k$ for $j < k < i$, then since R is only weakly u.f.p. via f, we lose all control to R-collapse y_i with any y_j for $j < i$. We overcome this obstruction by the use of auxiliary elements which we can use to enforce an R-collapse, even if some y_j and y_k do R-collapse. This is the role of the x_i^k below. As in the proof of Theorem 3.11, we use convergence of the φ_e we control to cause R-collapses and we will use the threat of Action ☠ in the construction below to enforce non-collapse.

In order to show that R is strongly u.m.c., we show in fact that for every ceer S, every assignment $(0, 1, \ldots, m) \mapsto (a_0, a_1, \ldots, a_m)$ with $m > 0$ inducing a partial monomorphism can be extended, uniformly in $a_0, a_1, \ldots, a_m$, to a total computable function inducing a monomorphism from S to R, provided that $[i]_S \neq [j]_S$ whenever $i \neq j$. (Uniformity extends also to the case in which there are pairs $i \neq j$ with $[i]_S = [j]_S$. The definition of a strongly u.m.c. ceer is thus just the case $m = 1$.) Our goal (under the assumption that the i's are pairwise not S-equivalent, for $i \leq m$) is to extend this to a total monomorphism by specifying a computable sequence of points $(a_{m+1}, a_{m+2}, \ldots)$ where for every pair i, j such that one of i or j is larger than m, we can force a_i to R-collapse to a_j. By the Recursion Theorem (or, more precisely, the Case Functional Recursion Theorem), we assume that we control φ_{e_i} for a computable sequence $\{e_i\}_{i \in \omega}$ of indices.

We will define computable arrays with the purpose that we can choose to cause R-collapses of pairs of y's independently. We first informally describe the uses of the elements x_i^k and y_i. For each k, we will build the element x_0^k to be R-collapsible to any element of $\{a_0, \ldots, a_m\}$. These x_0^k will be used to R-collapse any other x_i^k or y_k into $\{a_0, \ldots, a_m\}$. The role of x_i^k will be to allow y_k to R-collapse with y_i. In particular, y_k will be built to be R-collapsible with x_i^k, and x_i^k will be built to be R-collapsible with y_i. Note that each y_k needs to be able to R-collapse to any of the x_i^k for $0 < i < k$ and also at least one x_0^l (we use x_0^{2k} for this below) in order to R-collapse with an element of $\{a_0, \ldots, a_m\}$. Also x_i^k needs to be able to R-collapse to y_i as well as some x_0^l (we use x_0^{2i+1} below), in case it needs to later R-collapse with some element of $\{a_0, \ldots, a_m\}$.

Formally, define the computable arrays $\{x_i^k\}_{i,k \in \omega}$ and $\{y_i\}_{i \in \omega}$ as follows: Let $x_0^k = f(\{a_0, \ldots, a_m\}, e_1, k)$. Given $\{x_i^k\}_{i < n+1, k \in \omega}$, define

$$y_{n+1} = f(Y_{n+1}, e_{2n+2}, 0),$$

where $Y_{n+1} = \{x_i^{n+1} \mid 0 < i < n+1\} \cup \{x_0^{2n+2}\}$, and

$$x_{n+1}^k = f(\{y_{n+1}, x_0^{2n+3}\}, e_{2n+3}, k), \qquad \text{for } k \in \omega.$$

We let $a_j = y_j$ for $j > m$. The a_j's will be the markers that code S in R, the other numbers x_i^j, y_i are simply representatives of auxiliary classes which will assist in R-collapsing the a_j's.

We assume (see Corollary 1.8) that during each stage of the construction, exactly one pair of distinct S-equivalence classes collapses. At the beginning of the construction we assume $[i]_S = \{i\}$. $[i]_S$ represents an S-equivalence class with smallest member i. During the construction, to *identify* y_n *with* c means to define $\varphi_{e_{2n}}(0)\downarrow = c$, and similarly for x_n^k, in which case it means to define $\varphi_{e_{2n+1}}(k)\downarrow = c$.

Construction of φ_{e_i}. During the construction, if we ever discover that $a_i \; R \; a_j$, for some $i < j \leq m$, then we can ignore the rest of the construction below, and continue the construction trivially for the sake of uniformity, since the working assumption that $a_i \not R \; a_j$, for all $i < j \leq m$, is violated. At stage s of the construction, let $[i]_S$ and $[j]_S$ be the pair of collapsing S-classes. If $i, j \leq m$, we can ignore the rest of the construction below and continue trivially, otherwise there are two cases.

Case 1: $i \leq m < j$. We identify $a_j = y_j$ with x_0^{2j} (we will verify later that a_j cannot have been previously identified) and wait for either $y_j \; R \; x_0^{2j}$ or two elements of Y_j to R-collapse. If the latter happens first take Action ☠, otherwise we identify x_0^{2j} with a_i, and wait for $x_0^{2j} \; R \; a_i$.

Case 2: $m < i < j$. Identify $a_j = y_j$ with x_i^j, and wait for $y_j \; R \; x_i^j$ or two elements of Y_j to R-collapse. Again, if the latter happens first, take Action ☠, otherwise we now identify x_i^j with $a_i = y_i$. Wait for either $x_i^j \; R \; y_i$ or $y_i \; R \; x_0^{2i+1}$. If the latter happens first, take Action ☠, otherwise we achieve $a_j \; R \; a_i$.

Action ☠: We arrived here because we found $y_i \; R \; x_0^{2i+1}$ or two elements of Y_j have R-collapsed (and no element of Y_j has previously been identified). We describe two procedures P_k and Q_k which will call each other recursively until we force an R-collapse in $a_0, \ldots, a_m$.

Procedure P_k: This is called when $y_k \; R \; x_0^{2k+1}$. Perform the following steps.

(Step i) Check if y_k has been previously identified. If so, then by construction $y_k \; R \; a_{k'}$ for some least $k' < k$. If $k' \leq m$ then go to Step (ii). Otherwise, $y_{k'}$ has not been previously identified, and we now identify $y_{k'}$ with $x_0^{2k'}$. Wait for either $y_{k'} \; R \; x_0^{2k'}$ or two elements of $Y_{k'}$ to R-collapse. In the latter case we call $Q_{k'}$ (noting that no element in $Y_{k'}$ has been previously identified since $y_{k'}$ has not), otherwise we identify $x_0^{2k'}$ with a_0 and wait for $x_0^{2k'} \; R \; a_0$. Lastly, if y_k has not been previously identified, we identify y_k with x_0^{2k} and proceed as above, where we either call Q_k or we get $x_0^{2k} \; R \; a_0$. In either case, now continue with Step (ii).

(Step ii) If this step is reached then we have $y_k \; R \; a_{k'}$ for some $k' \leq m$. Clearly x_0^{2k+1} has not previously been identified. We now identify x_0^{2k+1} with $a_{k''}$

for any $k'' \neq k'$, $k'' \leq m$. We then obtain $a_{k'}\ R\ a_{k''}$ and continue the construction trivially for the sake of uniformity.

Procedure Q_k: This is called when two (least) elements in Y_k are R-collapsed, and no element of Y_k has been previously identified. There are two cases.

(Case i) The two elements are x_0^{2k} and x_i^k, $0 < i < k$. Identify x_i^k with x_0^{2i+1} and wait for $x_i^k\ R\ x_0^{2i+1}$ or $y_i\ R\ x_0^{2i+1}$. In the latter case call P_i, otherwise we now identify x_0^{2i+1} with a_1 and wait for the R-collapse. Now identify x_0^{2k} with a_0 and wait for the R-collapse. We succeed in forcing $a_0\ R\ a_1$.

(Case ii) The two elements are x_i^k and x_j^k, $0 < i < j < k$. Follow Case i to force $x_i^k\ R\ a_0$ and $x_j^k\ R\ a_1$, or call P_i or P_j.

This ends the description of the procedures P_k and Q_k. Suppose we arrive at this action because we found $y_i\ R\ x_0^{2i+1}$. We call P_i. On the other hand, if we arrive because two elements of Y_j have R-collapsed then we call Q_j. Clearly only finitely many different procedures can be called, and we either end up waiting forever at some step or provoking an R-collapse within $a_0, \dots, a_m$.

Enforcing non-collapse. At the end of stage s, check if there exist two elements of $\{a_n, x_n^k\}_{n,k\in\omega}$ which have R-collapsed but not yet been identified.

(Case i) The two elements are a_i and a_j for $i < j \leq m$. Continue the construction trivially for the sake of uniformity.

(Case ii) The two elements are a_i and a_j for $i \leq m < j$. Identify y_j with x_0^{2j} and wait for the desired R-collapse, where we will identify x_0^{2j} with $a_{i'}$ for any $i' \neq i$, $i' \leq m$. If we instead find that two elements of Y_j have R-collapsed, we take Action ☠.

(Case iii) The two elements are a_i and x_j^k for $i \leq m$. If $j > 0$ we identify x_j^k with x_0^{2j+1} and wait for the R-collapse. We then identify x_0^{2j+1} with $a_{i'}$ for any $i' \leq m$ with $i' \neq i$. If we instead find that $y_j\ R\ x_0^{2j+1}$, we take Action ☠. If $j = 0$, proceed similarly.

(Otherwise) For each of the remaining cases, we can follow Case ii or Case iii to force an R-collapse in $a_0, \dots, a_m$.

Verification. We list some easy facts about the construction.

- If any element of Y_j is identified then the same action must identify y_j (or continue the construction trivially for the sake of uniformity).
- If y_j is identified during the construction we will either continue the construction trivially for the sake of uniformity or force $y_j\ R\ a_i$ for some $i < j$ where $i\ S\ j$.
- If x_0^{2k+1} is ever identified during the construction then the same action will continue the construction trivially for the sake of uniformity.
- Therefore, any call to identify y_j or x_i^j during the construction must be successful in R-collapsing among $a_0, \dots, a_m$.

