Rudimentary and Arithmetical Constructive Set Theory

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1 Introduction

This paper is part of ongoing research to develop constructive mathematics in the conceptual framework of constructive set theory (CST). The aim is to highlight the various formal systems for CST, as weak as seems appropriate for the subject matter, in which significant mathematical topics can be developed.

Some Weak Axiom Systems for CST

The CST conceptual framework is a set theoretical approach to constructive mathematics initiated by Myhill in [Myh75]. It has been given a philosophical foundation via formal interpretations into versions of Martin-Löf's Intuitionistic Type Theory, [GA06, Acz86, Acz82, Acz78]. There are several axiom systems for Constructive Set Theory of varying logical strength. Perhaps the most familiar ones are **CZF** and $\mathbf{CZF}^+ \equiv \mathbf{CZF} + \mathbf{REA}$, see [AR01]. The axiom system **CZF** is formulated in the first order language $\mathcal{L}_\in$ for intuitionistic logic with equality having $\in$, an infix binary relation symbol, as the only non-logical symbol. So the logical symbols are $\bot, \wedge, \vee, \rightarrow, \forall, \exists, =$. We use the standard abbreviations for $\leftrightarrow$, $\neg$ and the bounded quantifiers $(\forall x \in t)$ and $(\exists x \in t)$. A formula is bounded if all its quantifiers are bounded.

We assume a standard axiom system for intuitionistic logic with equality. The non-logical axioms and schemes of **CZF** are the axioms of Extensionality, Emptyset, Union, Pairing and Infinity and the axiom schemes of Bounded

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Separation, Strong Collection, Subset Collection and Set Induction. The axiom system **CZF** is much weaker than **ZF**. Nevertheless when the law of excluded middle is added the resulting axiom system has the same theorems as **ZF**. Moreover when the powerset axiom and the full Separation scheme are added an axiom system is obtained that has the same theorems as **IZF**, an axiom system that has the same logical strength as **ZF** in virtue of a double negation interpretation of **ZF** into **IZF** due to Harvey Friedman.

The main aim of this paper is to formulate and study a weak axiom system for Arithmetical CST, **ACST**, that is strong enough to represent the class Nat of von Neumann natural numbers and its arithmetic so that Heyting Arithmetic can be interpreted. A significant feature of **CZF** is the role of, possibly infinitary, class inductive definitions that define classes that may not be sets. We will see a similar role for finitary inductive definitions in **ACST**.

A first approach to an axiom system for Arithmetical CST is the axiom system **BCST**$+MathInd(Nat)$. Here (i) the axiom system **BCST** for a basic CST is obtained by leaving out from **CZF** the axiom of Infinity and the axiom schemes of Strong Collection, Subset Collection and Set Induction, while adding the axiom scheme of Replacement, and (ii) $MathInd(Nat)$ is the axiom scheme of mathematical induction for a suitably defined class Nat of the von Neumann natural numbers. The axiom system **BCST**$+MathInd(Nat)$ does not assume that Nat is a set. An alternative basic axiom system for arithmetic that has been considered is **ECST**, which is obtained from **BCST** by adding the axiom of Strong Infinity, the axiom that expresses the existence of the smallest inductive set, ω. In contrast to **BCST**$+MathInd(Nat)$ the axiom system **ECST** does not have full mathematical induction, but can only derive mathematical induction for bounded formulae.

The Union Axiom and the Replacement Scheme can be combined into a single scheme, the Union Replacement Scheme. The full strength of the Union Replacement Scheme seems not to be needed for our purposes. It turns out that a rule of inference, the Global Union Replacement Rule (**GURR**) can be used instead and we will see that this rule provides exactly enough power to enable definitions of the rudimentary functions on sets. The rudimentary functions were originally introduced by Ronald Jensen, see [Jen72], in the context of classical set theory, in order to develop a good fine structure theory for Goedel's constructible sets.

So we are led to consider the very weak axiom system, **RCST**, of Rudimentary CST. This axiom system has a standard system of axioms and rules for intuitionistic logic in the language $\mathcal{L}_\in$, the rule **GURR**, the axiom of extensionality and the set existence axioms, Emptyset, Binary Intersection and Pairing for the existence of the sets $\emptyset, x_1 \cap x_2, \{x_1, x_2\}$ respectively, for sets x_1, x_2. Then $\mathbf{ACST} \equiv \mathbf{RCST}_0 + MathInd(Nat)$ will be our preferred axiom system for Arithmetical CST, where $\mathbf{RCST}_0$ is an axiom system that has the same theorems as **RCST**, but has the advantage that it does not use any non-logical rule of inference.

Although **RCST** is very weak it is strong enough to allow the derivation of every instance of the Bounded Separation Scheme. Also each rudimentary function is a total function $V^n \to V$ on the universe of sets which can be defined by a bounded formula $\phi[x_1, \ldots, x_n, y]$ such that

$$\mathbf{RCST} \vdash (\forall x_1, \ldots, x_n)(\exists! y)\phi[x_1, \ldots, x_n, y].$$

So each rudimentary function can be given in **RCST** by a provably total single valued class relation. It is natural to extend the language $\mathcal{L}_\in$ to a language $\mathcal{L}^*_\in$ with individual terms to represent the rudimentary functions. We are led to a simple axiom system $\mathbf{RCST}^*$ in the language $\mathcal{L}^*_\in$ which no longer needs the rule **GURR** and just has the non-logical axioms of extensionality and the term comprehension axioms for each form of term that is not a variable. We show that $\mathbf{RCST}^*$ is a conservative extension of **RCST** and we could use $\mathbf{ACST}^* \equiv \mathbf{RCST}^* + MathInd(Nat)$ as our axiom system for Arithmetical CST. As $\mathbf{ACST}^*$ is a conservative extension of **ACST** we could just as well use **ACST**. As **ACST** is in the standard language $\mathcal{L}_\in$ for set theory it is our preferred axiom system for arithmetical CST.

Outline of paper

The paper is in two parts. Sections 2-7 form Part I on Rudimentary CST and sections 8-10 form Part II on Arithmetical CST. Jensen's classical definition of the rudimentary functions are reviewed in section 2 along with a classically equivalent definition that is appropriate for CST. The language $\mathcal{L}^*_\in$ and axiom system $\mathbf{RCST}^*$ are introduced in section 3 where it is shown how the rudimentary functions are exactly the functions that can be defined by a term in $\mathbf{RCST}^*$. In section 4 it is shown that each instance of Bounded Separation can be derived in $\mathbf{RCST}^*$. In section 5 it is shown that every bounded formula of $\mathcal{L}^*_\in$ is equivalent in $\mathbf{RCST}^*$ to a bounded formula of $\mathcal{L}_\in$. The special case when the bounded formula is $t[x_1, \ldots, x_n] = y$ yields that the graph of each rudimentary function can be defined in $\mathbf{RCST}^*$ by a bounded formula of $\mathcal{L}_\in$. Section 6 introduces the axiom system $\mathbf{RCST}_0$, a rather useful, but unnatural axiom system for Rudimentary CST formulated in the language $\mathcal{L}_\in$. It is shown that $\mathbf{RCST}^*$ is a conservative extension of $\mathbf{RCST}_0$. The axiom system **RCST** is introduced in section 7 and shown to have the same theorems as $\mathbf{RCST}_0$ using a result, the Term Existence Theorem for $\mathbf{RCST}^*$, whose proof has been left for another occasion.

The axiom system $\mathbf{ACST} = \mathbf{RCST}_0 + MathInd(Nat)$ for Arithmetical CST is introduced in section 8 and the Finite AC Theorem is proved in section 9 with Finitary Strong Collection derived as a corollary. The theory of finitary inductive definitions of classes is developed in section 10.

In section 11 we compare various axiom systems for finite set theory with weak axiom systems for set theories which have an axiom of Infinity. We have placed in the appendix some definitions concerning the concept of an interpretation that are used in section 11.

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Part I: Rudimentary CST

2 The Rudimentary Functions on Sets

The Rudimentary functions on sets were introduced by Ronald Jensen in his famous paper [Jen72].

The definition makes sense in any sufficiently strong axiom system for set theory. The rudimentary functions are total functions defined on the class V of all sets.

Definition: 2.1 (Ronald Jensen (1972)) *A total function f on V is* rudimentary (à la Jensen) *if it is generated using the following schemata, where $\mathbf{x} = x_1, \ldots, x_n$ is a list of n distinct variables and $1 \leq i, j \leq n$.*

(a) $f(\mathbf{x}) = x_i$,

(b) $f(\mathbf{x}) = x_i - x_j$[1],

(c) $f(\mathbf{x}) = \{x_i, x_j\}$,

(d) $f(\mathbf{x}) = h(g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))$,

(e) $f(y, \mathbf{x}) = \cup_{z \in y} g(z, \mathbf{x})$,

where $h : V^m \to V$, $g_1, \ldots, g_m : V^n \to V$ and $g : V^{n+1} \to V$ are rudimentary.

Note that $f(\mathbf{x}) = \emptyset = x_i - x_i$ is rudimentary; and so is $f(\mathbf{x}) = x_i \cap x_j = x_i - (x_i - x_j)$ using classical logic. It follows that every rudimentary (à la Jensen) function is rudimentary (à la CST), where the rudimentary (à la CST) functions are generated using the schemata (**a**), (**b1**), (**b2**), (**c**), (**d**), (**e**), where

(b1) $f(\mathbf{x}) = \emptyset$,

(b2) $f(\mathbf{x}) = x_i \cap x_j$.

Conversely, the function $f(\mathbf{x}) = x_i - x_j$ is rudimentary (à la CST). To see this observe that $\{z\} = \{z, z\}$ and $x_i - x_j = \bigcup_{z \in x_i} \bigcup_{z' \in g(z, \mathbf{x})} \{z\}$, where

$$g(z, \mathbf{x}) = \{\{z\} \cap x_j\} \cap \{\emptyset\}.$$

It follows that the rudimentary (à la CST) functions are the same as the rudimentary (à la Jensen) functions in classical set theory, but it seems not

[1] $x - y = \{z \in x \mid z \notin y\}$

in constructive set theory. So, in the constructive context we prefer the definition for the rudimentary (à la CST) functions; i.e. using the schemes (**b1**), (**b2**) instead of the scheme (**b**). We can avoid scheme **(d)** by building function composition into the other schemes, except for **(a)**. So we are led to our 'official' definition of the rudimentary functions that is appropriate for constructive set theory and coincides with Jensen's definition in classical set theory.

Definition: 2.2 *A total function $f : V^n \to V$ is* rudimentary *if it is generated using the following schemata, where $\mathbf{x} = x_1, \ldots, x_n$ is a list of distinct variables and $1 \leq i \leq n$.*

1. $f(\mathbf{x}) = x_i$,
2. $f(\mathbf{x}) = \emptyset$,
3. $f(\mathbf{x}) = f_1(\mathbf{x}) \cap f_2(\mathbf{x})$,
4. $f(\mathbf{x}) = \{f_1(\mathbf{x}), f_2(\mathbf{x})\}$,
5. $f(\mathbf{x}) = \bigcup_{z \in f_1(\mathbf{x})} g(z, \mathbf{x})$

where $f_1, f_2 : V^n \to V$, and $g : V^{n+1} \to V$ are rudimentary.

