

CONNECTED CHOICE AND THE BROUWER FIXED POINT THEOREM

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ABSTRACT. We study the computational content of the Brouwer Fixed Point Theorem in the Weihrauch lattice. One of our main results is that for any fixed dimension the Brouwer Fixed Point Theorem of that dimension is computably equivalent to connected choice of the Euclidean unit cube of the same dimension. Connected choice is the operation that finds a point in a non-empty connected closed set given by negative information. Another main result is that connected choice is complete for dimension greater or equal to three in the sense that it is computably equivalent to Weak König's Lemma. In contrast to this, the connected choice operations in dimensions zero, one and two form a strictly increasing sequence of Weihrauch degrees, where connected choice of dimension one is known to be equivalent to the Intermediate Value Theorem. Whether connected choice of dimension two is strictly below connected choice of dimension three or equivalent to it is unknown, but we conjecture that the reduction is strict. As a side result we also prove that finding a connectedness component in a closed subset of the Euclidean unit cube of any dimension greater or equal to one is equivalent to Weak König's Lemma. In order to describe all these results we introduce a representation of closed subsets of the unit cube by trees of rational complexes.

Keywords: Computable analysis, Weihrauch lattice, reverse mathematics, choice principles, connected sets, fixed point theorems.

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Date: June 21, 2012.

This research was supported by a Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme and by the National Research Foundation of South Africa.

1. INTRODUCTION

In this paper we continue with the programme to classify the computational content of mathematical theorems in the Weihrauch lattice (see [GM09, BG11b, BG11a, Pau10b, Pau10a, BGM12, HRW12]). This lattice is induced by Weihrauch reducibility, which is a reducibility for multi-valued partial functions $f : \subseteq X \rightrightarrows Y$ on represented spaces X, Y . Intuitively, $f \leq_w g$ reflects the fact that f can be realized with a single application of g as an oracle. Hence, if two multi-valued functions are equivalent in the sense that they are mutually reducible to each other, then they are equivalent as computational resources, as far as computability is concerned.

Many theorems in mathematics are actually of the logical form

$$(\forall x \in X)(\exists y \in Y) P(x, y)$$

and such theorems can straightforwardly be represented by a multi-valued function $f : X \rightrightarrows Y$ with $f(x) := \{y \in Y : P(x, y)\}$ (sometimes partial f are needed, where the domain captures additional requirements that the input x has to satisfy). In some sense the multi-valued operation f directly reflects the computational task of the theorem to find some suitable y for any x . Hence, in a very natural way the classification of a theorem can be achieved via a classification of the corresponding multi-valued function that represents the theorem.

Theorems that have been compared and classified in this sense include Weak König's Lemma WKL, the Hahn-Banach Theorem [GM09], the Baire Category Theorem, Banach's Inverse Mapping Theorem, the Open Mapping Theorem, the Uniform Boundedness Theorem, the Intermediate Value Theorem [BG11a], the Bolzano-Weierstraß Theorem [BGM12], Nash Equilibria [Pau10a] and the Radon-Nikodym Theorem [HRW12]. In this paper we attempt to classify the Brouwer Fixed Point Theorem.

Theorem 1.1 (Brouwer Fixed Point Theorem 1911). *Every continuous function $f : [0, 1]^n \rightarrow [0, 1]^n$ has a fixed point $x \in [0, 1]^n$, i.e. a point such that $f(x) = x$.*

This theorem was first proved by Hadamard in 1910 and later by Brouwer [Bro11], after whom it is named the Brouwer Fixed Point Theorem. Brouwer is known as one of the founders of intuitionism, which is one of the well-studied varieties of constructive mathematics and ironically, the theorem that he is best known for does not admit any constructive proof.¹ This fact has been confirmed in many different ways, most relevant for us is the counterexample in Russian constructive analysis by Orevkov [Ore63], which was transferred into computable analysis by Baigge [Bai85]. Baigge's counterexample shows that from dimension two upwards (i.e. $n \geq 2$) there are computable functions $f : [0, 1]^n \rightarrow [0, 1]^n$ without computable fixed point x . Baigge's proof actually proceeds by encoding a Kleene tree (implicitly via a pair of computably inseparable sets) into a suitable computable function f and hence it can be seen as a reduction of Weak König's Lemma to the Brouwer Fixed Point Theorem.² The essential geometrical content of this construction is that the map

$$A \mapsto (A \times [0, 1]) \cup ([0, 1] \times A)$$

maps *arbitrary* non-empty closed sets $A \subseteq [0, 1]$ to *connected* non-empty closed subsets of $[0, 1]^2$ such that any pair in the resulting set has at least one component that is in A .