Now we assume that $a_0, \dots, a_m$ are in distinct R-equivalence classes. Then the construction is never continued trivially and we never take Action ☠. At the end of every stage s, we have that $i\ S\ j$ if and only if $a_i\ R\ a_j$. The left to right direction is ensured by Case 1 and Case 2 of the construction, while the right to left is ensured by the "enforcing non-collapse" action. □

Lemma 4.11. *Every strongly u.m.c. ceer is u.e.i.*

Proof. Let R be a strongly u.m.c. ceer. Let U, V be a pair of e.i. sets; fix $u \in U$ and $v \in V$; finally, let $S_{a,b}$ be the ceer defined by

$$x\ S_{a,b}\ y \Leftrightarrow \big[x = y \text{ or } x, y \in U \text{ or } x, y \in V \text{ or } [a\ R\ b \text{ and } x, y \in U \cup V]\big].$$

Notice that $S_{a,b}$ uniformly depends on a, b; moreover, if $a \not R\ b$, then $[u]_{S_{a,b}} = U$ and $[v]_{S_{a,b}} = V$, whereas if $a\ R\ b$, then $[u]_{S_{a,b}} = [v]_{S_{a,b}} = U \cup V$. Given a, b, consider the partial monomorphism π, defined only on the set $\{[u]_{S_{a,b}}, [v]_{S_{a,b}}\}$ by $\pi([u]_{S_{a,b}}) = [a]_R$ and $\pi([v]_{S_{a,b}}) = [b]_R$ (induced, say, by $\varphi(x) = a$, for all $x \in [u]_{S_{a,b}}$, and $\varphi(x) = b$, for all $x \in [v]_{S_{a,b}}$), and uniformly extend it to a monomorphism from $S_{a,b}$ to R, induced, say, by a computable function $f_{a,b}$. If $[a]_R \cap [b]_R = \emptyset$, then the uniformly given function $f_{a,b}$ m-reduces the e.i. pair (U, V) to the pair $([a]_R, [b]_R)$, showing that the latter is e.i. (for this property of e.i. pairs, see, e.g., Rogers [20]). □

The following is a natural companion to Lemma 4.7.

Corollary 4.12. *A ceer R is strongly u.m.c. if for every ceer S and every finite tuple of numbers $a_0, \ldots, a_m$, $m > 0$, we have that every partial monomorphism $\pi : S \longrightarrow R$ defined only on $\{[a_i]_S : i \leq m\}$ can be extended uniformly (in $a_0, \ldots, a_m$ and indices of partial computable functions inducing π) to a total monomorphism, provided $[a_i]_S \neq [a_j]_S$ for every $i \neq j$.*

Proof. By Theorem 4.8 and the proof of Lemma 4.10. □

We note that in the above corollary and in the definition of a strongly u.m.c. ceer, the condition $m > 0$ is necessary as removing this condition implies that R has a total diagonal function g defined in the following way: Given a, consider the ceer Id_2 having only two equivalence classes $[0]_{\mathrm{Id}_2}$, $[1]_{\mathrm{Id}_2}$, and extend the assignment $0 \mapsto a$ to a total computable function f inducing a monomorphism from Id_2 to R; finally take $g(a) = f(1)$. On the other hand, the property of having a total diagonal function is not necessarily possessed by u.e.i. and weakly u.f.p. ceers, as shown for instance by the ceer $\sim_T$ given by the provably equivalence relation of any theory T as in Example 3.5.

Proposition 4.13. *Each strongly u.m.c. ceer is u.m.c., but there are u.m.c. ceers that are not strongly u.m.c.*

Proof. It is clear that every strongly u.m.c. ceer is u.m.c., so by Theorem 4.8 it is enough to argue that some u.m.c. R is not u.e.i. The proof proceeds by a finite injury argument, which builds a ceer R satisfying the following requirements:

$\mathcal{R}_e$: For the ceer R_e and partial monomorphism $(a'_0, a'_1) \mapsto (a_0, a_1)$, there is a total extension reducing R_e to R.

$\mathcal{Q}_e$: R is not u.e.i. via the function φ_e.

The priority ordering is $\mathcal{R}_0 < \mathcal{Q}_0 < \mathcal{R}_1 < \cdots$. We use the fact that no effectivity is required in satisfying $\mathcal{R}_e$ by allowing the requirements to be injured. That is, we can change our mind about the extension, for each $\mathcal{R}_e$, finitely many times. Without loss of generality we assume that $a'_i = i$ for $i \leq 1$, and denote the pair (a_0, a_1) (for requirement $\mathcal{R}_e$) by $\vec{a}_e$. R will be nontrivial since R does not have any uniform productive function.

We denote by $X^{e,s}$ the (modified) i^{th} column of ω, i.e., the set of elements $X^{e,s}(j)$ with $X^{e,s}(j) = \langle i, j\rangle$ where $i = \langle e, s+1, \vec{a}_e\rangle$, if $j > 1$, and $X^{e,s}(j) = a_j$ if $j \leq 1$. (Note that thus the modified columns are not necessarily pairwise disjoint, and any possible overlap between columns are at the first two elements.) This will be used by $\mathcal{R}_e$ to code R_e: To code R_e into $X^{e,t}$ at some stage s of the construction means to R-collapse $X^{e,t}(i)$ and $X^{e,t}(j)$ if $[i]_{R_e} = [j]_{R_e}$ unless $i, j \leq 1$. We write X^e instead of $X^{e,t}$ when the context is clear. For a tuple $\vec{a}$ we say that $\vec{a}\ R\ x$ if there exist some $a_i \in \vec{a}$ such that $a_i\ R\ x$.

Construction of R. At each stage s, perform the following two steps.

Step 1: Pick the least e such that $\mathcal{Q}_e$ requires attention. This means either the pair of $\mathcal{Q}_e$-followers y^0_e, y^1_e is not yet picked, or else $\varphi_e(y^0_e, y^1_e, z^0_e, z^1_e)\downarrow \notin W_{z^0_e} \cup W_{z^1_e}$. Here, $W_{z^i_e}$ denotes the current R-equivalence class of y^i_e. For convenience we denote $\varphi_e(y^0_e, y^1_e, z^0_e, z^1_e)$ by f_e.

Clearly some $e < s$ must be found. If the $\mathcal{Q}_e$-followers are not defined, we pick a distinct fresh pair y^0_e, y^1_e from $X^{0,-1}$ (in particular, larger than any element of $\vec{a}_i$ for a higher-priority $\mathcal{R}_i$-requirement, or follower y^j_i for a higher-priority $\mathcal{Q}_i$-requirement). If the second case holds we R-collapse the classes of f_e and y^0_e. In either case we initialize all lower-priority requirements, i.e., for a $\mathcal{Q}_j$-requirement we reset the followers, and for an $\mathcal{R}_j$-requirement we are now ready to code R_j into $X^{j,t}$ for a fresh number t.

Step 2: For each $j < s$ we code R_j into X^j. If there exists a least $j < s$ such that the action at Step 1 or 2 collapses the two classes in $\vec{a}_j$, we make $\mathcal{R}_j$ inactive and initialize all lower-priority requirements.

We now verify that the requirements are satisfied. Clearly each requirement is initialized finitely often. The key lemma below says that during the construction, each class targeted for a $\mathcal{Q}$- or $\mathcal{R}$-requirement does not contain elements which are "bad" for the requirement.

Lemma 4.14. *Let s be a stage of the construction. The following hold at s:*

(i) *If $\vec{a}_i \not R\ X^i(n_i)$ for some n_i, then for every $x\ R\ X^i(n_i)$, we have $x \geq X^i(2)$.*
(ii) *If $f_i \not R\ y^{n_i}_i$ then for every $x\ R\ y^{n_i}_i$, we have $x \geq y^{n_i}_i$.*
(iii) *Suppose $X^i(n_i)\ R\ X^j(n_j)$ for some i, j, n_i, n_j. Then*

$$i \neq j \Rightarrow \vec{a}_i\vec{a}_j\ R\ X^i(n_i).$$

(iv) *Suppose $X^i(n_i)\ R\ y^{n_j}_j$ for some i, j, n_i, n_j. Then*

$$f_j \not R\ X^i(n_i) \Rightarrow \vec{a}_i\ R\ X^i(n_i) \text{ and } j < i.$$

(v) *Suppose $y_i^{n_i} \; R \; y_j^{n_j}$ and $y_i^{n_i} \neq y_j^{n_j}$ for some $i \leq j, n_i, n_j$. Then $i \neq j$ and we have*

$$f_j \; R \; y_j^{n_j}.$$

We explain what each of the items (i) through (v) mean. (i) says that for each class $[X^i(n_i)]_R$ targeted for an $\mathcal{R}_i$-requirement, if it contains an element x not initially in the class, then x must necessarily belong to a column of lower priority, unless $\mathcal{R}_i$ itself has already acted to collapse $X^i(n_i)$ to $\vec{a}_i$. (ii) expresses a similar fact for a $\mathcal{Q}_i$-requirement. It says that if the $\mathcal{Q}_i$-follower $y_i^{n_i}$ is related to a number x which is new, then this number must belong to a column of lower priority, unless $\mathcal{Q}_i$ has already acted to collapse f_i with $y_i^{n_i}$.

(iii), (iv) and (v) give specific details about the kind of elements which may be allowed to be collapsed to a given column. (iii) says that if some $X^i(n_i)$, in the $\mathcal{R}_i$-column, is collapsed with some $X^j(n_j)$ in the $\mathcal{R}_j$-column, where $i \neq j$, then it must be that either $\mathcal{R}_i$ or $\mathcal{R}_j$ has previously acted to collapse $X^i(n_i)$ with $\vec{a}_i$, or $X^i(n_j)$ with $\vec{a}_j$. Hence (iii) says that it is impossible for two $\mathcal{R}$-columns working for different requirements to be collapsed unless one of the two $\mathcal{R}$-requirements has already performed coding into the column. (iv) investigates when elements of an $\mathcal{R}_i$-column can collapse with elements in a column containing a $\mathcal{Q}_j$-follower. It says that the only way this can happen is if either the $\mathcal{Q}_j$-requirement has acted to collapse $y_j^{n_j}$ with f_j (and hence will never act again in the future), or if $\mathcal{R}_i$ has already collapsed $X^i(n_i)$ with $\vec{a}_i$. Lastly (v) asserts that the only way for a $\mathcal{Q}_i$-follower to be collapsed with a $\mathcal{Q}_j$-follower for $i \neq j$, is when the lower-priority one of the two acts to cause the collapse.