It is routine, following the inductive generation of a rudimentary function h to show the following result.

Proposition: 2.3 *If $h : V^m \to V$ and $g_1, \ldots, g_m : V^n \to V$ are rudimentary functions then so is $f : V^n \to V$, where, for $\mathbf{x} \in V^n$,*

$$f(\mathbf{x}) = h(g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})).$$

It follows that adding scheme **(d)** to the above schemes would not change the functions that can be generated and so, classically, our official definition of rudimentary function generates the same functions as the original Jensen definition.

3 The axiom system RCST*

The standard axiom system **CZF** for CST is formulated in the first order language $\mathcal{L}_\in$ for intuitionistic logic, with equality, having $\in$, an infix binary relation symbol, as the only non-logical symbol. So the logical symbols are $=, \bot, \wedge, \vee, \to, \forall, \exists$. We use the standard abbreviations for $\leftrightarrow$ and $\neg$. We assume a standard axiom system for intuitionistic logic.

In order to formulate **RCST*** we need to extend the language $\mathcal{L}_\in$ by allowing individual terms for the sets asserted to exist. We obtain $\mathcal{L}_\in^*$ by adding to $\mathcal{L}_\in$ the individual terms of $\mathcal{T}^*$, where the terms $t \in \mathcal{T}^*$ are inductively generated using the following syntax equation.

$$t ::= z \mid \emptyset \mid \{t_1, t_2\} \mid t_1 \cap t_2 \mid \cup_{z \in t_1} t_2[z]$$

Note that z here represents a variable and free occurences of z in $t_2[z]$ become bound in the term $\bigcup_{z \in t_1} t_2[z]$. We use the following convention for displaying variables in a term. If we write a term as $t[z]$ then this indicates that z may occur free in the term, and if t' is also a term then we write $t[t']$ for the result of substituting t' for the free occurences of z, relabelling bound variables when necessary to avoid variable capture. We use such a convention when displaying several distinct variables in a term. We carry over this convention for displaying variables in terms to displaying variables in formulae and substituting terms for the displayed variables in formulae.

We use the standard abbreviations for the bounded quantifiers; i.e. $(\forall x \in t)\phi[x]$ abbreviates $\forall x(x \in t \to \phi[x])$ and $(\exists x \in t)\phi[x]$ abbreviates $\exists x(x \in t \wedge \phi[x])$. A formula is *bounded* if every occurence of a quantifier in the formula is bounded.

The axiom system **RCST*** has, as non-logical axioms, the following Extensionality Axiom and Term Comprehension Axioms.

Extensionality Axiom: $\quad x = y \to (x \in z \to y \in z)$,

Term Comprehension Axioms:

$$\begin{array}{lll} A1) & u \in \emptyset & \leftrightarrow \ \bot \\ A2) & u \in t_1 \cap t_2 & \leftrightarrow \ (u \in t_1 \wedge u \in t_2) \\ A3) & u \in \{t_1, t_2\} & \leftrightarrow \ (u = t_1 \vee u = t_2) \\ A4) & u \in \cup_{z \in t_1} t_2[z] & \leftrightarrow \ (\exists z \in t_1)\ (u \in t_2[z]) \end{array}$$

Each term t whose free variables are taken from the list $\mathbf{x} = x_1, \ldots, x_n$ of distinct variables defines in an obvious way, an n-place rudimentary function $F_t^{\mathbf{x}}$ on sets where, for $\mathbf{a} = (a_1, \ldots, a_n) \in V^n$, $F_t^{\mathbf{x}}(\mathbf{a})$ is given by the following table.

t	$F_t^{\mathbf{x}}(\mathbf{a})$
x_i	a_i
$\emptyset$	$\emptyset$
$t_1 \cap t_2$	$F_{t_1}^{\mathbf{x}}(\mathbf{a}) \cap F_{t_1}^{\mathbf{x}}(\mathbf{a})$
$\{t_1, t_2\}$	$\{F_{t_1}^{\mathbf{x}}(\mathbf{a}), F_{t_1}^{\mathbf{x}}(\mathbf{a})\}$
$\bigcup_{z \in t_1} t_2[z]$	$\bigcup_{v \in F_{t_1}^{\mathbf{x}}(\mathbf{a})} F_{t_2[z]}^{\mathbf{x}}(v, \mathbf{a})$

Proposition: 3.1 *An n-place function f on sets is rudimentary iff $f = F_t^{\mathbf{x}}$ for some term t of $\mathcal{L}_\in^*$ whose free variables are taken from the list $\mathbf{x} = x_1, \ldots, x_n$ of distinct variables.*

Proof: Routine.
■

4 Bounded Separation in RCST*

We wish to show that for each bounded formula $\theta[x]$ of $\mathcal{L}_\in^*$ and each term t of $\mathcal{L}_\in^*$ in which x does not occur free there is a term t' in which x is not free such that

$$\mathbf{RCST}^* \vdash [x \in t' \ \leftrightarrow \ (x \in t \wedge \theta[x])].$$

The following definitions introduce notations for terms that behave in the expected way. Let t, t_1, t_2 be terms.

$$\{t\} \equiv \{t, t\}, \qquad (t_1, t_2) \equiv \{\{t_1\}, \{t_1, t_2\}\}, \qquad \cup t \equiv \textstyle\bigcup_{x \in t} x,$$

$$t_1 \cup t_2 \equiv \cup\{t_1, t_2\}, \quad t_1 \times t_2 \equiv \textstyle\bigcup_{x_1 \in t_1} \bigcup_{x_2 \in t_2} \{(x_1, x_2)\}, \quad t_1 \delta t_2 \equiv \{t_1\} \cap \{t_2\}$$

Note that

$$x \in t_1 \delta t_2 \quad \leftrightarrow \quad [x = t_1 = t_2].$$

Definition: 4.1 *For each term t let $(\exists \in t)$ be the formula $(\exists x \in t)(x = x)$, where x is chosen not free in t.*

If t_1 and $t_2[x]$ are terms, with x not free in t_1, let

$$[x \in t_1 \mid \exists \in t_2[x]] \quad \equiv \quad \bigcup_{x \in t_1} \bigcup_{z \in t_2[x]} \{x\}$$

and let

$$\Delta_{x \in t_1} t_2[x] \quad \equiv \quad [x \in t_1 \mid \exists \in t_2[x]] \delta t_1.$$

Observe that

$$y \in [x \in t_1 \mid \exists \in t_2[x]] \quad \leftrightarrow \quad y \in t_1 \wedge (\exists \in t_2[y])$$

so that,

$$(\exists \in \Delta_{x \in t_1} t_2[x]) \quad \leftrightarrow \quad (\forall x \in t_1)(\exists \in t_2[x]).$$

For terms t_1, t_2, with x not free in t_2 let

$$t_1 \to t_2 \quad \equiv \quad \Delta_{x \in t_1} t_2.$$

Then

$$(\exists \in t_1 \to t_2) \quad \leftrightarrow \quad (\exists \in t_1) \to (\exists \in t_2).$$

By structural recursion on the bounded formula θ of $\mathcal{L}^*_\in$ we may associate with θ a term t_θ using the following table.

θ	t_θ
$\bot$	$\emptyset$
$t_1 = t_2$	$t_1 \delta t_2$
$t_1 \in t_2$	$(\{t_1\} \cup t_2)\delta t_2$
$\theta_1 \vee \theta_2$	$t_{\theta_1} \cup t_{\theta_2}$
$\theta_1 \wedge \theta_2$	$t_{\theta_1} \times t_{\theta_2}$
$\theta_1 \to \theta_2$	$t_{\theta_1} \to t_{\theta_2}$
$(\exists x \in t_0)\theta_0[x]$	$\bigcup_{x \in t_0} t_{\theta_0[x]}$
$(\forall x \in t_0)\theta_0[x]$	$\Delta_{x \in t_0} t_{\theta_0[x]}$

Proposition: 4.2 *For each bounded formula θ of $\mathcal{L}^*_\in$*

$$\mathbf{RCST}^* \vdash [\theta \quad \leftrightarrow \quad (\exists \in t_\theta)].$$

Proof: By a routine structural induction on the bounded formula θ.
■

Corollary: 4.3 *Given a bounded formula $\theta[x]$ and a term t of $\mathcal{L}^*_\in$ in which x is not free*

$$\mathbf{RCST}^* \vdash x \in [x \in t \mid \ \exists \in t_{\theta[x]}] \quad \leftrightarrow \quad (x \in t \wedge \theta[x])$$

so that $\mathbf{RCST}^* \vdash (\exists y)[x \in y \quad \leftrightarrow \quad (x \in t \wedge \theta[x])]$.

5 A metatheorem

We wish to show that for each bounded formula θ of $\mathcal{L}^*_\in$ there is a bounded formula θ' of $\mathcal{L}_\in$ such that

$$\mathbf{RCST}^* \vdash [\theta \quad \leftrightarrow \quad \theta'].$$

We use a method of proof taken from [Jen72].

Definition: 5.1 *Let t be a term of $\mathcal{L}_{\in}^{*}$.*

1. *The term t is* simple *if, for every bounded formula $\theta[x]$ of $\mathcal{L}_{\in}$ there is a bounded formula θ' of $\mathcal{L}_{\in}$ such that*

$$\mathbf{RCST}^{*} \vdash [\theta[t] \quad \leftrightarrow \quad \theta'].$$

2. *The term t is* almost simple *if, for each bounded formula $\theta[x]$ of $\mathcal{L}_{\in}$ there are bounded formulae θ_1, θ_2 of $\mathcal{L}_{\in}$ such that*

$$\mathbf{RCST}^{*} \vdash [(\forall x \in t)\theta[x] \quad \leftrightarrow \quad \theta_1] \textit{ and}$$
$$\mathbf{RCST}^{*} \vdash [(\exists x \in t)\theta[x] \quad \leftrightarrow \quad \theta_2.]$$

Proposition: 5.2 *Every almost simple term is simple.*

Proof: First observe that, given a bounded formula θ of $\mathcal{L}_{\in}$ we may find a bounded formula θ' of $\mathcal{L}_{\in}$ that has the following property and can be proved equivalent to θ using intuitionistic logic with equality and the extensionality axiom. The required property of θ' is that every variable x that occurs free in θ' only occurs free as a bound in a bounded quantifier $(\forall y \in x)$ or $(\exists y \in x)$. To see this it suffices to notice that each atomic subformula $x \in y$ of θ can be replaced by the equivalent bounded formula $(\exists z \in y)(x = z)$ and then each atomic subformula $x = y$ can be replaced by the equivalent bounded formula

$$(\forall x_1 \in x)(\exists y_1 \in y)(x_1 = y_1) \ \wedge \ (\forall y_1 \in y)(\exists x_1 \in x)(y_1 = x_1).$$