¹However, as noticed already by Brouwer himself, the theorem admits an approximative constructive version [Bro52].

²See [Pot08] for a discussion of these counterexamples.

Constructions similar to those used for the above counterexamples have been utilized in order to prove that the Brouwer Fixed Point Theorem is equivalent to Weak König's Lemma in reverse mathematics [Sim99, ST90, Koh05] and to analyze computability properties of fixable sets [Mil02], but a careful analysis of these reductions reveals that none of them can be straightforwardly transferred into a *uniform* reduction in the sense that we are seeking here. The problem is that there is no uniform way to select a component x_i of a pair (x_1, x_2) such that $x_i \in A$, given that at least one of the components has this property. The results cited above essentially characterize the complexity of the fixed points themselves, whereas we want to characterize the complexity of finding a fixed point, given the function. This requires full uniformity.

In the Weihrauch lattice the Brouwer Fixed Point Theorem of dimension n is represented by the multi-valued function $\text{BFT}_n : \mathcal{C}([0, 1]^n, [0, 1]^n) \rightrightarrows [0, 1]^n$ that maps any continuous function $f : [0, 1]^n \rightarrow [0, 1]^n$ to the set of its fixed points $\text{BFT}_n(f) \subseteq [0, 1]^n$. The question now is where BFT_n is located in the Weihrauch lattice? It easily follows from a meta theorem presented in [BG11a] that the Brouwer Fixed Point Theorem BFT_n is reducible to Weak König's Lemma WKL for any dimension n , i.e. $\text{BFT}_n \leq_W \text{WKL}$. However, for which dimensions n do we also obtain the inverse reduction? Clearly not for $n = 0$, since BFT_0 is computable, and clearly not for $n = 1$, since BFT_1 is equivalent to the Intermediate Value Theorem IVT and hence not equivalent to WKL, as proved in [BG11a].³

In order to approach this question for a general dimension n , we introduce a choice principle CC_n that we call *connected choice* and which is just the closed choice operation restricted to connected subsets. That is, in the sense discussed above CC_n represents the following mathematical statement: every non-empty connected closed set $A \subseteq [0, 1]^n$ has a point $x \in [0, 1]^n$. Since closed sets are represented by negative information (i.e. by an enumeration of open balls that exhaust the complement), the computational task of CC_n consists in finding a point in a closed set $A \subseteq [0, 1]^n$ that is promised to be non-empty and connected and that is given by negative information.

One of our main results, proved in Section 4, is that the Brouwer Fixed Point Theorem is equivalent to connected choice for each fixed dimension n , i.e.

$$\text{BFT}_n \equiv_W \text{CC}_n.$$

This result allows us to study the Brouwer Fixed Point Theorem in terms of the operation CC_n that is easier to handle since it involves neither function spaces nor fixed points. This is also another instance of the observation that several important theorems are equivalent to certain choice principles (see [BG11a]) and many important classes of computable functions can be calibrated in terms of choice (see [BdBP12]). For instance, closed choice on Cantor space $\mathcal{C}_{\{0,1\}^{\mathbb{N}}}$ and on the unit cube $\mathcal{C}_{[0,1]^{\mathbb{N}}}$ are both easily seen to be equivalent to Weak König's Lemma WKL, i.e. $\text{WKL} \equiv_W \mathcal{C}_{\{0,1\}^{\mathbb{N}}} \equiv_W \mathcal{C}_{[0,1]^{\mathbb{N}}}$ for any $n \geq 1$. Studying the Brouwer Fixed Point Theorem in form of CC_n now amounts to compare $\mathcal{C}_{[0,1]^n}$ with its restriction CC_n .

Our second main result, proved in Section 5, is that from dimension three upwards connected choice is equivalent to Weak König's Lemma, i.e. $\text{CC}_n \equiv_W \mathcal{C}_{[0,1]^n}$ for $n \geq 3$. The backwards reduction is based on the geometrical construction

$$A \mapsto (A \times [0, 1] \times \{0\}) \cup (A \times A \times [0, 1]) \cup ([0, 1] \times A \times \{1\})$$

that maps an *arbitrary* non-empty closed set $A \subseteq [0, 1]$ to a *connected* non-empty closed subset of $[0, 1]^3$ that has the property that any of its points allows to compute a point of the original set A in a uniform sense. This construction seems to require

³We mention that in constructive reverse mathematics the Intermediate Value Theorem is equivalent to Weak König's Lemma [Ish06], as parallelization is freely available in this framework.

at least dimension three in a crucial sense. It is easy to see that the connected choice operations for dimensions 0, 1 and 2 form a strictly increasing sequence of Weihrauch degrees, i.e.

$$\text{CC}_0 <_{\text{W}} \text{CC}_1 <_{\text{W}} \text{CC}_2 \leq_{\text{W}} \text{CC}_3 \equiv_{\text{W}} \text{C}_{[0,1]}.$$

The status of connected choice CC_2 of dimension two remains unresolved and we conjecture that it is strictly weaker than choice of dimension three, i.e. $\text{CC}_2 <_{\text{W}} \text{CC}_3$.