This lemma then permits us to later verify that the $\mathcal{R}$-requirements are met. To see this, consider two $\mathcal{R}$-columns which the $\mathcal{R}$-requirements want to keep distinct. To argue that these two columns are never unintentionally collapsed during the construction, note that parts (iii), (iv) and (v) of the lemma say that the foreign elements introduced into these columns during the construction must be targeted for other requirements, say $\mathcal{R}'$ or $\mathcal{Q}'$, that have already acted for these columns. Hence neither $\mathcal{R}'$ nor $\mathcal{Q}'$ will ever again do anything directly with these columns.

Proof of Lemma 4.14. At each stage s of the construction we take finitely many actions. We proceed by induction on this sequence of actions. At stage $s = 0$ before any action is taken, every equivalence class starts off as a singleton, so (i)-(v) are clearly true. Suppose (i)-(v) holds at a certain point at stage s. We consider the next action and argue that (i)-(v) still holds after this action. We consider the different cases.

Suppose we are collapsing f_i and y_i^0 in Step 1 (henceforth, while analyzing Step 1, this will be known as the "action"). Since $f_i \not R \; y_i^0$ holds before this action, this means (by induction hypothesis on (iv)) that for every j, n_j, if $X^j(n_j) \; R \; y_i^0$ then $\vec{a}_j \; R \; y_i^0$, and $i < j$. Let us now verify that each of (i)-(v) holds after this next action.

(iii) Fix $X^j(n_j) \; R \; X^k(n_k)$ where $j \neq k$. If this was true before the action then we apply the induction hypothesis. Let us assume otherwise that

$X^j(n_j)$ and $X^k(n_k)$ are collapsed by the action. Hence we must have (without loss of generality) $X^j(n_j)\ R\ y_i^0$ and $X^k(n_k)\ R\ f_i$ holds before the action. By induction hypothesis (iv) on $X^j(n_j)\ R\ y_i^0$, we conclude that $\vec{a}_j\ R\ X^j(n_j)$. Hence (iii) holds after the action.

(iv) In a similar way we verify that (iv) holds after this action. Suppose that $X^j(n_j)\ R\ y_k^{n_k}$. We assume that these two elements are collapsed by the action, hence we either have $X^j(n_j)\ R\ f_i$ and $y_k^{n_k}\ R\ y_i^0$, or the symmetric case $X^j(n_j)\ R\ y_i^0$ and $y_k^{n_k}\ R\ f_i$ holds. Observe that $k \leq i$ because otherwise the action will cause $\mathcal{Q}_k$ to be initialized.

In the first case we may assume that $y_k^{n_k} \neq y_i^0$ because otherwise $k = i$ and we are immediately done. Hence we can apply the induction hypothesis (v) on $y_k^{n_k}\ R\ y_i^0$ to conclude that $f_i\ R\ y_i^0$ before the action, but this is impossible. Let us assume now that the latter symmetric case holds, i.e., $X^j(n_j)\ R\ y_i^0$ and $y_k^{n_k}\ R\ f_i$. We apply the induction hypothesis (iv) on $X^j(n_j)\ R\ y_i^0$ to conclude that $\vec{a}_j\ R\ X^j(n_j)$ and $j > i$. We already remarked above that $k \leq i$ must be true. Hence $k < j$ and (iv) is verified.

(v) Suppose that $y_j^{n_j}\ R\ y_k^{n_k}$ and these two numbers are different. Without loss of generality assume that $y_j^{n_j}\ R\ f_i$ and $y_k^{n_k}\ R\ y_i^0$. Since this action causes all lower-priority requirements to be initialized, we must have $j, k \leq i$. Clearly $j \neq i$ because otherwise the construction would not have collapsed f_i and y_i^0. If $k = i$ then we are immediately done for (v). Hence we assume that $k < i$, and applying induction hypothesis on $y_k^{n_k}\ R\ y_i^0$, we get that $f_i\ R\ y_i^0$ before the action, a contradiction.

(i) We fix j, n_j such that $\vec{a}_j \not R\ X^j(n_j)$. If $X^j(n_j)$ is related to neither f_i nor y_i^0 before the action, then once again we have that (i) holds by applying the induction hypothesis. $X^j(n_j)\ R\ y_i^0$ before the action is not possible by the induction hypothesis (iv). Hence it must be that $X^j(n_j)\ R\ f_i$ before the action. If $\mathcal{R}_j$ is of lower priority than $\mathcal{Q}_i$ then $\mathcal{R}_j$ is initialized after this action and so (i) is trivially true (since each fresh equivalence class is a singleton). Otherwise, $\mathcal{R}_j$ is of higher priority, which means that $X^j(2) \leq y_i^0$, so by induction hypothesis (i)-(ii), we obtain (i).

(ii) We proceed similarly as in (i).

We now consider the next action in step 2. Fix $i < s$ and consider the action of coding R_i into X^i (henceforth "action" refers to this). There are two cases.

Case 1: We have $X^i(n)$ and $X^i(n')$ being collapsed, where $\vec{a}_i \not R\ X^i(n)$ and $\vec{a}_i \not R\ X^i(n')$ (before the action). We run through each case.

(iii) Consider $X^j(n_j)\ R\ X^i(n)$ and $X^k(n_k)\ R\ X^i(n')$ where $j \neq k$. Hence i is not equal to one of j or k. Apply the induction hypothesis (iii) on the appropriate pair.

(iv) Apply induction hypothesis (iv).

(v) Consider $y_j^{n_j}\ R\ X^i(n)$ and $y_k^{n_k}\ R\ X^i(n')$. By induction hypothesis (iv), we have $f_j\ R\ y_j^{n_j}$ and $f_k\ R\ y_k^{n_k}$, and hence we have $j \neq k$.

(i) Consider j, n_j such that $\vec{a}_j \not R\ X^j(n_j)$, and $X^j(n_j)\ R\ X^i(n)$. The case $j \neq i$ is impossible by induction hypothesis (iii). So assume $j = i$. By induction hypothesis, both classes $[X^i(n)]_R$ and $[X^i(n')]_R$ contain only numbers no smaller than $X^i(2)$, so we are again done.

(ii) Trivially true.

Case 2: We have $\vec{a}_i \; R \; X^i(n)$ and $\vec{a}_i \not R \; X^i(n')$. Again we consider each case separately.

(iii) A straightforward application of the induction hypothesis (iii).

(iv) Consider $y_j^{n_j} \; R \; X^k(n_k)$. If $y_j^{n_j} \; R \; X^i(n')$ then $f_j \; R \; X^i(n')$ by induction hypothesis (iv), which means (iv) must be true. Hence we may assume that $y_j^{n_j} \; R \; X^i(n)$, and that $f_j \not R \; y_j^{n_j}$. By the induction hypothesis (iv) on $y_j^{n_j} \; R \; X^i(n)$, we must have $j < i$. We have $X^k(n_k) \; R \; X^i(n')$. If $k = i$ then we are done, so assume $k \neq i$. Hence by induction hypothesis (iii), we have $\vec{a}_k \; R \; X^i(n')$. If $k < i$ then by construction $X^i(2) > \max \vec{a}_k$, contradicting the induction hypothesis (i). Hence we must have $k > i > j$, so we have (iv).

(v) Fix $y_j^{n_j} \; R \; X^i(n)$ and $y_k^{n_k} \; R \; X^i(n')$ and $y_j^{n_j} \neq y_k^{n_k}$. We have, by the induction hypothesis (iv), $f_k \; R \; X^i(n')$. If $f_j \; R \; X^i(n)$ then $j \neq k$ and (v) holds. So suppose that $f_j \not R \; X^i(n)$. By the induction hypothesis (i)-(ii), we get that $\mathcal{Q}_j < \mathcal{R}_i < \mathcal{Q}_k$. To wit, by induction hypothesis on (i) and the fact that $\vec{a}_i \not R \; X^i(n')$ and $y_k^{n_k} \; R \; X^i(n')$, we conclude that $y_k^{n_k} \geq X^i(2)$. Therefore $\mathcal{Q}_k$ is of lower priority than $\mathcal{R}_i$ because otherwise, $\mathcal{R}_i$ would pick $X^i(2)$ to be larger than $y_k^{n_k}$. To conclude that $\mathcal{Q}_j$ is of higher priority than $\mathcal{R}_i$, we apply the induction hypothesis (ii) to conclude that $\vec{a}_i \geq y_j^{n_j}$, whereas, if $\mathcal{Q}_j$ were of higher priority, then it would pick $y_j^{n_j}$ larger than $X^i(2)$. Hence we have (v).

(i) Fix $X^j(n_j)$ where $\vec{a}_j \not R \; X^j(n_j)$. If $j = i$ then it is trivial, so assume $j \neq i$. By induction hypothesis (iii), we conclude that $X^j(n_j) \not R \; X^i(n')$ and hence $X^j(n_j) \; R \; X^i(n)$. By the induction hypothesis (i) applied to $X^j(n_j)$, we have $j < i$. Now by the induction hypothesis (i), this time applied to $X^i(n')$, and the fact that $X^i(2) > X^j(2)$, we obtain (i).

(ii) Fix $y_j^{n_j} \; R \; X^i(n)$ where $f_j \not R \; y_j^{n_j}$ (again by induction hypothesis (iv), $y_j^{n_j} \; R \; X^i(n')$ is impossible). Then by the induction hypothesis (ii), $\mathcal{Q}_j$ is of higher priority than $\mathcal{R}_i$, which means that $X^i(2) > y_j^{n_j}$. Thus by the induction hypothesis (i) applied to $X^i(n')$, we have (ii). □

We now argue that each $\mathcal{Q}_e$ is met. Fix a stage after which $\mathcal{Q}_e$ is never initialized, and let y_e^0, y_e^1 be the final $\mathcal{Q}_e$ followers. By Lemma 4.14(v), $y_e^0 \not R \; y_e^1$. Thus if $f_e\downarrow$ then $y_e^0 y_e^1 \; R \; f_e$, hence φ_e cannot be the function witnessing that the ceer is u.e.i.