Now let t be an almost simple term. We must show that, given a bounded formula $\theta[x]$ of $\mathcal{L}_{\in}$, there is a bounded formula θ' of $\mathcal{L}_{\in}$ such that

$$\mathbf{RCST}^{*} \vdash [\theta[t] \quad \leftrightarrow \quad \theta'].$$

By the above observation we may assume that $\theta[x]$ has the property that each occurence of x is the bound of a bounded quantifier. We can now easily show, by structural induction on the subformula, that for each subformula $\psi[x]$ of $\theta[x]$ there is a formula ψ' of $\mathcal{L}_{\in}$ such that

$$\mathbf{RCST}^{*} \vdash [\psi[t] \quad \leftrightarrow \quad \psi'].$$

using the assumption on t for the bounded quantifier induction steps where the bound is x.
■

Proposition: 5.3 *Every term is almost simple.*

Proof:

We show, by structural induction on terms t that t is almost simple. To see this it suffices to examine the following table where, in the cases of the terms $t_1 \cap t_2$ and $\{t_1, t_2\}$ the previous proposition is used.

t	$(\forall x \in t)\theta[x]$	$(\exists x \in t)\theta[x]$
z	$(\forall x \in z)\theta[x]$	$(\exists x \in z)\theta[x]$
$\emptyset$	$\bot \to \bot$	$\bot$
$t_1 \cap t_2$	$(\forall x \in t_1)(x \in t_2 \to \theta[x])$	$(\exists x \in t_1)(x \in t_2 \wedge \theta[x])$
$\{t_1, t_2\}$	$\theta[t_1] \wedge \theta[t_2]$	$\theta[t_1] \vee \theta[t_2]$
$\bigcup_{z \in t_1} t_2[z]$	$(\forall z \in t_1)(\forall x \in t_2[z])\theta[x]$	$(\exists z \in t_1)(\exists x \in t_2[z])\theta[x]$

■

Theorem: 5.4 *For every bounded formula ϕ of $\mathcal{L}^*_\in$ there is a bounded formula ϕ' of $\mathcal{L}_\in$ such that*

$$\mathbf{RCST}^* \vdash [\phi \;\leftrightarrow\; \phi'].$$

Proof: As in the proof of Proposition 5.2 we may assume that each occurence of a term t in ϕ that is not a variable is a bound in a bounded quantifier $(\forall x \in t)$ or $(\exists x \in t)$. So it is straightforward to prove the result for each subformula of ϕ by structural induction. We just consider the induction step for a subformula, ψ, of the form $(\forall x \in t)\theta[x]$. The formula $\theta[x]$ is a bounded formula of $\mathcal{L}^*_\in$ and, by the induction hypothesis, there is a bounded formula $\theta'[x]$ of $\mathcal{L}_\in$ such that $\mathbf{RCST}^* \vdash (\theta[x] \;\leftrightarrow\; \theta'[x])$. As each term is almost simple there is a bounded formula ψ' of $\mathcal{L}_\in$ such that $\mathbf{RCST}^* \vdash ((\forall x \in t)\theta'[x] \;\leftrightarrow\; \psi')$. It follows that $\mathbf{RCST}^* \vdash (\psi \;\leftrightarrow\; \psi')$.
■

6 The Axiom System $\mathbf{RCST}_0$

The axiom system $\mathbf{RCST}^*$ is in the language $\mathcal{L}^*_\in$ having terms for the rudimentary functions. We wish to formulate an axiom system for the rudimentary functions in the basic set theoretic language $\mathcal{L}_\in$ that has $\mathbf{RCST}^*$ as a conservative extension. In order to do so it will be convenient to summarize the Term Comprehension Axioms of $\mathbf{RCST}^*$ as the formulae of the form

$$u \in t \ \leftrightarrow \ \theta_t[u],$$

for each non-variable term t where, for each term t the formula $\theta_t[u]$ is given by the following table.

t	$\theta_t[u]$
z	$u \in z$
$\emptyset$	$\bot$
$t_1 \cap t_2$	$u \in t_1 \wedge u \in t_2$
$\{t_1, t_2\}$	$u = t_1 \vee u = t_2$
$\bigcup_{z \in t_1} t_2[z]$	$(\exists z \in t_1)(u \in t_2[z])$

By structural recursion on the terms of $\mathcal{L}^*_\in$ we define a formula $\phi_t[u]$ of $\mathcal{L}_\in$ for each term t using the following table.

t	$\phi_t[u]$
z	$u \in z$
$\emptyset$	$\bot$
$t_1 \cap t_2$	$\phi_{t_1}[u] \wedge \phi_{t_2}[u]$
$\{t_1, t_2\}$	$\psi_{t_1}[u] \vee \psi_{t_2}[u]$
$\bigcup_{z \in t_1} t_2[z]$	$\exists z(\phi_{t_1}[z] \wedge \phi_{t_2[z]}[u])$

where, for each term t, $\psi_t[y]$ is defined to be the formula

$$\forall u(u \in y \ \leftrightarrow \ \phi_t[u])$$

of $\mathcal{L}_\in$.

Definition: 6.1 *Let* **EXT** *be the axiom system formulated in the language* $\mathcal{L}_\in$ *having the logical axioms and rules of a standard axiomatisation of intuitionistic logic with equality and the single non-logical axiom of Extensionality. Let* $\mathbf{RCST}_0$ *be obtained from* **EXT** *by adding a non-logical axiom* $\exists y\ \psi_t[y]$ *for each term* t *of* $\mathcal{L}^*_\in$.

Lemma: 6.2 *For all terms* t *of* $\mathcal{L}^*_\in$ *and variables* u, y *not free in* t *the following are theorems of* $\mathbf{RCST}^*$.

1. $\phi_t[u] \leftrightarrow u \in t$

2. $\psi_t[y] \leftrightarrow y = t$

3. $\exists y \psi_t[y]$

Proof: The derivation in $\mathbf{RCST}^*$ of 1 for each term t is obtained by an easy structural induction on t. 2 follows immediately from 1 and 3 follows immediately from 2.
■

Proposition: 6.3 *Every theorem of* $\mathbf{RCST}_0$ *is a theorem of* $\mathbf{RCST}^*$.

Proof: This is an immediate consequence of the previous lemma.
■

We wish to give an interpretation of $\mathbf{RCST}^*$ in $\mathbf{RCST}_0$. For each formula ϕ of $\mathcal{L}^*_\in$ let $\phi^\sharp$ be the formula of $\mathcal{L}_\in$ obtained from ϕ by replacing each atomic subformula $t_1 \in t_2$ by $\exists y(\psi_{t_1}[y] \wedge \phi_{t_2}[y])$, where y is chosen to be a variable that is not free in either t_1 or t_2.

Proposition: 6.4 *For each formula* ϕ *of* $\mathcal{L}_\in$, $\mathbf{EXT} \vdash (\phi \leftrightarrow \phi^\sharp)$.

Lemma: 6.5 *For each term* t *of* $\mathcal{L}^*_\in$ *and variable* u *not free in* t

1. $\mathbf{EXT} \vdash (u \in t)^\sharp \leftrightarrow \phi_t[u]$,

2. $\mathbf{EXT} \vdash (u = t)^\sharp \leftrightarrow \psi_t[u]$ *and*

3. $\mathbf{EXT} \vdash (\theta_t[u])^\sharp \leftrightarrow \phi_t[u]$.

Proof: We work informally in **EXT**.

1. It suffices to observe that

$$\begin{aligned} (u \in t)^\sharp &\leftrightarrow \exists y(u = y \wedge \phi_t[y]) \\ &\leftrightarrow \phi_t[u]. \end{aligned}$$

2.

$$\begin{aligned} (u = t)^\sharp &\equiv (\forall x \in u)(x \in t)^\sharp \wedge \forall x((x \in t)^\sharp \rightarrow x \in u) \\ &\leftrightarrow (\forall x \in u)\phi_t[x] \wedge \forall x(\phi_t[x] \rightarrow x \in u) \\ &\leftrightarrow \forall x(x \in u \leftrightarrow \phi_t[x]) \\ &\equiv \psi_t[u]. \end{aligned}$$

3. We use structural induction on the term t.

$$
\begin{array}{lrcl}
\mathbf{t \equiv z}: & \theta_t[u]^\sharp & \equiv & (u \in z) \\
 & & \equiv & \phi_t[u] \\
\mathbf{t \equiv \bot}: & \theta_t[u]^\sharp & \equiv & \bot \\
 & & \equiv & \phi_t[u] \\
\mathbf{t \equiv t_1 \cap t_2}: & \theta_t[u]^\sharp & \equiv & (u \in t_1)^\sharp \wedge (u \in t_2)^\sharp \\
 & & \leftrightarrow & (\phi_{t_1}[u] \wedge \phi_{t_2}[u]) \\
 & & \equiv & \phi_t[u] \\
\mathbf{t \equiv \{t_1, t_2\}}: & \theta_t[u]^\sharp & \equiv & \psi_{t_1}[u]^\sharp \vee \psi_{t_2}[u]^\sharp \\
 & & \leftrightarrow & \phi_t[u] \\
\mathbf{t \equiv \bigcup_{z \in t_1} t_2[z]}: & \theta_t[u]^\sharp & \leftrightarrow & \exists z(\phi_{t_1}[z] \wedge \phi_{t_2[z]}[u]) \\
 & & \equiv & \phi_t[u]
\end{array}
$$

■

We show that $\sharp$ translates $\mathbf{RCST}^*$ into $\mathbf{RCST}_0$. To deal with substitution we need the following result.

Lemma: 6.6 *For all formulae $\phi[x]$ and terms s of $\mathcal{L}^*_\in$ the following formulae are theorems of $\mathbf{RCST}_0$.*

1. $(\forall x \phi[x] \to \phi[s])^\sharp$
2. $(\phi[s] \to \exists x \phi[x])^\sharp$

Proof: Observe that it suffices to show that

$$(*) \quad \mathbf{RCST}_0 \vdash \exists x(\phi[s]^\sharp \leftrightarrow \phi[x]^\sharp)$$

so that both $\mathbf{RCST}_0 \vdash \forall x \phi[x]^\sharp \to \phi[s]^\sharp$ and $\mathbf{RCST}_0 \vdash \phi[s]^\sharp \to \exists x \phi[x]^\sharp$; i.e. 1 and 2.

We prove (*) by structural induction on the formula $\phi[x]$. The induction steps are easy so that we are left with the base case when $\phi[x]$ is an atomic formula $t_1[x] \in t_2[x]$. In that case it suffices to show that

$$\mathbf{EXT} \vdash \psi_s[x] \to ((t_1[s] \in t_2[s])^\sharp \leftrightarrow (t_1[x] \in t_2[x])^\sharp).$$

As $(t_1[x] \in t_2[x])^\sharp)$ is

$$\exists y(\psi_{t_1[x]}[y] \wedge \phi_{t_2[x]}[y])$$

and $(t_1[s] \in t_2[s])^\sharp)$ is

$$\exists y(\psi_{t_1[s]}[y] \wedge \phi_{t_2[s]}[y])$$

it suffices to show that, for each term t of $\mathcal{L}^*_\in$ the following formulae are theorems of $\mathbf{EXT}$.