In order to prove our results, we use a representation of closed sets by trees of so-called rational complexes, which we introduce in Section 3. It can be seen as a generalization of the well-known representation of co-c.e. closed subsets of Cantor space $\{0, 1\}^{\mathbb{N}}$ by trees. As a side result we prove that finding a connectedness component in a closed set for any fixed dimension from one upwards is equivalent to Weak König's Lemma. This yields conclusions along the line of earlier studies of connected components in [LRZ08]. In the following Section 2 we start with a short summary of relevant definitions and results regarding the Weihrauch lattice.

2. THE WEIHRAUCH LATTICE

In this section we briefly recall some basic results and definitions regarding the Weihrauch lattice. The original definition of Weihrauch reducibility is due to Weihrauch and has been studied for many years (see [Ste89, Wei92a, Wei92b, Her96, Bra99, Bra05]). Only recently it has been noticed that a certain variant of this reducibility yields a lattice that is very suitable for the classification of mathematical theorems (see [GM09, Pau10b, Pau10a, BG11b, BG11a, BdBP12, BGM12]). The basic reference for all notions from computable analysis is [Wei00]. The Weihrauch lattice is a lattice of multi-valued functions on represented spaces. A *representation* δ of a set X is just a surjective partial map $\delta : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$. In this situation we call (X, δ) a *represented space*. In general we use the symbol " \subseteq " in order to indicate that a function is potentially partial. Using represented spaces we can define the concept of a realizer. We denote the composition of two (multi-valued) functions f and g either by $f \circ g$ or by fg .

Definition 2.1 (Realizer). Let $f : \subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces. A function $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ is called a *realizer* of f , in symbols $F \vdash f$, if $\delta_Y F(p) \in f \delta_X(p)$ for all $p \in \text{dom}(f \delta_X)$.

Realizers allow us to transfer the notions of computability and continuity and other notions available for Baire space to any represented space; a function between represented spaces will be called *computable*, if it has a computable realizer, etc. Now we can define Weihrauch reducibility. By $\langle \cdot, \cdot \rangle : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ we denote the standard pairing function, defined by $\langle p, q \rangle(2n) := p(n)$ and $\langle p, q \rangle(2n + 1) := q(n)$ for all $p, q \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

Definition 2.2 (Weihrauch reducibility). Let f, g be multi-valued functions on represented spaces. Then f is said to be *Weihrauch reducible* to g , in symbols $f \leq_{\text{W}} g$, if there are computable functions $K, H : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that $K \langle \text{id}, GH \rangle \vdash f$ for all $G \vdash g$. Moreover, f is said to be *strongly Weihrauch reducible* to g , in symbols $f \leq_{\text{sW}} g$, if there are computable functions K, H such that $KGH \vdash f$ for all $G \vdash g$.

The difference between ordinary and strong Weihrauch reducibility is that the "output modifier" K has direct access to the original input in case of ordinary Weihrauch reducibility, but not in case of strong Weihrauch reducibility. We note that the relations \leq_{W} , \leq_{sW} and \vdash implicitly refer to the underlying representations, which we will only mention explicitly if necessary. It is known that these relations only depend on the underlying equivalence classes of representations, but not on the

specific representatives (see Lemma 2.11 in [BG11b]). The relations \leq_W and \leq_{sW} are reflexive and transitive, thus they induces corresponding partial orders on the sets of their equivalence classes (which we refer to as *Weihrauch degrees* or *strong Weihrauch degrees*, respectively). These partial orders will be denoted by \leq_W and \leq_{sW} as well. In this way one obtains distributive bounded lattices (for details see [Pau10b] and [BG11b]). We use \equiv_W and \equiv_{sW} to denote the respective equivalences regarding \leq_W and \leq_{sW} , and by $<_W$ and $<_{sW}$ we denote strict reducibility.⁴