Now consider $\mathcal{R}_e$, and a stage after which it is never initialized. Let X^e be the final version. We claim that for i or $j \geq 2$, $i \; R_e \; j$ if and only if $X^e(i) \; R \; X^e(j)$. The left to right implication is explicitly ensured by the construction. Suppose that $X^e(i)$ is collapsed with $X^e(j)$ at some stage s in the construction, by some action which is not the coding of R_e. There are again two cases.

Case 1: Coding of R_k for $k \neq e$. We may assume that $X^k(l)\ R\ X^e(i)$ and $X^k(l')\ R\ X^e(j)$. We do not worry about the case when $\vec{a}_e\ R\ X^e(i)$ and $\vec{a}_e\ R\ X^e(j)$, since we would make $\mathcal{R}_e$ inactive after this action. Assume that $\vec{a}_e \not R\ X^e(i)$ and $\vec{a}_e \not R\ X^e(j)$. By Lemma 4.14(iii) we have $\vec{a}_k\ R\ X^k(l)$ and $\vec{a}_k\ R\ X^k(l')$, but by construction we would not have collapsed $X^k(l)$ and $X^k(l')$. Now assume that $\vec{a}_e \not R\ X^e(i)$ and $\vec{a}_e\ R\ X^e(j)$. By Lemma 4.14(i) and (iii) we get that $e < k$. Now since $\vec{a}_k \not R\ X^k(l')$ by Lemma 4.14(i) and (iii) again we get that $k < e$, a contradiction.

Case 2: Action in Step 1. Assume we collapsed $y_k^0\ R\ X^e(i)$ with $f_k\ R\ X^e(j)$. Since $f_k \not R\ y_k^0$ before this action, by Lemma 4.14(iv) we have $\vec{a}_e\ R\ X^e(i)$. By Lemma 4.14(ii), we have that $\mathcal{Q}_k$ is of higher priority than $\mathcal{R}_e$, hence $\mathcal{R}_e$ will get initialized, a contradiction. □

Question 2. *Do the u.f.p. ceers coincide with the weakly u.f.p. ceers?*

Corollary 4.15 below subsumes all universality results seen so far and is a natural companion of the following classical results:

- Every creative set is m-complete (Myhill [17]).
- Every pair of effectively inseparable sets is m-complete (Smullyan [24]).
- All creative sequences are m-complete (Cleave [6]).

Corollary 4.15. *Every u.e.i. ceer is universal.*

Proof. Immediate by Theorem 4.8, as every strongly u.m.c. ceer is clearly universal: If R is a strongly u.m.c. ceer, and S is any ceer with two distinct equivalence classes, then start off with a partial monomorphism $\pi : S \longrightarrow R$, defined on these two equivalence classes, and extend it to a full monomorphism. □

We continue with some easy observations.

Theorem 4.16. *A ceer R is universal if and only if R has a u.e.i. subobject $S \leq R$.*

Proof. If R is universal and S is u.e.i., then trivially $S \leq R$. Conversely, if S is u.e.i. and $S \leq R$, then R is universal since so is S, by Corollary 4.15.

□

Corollary 4.17. *A ceer R is universal if and only if there exist a c.e. set A which is R-closed, and a computable bijection $f : \omega \longrightarrow A$ such that the ceer S given by*

$$x\ S\ y \Leftrightarrow f(x)\ R\ f(y),$$

is u.e.i.

Proof. The right-to-left implication follows from the fact that $S \leq R$. For the other direction, assume that R is universal, and let E be a u.e.i. ceer (with $f_E(a, b, u, v)$ as its uniform productive function), with $E \leq R$ via a computable function g. Let $A = [\text{range}(g)]_R$, and $f : \omega \longrightarrow A$ be a computable bijection. By Lemma 1.3, we can also assume that g is 1-1, since every $[g(x)]_R$ is infinite. Finally, let S be defined as in the statement of the theorem. A uniform productive function for S is then

$$f_S(a, b, u, v) = f^{-1}(g(f_E(g^{-1}(f(a)), g^{-1}(f(b)), u', v')))$$

where u', v' are uniformly found indices for

$$W_{u'} = g^{-1}[f[W_u]] \qquad W_{v'} = g^{-1}[f[W_v]].$$

□

Remark 4.18. Of course, if R is a universal ceer, then for every ceer S, we have that $R \oplus S$ is also universal. So there are universal ceers that are not u.e.i., not even e.i.

The following theorem shows that uniformity is essential in showing that u.e.i. ceers are universal:

Theorem 4.19. *There exists an e.i. ceer that is not universal.*

Proof. To show the result, we build, stage by stage, a (nontrivial) e.i. ceer R and a ceer S such that $S \not\leq R$. Although not explicitly mentioned below, whenever at any stage $s+1$ we extend R^s to R^{s+1} by adding some pair to R^{s+1}, it is understood that then we further enlarge R^{s+1} to the equivalence relation generated by the set of pairs which have been enumerated so far, by taking the transitive closure to guarantee that we have an equivalence relation (and similarly for S). At any stage, a number is *new* if it has never been mentioned so far in the construction.

The construction of R and S will satisfy the following requirements, for all numbers a, b, k with $a \neq b$:

$P_{a,b}$: $[a]_R \cap [b]_R = \emptyset \Rightarrow f_{a,b}$ is a productive function for the pair $([a]_R, [b]_R)$,

N_k : φ_k does not witness $S \leq R$,

where $f_{a,b}$ is a total computable function built by us. In fact, $P_{a,b}$ should be written as $P_{\{a,b\}}$, and one should think of the P-requirements as linearly ordered according to the canonical index of $\{a, b\}$. In order to achieve that φ_k does not witness that $S \leq R$, strategy N_k will use four witnesses $a_0(k), a_1(k), a_2(k), a_3(k)$.

At any stage we say that we *initialize* a strategy, if the strategy is either a P-strategy, working, say, for requirement $P_{a,b}$, and at the given stage we set $f_{a,b} = \emptyset$, or the strategy is an N-strategy, working, say, for N_k and we set the witnesses $a_0(k), a_1(k), a_2(k), a_3(k)$ to be undefined

We must also make sure that R is not trivial.

Strategy for $P_{a,b}$. We say that $P_{a,b}$ becomes *inactive at stage* $s + 1$ (and stays inactive ever after) if either a is not the least element of $[a]_{R^{s+1}}$ or b is not the least element of $[b]_{R^{s+1}}$. If $P_{a,b}$ first becomes inactive at $s + 1$, then it initializes all strategies of lower priority.

If $P_{a,b}$ is not inactive at stage $s + 1$, then we extend the definition of $f_{a,b}$ to the next pair (u, v) and correct the already defined values of $f_{a,b}$ as follows:

(1) Define $f^{s+1}_{a,b}(u, v) = m$, where m is new;
(2) If $f^s_{a,b}(u', v') = m'$, then
    (a) let $m' \; R^{s+1} \; b$, if $m' \in W_{u',s+1}$;
    (b) let $m' \; R^{s+1} \; a$, if $m' \in W_{v',s+1}$;
    (c) do nothing, otherwise.

Outcomes of strategy $P_{a,b}$. Notice that in (2a) we make $W_{u'} \cap W_{v'} \neq \emptyset$, if $[a]_R \subseteq W_{u'}$ and $[b]_R \subseteq W_{v'}$; similarly, under the same assumptions, we make $W_{u'} \cap W_{v'} \neq \emptyset$ in (2b). Thus if $[a]_R \subseteq W_u$, $[b]_R \subseteq W_v$, and $W_u \cap W_v = \emptyset$ then $f_{a,b}(u,v) \notin W_u \cup W_v$. So if $P_{a,b}$ requires attention infinitely often, eventually without being initialized, then $f_{a,b}$ is a total productive function for the pair $([a]_R, [b]_R)$.

Strategy for N_k. The strategy aims at making S not reducible to R via φ_k:

(1) Appoint numbers a_0, a_1, b_0, b_1, which are new (hence, for every $x, y \in \{a_0, a_1, b_0, b_1\}$ such that $x \neq y$, we have $x \not\mathrel{S} y$, at the current stage).
(2) Wait for $\varphi_k(a_0)\downarrow$ and $\varphi_k(b_0)\downarrow$:
  (a) If already $\varphi_k(a_0) \mathrel{R} \varphi_k(b_0)$, then do nothing.
  (b) Otherwise, let $a_0 \mathrel{S} b_0$, and initialize lower-priority strategies; and
  (c) if later $\varphi_k(a_0) \mathrel{R} \varphi_k(b_0)$ (notice that, by the initialization undertaken in the previous item, this can happen only due to the action of higher-priority strategies), then repeat the previous steps with a_1, b_1 in place of a_0, b_0 respectively; more specifically, go to stage (2a) with a_1, b_1 in place of a_0, b_0 respectively.
(3) After completing (2c) for a_1, b_1, if already
$$\varphi_k(a_0) \mathrel{R} \varphi_k(b_0) \mathrel{R} \varphi_k(a_1) \mathrel{R} \varphi_k(b_1),$$
then do nothing;
(4) otherwise, let
$$a_0 \mathrel{S} b_0 \mathrel{S} a_1 \mathrel{S} b_1,$$
and initialize lower-priority requirements.

We say that N_k *requires attention* at stage $s+1$, if N_k is ready to act according to (1), or (2b) for a_0, b_0, or (2b) for a_1, b_1, or (4).