1. $\psi_s[x] \to (\phi_{t[x]}[y] \leftrightarrow \phi_{t[s]}[y])$
2. $\psi_s[x] \to (\psi_{t[x]}[y] \leftrightarrow \psi_{t[s]}[y])$

For each term $t[x]$ 2 is an easy consequence of 1. We prove 1 by structural induction on $t[x]$. For the base case $t[x]$ is a variable or is $\emptyset$. If it is a variable z distinct from x then 1 is $\psi_s[x] \to (y \in z \leftrightarrow y \in z)$ and if it is $\emptyset$ then 1 is $\psi_s[x] \to (\bot \to \bot)$. If it is the variable x then 1 is $\psi_s[x] \to (y \in x \leftrightarrow y \in s)$. In all three cases 1 is a theorem of **EXT**. If $t[x]$ is $\{t_1[x], t_2[x]\}$ then, assuming $\psi_s[x]$, using the induction hypotheses on $t_1[x]$ and $t_2[x]$ and the fact that 1 implies 2, we get the following in **EXT**.

$$\begin{aligned} \phi_{t[s]}[y] &\equiv \psi_{t_1[s]}[y] \vee \psi_{t_2[s]}[y] \\ &\leftrightarrow \psi_{t_1[x]}[y] \vee \psi_{t_2[x]}[y] \\ &\leftrightarrow \phi_{t[x]}[y] \end{aligned}$$

The other induction steps are similar.
■

Theorem: 6.7 *For all formulae ϕ of $\mathcal{L}^*_\in$*

$$\mathbf{RCST}^* \vdash \phi \quad \textit{implies} \quad \mathbf{RCST}_0 \vdash \phi^\sharp.$$

Proof: As the sharp translation preserves the logical operations and Lemma 6.6 takes care of the logical axioms dealing with substitution the translation preserves all the logical axioms and rules of inference. If θ is a non-logical axiom of $\mathbf{RCST}^*$ then it is either the extensionality axiom or else it is one of the term comprehension axioms. If θ is the extensionality axiom then it is a formula of $\mathcal{L}_\in$ so that, by Proposition 6.4, $\mathbf{EXT} \vdash \theta^\sharp \leftrightarrow \theta$ and hence $\mathbf{RCST}_0 \vdash \theta^\sharp$. If θ is the term comprehension axiom $u \in t \leftrightarrow \theta_t[u]$ then, by Lemma 6.5,

$$\mathbf{RCST}_0 \vdash \theta^\sharp \leftrightarrow (\phi_t[u] \leftrightarrow \phi_t[u]).$$

so that $\mathbf{RCST}_0 \vdash \theta^\sharp$.
■

Corollary: 6.8 $\mathbf{RCST}^*$ *is a conservative extension of* $\mathbf{RCST}_0$.

Proof: By propositions 6.3 and 6.4 and Theorem 6.7.
■

7 The Axiom System RCST

The axiom system $\mathbf{RCST}_0$ is not a particularly natural one. By using a non-logical rule of inference we can obtain a more natural one, **RCST**, that has the same theorems as $\mathbf{RCST}_0$. For the non-logical axioms of **RCST** we need the usual extensionality axiom and set existence axioms corresponding to the different forms of term. For the forms of term for $\emptyset, \{t_1, t_2\}, t_1 \cap t_2$ we have the following axioms.

Emptyset $(\exists y)(\forall z)(z \in y \ \leftrightarrow \ \bot)$,

Pairing $(\exists y)(\forall z)(z \in y \ \leftrightarrow \ (z = x_1 \vee z = x_2))$,

Binary Intersection $(\exists y)(\forall z)(z \in y \ \leftrightarrow \ (z \in x_1 \wedge z \in x_2))$.

Dealing with the form of term $\bigcup_{z \in t_1} t_2[z]$ is not so easy. A first thought is to add the following Union-Replacement scheme.

Union-Replacement Scheme: For each formula $\phi[u, v]$,

$$\forall x[(\forall u \in x)(\exists! v)\phi[u, v] \ \rightarrow \ \exists y \phi'[x, y]],$$

where $\phi'[x, y]$ is $\forall z(z \in y \ \leftrightarrow \ \exists v(z \in v \wedge (\exists u \in x)\phi[u, v])$. Adding this scheme is certainly strong enough to be able to define the graphs of all rudimentary functions. But it seem to be too strong for two reasons. Firstly, only a global version of the scheme seems to be needed, as each rudimentary function is totally defined on the universe of sets. So the following seemingly weaker scheme is strong enough.

Global Union-Replacement Scheme: For each formula $\phi[u, v]$,

$$(\forall u)(\exists! v)\phi[u, v] \ \rightarrow \ \forall x \exists y \phi'[x, y].$$

But even this scheme may be too strong, as the following seemingly weaker rule version seems strong enough.

Global Union-Replacement Rule (GURR): For each formula $\phi[u, v]$,

$$\frac{(\forall u)(\exists! v)\phi[u, v]}{\forall x \exists y \phi'[x, y]},$$

where $\phi'[x, y]$ was defined above. Note that in the schemes and the rule the formula $\phi[u, v]$ may have other variables than u, v occuring free. So, in both the premiss and the conclusion of the rule those additional free variables are implicitly universally quantified.

Definition: 7.1 *Let* **RCST** *be the axiom system formulated in the language* $\mathcal{L}_\in$ *having the logical axioms and rules of a standard axiomatisation of intuitionistic logic, the non-logical axioms of* **Extensionality, Emptyset, Pairing** *and* **Binary Intersection** *and the rule* **GURR** *for formulae* $\phi[u, v]$ *of* $\mathcal{L}_\in$.

Theorem: 7.2 RCST *has the same theorems as* $\mathbf{RCST}_0$ *so that* $\mathbf{RCST}^*$ *is a conservative extension of* **RCST**.

Proof: We first show that every theorem of $\mathbf{RCST}_0$ is a theorem of **RCST**. So we need to show that $\exists y \psi_t[y]$ is a theorem of **RCST** for each term t of $\mathcal{L}^*_\in$. We do this by structural induction on the term t. The base cases when t is a variable or when t is $\emptyset$ are trivial, using the emptyset axiom of **RCST** in the second case and the induction steps when t is one of $t_1 \cap t_2, \{t_1, t_2\}$ are easy, using the binary intersection and pairing axioms of **RCST**. The induction step when t is $\bigcup_{z \in t_1} t_2[z]$ uses **GURR**. By the induction hypotheses for t_1 and $t_2[z]$

1. $\exists x \psi_{t_1}[x]$,
2. $\forall z \exists u \psi_{t_2[z]}[u]$.

By **EXT** and 2, $\forall z \exists! u \psi_{t_2[z]}[u]$ so that, by **GURR**,

$$(*) \qquad \forall x \exists y \forall z' (z' \in y \;\leftrightarrow\; \xi[x, z'])$$

where

$$\xi[x, z'] \equiv \exists u (z' \in u \wedge (\exists z \in x) \psi_{t_2[z]}[u])$$

Assuming $\psi_{t_1}[x]$,

$$\begin{array}{rcl} \xi[x, z'] & \leftrightarrow & \exists z (\phi_{t_1}[z] \wedge \exists u (z' \in u \wedge \psi_{t_2[z]}[u])) \\ & \leftrightarrow & \exists z (\phi_{t_1}[z] \wedge \phi_{t_2[z]}[z']) \\ & \equiv & \phi_t[z'] \end{array}$$

So, by (*) $\exists y \forall z' (z' \in y \;\leftrightarrow\; \phi_t[z'])$; i.e. $\exists y \psi_t[y]$.

Thus we have proved $\exists y \psi_t[y]$ assuming $\psi_{t_1}[x]$. So, by 1, $\exists y \psi_t[y]$.

It remains to show that every theorem of **RCST** is a theorem of $\mathbf{RCST}_0$. This is not so easy and uses the following important fact about $\mathbf{RCST}^*$.

> **Theorem: 7.3 (The Term Existence Property for $\mathbf{RCST}^*$)** *If* $\mathbf{RCST}^* \vdash \exists v \phi[v]$ *then there is a term* t *of* $\mathcal{L}^*_\in$ *such that* $\mathbf{RCST}^* \vdash \phi[t]$.

The proof of this result uses the so called Friedman realizability, which was used by Myhill in [Myh73]. We leave the proof for another occasion.

The extensionality axiom and the emptyset, binary intersection and pairing axioms of **RCST** are easy to deal with using the extensionality axiom and the axioms $\exists y \psi_t[y]$ of $\mathbf{RCST}_0$ for t the terms $\emptyset$, $x_1 \cap x_2$ and $\{x_1, x_2\}$ respectively. So it only remains to show that **GURR** is an admissible rule of $\mathbf{RCST}_0$. Assume that $\mathbf{RCST}_0 \vdash \forall u \exists! v \phi[u, v]$. We must show that

$$\mathbf{RCST}_0 \vdash \forall x \exists y \phi'[x, y]$$

where $\phi'[x, y]$ is

$$\forall z(z \in y \ \leftrightarrow \ \exists v(z \in v \wedge (\exists u \in x)\phi[u, v])).$$

By Proposition 6.3, as $\mathbf{RCST}_0 \vdash \exists! v \phi[u, v]$

$$\mathbf{RCST}^* \vdash \exists! v \phi[u, v]$$

so that, by Theorem 7.3, there is a term $t[u]$ of $\mathcal{L}^*_\in$ such that $\mathbf{RCST}^* \vdash \phi[u, t[u]]$. In fact, by $(*)$,

$$\mathbf{RCST}^* \vdash (\phi[u, v] \ \leftrightarrow \ (v = t[u])$$

so that, by Theorem 6.7, Proposition 6.4 and part 2 of Lemma 6.5,

$$\mathbf{RCST}_0 \vdash (\phi[u, v] \ \leftrightarrow \ \psi_{t[u]}[v]).$$

Let $t'[x] \equiv \bigcup_{z \in x} t[z]$. Then

$$\phi_{t'[x]}[z] \equiv \exists u(\phi_x[u] \wedge \phi_{t[u]}[z]) \equiv (\exists u \in x)\phi_{t[u]}[z].$$

So, working in $\mathbf{RCST}_0$, as $\phi_{t[u]}[z] \ \leftrightarrow \ \exists v(z \in v \wedge \psi_{t[u]}[v])$,

$$\begin{aligned} \phi_{t'[x]}[z] \quad &\leftrightarrow \ (\exists u \in x)\exists v(z \in v \wedge \psi_{t[u]}[v]) \\ &\leftrightarrow \ \exists v(z \in v \wedge (\exists u \in x)\phi[u, v]). \end{aligned}$$

Hence

$$\begin{aligned} \psi_{t'[x]}[y] \quad &\equiv \forall z(z \in y \ \leftrightarrow \ \phi_{t'[x]}[z]) \\ &\leftrightarrow \ \forall z(z \in y \ \leftrightarrow \ \exists v(z \in v \wedge (\exists u \in x)\phi[u, v])) \\ &\equiv \phi'[x, y]. \end{aligned}$$

As $\forall x \exists y \psi_{t'[x]}[y]$ we get $\forall x \exists y \phi'[x, y]$, as desired.
■

Corollary: 7.4 (Definable Existence Property for RCST)
If $\mathbf{RCST} \vdash \exists x \phi[x]$ *then*

$$\mathbf{RCST} \vdash \exists! x \psi[x] \wedge \forall x(\psi[x] \rightarrow \phi[x])$$

for some bounded formula $\psi[x]$*.*

Proof: Use Theorems 7.2 and 7.3.
■

Part II: Arithmetical CST

We are concerned to investigate set theories in which there is a class of natural numbers which may not be a set. But how should the class be defined? Here we choose to use the class

$$Nat = \{\alpha \in On \mid \alpha^+ \subseteq \{0\} \cup \{\gamma^+ \mid \gamma \in On\}\}$$

where $0 = \emptyset$, $x^+ = x \cup \{x\}$ for any set x and On is the class of transitive sets of transitive sets; i.e. $On = \{x \mid (\forall y \in x^+) \ \cup y \subseteq y\}$. We can now define the class of finite sets as the class

$$Fin(Nat) = \{x \mid (\exists n \in Nat) \ n \sim x\},$$

where, for sets X, Y,

$$X \sim Y \equiv (\exists f : X \to Y) \ (f \text{ is a bijection}).$$

For each class X let $Ind(X) \equiv (0 \in X \wedge (\forall y \in X) \ y^+ \in X)$. We call a class X *inductive* if $Ind(X)$ holds.