The Weihrauch lattice is equipped with a number of useful algebraic operations that we summarize in the next definition. We use $X \times Y$ to denote the ordinary set-theoretic *product*, $X \sqcup Y := (\{0\} \times X) \cup (\{1\} \times Y)$ in order to denote *disjoint sums* or *coproducts*, by $\bigsqcup_{i=0}^{\infty} X_i := \bigcup_{i=0}^{\infty} (\{i\} \times X_i)$ we denote the *infinite coproduct*. By X^i we denote the i -fold product of a set X with itself, where $X^0 = \{()\}$ is some canonical computable singleton. By $X^* := \bigsqcup_{i=0}^{\infty} X^i$ we denote the set of all *finite sequences over X* and by $X^{\mathbb{N}}$ the set of all *infinite sequences over X* . All these constructions have parallel canonical constructions on representations and the corresponding representations are denoted by $[\delta_X, \delta_Y]$ for the product of (X, δ_X) and (Y, δ_Y) , $\delta_X \sqcup \delta_Y$ for the coproduct and δ_X^* for the representation of X^* and $\delta_X^{\mathbb{N}}$ for the representation of $X^{\mathbb{N}}$ (see [BG11b, Pau10b, BdBP12] for details). We will always assume that these canonical representations are used, if not mentioned otherwise.

Definition 2.3 (Algebraic operations). Let $f : \subseteq X \rightrightarrows Y$ and $g : \subseteq Z \rightrightarrows W$ are multi-valued functions on represented spaces. Then we define the following operations:

- (1) $f \times g : \subseteq X \times Z \rightrightarrows Y \times W, (f \times g)(x, z) := f(x) \times g(z)$ (product)
- (2) $f \sqcap g : X \times Z \rightrightarrows Y \sqcup W, (f \sqcap g)(x, z) := f(x) \sqcup g(z)$ (sum)
- (3) $f \sqcup g : \subseteq X \sqcup Z \rightrightarrows Y \sqcup W$, with $(f \sqcup g)(0, x) := \{0\} \times f(x)$ and $(f \sqcup g)(1, z) := \{1\} \times g(z)$ (coproduct)
- (4) $f^* : X^* \rightrightarrows Y^*, f(i, x) := \{i\} \times f^i(x)$ (finite parallelization)
- (5) $\widehat{f} : X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, f(x_n) := \times_{i=0}^{\infty} f(x_i)$ (parallelization)

In this definition and in general we denote by $f^i : \subseteq X^i \rightrightarrows Y^i$ the i -th fold product of the multi-valued map f with itself. For f^0 we assume that $X^0 := \{()\}$ is a canonical singleton for each set X and hence f^0 is just the constant operation on that set. It is known that $f \sqcap g$ is the *infimum* of f and g with respect to strong as well as ordinary Weihrauch reducibility (see [BG11b], where this operation was denoted by $f \oplus g$). Correspondingly, $f \sqcup g$ is known to be the *supremum* of f and g (see [Pau10b]). The two operations $f \mapsto \widehat{f}$ and $f \mapsto f^*$ are known to be *closure operators* in the corresponding lattices, which means $f \leq_W \widehat{f}$ and $\widehat{f} \equiv_W \widehat{\widehat{f}}$, and $f \leq_W g$ implies $\widehat{f} \leq_W \widehat{g}$ and analogously for finite parallelization (see [BG11b, Pau10b]).

There is some terminology related to these algebraic operations. We say that f is a *cylinder* if $f \equiv_{sW} \text{id} \times f$ where $\text{id} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ always denotes the identity on Baire space, if not mentioned otherwise. Cylinders f have the property that $g \leq_W f$ is equivalent to $g \leq_{sW} f$ (see [BG11b]). We say that f is *idempotent* if $f \equiv_W f \times f$. We say that a multi-valued function on represented spaces is *pointed*, if it has a computable point in its domain. For pointed f and g we obtain $f \sqcup g \leq_{sW} f \times g$. If $f \sqcup g$ is idempotent, then we also obtain the inverse reduction. The finite parallelization

⁴It is interesting to mention that some fragments of the theory of (continuous) Weihrauch degrees have recently been studied with regards to decidability and computational time complexity (see [KSZ10, HS11]).

f^* can also be considered as *idempotent closure* as $f \equiv_{\text{W}} f^*$ holds if and only if f is idempotent and pointed. We call f *parallelizable* if $f \equiv_{\text{W}} \widehat{f}$ and it is easy to see that \widehat{f} is always idempotent. The properties of pointedness and idempotency are both preserved under equivalence and hence they can be considered as properties of the respective degrees.

A particularly useful multi-valued function in the Weihrauch lattice is closed choice (see [GM09, BG11b, BG11a, BdBP12]) and it is known that many notions of computability can be calibrated using the right version of choice. We will focus on closed choice for computable metric spaces, which are separable metric spaces such that the distance function is computable on the given dense subset. We assume that computable metric spaces are represented via their Cauchy representation (see [Wei00] for details).