Outcomes of strategy N_k. The strategy has the following outcomes:

(1) If the strategy stops at (2) before reaching (2a), either for the pair a_0, b_0 or for the pair a_1, b_1, then φ_k is not total, and therefore the requirement is satisfied.
(2) If (2a) holds for the pair a_0, b_0 then
$$a_0 \not\mathrel{S} b_0 \text{ and } \varphi_k(a_0) \mathrel{R} \varphi_k(b_0);$$
similarly, if (2a) holds for the pair a_1, b_1 then
$$a_1 \not\mathrel{S} b_1 \text{ and } \varphi_k(a_1) \mathrel{R} \varphi_k(b_1).$$
(3) If we wait forever at (2c) for the pair a_0, b_0 then
$$a_0 \mathrel{S} b_0 \text{ and } \varphi_k(a_0) \not\mathrel{R} \varphi_k(b_0);$$
similarly, if we wait forever at (2c) for the pair a_1, b_1 then
$$a_1 \mathrel{S} b_1 \text{ and } \varphi_k(a_1) \not\mathrel{R} \varphi_k(b_1).$$
(4) Otherwise, at some point, the strategy yields
$$a_0 \mathrel{S} b_0 \text{ and } \varphi_k(a_0) \mathrel{R} \varphi_k(b_0),$$
$$a_1 \mathrel{S} b_1 \text{ and } \varphi_k(a_1) \mathrel{R} \varphi_k(b_1).$$
When this happens,

(a) if already $\varphi_k(b_0)\ R\ \varphi_k(a_1)$, then we keep $b_0 \not S\ a_1$;
(b) if $\varphi_k(b_0) \not R\ \varphi_k(a_1)$, then our action in (4) makes $b_0\ S\ a_1$, and, by initialization, keeps $\varphi_k(b_0) \not R\ \varphi_k(a_1)$.

The outcomes considered so far are all winning outcomes for N_k. There is another possibility that needs to be considered, detailed in the next item, which results from higher-priority requirements injuring N_k. Every time N_k is injured, it changes by initialization its quadruple of witnesses. We will argue, however, that eventually N_k is not injured anymore, so there is a final choice of the witnesses which allows for N_k only winning outcomes.

(5) The last thing to consider is when

$$\varphi_k(a_0)\ R\ \varphi_k(b_0)\ R\ \varphi_k(a_1)\ R\ \varphi_k(b_1)$$

and we have already defined

$$a_0\ S\ b_0\ S\ a_1\ S\ b_1.$$

Notice that when we defined $a_0\ S\ b_0$ we had $\varphi_k(a_0) \not R\ \varphi_k(b_0)$ by (2a). The R-collapse of $\varphi_k(a_0)$ and $\varphi_k(b_0)$ to, say, a number a must be the effect of later actions of higher-priority requirements of the form $P_{a,b}$: After convergence of $\varphi_k(a_0)$ and $\varphi_k(b_0)$, the lower-priority P-requirements are initialized, and thus they cannot move $\varphi_k(a_0)$ or $\varphi_k(b_0)$ to new equivalence classes, since they can only move their markers m, but these by initialization are chosen to be different from $\varphi_k(a_0)$ and $\varphi_k(b_0)$. Similarly, when we defined $a_1\ S\ b_1$, we had $\varphi_k(a_1) \not R\ \varphi_k(b_1)$. The R-collapse of $\varphi_k(a_1)$ and $\varphi_k(b_1)$ to, say, c must be the effect of later actions of higher-priority requirements $P_{c,d}$. When we defined $a_0\ S\ b_0\ S\ a_1\ S\ b_1$, we had $\varphi_k(b_0) \not R\ \varphi_k(a_1)$, hence $a \not R\ c$. When later we R-collapse a and c, either a or c stops being the least representative in its equivalence class, and so either $P_{a,b}$ or $P_{c,d}$ becomes inactive, and it initializes N_k. We will see that it only happens finitely often that a P-requirement, of priority higher than N_k, becomes inactive.

Construction. The construction at stage s proceeds in substages $t \leq s$. At stage 0, all strategies are initialized. At a substage $t \leq s$ of a stage $s > 0$, if $t = s$ then we end the stage. If $t < s$, then we attack the requirement $Q = Q_t$ with priority rank t. If Q is a P-requirement that was not inactive at the previous stage, but is now inactive, then we end the stage; otherwise, if Q is not inactive, then we act as described above (in the section Strategy for $P_{a,b}$). If Q is an N-requirement that requires attention then we act as described above, and we end the stage. In all other cases for $t < s$, after completing substage t we move on to substage $t + 1$.

After completing stage s, with say t the last substage before completing the stage, then we initialize all requirements having lower priority than Q_t.

Let $S = \bigcup_s S^s$, $R = \bigcup_s R^s$.

Verification. The verification is based on the following

Lemma 4.20. *Each N_k requires attention finitely often and initializes lower-priority strategies only finitely often. Each $P_{a,b}$ initializes lower-priority strategies only finitely often.*

Proof. This follows by a simple inductive argument. Suppose that the claim is true of every requirement Q, with $Q < N_k$. After all Q, with $Q < N_k$, stop initializing, we have that N_k cannot be further initialized and requires attention only finitely often, since all outcomes are finitary, and thus it initializes only finitely often. Similarly, if we assume that the claim is true of every requirement Q, with $Q < P_{a,b}$: After its last initialization, $P_{a,b}$ may initialize lower-priority strategies at most once, upon becoming inactive. □

Lemma 4.21. *Let a, b be such that $P_{a,b}$ is the highest priority P-requirement. Then $[a]_R \cap [b]_R = \emptyset$. Thus R is nontrivial.*

Proof. If a P-requirement $P_{c,d}$ appoints $f_{c,d}(u, v) = m$, then $m \neq a, b$, and only $P_{c,d}$ can enumerate m at some point in either $[a]_R$ or $[b]_R$, but not in both: After this enumeration, m is not moved any more to any equivalence class. □

Lemma 4.22. *Each P-requirement is satisfied.*

Proof. Let $P_{a,b}$ be given. By Lemma 4.20, there is a least stage after which $P_{a,b}$ is not initialized any more. Then after this stage we may construct $f_{a,b}$ witnessing that $([a]_R, [b]_R)$ is an e.i. pair if $[a]_R \cap [b]_R = \emptyset$. □

Lemma 4.23. *Each N_k is satisfied.*

Proof. Let N_k be given, and let s_0 be a stage after which N_k is never again initialized, so no higher-priority N-requirement requires attention after s_0, nor does any higher-priority P-requirement become inactive after s_0. After its last initialization, N_k appoints four permanent witnesses $a_0(k)$, $b_0(k)$, $a_1(k)$, $b_1(k)$. For simplicity, for $i = 0, 1$, write $a_i = a_i(k)$ and $b_i = b_i(k)$. We may suppose that for every $i = 0, 1$, $\varphi_k(a_i)$ and $\varphi_k(b_i)$ converge, otherwise N_k is trivially satisfied. Moreover we may suppose that action taken by N_k makes $a_0 S b_0$ and $a_1 S b_1$; otherwise, again N_k is satisfied. We must exclude the possibility

$$\varphi_k(a_0)\ R\ \varphi_k(b_0)\ R\ \varphi_k(a_1)\ R\ \varphi_k(b_1)$$

and

$$a_0\ S\ b_0\ S\ a_1\ S\ b_1.$$

But, as explained in the informal description of the outcomes of N_k, this possibility would require some $P < N_k$ to become inactive at some stage after s_0, thus providing one more initialization of N_k, which is impossible by the choice of s_0 □

This completes the proof. □

Figure 1 summarizes the inclusion relationships between the various classes of universal ceers which we have been dealing with in the paper. Notice that, except for the inclusion u.f.p. $\subseteq$ u.e.i. (for which properness is still open, see Question 2),

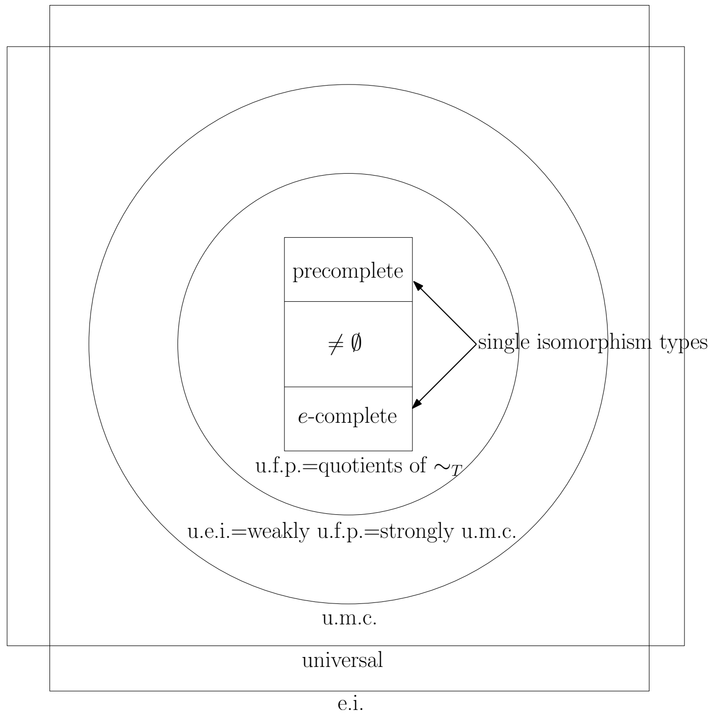

FIGURE 1. Classes of universal ceers under inclusion.

all other inclusions have been shown to be proper. Moreover universal $-$ e.i. $\neq \emptyset$ (Remark 4.18), and e.i. $-$ universal $\neq \emptyset$ (Theorem 4.19).

5. Characterizing universal ceers through a jump operator

5.1. A jump operator on ceers.

The following definition is due to Gao and Gerdes [10]; the defined operation is called the *halting jump operation.*

Definition 5.1. Given a ceer R, define

$$xR'y \Leftrightarrow x = y \text{ or } \varphi_x(x)\!\downarrow\! R\, \varphi_y(y)\!\downarrow .$$

Lemma 5.2 (Gao and Gerdes [10])**.** *The following properties hold:*

- $R \leq R'$;
- $R \leq S \Leftrightarrow R' \leq S'$;
- *If R is not universal then R' is not universal.*

One can thus introduce a well-defined operation on $\mathcal{P}$, by

$$(\deg(R))' = \deg(R').$$

Notice that $(\mathrm{Id}_1)' = R_K$, that is the equivalence relation having the halting set K as its unique nontrivial equivalence class, and $(\mathrm{Id})'$ is the ceer yielding the partition

$$\{K_i : i \in \omega\} \cup \{\{x\} : x \notin K\},$$

where $K_i = \{x : \varphi_x(x)\!\downarrow = i\}$.