The Mathematical Induction axiom scheme $MathInd(Nat)$**:**

For each class X,

$$Ind(X) \to Nat \subseteq X.$$

We start by formulating classical finite set theory.

The Finite Set Theories $\mathbf{ZF}^{fin}$, $\mathbf{CZF}^{fin}$ and $\mathbf{IZF}^{fin}$

Let $\mathbf{ZF}^{fin}$ be the axiom system **ZF** with the axiom of Foundation replaced by the Set Induction Scheme and the axiom of Infinity replaced by the axiom $V = Fin(Nat)$ that expresses that every set is finite.

Working informally in $\mathbf{ZF}^{fin}$ it is routine to show that Nat is inductive and derive each instance of $MathInd(Nat)$ so that Nat is the smallest inductive class. Also we can define the successor, S, addition, $+$, and multiplication, $\times$, on Nat in $\mathbf{ZF}^{fin}$ so that, in $\mathbf{ZF}^{fin}$, the class structure $(Nat, 0, S, +, \times)$ satisfies each axiom of **PA**. It follows that we get an interpretation[2] $nat : \mathbf{PA} \to \mathbf{ZF}^{fin}$, where $Dom^{nat}[x]$ is $x \in Nat$.

[2]See the Appendix for some definitions concerning the concept of an interpretation and [Vis06] for a more thorough treatment.

There is also the Ackerman interpretation[3] $ack : \mathbf{ZF}^{fin} \to \mathbf{PA}$ and it is presumably true that $(ack \circ nat) \sim id_{\mathcal{L}_{\mathbf{PA}}} : \mathbf{PA} \to \mathbf{PA}$, so that $nat : \mathbf{PA} \tilde{\to} \mathbf{T}$ for any subtheory $\mathbf{T}$ of $\mathbf{ZF}^{fin}$ such that $nat : \mathbf{PA} \to \mathbf{T}$.

Note that it is the Set Induction Scheme of $\mathbf{ZF}^{fin}$ that is used to derive each instance of $MathInd(Nat)$. The standard proof in **ZF** of each instance of Set Induction uses the result of **ZF** that each set has a transitive closure and that result makes essential use of the Infinity axiom. See [KW07].

As with $\mathbf{ZF}^{fin}$ we can define the constructive set theory $\mathbf{CZF}^{fin}$ to be obtained from **CZF** by leaving out the axiom of Infinity and replacing it with the axiom $V = Fin(Nat)$. We can define $\mathbf{IZF}^{fin}$ from **IZF** in the same way.

8 The Axiom System ACST

Arithmetical CST is Rudimentary CST together with the scheme $MathInd(Nat)$.

Definition: 8.1 $\mathbf{ACST} = \mathbf{RCST_0} + MathInd(Nat)$.

In this and the next two sections, unless otherwise indicated, we work informally in the axiom system **ACST**.

Finite Powers of classes and sets

For each class A, if $n \in Nat$ let ${}^nA$ be the class of functions $n \to A$.

Definition: 8.2 (Finite Powers Axiom (FPA)) *For each set A the class ${}^nA$ is a set for all $n \in Nat$.*

Proposition: 8.3 FPA *is a theorem of ACST.*

Proof: Let A be a set. Note that ${}^0A = \{\emptyset\}$ is a set and, for each $n \in Nat$, if ${}^nA$ is a set then so is ${}^{n^+}A$, as

$$ {}^{n^+}A = \bigcup_{x \in A} \{f \cup \{(n, x)\} \mid f \in {}^n A\}. $$

Hence, using $MathInd(Nat)$, we get the result.
■

[3] See, for example, section 6.4 of [Vis06]

8.1 Decidability on Nat

A formula is decidable if $D\phi$ holds, where $D\phi \equiv (\phi \vee \neg\phi)$. The following proposition gives some standard facts concerning the natural numbers in constructive mathematics. We leave their proof in **ACST** as an exercise for the reader.

Proposition: 8.4

1. *For all* $n, m \in Nat$, $D(n \in m) \wedge D(n = m)$.
2. *For each formula* $\phi[x]$ *of* $\mathcal{L}^*_\in$, *if* $n \in Nat$ *such that* $(\forall x \in n) D\phi[x]$ *then the following hold*
   (a) $(\ D(\exists x \in n)\phi[x]\ \wedge\ D(\forall x \in n)\phi[x]\ )$ *and*
   (b) $(\exists x \in n)\phi[x]\ \rightarrow\ (\exists x \in n)(\phi[x] \wedge (\forall y \in x)\neg\phi[y])$.
3. *For each formula* $\phi[x]$ *of* $\mathcal{L}^*_\in$, *if* $(\forall x \in Nat) D\phi[x]$ *then*

$$(\exists x \in Nat)\phi[x]\ \rightarrow\ (\exists x \in Nat)(\phi[x] \wedge (\forall y \in x)\neg\phi[y]).$$

Note that part 3 is the least number principle for decidable definable properties of natural numbers. Also note that when $\phi[x]$ is a bounded formula mathematical induction is only needed for bounded formulae.

8.2 The Finite and Finitely Enumerable Sets

Definition: 8.5 *A set* A *is* finite *if there is a bijection* $n \rightarrow A$ *for some* $n \in Nat$ *and is* finitely enumerable (f.e.) *if there is a surjection* $n \rightarrow A$ *for some* $n \in Nat$.

Proposition: 8.6 *For each f.e. set* A

$$(\forall x \in A) D\phi[x]\ \rightarrow\ D(\exists x \in A)\phi[x]\ \ \wedge\ \ D(\forall x \in A)\phi[x].$$

Proof: Let $g : n \rightarrow A$ be surjective, with $n \in Nat$, and assume that $D\phi[x]$. Then $(\forall x \in n) D\phi[gx]$ so that, by part 2 of Proposition 8.4,

$$D(\exists x \in n) D\phi[gx] \wedge D(\forall x \in n) D\phi[gx]$$

and hence

$$D(\exists x \in A) D\phi[x] \wedge D(\forall x \in A) D\phi[x].$$

■

8.2.1 Discrete Classes

Definition: 8.7 *A class A is* discrete if $(\forall x, y \in A)\ D(x = y)$.

Proposition: 8.8 *A set is* finite *iff it is an f.e. discrete set.*

Proof: For the implication from right to left let A be an f.e. discrete set. So let $g : n \to A$ be surjective, with $n \in Nat$. It suffices to show, by mathematical induction on k that, for all $k \in Nat$,

$$(*) \quad k \in n^+ \to (\exists k_0 \in k^+)(\exists f_0)\ f_0 : k_0 \sim X_k,$$

where $X_k = \{gx \mid x \in k\}$, as then, putting $k = n$ we get that $k_0 \sim A$ for some $k_0 \in n^+$, so that A is finite.

Base Case: When $k = 0$ let $k_0 = 0 \in k^+$ and $f_0 = \emptyset : k_0 \sim X_0$.

Induction Step: Assume $k \in Nat$ such that (*) and let $k^+ \in n^+$. We must show that there is $f_1 : k_1 \sim X_{k^+}$ for some $k_1 \in k^{++}$.

Observe that $D\theta$, where $\theta \equiv (gk \in X_k)$. This is because

$$\theta \ \leftrightarrow\ (\exists k' \in k)(gk = gk')$$

and, as A is discrete, $(\forall k' \in k)D(gk = gk')$ so that, by part 2 of Proposition 8.4, $D(\forall k' \in k)(gk = gk')$ and hence $D\theta$.

As $D\theta$, i.e. $\theta \vee \neg\theta$, we can argue by cases. If θ let $k_1 \equiv k_0^+ \in k^{++}$ and $f_1 \equiv f_0 : k_1 \sim X_{k^+}$. If $\neg\theta$ let $k_1 \equiv k_0^+ \in k^{++}$ and $f_1 \equiv f_0 \cup \{(k_0, gk)\} : k_1 \sim X_{k^+}$.

For the converse implication let A be a finite set. So let $g : n \to A$ be a bijection for some $n \in Nat$. Then g is a surjection and so A is an f.e. set. Also, for $x, y \in A$, $x = y$ iff $g^{-1}x = g^{-1}y$ so that A is discrete, as n is discrete. ■

Definition: 8.9 *For each class A, $Fin(A)$ is defined to be the class of finite subsets of A.*

Proposition: 8.10 *If A is a discrete class then so is $Fin(A)$.*

Proof: Let A be a discrete class. To show that $Fin(A)$ is discrete we must show that for $X, Y \in Fin(A)$, $D(X = Y)$. As $(\forall x, y \in A)D(x = y)$

and X, Y are finite we get that $(\forall x \in X)D(\exists y \in Y)(x = y)$ and hence $D(\forall x \in X)(\exists y \in Y)(x = y)$. Similarily $D(\forall y \in Y)(\exists x \in X)(x = y)$. As

$$X = Y \quad \leftrightarrow \quad (\forall x \in X)(\exists y \in Y)(x = y) \wedge (\forall y \in Y)(\exists x \in X)(x = y)$$

so that

$$D(X = Y) \quad \leftrightarrow \quad D(\forall x \in X)(\exists y \in Y)(x = y) \ \wedge \ D(\forall y \in Y)(\exists x \in X)(x = y)$$

we have $D(X = Y)$.
■

9 The Finite AC Theorem

We show the familiar result that choice functions can always be defined on a finite set.

Theorem: 9.1 *For each formula $\phi[x, y]$, if A is a finite set such that*

$$(\forall x \in A)(\exists y)\phi[x, y]$$

then there is a set f that is a function defined on A such that

$$(\forall x \in A)\phi[x, fx].$$

Proof: Given the formula $\phi[x, y]$ and the finite set A such that $(\forall x \in A)(\exists y)\phi[x, y]$ let $n \in Nat$ with a bijection $g : n \to A$. So,

$$(\forall k \in n)(\exists y)\phi[gk, y].$$

Let X be the class of $m \in Nat$ such that if $m \in n^+$ then there is a function h defined on m such that

$$(*) \quad (\forall k \in m)\phi[gk, hk].$$

We show that X is inductive. Trivially $0 \in X$ as $0 \in n^+$ and $\emptyset$ is the required function defined on 0. Now assume that $m \in X$, to show that $m^+ \in X$. If $m^+ \in n^+$ then $m \in n^+$ so that there is a function h defined on m such that $(*)$. Let $h' = h \cup \{(m, y)\}$, where y is such that $\phi[gm, y]$. Then h' is the required function defined on m^+ showing that $m^+ \in X$.