By $\mathcal{A}_-(X)$ we denote the set of closed subsets of a metric space X and the index “ $-$ ” indicates that we work with negative information given by representation $\psi_- : \mathbb{N}^{\mathbb{N}} \rightarrow \mathcal{A}_-(X)$, defined by

$$\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_{p(i)},$$

where B_n is some standard enumeration of the open balls of X with center in the dense subset and rational radius. The computable points in $\mathcal{A}_-(X)$ are called *c.o.c.e. closed sets*. We now define closed choice for the case of computable metric spaces.

Definition 2.4 (Closed Choice). Let X be a computable metric space. Then the *closed choice* operation of this space X is defined by

$$C_X : \subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$$

with $\text{dom}(C_X) := \{A \in \mathcal{A}_-(X) : A \neq \emptyset\}$.

Intuitively, C_X takes as input a non-empty closed set in negative description (i.e. given by ψ_-) and it produces an arbitrary point of this set as output. Hence, $A \mapsto A$ means that the multi-valued map C_X maps the input $A \in \mathcal{A}_-(X)$ to the set $A \subseteq X$ as a set of possible outputs. We mention a couple of properties of closed choice for specific spaces.

The omniscience principle LLPO turned out to be very useful and it is closely related to closed choice. We recall the definition.

Definition 2.5 (Omniscience principle). We define $\text{LLPO} : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ by

$$j \in \text{LLPO}(p) \iff (\forall n \in \mathbb{N}) p(2n + j) = 0$$

for all $j \in \{0, 1\}$, where $\text{dom}(\text{LLPO}) := \{p \in \mathbb{N}^{\mathbb{N}} : p(k) \neq 0 \text{ for at most one } k\}$.

It is easy to see that $C_{\{0,1\}} \equiv_{\text{sW}} \text{LLPO}$. We mention that closed choice can also be used to characterize the computational content of many theorems. By WKL we denote the straightforward formalization of Weak König’s Lemma. Since we will not use WKL in any formal sense here, we refer the reader to [GM09, BG11b] for precise definitions.

Fact 2.6 (Weak König’s Lemma). $\text{WKL} \equiv_{\text{sW}} C_{\{0,1\}^{\mathbb{N}}} \equiv_{\text{sW}} C_{[0,1]^n} \equiv_{\text{sW}} \widehat{\text{LLPO}}$ for all $n \geq 1$.

Finally, we mention that in Corollary 8.12 and Theorem 8.10, both in [BdBP12], a uniform version of the Low Basis Theorem was proved. We state this result for further reference here as well.

Fact 2.7 (Uniform Low Basis Theorem). $C_{\{0,1\}^{\mathbb{N}}} \leq_{\text{sW}} \text{L}$.

We recall that $L := J^{-1} \circ \lim$. Here $J : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, p \mapsto p'$ denotes the *Turing jump operator* and by $\lim : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, p_2, \dots \rangle \mapsto \lim_{i \rightarrow \infty} p_i$ we denote the usual limit operation on Baire space (with the input sequence encoded in a single sequence). A point $p \in \mathbb{N}^{\mathbb{N}}$ is *low* if and only if there is a computable q such that $L(q) = p$. The classical Low Basis Theorem of Jockusch and Soare [JS72] states that any non-empty co-c.e. closed set $A \subseteq \{0, 1\}^{\mathbb{N}}$ has a low member and Fact 2.7 can be seen as a uniform version of this result (see [BdBP12] for a further discussion of this theorem).

3. CLOSED SETS AND TREES OF RATIONAL COMPLEXES

In this section we want to describe a representation of closed sets $A \subseteq [0, 1]^n$ that is useful for the study of connectedness. It is well-known that closed subsets of Cantor space can be characterized exactly as sets of infinite paths of trees (see for instance [CR98]). We describe a similar representation of closed subsets of the unit cube $[0, 1]^n$ of the Euclidean space. While in the case of Cantor space clopen balls are associated to each node of the tree, we now associate finite complexes of rational balls to each node. While infinite paths lead to points of the closed set in case of Cantor space, they now lead to connectedness components (which can be seen as a generalization, since the connectedness components in Cantor space are singletons).

This representation of closed subsets $A \subseteq [0, 1]^n$ of the unit cube will enable us to analyze the relation between connected choice and the Brouwer Fixed Point Theorem in the next section. In this section we will use this representation in order to prove the result that finding a connectedness component of a closed set A is computationally exactly as difficult as Weak König's Lemma.