The following theorem answers Problem 10.2 of Gao and Gerdes [10].

Theorem 5.3. *For every ceer E, if $E' \leq E$ then E is universal.*

Proof. Assume that h is a computable function that induces a monomorphism from E' to E. Let R be a ceer, with computable approximations $\{R^s : s \in \omega\}$; similarly, we will work with computable approximations $\{E^s : s \in \omega\}$ to E.

We first outline the idea of the proof through a particular example. We use an infinite computable sequence of indices $e_0, e_1, \ldots$, which we control by the Recursion Theorem. Eventually we define $f(i) = h(e_i)$, and show

$$i \, R \, j \Leftrightarrow e_i \, E' \, e_j (\Leftrightarrow h(e_i) \, E \, h(e_j)).$$

Our choice of these indices will make us able to E'-collapse any pair of them as needed. Suppose for instance that we want to make $e_0 \; E' \; e_1$ because we see at some point that $0 \; R \; 1$. The basic module for this is the following:

(1) Keep $\varphi_{e_0}(e_0)$ and $\varphi_{e_1}(e_1)$ undefined until we see $0 \; R \; 1$.
(2) Define $\varphi_{e_0}(e_0) = \varphi_{e_1}(e_1) = h(e')$ for another suitably chosen fixed point e' (while keeping $\varphi_{e'}(e')\uparrow$).

Suppose that even later we want to E'-collapse e_1 and e_2:

(1) Keep $\varphi_{e'}(e')$ and $\varphi_{e_2}(e_2)$ undefined, until $1 \; R \; 2$.
(2) Define $\varphi_{e_2}(e_2) = h(e'')$ and $\varphi_{e'}(e') = \varphi_{e''}(e'') = h(e''')$ (while keeping $\varphi_{e'''}(e''')\uparrow$), where e'' and e''' are further suitably chosen fixed points.

Notice that

$$\begin{aligned} \varphi_{e'}(e')\downarrow = \varphi_{e''}(e'')\downarrow &\Rightarrow e' \; E' \; e'' \\ &\Rightarrow h(e') \; E \; h(e'') \\ &\Rightarrow e_1 \; E' \; e_2. \end{aligned}$$

Care must be taken (by carefully controlling convergence of the various computations $\varphi_e(e)$), to collapse only what we *need* to collapse. In particular, if we see that E is threatening to E-collapse values, say, $h(e_i)$ and $h(e_j)$, without having $i \; R \; j$, then we threaten in our turn to stop the construction leaving certain computations divergent (exploiting the fact that if $u \neq v$ and $\varphi_u(u)$ and $\varphi_v(v)$ do not converge, then $u \not\!E' \; v$, and thus $h(u) \not\!E \; h(v)$), therefore forcing E to remove its threat if it wants to avoid a contradiction.

If D is a finite set, and n is a number, then $\langle D, n\rangle$ denotes the code $\langle u, n\rangle$ where u is the canonical index of D. A pair $\alpha = \langle D, n\rangle$ will be called a *node*: We sometimes denote the components of a node α by D_α and n_α. Our formal implementation of the above idea uses the Case Functional Recursion Theorem as a tool to find infinitely many synchronized fixed points. Thus, we assume that we are working with a computable sequence of indices $\{e_\alpha : \alpha \text{ node}\}$, which we control by the Case Functional Recursion Theorem.

It might be instructive to see how the two-steps example above is formally implemented.

(1) Keep $\varphi_{e_{\langle\{0\},0\rangle}}(e_{\langle\{0\},0\rangle})$ and $\varphi_{e_{\langle\{1\},0\rangle}}(e_{\langle\{1\},0\rangle})$ undefined, until we see $0 \; R \; 1$.

(2) Define

$$\varphi_{e_{\langle\{0\},0\rangle}}(e_{\langle\{0\},0\rangle}) = \varphi_{e_{\langle\{1\},0\rangle}}(e_{\langle\{1\},0\rangle}) = h(e_{\langle\{0,1\},1\rangle}),$$

still keeping $\varphi_{e_{\langle\{0,1\},1\rangle}}((e_{\langle\{0,1\},1\rangle})$ undefined (notice that $\{0\}$ and $\{1\}$ merge into $\{0,1\}$) so that, in the two-steps example above, we take $e_0 = e_{\langle\{0\},0\rangle}$, $e_1 = e_{\langle\{1\},0\rangle}$, and $e' = e_{\langle\{0,1\},1\rangle}$.

Suppose that even later we want to E'-collapse e_1 and $e_2 = e_{\langle\{2\},0\rangle}$:

(1) Keep $\varphi_{e_{\langle\{0,1\},1\rangle}}(e_{\langle\{0,1\},1\rangle})$ and $\varphi_{e_{\langle\{2\},0\rangle}}(e_{\langle\{2\},0\rangle})$ undefined, until $1\ R\ 2$.

(2) Define $\varphi_{e_{\langle\{2\},0\rangle}}(e_{\langle\{2\},0\rangle}) = h(e_{\langle\{2\},1\rangle})$ (notice, since we want to merge $\{2\}$ and $\{0,1\}$ into $\{0,1,2\}$, and since the node $\alpha = \langle\{0,1\},1\rangle$ has level 1, i.e., $n_\alpha = 1$, we first transform $\langle\{2\},0\rangle$ into a node $\langle\{2\},1\rangle$ with level 1: This transformation procedure will be called the synchronization procedure in the formal construction given below), and set

$$\varphi_{e_{\langle\{2\},1\rangle}}(e_{\langle\{2\},1\rangle}) = \varphi_{e_{\langle\{0,1\},1\rangle}}(e_{\langle\{0,1\},1\rangle}) = h(e_{\{0,1,2\},2\rangle}).$$

Thus, taking $e'' = e_{\langle\{2\},1\rangle}$, and $e''' = e_{\langle\{0,1,2\},2\rangle}$ we have that $\varphi_{e_2}(e_2) = h(e'')$ and $\varphi_{e'}(e') = \varphi_{e''}(e'') = h(e''')$ (still keeping $\varphi_{e'''}(e''')\uparrow$).

We see that the desired numbers e_i are taken to be $e_i = e_{\langle\{i\},0\rangle}$.

We say that a node β is a *parent* of a node α, if

- $n_\alpha = n_\beta + 1$; and
- $\varphi_{e_\beta}(e_\beta) = h(e_\alpha)$.

The construction will make sure that every node has at most two parents. A node α has only one parent β if α is the result of a definition due to the synchronization procedure, described below, i.e., $\alpha = \langle D_\beta, n_\beta + 1\rangle$ and $\varphi_{e_\beta}(e_\beta) = h(e_\alpha)$.

Given a node α, let T_α be the finite tree, defined as the smallest set of nodes such that:

- $\alpha \in T_\alpha$;
- if $\beta \in T_\alpha$ and γ is a parent of β then $\gamma \in T_\alpha$.

Finiteness of T_α follows from the fact that if γ is a parent of β, then $n_\gamma < n_\beta$.

We say that a node α is *realized*, if $n_\alpha = 0$, or $T_\alpha \neq \{\alpha\}$ (i.e., in the latter case, α has parents).

The above notions (a node β is a parent of a node α; the tree T_α; and α is realized) can be approximated at each stage s in the obvious way, by approximating at stage s the relevant computations $\varphi_e(e)$. In fact, if α is realized at s, then $T_{\alpha,s} = T_\alpha$, as can be easily seen. The guiding idea is that if α is realized at s, and $\varphi_{e_\alpha}(e_\alpha)$ is still undefined, then D_α is a block of R^s; if at some later stage $t > s$, R collapses D_α with another block D_β, relative to a similarly realized β, with $\varphi_{e_\beta}(e_\beta)$ still undefined, and $n = n_\alpha = n_\beta$, then we will define

$$\varphi_{e_\alpha}(e_\alpha) = \varphi_{e_\beta}(e_\beta) = h(e_{\langle D_\alpha \cup D_\beta, n+1\rangle}).$$

(We say that these convergent computations *code R into E*.)

Lemma 5.4. *Let α be a realized node, with $n_\alpha = n$. For every $i \leq n$, for every $\beta, \gamma \in T_\alpha$, if $n_\beta = n_\gamma = i$ then $h(e_\beta)\ E\ h(e_\gamma)$.*

Proof. We may assume $n > 0$, otherwise the claim is trivial. We will prove the claim by reverse induction. Assume $i = n$: The only node $\beta \in T_\alpha$ with $n_\beta = n$ is α. Thus the claim trivially holds for $i = n$.