As X is inductive and $n \in Nat$, $n \in X$ so that there is a function h defined on n such that

$$(\forall k \in n)\phi[gk, hk].$$

As $g : n \to A$ is a bijection $f = \{(g(k), h(k)) \mid k \in n\}$ is the required function defined on A.
■

Corollary: 9.2 (Finitary Strong Collection) *If A is an f.e. set such that $(\forall x \in A)(\exists y)\ \phi[x, y]$ then there is an f.e. set B such that*

$$(\forall x \in A)(\exists y \in B)\phi[x, y]\ \&\ (\forall y \in B)(\exists x \in A)\phi[x, y]$$

Proof: Let A be an $f.e.$ set such that $(\forall x \in A)\exists y\ \phi[x, y]$. As A is $f.e.$ there is $n \in Nat$ and a surjection $g : n \to A$ so that

$$(\forall m \in n)(\exists y)\ \phi[gm, y].$$

Using AC for finite sets there is a function $f : n \to V$ such that, for all $m \in n$, $\phi[gm, fm]$. The desired $f.e.$ set B is the set $ranf$.
■

10 Finitary Inductive Definitions of Classes

10.1 Inductive Definitions

In constructive set theory any class Φ can be viewed as an inductive definition. Each pair (Y, a) in Φ is an *(inference) step* of the inductive definition and is called a Φ-*step* and written Y/a, the set Y being the set of *premises* of the step and a being the *conclusion* of the step.

Definition: 10.1

- *A class X is* Φ-closed *if, for every Φ-step Y/a,*

$$Y \subseteq X \Rightarrow a \in X.$$

- *A Φ-closed class I is* the class inductively defined by Φ *if it is the smallest Φ-closed class; i.e. $I \subseteq X$ for each Φ-closed class X. It is clearly unique if it exists and will then be written $I(\Phi)$.*

10.2 Finitary Inductive Definitions

We aim to focus on finitary inductive definitions.

Definition: 10.2 *An inductive definition* Φ *is defined to be* finitary *if, for each* Φ*-step* Y/a*, the set* Y *is* f.e.; i.e. finitely enumerable.

A fundamental example of a finitary inductive definition is the inductive definition of the class of von Neumann natural numbers. This has the class Φ_{Nat} whose steps are $\emptyset/0$ and $\{a\}/a^+$ for arbitrary a. Thus $Nat = I(\Phi_{Nat})$ and $MathInd(Nat)$ is the axiom scheme that expresses that Φ_{Nat} is a generating inductive definition.

10.3 The finitary Inductive Definition Theorem

We will need the following proposition.

Proposition: 10.3 *Every* $f.e.$ *subset of* Nat *is a subset of* m *for some* $m \in Nat$.

Proof: An easy proof by mathematical induction shows that for all $n \in Nat$, if $f : n \to Nat$ then there is $m \in Nat$ such that $f : n \to m$. For the base case when $n = 0$ we can let $m = 0$. For the induction step, if $f : n^+ \to Nat$ then $f' = \{(j, k) \in f \mid j \in n\} : n \to Nat$ so that, by the induction hypothesis there is $m \in Nat$ such that $f' : n \to m$. So $f = f' \cup \{(n, f(n)\}$ so that $f : n^+ \to m_0$, provided that $m_0 \in Nat$ such that $m \subseteq m_0$ and $f(n) \in m_0$. The existence of such an m_0 is a consequence of the following claim.

> **Claim:** For all $k \in Nat$, if $m \in Nat$ then there is $m_0 \in Nat$ such that $m \subseteq m_0$ and $k \in m_0$.

This can be proved by induction on k. For the base case use (i) $0 \in m_0^+$ and for the induction step use (ii) $k \in m \to k^+ \in m^+$ for all $k, m \in Nat$. We leave the proofs of (i) and (ii) as exercises.
■

If Φ is a finitary inductive definition then for each class X let

$$\Gamma_\Phi X = \{y \mid (\exists Y \in Pow(X))\ [Y/y \text{ is a step in } \Phi]\}.$$

Theorem: 10.4 *If* Φ *is a finitary inductive definition then there is a smallest* Φ*-closed class* $I(\Phi)$*.*

Proof: For G a subclass of $Nat \times V$ and $n \in Nat$ let

$$G^n = \{y \mid (n, y) \in G\} \text{ and } G^{<n} = \bigcup_{m \in n} G^m.$$

Call such a class G *good* if $G^n \subseteq \Gamma_\Phi G^{<n}$ for all $n \in Nat$, and let

$$J = \bigcup \{G \mid G \text{ is a good set}\}.$$

Claim 1: J is a good class.
Proof: Let $y \in J^n$, with $n \in Nat$. Then $y \in G^n \subseteq \Gamma_\Phi G^{<n}$ for some good set G. As Γ_Φ is monotone $y \in \Gamma_\Phi J^{<n}$. Thus $J^n \subseteq \Gamma_\Phi J^{<n}$.
■

Let $I = \bigcup_{n \in Nat} J^n$

Corollary to Claim 1: If X is a Φ-closed class then $I \subseteq X$.
Proof: Assume that X is Φ-closed; i.e. $\Gamma_\Phi X \subseteq X$. Then, by the claim, using $MathInd(Nat)$, $J^n \subseteq X$ for all $n \in Nat$ and hence $I \subseteq X$.
■

Claim 2: I is Φ-closed.
Proof: Let Y/a be a Φ-step for some $Y \subseteq I$; i.e.

$$(\forall y \in Y)(\exists G)[G \text{ is a good set and } (\exists n \in Nat)\ y \in G^n].$$

By Finitary Strong Collection, Corollary 9.2 above, as Y is $f.e.$ there is an $f.e.$ set $\mathcal{Y}$ of good sets such that

$$(\forall y \in Y)(\exists G \in \mathcal{Y})(\exists n \in Nat)\ y \in G^n.$$

It follows that

$$(\forall y \in Y)(\exists n \in Nat)(\exists G \in \mathcal{Y})\ y \in G^n.$$

So, by Finitary Strong Collection again there is a finitely enumerable subset P of Nat such that

$$(\forall y \in Y)(\exists n \in P)(\exists G \in \mathcal{Y})\ y \in G^n.$$

So, by Proposition 10.3, $P \subseteq m$ for some $m \in Nat$. It follows that $Y \subseteq G_0^{<m}$ where $G_0 = \bigcup \mathcal{Y}$ is the union of a set of good sets and so, as in the proof of Claim 2, is itself a good set. As Y/a is a Φ-step, $a \in \Gamma_\Phi G_0^{<m}$. Hence $G = G_0 \cup \{(m, a)\}$ is good, so that $a \in G^m \subseteq J^m \subseteq I$.

■

By Claim 2 and the corollary to Claim 1, the class I is the required class $I(\Phi)$.

■

Proposition: 10.5 *If Φ is a finitary inductive definition and J is as in the previous proof then $J^n = \Gamma_\Phi J^{<n}$ for all $n \in Nat$.*

Proof: In the previous proof we showed that, for $n \in Nat$, $J^n \subseteq \Gamma_\Phi J^{<n}$. To show that $\Gamma_\Phi J^{<n} \subseteq J^n$ let $a \in \Gamma_\Phi J^{<n}$. So Y/a is a step of Φ for some $Y \subseteq J^{<n}$. We have

$$(\forall y \in Y)\exists G\ [G \text{ is a good set and } y \in G^{<n}].$$

So, by Finitary Strong Collection there is a good set G_0 such that $Y \subseteq G_0^{<n}$. It follows that $G = G_0 \cup \{(n, a)\}$ is a good set with $a \in G^n \subseteq J^n$. Thus $\Gamma_\Phi J^{<n} \subseteq J^n$.

■

10.4 The Primitive Recursion Theorem

Theorem: 10.6 *Let $G_0 : B \to A$ and $F : Nat \times B \times A \to A$ be class functions, where A, B are classes. Then there is a unique class function $G : Nat \times B \to A$ such that, for all $b \in B$ and $n \in Nat$,*

$$(*) \quad \begin{cases} G(0, b) &= G_0(b), \\ G(n^+, b) &= F(n, b, G(n, b)), \end{cases}$$

Proof: Let $G = I(\Phi)$, where Φ is the inductive definition with steps $\emptyset/((0, b), G_0(b))$, for $b \in B$, and $\{((n, b), x)\}/(n^+, F(n, b, x))$ for $b \in B$ and $x \in A$. It is routine to show that G is the unique required class function.

■

Corollary: 10.7 *There are unique binary class functions* $Add, Mult : Nat \times Nat \to Nat$ *such that, for* $n, m \in Nat$,

1. $Plus(n, 0) = n$,
2. $Plus(n, m^+) = Plus(n, m)^+$,
3. $Mult(n, 0) = 0$,
4. $Mult(n, m^+) = Plus(Mult(n, m), n)$.

Proof: Apply the theorem with $A = B = Nat$, first with $F(n, m, k) = k^+$ to obtain $Plus$ and then with $F(n, m, k) = Plus(k, n)$ to obtain $Mult$.
■

Using this result it is clear that there is an obvious standard interpretation of Heyting Arithmetic in **ACST**.

10.5 The Hereditarily Finite sets

The hereditarily finite sets form the smallest class **HF** of sets such that every finite subset of **HF** is in **HF**; i.e. **HF** has the following finitary inductive definition, where we also give two other finitary inductive definitions of what turn out to be the same class.

Definition: 10.8

1. $\mathbf{HF} \equiv I(\Phi_{finite})$, *where* $\Phi_{finite} \equiv \{a/a \mid a \text{ is a finite set}\}$,
2. $\mathbf{HF_{f.e.}} \equiv I(\Phi_{f.e.})$, *where* $\Phi_{f.e.} \equiv \{a/a \mid a \text{ is f.e.}\}$,
3. $\mathbf{HF_0} \equiv I(\Phi_0)$, *where* $\Phi_0 \equiv \{\emptyset/\emptyset\} \cup \{\{a, b\}/(a \cup \{b\}) \mid a, b \in V\}$.

Proposition: 10.9

1. **HF** *is transitive; i.e.* $(\forall x \in \mathbf{HF})\ x \subseteq \mathbf{HF}$.
2. *For each formula* $\phi[x]$, *in order to prove* $(\forall x \in \mathbf{HF})\phi[x]$ *it suffices to show that* $(\forall x \in \mathbf{HF})((\forall x' \in x)\phi[x'] \to \phi[x])$.
3. *For each formula* $\phi[x, y]$, *in order to prove* $(\forall x, y \in \mathbf{HF})\phi[x, y]$ *it suffices to show that*

$$(\forall x, y \in \mathbf{HF})((\forall x' \in x)(\forall y' \in y)\phi[x', y'] \to \phi[x, y]).$$

Proof:

1. It suffices to observe that $\{x \in \mathbf{HF} \mid x \subseteq \mathbf{HF}\}$ is Φ_{finite}-closed.

2. It suffices to observe that if $(\forall x \in \mathbf{HF})((\forall x' \in x)\phi[x'] \rightarrow \phi[x]$ then $\{x \mid \phi[x]\}$ is Φ_{finite}-closed.