We first fix some topological terminology that we are going to use. We work with the *maximum norm* $\|\cdot\|$ on \mathbb{R}^n , defined by $\|(x_1, \dots, x_n)\| := \max\{|x_i| : i = 1, \dots, n\}$. By $d(x, A) := \inf_{a \in A} \|x - a\|$ we denote the *distance* of $x \in \mathbb{R}^n$ to $A \subseteq \mathbb{R}^n$. By $d_A : \mathbb{R}^n \rightarrow \mathbb{R}$ we denote the corresponding *distance function* given by $d_A(x) := d(x, A)$. By $B(x, r) := \{y \in \mathbb{R}^n : \|x - y\| < r\}$ we denote the *open ball* with center x and radius r and by $B[x, r] := \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$ the corresponding *closed ball*. Since we are using the maximum norm, all these balls are open or closed cubes, respectively (if the radius is positive). By ∂A we denote the topological *boundary*, by \bar{A} the *closure* and by A° the *interior* of a set A . If the underlying space X is clear from the context, then $A^c := X \setminus A$ denotes the *complement* of A .

We are now prepared to define rational complexes, which are just finite sets of rational closed balls whose union is connected and that pairwise intersect at most on their boundary.

Definition 3.1 (Rational complex). We call a set $R := \{B[c_1, r_1], \dots, B[c_k, r_k]\}$ of finitely many closed balls $B[c_i, r_i]$ with rational center $c_i \in \mathbb{Q}^n$ and positive rational radius $r_i \in \mathbb{Q}$ an (*n-dimensional*) *rational complex* if $\bigcup R$ is connected and $B_1, B_2 \in R$ with $B_1 \neq B_2$ implies $B_1^\circ \cap B_2^\circ = \emptyset$. We say that a rational complex is *non-empty*, if $\bigcup R \neq \emptyset$. By $\mathbb{C}\mathbb{Q}^n$ we denote the set of *n-dimensional* rational complexes.

Each rational complex R can be represented by a list of the corresponding rational numbers $c_1, r_1, \dots, c_k, r_k$ and we implicitly assume in the following that this representation is used for the set of rational complexes $\mathbb{C}\mathbb{Q}^n$.

In order to organize the rational complexes that are used to approximate sets it is suitable to use trees. We recall that a *tree* is a set $T \subseteq \mathbb{N}^*$ which is closed under prefix, i.e. $u \sqsubseteq v$ and $v \in T$ implies $u \in T$. A function $b : \mathbb{N} \rightarrow \mathbb{N}$ is called a *bound* of a tree T if $w \in T$ implies $w(i) \leq b(i)$ for all $i = 0, \dots, |w| - 1$, where $|w|$ denotes

the *length* of the word w . A tree is called *finitely branching*, if it has a bound. A tree of rational complexes is understood to be a finitely branching tree T (together with a bound) such that to each node of the tree a rational complex is associated, with the property that these complexes are compactly included in each other if we proceed along paths of the tree and they are disjoint on any particular level of the tree. We write $A \Subset B$ for two sets $A, B \subseteq \mathbb{R}^n$ if the closure \overline{A} of A is included in the interior B° of B and we say that A is *compactly included* in B in this case.

Definition 3.2 (Tree of rational complexes). We call (T, f) a *tree of rational complexes* if $T \subseteq \mathbb{N}^*$ is a finitely branching tree and $f : T \rightarrow \mathbb{C}\mathbb{Q}^n$ is a function such that for all $u, v \in T$ with $u \neq v$

- (1) $u \sqsubseteq v \implies \bigcup f(v) \Subset \bigcup f(u)$,
- (2) $|u| = |v| \implies \bigcup f(u) \cap \bigcup f(v) = \emptyset$.

In the following we assume that finitely branching trees T are represented as a pair (χ_T, b) , where $\chi_T : \mathbb{N}^* \rightarrow \{0, 1\}$ is the characteristic function of T and $b : \mathbb{N} \rightarrow \mathbb{N}$ is a bound of T . Correspondingly, trees (T, f) of rational complexes are then represented in a canonical way by (χ_T, b, f) . We now define which set $A \subseteq [0, 1]^n$ is represented by such a tree (T, f) of rational complexes.

Definition 3.3 (Closed sets and trees of rational complexes). We say that a closed set $A \subseteq \mathbb{R}^n$ is *represented* by a tree (T, f) of n -dimensional rational complexes if one obtains $A = \bigcap_{i=0}^{\infty} \bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w)$.