Suppose that the claim is true of i, with $0 < i$, and let us show it for $i - 1$: For every node γ with $n_\gamma = i - 1$, there is a node β with $n_\beta = i$ such that $\varphi_{e_\gamma}(e_\gamma) = h(e_\beta)$. But by the inductive assumption, all the nodes β with $n_\beta = i$ are such that the corresponding values $h(e_\beta)$ are all E-equivalent, hence if γ, δ are nodes such that $n_\gamma = n_\delta = i-1$, we have that $e_\gamma \; E' \; e_\delta$ and thus $h(e_\gamma) \; E \; h(e_\delta)$. □

Lemma 5.5. *If α and β are distinct realized nodes, with $n_\alpha = n_\beta = n$ such that $\varphi_{e_\alpha}(e_\alpha)$ and $\varphi_{e_\beta}(e_\beta)$ are undefined, then, for every $\gamma \in T_\alpha$, and $\delta \in T_\beta$ such that $n_\gamma = n_\delta$, we have that $e_\gamma \not{E}' \; e_\delta$.*

Proof. By hypothesis we have $\varphi_{e_\alpha}(e_\alpha)\uparrow$ and $\varphi_{e_\beta}(e_\beta)\uparrow$. So the claim is true of $i = n$ since $e_\alpha \not{E}' \; e_\beta$. Suppose now that the claim is true of $0 < i \leq n$, and let $\gamma \in T_\alpha$ and $\delta \in T_\beta$ be such that $n_\gamma = n_\delta = i - 1$. Then there are $\gamma' \in T_\alpha$ and $\delta' \in T_\beta$ such that $n_{\gamma'} = n_{\delta'} = i$, and $\varphi_{e_\gamma}(e_\gamma) = h(e_{\gamma'})$, and $\varphi_{e_\delta}(e_\delta) = h(e_{\delta'})$. By induction, $e_{\gamma'} \not{E}' \; e_{\delta'}$, so $h(e_{\gamma'}) \not{E} \; h(e_{\delta'})$, and thus we may conclude that $e_\gamma \not{E}' \; e_\delta$. □

The synchronization procedure for two nodes α, β at stage $s + 1$:

(1) If $n_\alpha = n_\beta$ then do nothing.
(2) If $n_\alpha < n_\beta$, then for every i with $n_\alpha \leq i < n_\beta - 1$, define (at stage $s + 1$),

$$\varphi_{e_{\langle D_\alpha, i\rangle}}(e_{\langle D_\alpha, i\rangle}) = h(e_{\langle D_\alpha, i+1\rangle}).$$

The purpose of the synchronization procedure can be described as follows: Suppose that we have two nodes α and β and we want to R-collapse D_α and D_β, by defining $\varphi_{e_\alpha}(e_\alpha)\downarrow = \varphi_{e_\beta}(e_\beta)\downarrow$. But, following the construction which is detailed later, this can be done only if $n_\alpha = n_\beta$: If $n_\alpha < n_\beta$, we keep transforming α into nodes γ, with $D_\gamma = D_\alpha$, but with bigger and bigger n_γ, until we catch up with n_β.

Construction. We are now ready to describe the construction, which basically consists of two main actions:

(1) Waiting for R to catch up with E, when we see at any stage that for some a, b, $h(e_{\langle\{a\},0\rangle}) \; E \; h(e_{\langle\{b\},0\rangle})$, but $a \not{R} \; b$: Then, by the End of Stage procedure, we stop the construction, and wait for $a \; R \; b$. By Lemma 5.5 we eventually stop waiting.
(2) Coding of R into E via Lemma 5.4, through suitable convergent computations that code R into E (see the remark preceding Lemma 5.4); coding is performed while we are not currently waiting as in the previous item.

By Corollary 1.8, we assume that the chosen computable approximation of R at each new stage collapses exactly one pair of equivalence classes.

Stage 0. Start off with $\varphi_{e_\alpha}(e_\alpha)$ undefined for every node α.

Stage $s + 1$. If we are waiting at $s + 1$ for some pair of nodes α, β, then do nothing and go to next stage. Otherwise, let s^- be the last stage, if any, at the end of which we started to wait for some pair of nodes: If there is no such stage, then let $s^- = s$. For the sake of coding R into E, we now consider all possible R-collapses performed by R on pairs of equivalence classes in the time interval between $s^- + 1$ and $s+1$: Proceed by substages $t = 1, \dots, s+1-s^-$: At substage t, if α and β are nodes such that R collapses D_α and D_β at $s^- + t$, then synchronize α and β to two new nodes α', β', so after synchronization, we may assume (by replacing α, β with α', β', respectively) $n = n_\alpha = n_\beta$, and define

$$\varphi_{e_\alpha}(e_\alpha) = h(e_{\langle G, n+1\rangle})$$
$$\varphi_{e_\beta}(e_\beta) = h(e_{\langle G, n+1\rangle})$$

where $G = D_\alpha \cup D_\beta$. Notice that G is a new block in the approximation to R at stage $s^- + t$. After completing substage $s + 1 - s^-$ go to the End of Stage procedure.

End of Stage. There are a, b such that $h(e_{\langle\{a\},0\rangle})\ E^{s+1}\ h(e_{\langle\{b\},0\rangle})$, but $a \not R^{s+1} b$. Pick the least such pair of numbers, and pick nodes α, β, realized at s, such that $a \in D_\alpha$, $b \in D_\beta$, and $\varphi_{e_\alpha}(e_\alpha)$ and $\varphi_{e_\beta}(e_\beta)$ are still undefined at stage $s + 1$. (These nodes exist and are unique by Lemma 5.6.) Synchronize α and β at stage $s + 1$ to get nodes α', β' of the same level. At any future stage we say that we are *waiting for* α, β, until the first stage at which R collapses D_α and D_β, when we say that we *are not waiting for* α, β. Go to next stage.

If there is no pair of numbers as above, then go to next stage.

Verification. For every n, let

$$g(n) = h(e_{\langle\{n\},0\rangle}).$$

We notice:

Lemma 5.6. *For every a and s, there exists exactly one node α, realized at s, such that $a \in D_\alpha$, which is a block of the equivalence relation R^s, and $\varphi_{e_\alpha}(e_\alpha)$ is undefined at stage s.*

Proof. For $s = 0$, the desired unique node α is $\alpha = \langle\{a\}, 0\rangle$. The full claim follows by an easy induction on s. □

Lemma 5.7. *If α is realized at s, then for every $i \leq n_\alpha$, T_α contains also nodes β with $n_\beta = i$, and contains all nodes $\langle\{a\}, 0\rangle$, for all $a \in D_\alpha$.*

Proof. By the synchronization procedure. The claim that T_α contains all nodes $\langle\{a\}, 0\rangle$, for all $a \in D_\alpha$, follows by induction on n_α. □

Finally, we claim that, for every a, b,

$$a\ R\ b \Leftrightarrow g(a)\ E\ g(b).$$

Assume first that there exists a pair α, β and s_0 which makes us wait at all stages $s \geq s_0$. The reason for this was that we saw at a previous stage that

$$h(e_{\langle\{a\},0\rangle})\ E\ h(e_{\langle\{b\},0\rangle}),$$

for some $a \in D_\alpha$, $b \in D_\beta$, but $a \not R\, b$, and D_α and D_β never collapse to the same R-equivalence class at any stage $s \geq s_0$. By construction, we have $\varphi_{e_\alpha}(e_\alpha)\uparrow$ and $\varphi_{e_\beta}(e_\beta)\uparrow$. So by Lemma 5.5 and Lemma 5.7 we would conclude

$$h(e_{\langle\{a\},0\rangle}) \not E\ h(e_{\langle\{b\},0\rangle})$$

a contradiction.

So, there is no permanent wait, and thus there is no pair a, b such that $g(a)\ E\ g(b)$, but $a \not R\ b$.

Let us now show the left-to-right implication. Assume that $a\ R\ b$, and let $s+1$ be the least stage at which some pair of equivalence classes containing a and b R-collapse. Then there is a unique pair α, β with $a \in D_\alpha$ and $b \in D_\beta$ such that $\varphi_{e_\alpha}(e_\alpha)$ and $\varphi_{e_\alpha}(e_\alpha)$ are still undefined at s. Since there is no permanent wait in the construction, there will be a later stage at which we process α, β, and thus we define $\varphi_{e_\alpha} = \varphi_{e_\beta} = h(e_\gamma)$, for some γ (or, rather, $\varphi_{e_{\alpha'}} = \varphi_{e_{\beta'}} = h(e_\gamma)$, where α' and β' are the results of synchronizing α and β). Then by Lemma 5.4, applied to γ and to the tree T_γ, and Lemma 5.7, we have for all c, d in $D_\gamma = D_\alpha \cup D_\beta$, thus including a and b, that $h(e_{\langle\{c\},0\rangle})\ E\ h(e_{\langle\{d\},0\rangle})$, hence $g(a)\ E\ g(b)$. □

Corollary 5.8. *The poset $\mathcal{P}$ of the degrees of ceers is upwards dense, i.e., for every $\mathbf{R} < \mathbf{1}$, there exists $\mathbf{S}$ such that $\mathbf{R} < \mathbf{S} < \mathbf{1}$ (where $\mathbf{1}$ denotes the greatest element of $\mathcal{P}$).*

Proof. Given $\mathbf{R} < \mathbf{1}$, take $\mathbf{S} = \mathbf{R}'$. □

6. Index sets

In this section we classify some index sets of collections of ceers which have been considered in the paper.

We use below that for every Σ^0_3-set S there uniformly exists a c.e. class $\{X_{\langle i,j\rangle} : i, j \in \omega\}$ such that

$$i \in S \Rightarrow (\exists j)[X_{\langle i,j\rangle} = \omega],$$
$$i \notin S \Rightarrow (\forall j)[X_{\langle i,j\rangle} \text{ finite}],$$

see Soare [25, Corollary IV.3.7].

The following answers Problem 10.1 of Gao and Gerdes [10]:

Theorem 6.1. *The index set $\{x : R_x$ is universal$\}$ is Σ^0_3-complete.*

Proof. Let Univ $= \{x : R_x \text{ is universal}\}$. An easy calculation, using the fact that a ceer R is universal if and only if $E \leq R$, for a fixed universal ceer E, shows that Univ $\in \Sigma^0_3$, namely,

$$x \in \text{Univ} \Leftrightarrow (\exists e)[\varphi_e \text{ is total and } \varphi_e \text{ reduces } E \text{ to } R_x].$$

Next, we show that for every $S \in \Sigma^0_3$, we have $S \leq_m$ Univ. Given S, fix again a universal ceer E and a c.e. class $\{X_{\langle i,j\rangle} : i, j \in \omega\}$ as above; uniformly in i, build a ceer R, such that, denoting by $R^{[j]}$ the ceer

$$x\ R^{[j]}\ y \Leftrightarrow \langle j, x\rangle\ R\ \langle j, y\rangle,$$

we have that

$$i \in S \Rightarrow (\exists j)[R^{[j]} = E],$$
$$i \notin S \Rightarrow R \text{ yields a partition into finite sets.}$$

This is enough to prove the claim, since a universal ceer has always (infinitely many) infinite equivalence classes; indeed, if E, T are ceers such that $E \leq T$ via a monomorphism induced by computable function f, and $[x]_E$ is an undecidable equivalence class, then so is $[h(x)]_T$.