3. Assume

$$(*) \quad (\forall x, y \in \mathbf{HF})((\forall x' \in x)(\forall y' \in y)\phi[x', y'] \rightarrow \phi[x, y])$$

and let $\psi[x] \equiv (\forall y \in \mathbf{HF})\phi[x, y]$. We want to show that$(\forall x \in \mathbf{HF})\psi[x]$. By 2 it suffices to show that, for $x \in \mathbf{HF}$, $(\forall x' \in x)\psi[x'] \rightarrow \psi[x])$. So let $x \in \mathbf{HF}$ such that $(\forall x' \in x)\psi[x']$; i.e.

$$(**) \quad (\forall x' \in x)(\forall y \in \mathbf{HF})\phi[x', y].$$

We want to show that $\psi[x]$; i.e. $(\forall y \in \mathbf{HF})\phi[x, y]$. So let $y \in \mathbf{HF}$. By $(*)$ it suffices to show that $(\forall x' \in x)(\forall y' \in y)\phi[x', y']$. By 1, if $y' \in y$ the $y' \in \mathbf{HF}$ so that $(**)$ yields this.

■

Proposition: 10.10

1. $(\forall x, y \in \mathbf{HF})\ D(x = y)$*; i.e.* **HF** *is discrete.*

2. $(\forall x, y \in \mathbf{HF})\ D(x \in y)$

3. $(\forall x, y \in \mathbf{HF})\ x \cup \{y\} \in \mathbf{HF}$*; i.e.* **HF** *is* Φ_0*-closed so that* $\mathbf{HF}_0 \subseteq \mathbf{HF}$.

Proof:

1. It suffices to show, for $x, y \in \mathbf{HF}$, that

$$(\forall x' \in x)(\forall y' \in y)D(x' = y') \ \rightarrow \ D(x = y).$$

Assuming the left hand side, as x, y are finite, by ...,

$$(\forall x' \in x)D(\exists y' \in y)(x' = y')$$

and hence $D\phi_1$ where

$$\phi_1 \equiv (\forall x' \in x)(\exists y' \in y)(x' = y').$$

Similarily $D\phi_2$ where

$$\phi_2 \equiv (\forall y' \in y)(\exists x' \in x)(x' = y').$$

As $(x = y) \ \leftrightarrow \ \phi_1 \wedge \phi_2$ and $D\phi_1 \wedge D\phi_2 \ \rightarrow \ D(\phi_1 \wedge \phi_2)$ we get $D(x = y)$, as desired.

2. For $x, y \in \mathbf{HF}$,
$$x \in y \quad \leftrightarrow \quad (\exists y' \in y)\; x = y'.$$
As y is finite and $D(x = y')$ for $y' \in y$, by ..., we get that $D(x \in y)$.

3. Let $x, y \in \mathbf{HF}$. As x is finite there is a bijection $f : n \to x$ for some $n \in Nat$. By 2 either $y \in x$ or $y \notin x$. If $y \in x$ then $x \cup \{y\} = x \in \mathbf{HF}$ and if $y \notin x$ then $(f \cup \{(n, y)\} : n^+ \to x \cup \{y\}$ is a bijection so that $x \cup \{y\}$ is a finite subset of $\mathbf{HF}$ and so is in $\mathbf{HF}$. Thus, in either case $x \cup \{y\} \in \mathbf{HF}$.

■

Proposition: 10.11 $\mathbf{HF} \subseteq \mathbf{HF}_{f.e.}$.

Proof: It suffices to observe that $(\forall x \subseteq \mathbf{HF}_{f.e.})(x \text{ is finite } \to x \in \mathbf{HF}_{f.e.})$; i.e. $\mathbf{HF}_{f.e.}$ is Φ_{finite}-closed.

■

Proposition: 10.12 $\mathbf{HF}_{f.e.} \subseteq \mathbf{HF}_0$.

Proof: It suffices to show that $\mathbf{HF}_0$ is $\Phi_{f.e.}$-closed; i.e. that
$$(\forall x \subseteq \mathbf{HF}_0)(x \text{ is f.e. } \to x \in \mathbf{HF}_0).$$
We will show, by mathematical induction on n, that for all $n \in Nat$.

if $f : n \to \mathbf{HF}_0$ then $ran(f) \in \mathbf{HF}_0$.

If $n = 0$ and$f : n \to \mathbf{HF}_0$ then $ran(f) = \emptyset \in \mathbf{HF}_0$. For the induction step, assume that $f : n^+ \to \mathbf{HF}_0$. Then $f \restriction n : n \to \mathbf{HF}_0$ so that, by the induction hypothesis, $ran(f \restriction n) \in \mathbf{HF}_0$. So $ran(f) = ran(f \restriction n) \cup \{fn\} \in \mathbf{HF}_0$.
■

Theorem: 10.13 $\mathbf{HF} = \mathbf{HF}_{f.e.} = \mathbf{HF}_0$.

Proof: Use Propositions 10.10, 10.11 and 10.12.
■

Definition: 10.14 *A class A is* regular *if it is transitive (i.e. every element is a subset) and, for each formula $\phi[x, y]$, for every $a \in A$, such that*

$$(\forall x \in a)(\exists y \in A)\phi[x, y]$$

there is $b \in A$ such that

$$(\forall x \in a)(\exists y \in b)\phi[x, y] \wedge (\forall y \in b)(\exists x \in a)\phi[x, y]$$

Note that the property of a class being regular is an assertion scheme and not a single assertion.

Proposition: 10.15 HF *is a regular class.*

Proof: **HF** is transitive, by Proposition 10.9. Let $a \in \mathbf{HF}$ such that

$$(\forall x \in a)(\exists y \in \mathbf{HF})\phi[x, y].$$

As $a \in \mathbf{HF}$ it is finite so that, by Finitary AC, there is $f : a \to \mathbf{HF}$ such that $(\forall x \in a)\phi[x, fx]$. Let $b = ran(f)$. Then b is an f.e. subset of **HF** so that, by Theorem 10.13, it is in **HF**. Clearly

$$(\forall x \in a)(\exists y \in b)\phi[x, y] \ \wedge \ (\forall y \in b)(\exists x \in a)\phi[x, y] \ \wedge .$$

Thus **HF** is a regular class.
■

Definition: 10.16 *A transitive class A is* functionally regular *if, for all $f : a \to A$, with $a \in A$, $ran(f) \in A$.*

Observe that each regular class is functionally regular.

Proposition: 10.17 *If A is a functionally regular class such that $Nat \subseteq A$ then $\mathbf{HF} \subseteq A$.*

Proof: Let A be a functionally regular class such that $Nat \subseteq A$. Observe that A is $\Phi_{f.e.}$-closed for if $f : n \to A$ with $n \in Nat \subseteq A$ then $ran(f) \in A$. It follows that $\mathbf{HF} = \mathbf{HF}_{f.e.} \subseteq A$.
■

Corollary: 10.18 HF *is the smallest (functionally) regular class which includes Nat.*

11 Further Remarks

In this final section we look at some other axiom systems for finite set theory and compare them with some weak axiom systems for set theory that have an axiom of infinity.

Intuitionistic Finite Set Theories

Recall that $\mathbf{ACST} \equiv \mathbf{RCST}_0 + MathInd(Nat)$ We let

$$\mathbf{ACST}^{fin} \equiv \mathbf{ACST} + (V = Fin(Nat)).$$

It is not hard to see, using the work in the previous section, that each instance of the axiom schemes AC and REM are theorems of $\mathbf{ACST}^{fin}$. Here REM is the Restricted Excluded Middle scheme, having instances $(\phi \vee \neg\phi)$ for bounded formulae ϕ. It follows that the Powerset axiom and each instance of Strong Collection are also theorems of $\mathbf{ACST}^{fin}$ so that $\mathbf{CZF}^{fin} \vdash\dashv$ [4]$(\mathbf{ACST}^{fin} + \textit{Set-Ind})$ and $\mathbf{IZF}^{fin} \vdash\dashv (\mathbf{CZF}^{fin} + Sep)$, where *Set-Ind* is the Set-Induction Scheme and *Sep* is the full Separation Scheme. Note that, using the schemes *Sep* and *REM* we get the full law of excluded middle; i.e. classical logic, so that $\mathbf{IZF}^{fin} \vdash\dashv \mathbf{ZF}^{fin}$.

It is also not hard to see that, by restricting quantifiers to **HF** we get a conservative interpretation[5] $hf : \mathbf{CZF}^{fin} \tilde{\to} \mathbf{ACST}$.

We have the interpretation $nat : \mathbf{HA} \to \mathbf{ACST}$ and probably also have the interpretation $ack : \mathbf{IZF}^{fin} \to \mathbf{HA}$ such that $ack \circ nat : \mathbf{HA} \tilde{\to} \mathbf{HA}$ which would give us that $nat : \mathbf{HA} \tilde{\to} \mathbf{IZF}^{fin}$ and hence $nat : \mathbf{HA} \tilde{\to} \mathbf{T}$ for any subtheory $\mathbf{T}$ of $\mathbf{IZF}^{fin}$ that has **ACST** as a subtheory.

Weak Intuitionistic set theories with a set ω

We consider some other weak intuitionistic set theories for CST that have been studied in the literature. The axiom system **BCST** has the non-logical axioms of Extensionality, Emptyset, Pairing, Union and the schemes of Bounded Separation and Replacement. We get **ECST** by adding to **BCST** the axiom of Strong Infinity, which states that there is a smallest inductive set ω. Let $\mathbf{ECST}^{coll}$ be obtained from **ECST** by adding the Strong Collection scheme and let $\mathbf{CZF}^{-}$ be obtained from $\mathbf{ECST}^{coll}$ by adding the Subset Collection Scheme. So $\mathbf{CZF}^{-}$ has the same theorems as the axiom system

[4]The relation $\vdash\dashv$ holds between two theories if they have the same theorems. See the Appendix.

[5]See the Appendix for this notion.

obtained from **CZF** by leaving out the Set Induction Scheme and using the Strong Infinity axiom instead of the Infinity axiom.

We summarise some results concerning these axiom systems that are explicit or implicit in [Rat08] and should be compared with the results in this paper.

The papers [KW07, Cam07] are also relevant.

In the theorem below we use the following.

1. The interpretation nat_ω is defined like the interpretation nat but uses ω instead of Nat.

2. If $\mathbf{T}$ is a theory in the language $\mathcal{L}_\in$ then $\mathbf{T}$ is Π^0_2-conservative over $\mathbf{HA}$ if $\mathbf{HA} \vdash \phi$ whenever ϕ is a Π^0_2-sentence of $\mathcal{L}_{\mathbf{HA}}$ such that $(\mathbf{CZF}^- + DC/PAx) \vdash \phi^{nat_\omega}$.