It is clear that in this way any tree (T, f) of rational complexes actually represents a compact set A . This is because $\bigcup f(w)$ is compact for each $w \in T$ and since T is finitely branching, the set $T \cap \mathbb{N}^i$ is finite for each i , hence $\bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w)$ is compact and hence A is compact too. Vice versa, every compact set $A \subseteq \mathbb{R}^n$ can be represented by a tree of n -dimensional rational complexes. For $[0, 1]^n$ we prove the uniform result that even the map $(T, f) \mapsto A$ is computable and has a computable multi-valued right inverse. We assume that trees of rational complexes are represented as specified above and closed sets A are represented as points in $\mathcal{A}_-([0, 1]^n)$. We recall that a *connectedness component* of a set A is a connected subset of A that is not included in any larger connected subset of A . Any connectedness component of a subset A is closed in A . If $A = \emptyset$, then the only connectedness component is the empty set, otherwise connectedness components are always non-empty.

Proposition 3.4 (Closed sets and complexes). *Let $n \geq 1$. The map $(T, f) \mapsto A$ that maps every tree of n -dimensional rational complexes (T, f) to the closed set $A \subseteq [0, 1]^n$ represented by it, is computable and has a multi-valued computable right inverse. An analogous result holds restricted to infinite trees of non-empty rational complexes and non-empty closed A .*

Proof. It is clear that, given (T, f) and a bound b of T we can actually compute $A \in \mathcal{A}_-([0, 1]^n)$. Firstly, we can explicitly determine all finitely many $w \in T \cap \mathbb{N}^i$ using the bound b and compute $C_i := \bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w) \in \mathcal{A}_-([0, 1]^n)$ for each i . Since intersection of sequences of closed sets is computable on $\mathcal{A}_-([0, 1]^n)$, we can also compute $A := \bigcap_{i=0}^{\infty} C_i$.

We note that if (T, f) is an infinite tree of non-empty rational complexes then the C_i form a decreasing chain of non-empty compact sets and hence $A = \bigcap_{i=0}^{\infty} C_i$ is non-empty too by Cantor's Intersection Theorem.

For the other direction, let us assume that $A \subseteq [0, 1]^n$ is given as the complement of a union of rational open balls $B(c_i, r_i)$. We use the larger cube $Q := [-1, 2]^n$ and we assume that $A = Q \cap (\bigcup_{i=0}^{\infty} B(c_i, r_i))^c$ with $B(c_i, r_i) \cap Q \neq \emptyset$ for all i . Now we show how we can compute a tree (T, f) of rational complexes together with a bound

b that represents A . We proceed inductively over the length $i = |w| = 0, 1, 2, \dots$ of words in the tree T .

We start with the empty node $\varepsilon \in T$ and we assign $f(\varepsilon) = \{Q\}$ to it. Let us now assume that $T \cap \mathbb{N}^i$ has been completely determined, $f(w)$ has been fixed for all $w \in T \cap \mathbb{N}^i$ and $b(j)$ has been determined for all $j < i$. We now determine $T \cap \mathbb{N}^{i+1}$, $f(w)$ for words $w \in T \cap \mathbb{N}^{i+1}$ and $b(i)$. The following is applied to each $w \in T \cap \mathbb{N}^i$:

- (1) Firstly, we copy each rational complex $f(w)$ into $f(w0)$.
- (2) Then the points $B := \{x : d(x, \partial \bigcup f(w)) < 2^{-i-1}\}$ which are close to the boundary are removed from $\bigcup f(w0)$. That is $f(w0)$ is refined such that the resulting union is the original one minus B and all new balls in $f(w0)$ intersect at most on their boundaries. This guarantees $\bigcup f(w0) \subseteq \bigcup f(w)$ (but it might destroy the property that $\bigcup f(w0)$ is connected).
- (3) In the next step $U := \bigcup_{j=0}^i B(c_j, r_j - 2^{-i})$ is removed from $\bigcup f(w0)$. This means that $f(w0)$ is refined such that the union is the original union minus U and all new balls in $f(w0)$ intersect at most on their boundaries. This guarantees that the tree of rational complexes will eventually represent A (we subtract 2^{-i} from the radius here in order to ensure that there is enough space for removing the boundary stripe B in the next step (2) of the induction without removing anything of A).
- (4) Now $\bigcup f(w0)$ is not necessarily connected, but it has only finitely many connectedness components C_0, \dots, C_k that can all be explicitly determined as rational complexes. We copy these rational complexes into $f(w0), \dots, f(wk)$ such that $\bigcup f(wj) = C_j$ for $j = 0, \dots, k$ afterwards. Then all the $\bigcup f(wj)$ are pairwise disjoint and $\bigcup f(wj) \subseteq \bigcup f(w)$ for all $j = 0, \dots, k$. Should the only connectedness component C_0 be the empty set, then we stop the tree T at this point and add no words wj to it.