Construction. Let $\{E_s\}_{s\in\omega}$ be a computable approximation to E, and consider a computable approximation $\{X_{\langle i,j\rangle,s}\}_{s\in\omega}$ to $\{X_{\langle i,j\rangle}\}_{i,j\in\omega}$ via finite sets: We say that $s+1$ is $\langle i,j\rangle$-*expansionary* if

$$X_{\langle i,j\rangle,s+1} - X_{\langle i,j\rangle,s} \neq \emptyset.$$

Stage by stage we define, uniformly in i, a finite set R^s so that, eventually, $R = \bigcup_s R^s$ is our desired ceer.

Stage 0. Let $R^0 = \emptyset$.

Stage $s+1$. Let j be the least number $\leq s$, if any, such that $s+1$ is $\langle i,j\rangle$-expansionary: Then carry out the following, with the understanding that if there is no such j, then only item (1) applies:

(1) For every $k \neq j$, $k \leq s$, and $x \leq s$, let $\langle\langle k,x\rangle,\langle k,x\rangle\rangle \in R^{s+1}$.
(2) Let $\langle\langle j,x\rangle,\langle j,y\rangle\rangle \in R^{s+1}$ for every $\langle x,y\rangle \in E_s$.

It is straightforward to verify that if $i \notin S$ then every j has only finitely many $\langle i,j\rangle$-expansionary stages, so the equivalence classes of R are finite, hence R is not universal. Otherwise, for the least j such that there are infinitely many $\langle i,j\rangle$-expansionary stages, we have that $R^{[j]} = E$ (where we define $x \; R^{[j]} \; y$ if and only $\langle j,x\rangle \; R \; \langle j,y\rangle$), hence $E \leq R$, i.e., R is universal. □

Theorem 6.2. *The set $\{x : R_x$ is u.e.i.$\}$ is Σ^0_3-complete.*

Proof. Let Uei $= \{x : R_x$ is u.e.i.$\}$. A simple calculation shows that Uei $\in \Sigma^0_3$.

We now show that for every $S \in \Sigma^0_3$, we have $S \leq_m$ Uei. Given S, fix a c.e. class $\{X_{\langle i,j\rangle} : i,j \in \omega\}$ as above; uniformly in i, build a (non-trivial) ceer R, such that

$$i \in S \Rightarrow R \text{ is u.e.i.},$$
$$i \notin S \Rightarrow R \text{ is not u.e.i.}$$

Fixing i, for every j we have the two requirements:

$$P_j : \quad \varphi_j \text{ is not a uniform total productive function for } R,$$
$$Q_j : \quad (\exists^\infty \ \langle i,j\rangle\text{-expansionary stages}) \Rightarrow$$
$$f_j \text{ is a uniform total productive function for } R,$$

where f_j is a partial computable function (which is total if there exist infinitely many $\langle i,j\rangle$-expansionary stages) built by us.

We will guarantee also that R is not trivial, as by definition a u.e.i. ceer is not trivial.

We give the requirements the following priority ordering:

$$P_0 < Q_0 < \cdots < P_n < Q_n < \cdots$$

Strategy for P_j. Pick two new parameters a, b for P_j: We also assume that we work with indices u, v (given by the Recursion Theorem) of $[a]_R$ and $[b]_R$, respectively. Then

(1) keep $[a]_R$ and $[b]_R$ disjoint;
(2) wait for $\varphi_j(a, b, u, v)$ to converge;
(3) if $\varphi_j(a, b, u, v)$ converges to m, say, then add m to $[a]_R$, unless already in $[b]_R$;
(4) initialize lower-priority requirements.

The outcomes for the strategy are clear: If we wait forever for $\varphi_j(a, b, u, v)$ to converge, then φ_j is not total and thus cannot be a uniform total productive function for R. Otherwise, $[a]_R \cap [b]_R = \emptyset$, but $\varphi_j(a, b, u, v) \in [a]_R \cup [b]_R \subseteq W_u \cup W_v$. Hence $\varphi_j(a, b, \_, \_)$ does not witness effective inseparability of the pair $[a]_R$ and $[b]_R$.

Strategy for Q_j. Suppose at a stage s, we have defined f_j on a finite set of quadruples, and $s + 1$ is $\langle i, j\rangle$-expansionary. Then

(1) we extend f_j to the least quadruple not yet in the domain of f_j. Suppose that this quadruple is (a, b, u, v): Define $f_j(a, b, u, v) = m$ where m is new (meaning that m is a number that has never appeared so far in the construction);
(2) for every already defined value $f_j(a', b', u', v') = m'$,
    (a) if $m' \in W_{u',s}$ then let $m' \; R^{s+1} \; b'$;
    (b) if $m' \in W_{v',s}$ then let $m' \; R^{s+1} \; a'$;
(3) initialize lower-priority requirements.

If we have infinitely many stages that are $\langle i, j\rangle$-expansionary, then f_j is a total computable function. Given a, b such that $[a]_R \cap [b]_R = \emptyset$ and $[a]_R \subseteq W_u$, $[b]_R \subseteq W_v$, we have that $f_j(a, b, u, v) \notin W_u \cup W_v$, otherwise the construction makes $W_u \cap W_v \neq \emptyset$.

Construction. We say that P_j *requires attention at stage* $s + 1$ if either $a_{j,s}$, $b_{j,s}$ are undefined, or $a = a_{j,s}$ and $b = b_{j,s}$ are defined and not R^s-equivalent and $\varphi_{j,s}(a, b, u, v)$ is defined and equal to m, say, but $m \notin W_{u,s} \cup W_{v,s}$; we say that P_j is *initialized at* s by letting $a_{j,s}$ and $b_{j,s}$ be undefined. We say that Q_j *requires attention at stage* $s+1$ if $s+1$ is $\langle i, j\rangle$-expansionary; we say that Q_j is *initialized at* s by letting $f_{j,s} = \emptyset$.

In this construction as well, it is understood that whenever we extend R^{s+1} by adding some pair, we extend in fact to the transitive closure of the enlarged set of pairs.

Stage 0. Let $R^0 = \emptyset$; $f_{j,0} = \emptyset$; and let $a_{j,0}$ and $b_{j,0}$ be undefined for all j;

Stage $s+1$. Let N be the least requirement with index $\leq s$ that requires attention at stage $s + 1$. If no such requirement exists then go to stage $s + 2$. Otherwise, pick the least such N:

- $N = P_j$. Perform in order the following items:
  (1) If $a_{j,s}$ and $b_{j,s}$ are undefined, then choose $a_{j,s+1}$ and $b_{j,s+1}$ to be new numbers.

(2) Otherwise, suppose $a = a_{j,s}$ and $b = b_{j,s}$, and $\varphi_{j,s}(a, b, u, v)$ converges to, say, $m \notin W_{u,s} \cup W_{v,s}$. Let $m \; R^{s+1} \; a$; stop the stage, initialize all lower-priority requirements, and go to the next stage.

- $N = Q_j$. Perform in order the following items:
  (1) We extend f_j to the least quadruple (by code) not yet in the domain of $f_{j,s}$. Suppose that this quadruple is (a, b, u, v): Define $f_{j,s+1}(a, b, u, v) = m$ where m is new;
  (2) for every already defined value $f_{j,s}(a', b', u', v') = m'$,
    (a) if $m' \in W_{u',s}$ then let $m' \; R^{s+1} \; b'$;
    (b) if $m' \in W_{v',s}$ then $m' \; R^{s+1} \; a'$.
  (3) End the stage, initialize all lower-priority requirements, and go to next stage.

Let $R = \bigcup_s R^s$.

Verification. The verification is based on the following lemmas.

Lemma 6.3. *If N is eventually never initialized, then N is satisfied.*

Proof. The claim is by induction on the priority rank of N. Suppose that the claim is true of every $N' < N$. If $N = P_j$ then clearly $a_j = \lim_s a_{j,s}$ and $b_j = \lim_s b_{j,s}$ exist, and if $[a_j]_R \cap [b]_R = \emptyset$, and $\varphi_j(a, b, u, v)$ is defined and equal to m (where u, v are indices of the R-equivalence classes of a and b, respectively), then $m \in W_u \cup W_v$, thus $\varphi_j(a, b, \_, \_)$ does not witness effective inseparability of $[a_j]_R$ and $[b_j]_R$. If $N = Q_j$ then Q_j, after its last initialization, is eventually able to build its own function f_j, which is total if there are infinitely many $\langle i, j\rangle$-expansionary stages. □

Lemma 6.4. *R is not trivial.*

Proof. Let a_0 and b_0 be the final witnesses of P_0. Then $[a_0]_R \cap [b_0]_R = \emptyset$, since by initialization no lower-priority requirements can collapse $[a_0]_R$ and $[b_0]_R$, whereas P_0 acts at most once. □

Lemma 6.5. *If $i \notin S$ then R is u.e.i.; otherwise R is not u.e.i.*

Proof. If $i \in S$ and j is the least number for which there exist infinitely many $\langle i, j\rangle$-expansionary stages, then f_j (the function built by Q_j after its last initialization) is the desired uniform effective inseparability function.

If there is no $\langle i, j\rangle$ with infinitely many expansionary stages, then all strategies in the construction are finitary, so every requirement is eventually not initialized, and by Lemma 6.3 above, every P_j is satisfied. □

This completes the proof of the theorem. □

We conclude with the following question, for which Theorem 4.19 seems to suggest an affirmative answer:

Question 3. *Is $\{x : R_x$ is e.i.$\}$ a Π^0_4-complete set?*

Note that by the proof of Theorem 6.2, $\{x : R_x$ is e.i.$\}$ is Σ^0_3-hard.

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