3. DC is the axiom of Dependent Choices and PAx is the Presentation axiom.

4. Δ_0-$ITER_\omega$

   is the scheme that allows the definition of a function $g : \omega \to A$, where A is a bounded class, by iterating a bounded class function $F : A \to A$ starting from some $a_0 \in A$; i.e. g is defined to be the unique function such that $g0 = a_0$ and $gn^+ = F(gn)$ for $n \in \omega$.

Theorem: 11.1 (Rathjen)

1. Addition on ω cannot be derived in $\mathbf{ECST}^{coll}$.

2. Addition and multiplication on ω can be defined in $\mathbf{CZF}^-$ so that $nat_\omega : \mathbf{HA} \to \mathbf{CZF}^-$.

3. $(\mathbf{CZF}^- + DC/PAx)$ is Π^0_2-conservative over $\mathbf{HA}$.

4. $(\mathbf{ECST} + \Delta_0\text{-}ITER_\omega) \vdash Consis(\mathbf{CZF}^-)$.

12 Appendix

In the following $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ etc. are first order languages with equality and have the logical symbols $=, \bot, \wedge, \vee, \rightarrow, \forall, \exists$, with $\neg$ and $\leftrightarrow$ defined in the usual way. $\mathcal{F}(\mathcal{L})$ is the set of formulae of a language $\mathcal{L}$. We assume a standard axiom system $\mathbf{Int}_{\mathcal{L}}$ for intuitionistic first order logic with equality for the language $\mathcal{L}$ and $\mathbf{Class}_{\mathcal{L}}$ for the classical axiom system obtained by adding the law of excluded middle, $\phi \vee \neg\phi$ for all formulae $\phi \in \mathcal{F}(\mathcal{L})$. Formulae ϕ, ψ of $\mathcal{L}$ are *logically equivalent*, written $\phi \equiv_{\mathcal{L}} \psi$, if $(\phi \leftrightarrow \psi)$ is a theorem of $\mathbf{Int}_{\mathcal{L}}$.

An $\mathcal{L}$-theory is an axiom system having axioms and rules of inference whose formulae are in the language $\mathcal{L}$. The theory will have a logical part which is either $\mathbf{Int}_{\mathcal{L}}$ or $\mathbf{Class}_{\mathcal{L}}$. Usually the non-logical part will be given by a set of formulae presented using axiom schemes. But occasionally there may be some non-logical rules of inference. Note that the axioms of a theory may be formulae that are implicitly universally quantified.

Given a theory $\mathbf{T}$, let $Thm(\mathbf{T}) = \{\phi \mid \mathbf{T} \vdash \phi\}$, where $\mathbf{T} \vdash \phi$, if the formula ϕ can be derived in the theory $\mathbf{T}$. If $\mathbf{T}'$ is a theory of the same language we write $\mathbf{T} \vdash \mathbf{T}'$ if $Thm(\mathbf{T}') \subseteq Thm(\mathbf{T})$ and $\mathbf{T} \vdash\dashv \mathbf{T}'$ if $Thm(\mathbf{T}) = Thm(\mathbf{T}')$.

Interpretations

Here we summarize some definitions concerning translations and interpretations that suffice for our purposes. A more thorough treatment can be found in [Vis06].

Definition: 12.1

1. *A* translation $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ *is an assignment of a formula* $\phi^f \in \mathcal{F}(\mathcal{L}_2)$, *with the same free variables as* ϕ, *to each formula* $\phi \in \mathcal{F}(\mathcal{L}_1)$ *together with a formula* $Dom^f[x]$ *of* $\mathcal{L}_2$, *having at most the variable* x *free, such that the following hold. Note that a translation* $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ *is uniquely determined by its action on atomic formulae and the formula* $Dom^f[x]$.

   (a) $\bot^f = \bot$,

   (b) $(\phi_1 \Box \phi_2)^f = (\phi_1^f \Box \phi_2^f)$, *for* $\Box \in \{\wedge, \vee, \rightarrow\}$,

   (c) $(\forall x \phi[x])^f = (\forall x)(Dom^f[x] \rightarrow \phi[x]^f)$, *and*

   (d) $(\exists x \phi[x])^f = (\exists x)(Dom^f[x] \ \wedge \ \phi[x]^f)$,

   where $Dom^f[x]$ *is a formula of* $\mathcal{L}_2$.

2. *$id_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ is the identity translation; i.e. it is the unique translation such that $\theta^{id_{\mathcal{L}}} = \theta$ for each atomic $\theta \in \mathcal{F}(\mathcal{L})$ and $Dom^{id_{\mathcal{L}}}[x]$ is $(x = x)$. It follows that $(\phi^{id_{\mathcal{L}}} \equiv_{\mathcal{L}} \phi)$ for each formula $\phi \in \mathcal{F}(\mathcal{L})$.*

3. *If $f : \mathcal{L}_1 \to \mathcal{L}_2$ and $g : \mathcal{L}_2 \to \mathcal{L}_3$ then $(g \circ f) : \mathcal{L}_1 \to \mathcal{L}_3$ is their composition; i.e. it is the unique translation where $\theta^{g \circ f} = (\theta^f)^g$ for each atomic $\theta \in \mathcal{F}(\mathcal{L}_1)$ and $Dom^{(g \circ f)}[x]$ is $Dom^g[x] \wedge (Dom^f[x])^g$. So $\phi^{g \circ f} \equiv_{\mathcal{L}_3} (\phi^f)^g$ for each formula $\phi \in \mathcal{F}(\mathcal{L}_1)$.*

4. *Let $\mathbf{T}_i$ be an $\mathcal{L}_i$-theory for $i = 1, 2$. A translation $f : \mathcal{L}_1 \to \mathcal{L}_2$ is an* interpretation *$f : \mathbf{T}_1 \to \mathbf{T}_2$ if*

$$\mathbf{T}_1 \vdash \phi \;\Rightarrow\; \mathbf{T}_2 \vdash \phi^f$$

for all formulae $\phi \in \mathcal{F}(\mathcal{L}_1)$. It is a conservative interpretation[6] ,written *$f : \mathbf{T}_1 \tilde{\to} \mathbf{T}_2$, if*

$$\mathbf{T}_1 \vdash \phi \;\Leftrightarrow\; \mathbf{T}_2 \vdash \phi^f$$

for all formulae $\phi \in \mathcal{F}(\mathcal{L}_1)$.

5. *If $f, f' : \mathbf{T}_1 \to \mathbf{T}_2$ we write $f \sim f' : \mathbf{T}_1 \to \mathbf{T}_2$ if*

$$\mathbf{T}_2 \vdash (Dom^f[x] \leftrightarrow Dom^{f'}[x]) \text{ and}$$

$$\mathbf{T}_2 \vdash (Dom^f[x_1] \wedge \cdots \wedge Dom^f[x_n]) \to (\phi^f \leftrightarrow \phi^{f'})$$

for all $\phi \in \mathcal{F}(\mathcal{L}_1)$, where $x_1, \ldots, x_n$ is a list of the distinct variables that occur free in ϕ.

6. *Interpretations $f : \mathbf{T}_1 \to \mathbf{T}_2$ and $g : \mathbf{T}_2 \to \mathbf{T}_1$ are* inverses of each other *if $(g \circ f) \sim id_{\mathcal{L}_1} : \mathbf{T}_1 \to \mathbf{T}_1$ and $(f \circ g) \sim id_{\mathcal{L}_2} : \mathbf{T}_2 \to \mathbf{T}_2$. When this holds the theories $\mathbf{T}_1, \mathbf{T}_2$ are said to be* synonymous or definitionally equal.

Proposition: 12.2

1. *For each $\mathcal{L}$-theory $\mathbf{T}$, $id_{\mathcal{L}} : \mathbf{T} \tilde{\to} \mathbf{T}$.*

2. *If $f : \mathbf{T}_1 \to (\tilde{\to})\mathbf{T}_2$ and $g : \mathbf{T}_2 \to (\tilde{\to})\mathbf{T}_3$ then $g \circ f : \mathbf{T}_1 \to (\tilde{\to})\mathbf{T}_3$.*

3. *Let $f : \mathbf{T}_1 \to \mathbf{T}_2$ and $g : \mathbf{T}_2 \to \mathbf{T}_1$.*

   (a) *If $(g \circ f) \sim id_{\mathcal{L}_1} : \mathbf{T}_1 \to \mathbf{T}_2$ then $g \circ f : \mathbf{T}_1 \tilde{\to} \mathbf{T}_1$.*

   (b) *If $g \circ f : \mathbf{T}_1 \tilde{\to} \mathbf{T}_1$ then $f : \mathbf{T}_1 \tilde{\to} \mathbf{T}_2$.*

[6] This is often called a faithful interpretation

References

[Acz78] Peter Aczel. The type theoretic interpretation of constructive set theory. In *Logic Colloquium '77 (Proc. Conf., Wrocław, 1977)*, volume 96 of *Stud. Logic Foundations Math.*, pages 55–66. North-Holland, Amsterdam, 1978.

[Acz82] Peter Aczel. The type theoretic interpretation of constructive set theory: choice principles. In *The L. E. J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*, volume 110 of *Stud. Logic Found. Math.*, pages 1–40. North-Holland, Amsterdam, 1982.

[Acz86] Peter Aczel. The type theoretic interpretation of constructive set theory: inductive definitions. In *Logic, methodology and philosophy of science, VII (Salzburg, 1983)*, volume 114 of *Stud. Logic Found. Math.*, pages 17–49. North-Holland, Amsterdam, 1986.

[AR01] Peter Aczel and Michael Rathjen. Notes on constructive set theory. Technical Report No. 40, 2000/2001, Mittag-Leffler, 2001.

[Cam07] Riccardo Camerlo. On Interpretations of Arithmetic and Set Theory. *Notre Dame Journal of Formal Logic*, 48(4):473–488, 2007.

[GA06] Nicola Gambino and Peter Aczel. The generalized type-theoretic interpretation of constructive set theory. *J. Symbolic Logic*, 71(1):67–103, 2006.

[Jen72] Ronald B. Jensen. The fine structure of the constructible hierarchy. *Annals of Mathematical Logic*, 4(?):229–308, 1972.

[KW07] Richard Kaye and Tin Lok Wong. Finiteness Axioms on Fragments of Intuitionistic Set Theory. *Notre Dame Journal of Formal Logic*, 48(4):497–510, 2007.

[Myh73] John R. Myhill. Some properties of Intuitionistic Zermelo-Fraenkel set theory. In A.R.D. Mathias and H. Rogers, editors, *Proceedings of the 1971 Cambridge Summer School in Mathematical Logic*, volume 337 of *Lecture Notes in Mathematics*, pages 206–231, Berlin, 1973. Springer.

[Myh75] John R. Myhill. Constructive set theory. *J. Symb. Log.*, 40(3):347–382, 1975.

[Rat08] Michael Rathjen. The natural numbers in constructive set theory. *Mathematical Logic Quarterly*, 54(1):83–97, 2008.

[Vis06] A. Visser. Categories of Theories and Interpretations. In Ali Enayat, Iraj Kalantari, and Mojtaba Moniri, editors, *Logic in Tehran. Proceedings of the workshop and conference on Logic, Algebra and Arithmetic, held October 18–22, 2003*, volume 26 of *Lecture Notes in Logic*, pages 284–341. ASL, A.K. Peters, Ltd., Wellesley, Mass., 2006.