After this procedure has been completed for all finitely many $w \in T \cap \mathbb{N}^i$, we choose $b(i)$ as the maximal number k (of connectedness components) that occurred for any of these words w . It is clear that then $v(i) \leq b(i)$ for all $v \in T \cap \mathbb{N}^{i+1}$. Moreover, we also have $\bigcup f(wj) \cap \bigcup f(vl) = \emptyset$ for all $w, v \in \mathbb{N}^{i+1}$ with $v \neq w$ since $\bigcup f(w) \cap \bigcup f(v) = \emptyset$ and $\bigcup f(wj) \subseteq \bigcup f(w)$ and $\bigcup f(vl) \subseteq \bigcup f(v)$.

Altogether, (T, f) as constructed here is a tree of rational complexes with bound b . We still need to prove that the set $A_{(T, f)}$ represented by (T, f) is actually A . Let us denote by $A_i := \bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w)$ the closed set represented by the union of all the complexes of height i . In particular $A_{(T, f)} = \bigcap_{i \in \mathbb{N}} A_i$. If $x \in Q \setminus A$, then there are some i, j such that $x \in B(c_j, r_j - 2^{-i})$ and hence x is removed from all the complexes of height i of the tree in step (3) above. Hence $x \notin A_i$, which implies $A_{(T, f)} \subseteq A$. Let, on the other hand, $x \in A$. Then clearly $x \in A_0 = Q$ and has distance from ∂A_0 at least 1. By induction one can show that for each i the distance $d(x, \partial A_i)$ is at least 2^{-i} and hence x cannot be removed in step (2) (and also not in step (3) since only points outside A are removed there). This induction shows that $x \in A_i$ for all i and hence $x \in A_{(T, f)}$. Altogether we have proved $A = A_{(T, f)}$.

We note that if A is a non-empty set, then there is always at least one non-empty connectedness component C_0 in step (4) of the algorithm and the computed tree is automatically an infinite tree of non-empty rational complexes. If A is the empty set, then the computed tree is a finite tree of non-empty rational complexes. \square

We note that this proof in particular shows that we can restrict the investigation in general to trees of non-empty rational complexes (even if we want to include the empty closed set). The previous result has a lot of interesting applications. For instance, if A is represented by (T, f) , then the sets $A_i := \bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w)$ of height i used in the previous proof are of very special form. They are finite unions

of connected sets that are themselves finite unions of rational closed balls. It is easy to see that for a co-c.e. closed A the resulting sequence $(A_i)_{i \in \mathbb{N}}$ is automatically a computable sequence of *bi-computable* sets A_i , which means that the sequences $(d_{A_i})_{i \in \mathbb{N}}$ and $(d_{A_i^c})_{i \in \mathbb{N}}$ of characteristic functions are computable (see [Her99] for more information on bi-computable sets). This is because the maps $R \mapsto d_{\bigcup R}$ and $R \mapsto d_{(\bigcup R)^c}$ of type $\mathcal{CQ}^n \rightarrow \mathcal{C}(\mathbb{R}^n, \mathbb{R})$ are easily seen to be computable. This leads to the following corollary.

Corollary 3.5. *For every non-empty co-c.e. closed set $A \subseteq [0, 1]^n$ there is a computable sequence $(A_i)_{i \in \mathbb{N}}$ of bi-computable compact sets $A_i \subseteq [-1, 2]^n$ that is compactly decreasing, i.e. $A_{i+1} \subseteq A_i$ for all $i \in \mathbb{N}$ and such that $A = \bigcap_{i \in \mathbb{N}} A_i$.*

The representation of closed sets $A \subseteq [0, 1]^n$ by trees of rational complexes also has the advantage that connectedness components of A can easily be expressed in terms of the tree structure. This is made precise by the following lemma. By $[T] := \{p \in \mathbb{N}^{\mathbb{N}} : (\forall i) p|_i \in T\}$ we denote the set of *infinite paths* of T , which is also called the *body* of T . Here $p|_i = p(0)\dots p(i-1) \in \mathbb{N}^*$ denotes the *prefix* of p of length i for each $i \in \mathbb{N}$. According to the following lemma there is bijection between $[T]$ and the set of connectedness components of a non-empty closed set $A \subseteq [0, 1]^n$.

Lemma 3.6 (Connectedness components). *Let (T, f) be an infinite tree of n -dimensional non-empty rational complexes and let $A \subseteq [0, 1]^n$ be the non-empty closed set represented by (T, f) . Then the sets $C_p := \bigcap_{i=0}^{\infty} \bigcup f(p|_i)$ for $p \in [T]$ are exactly all connectedness components of A (without repetitions).*

Proof. Let $A \subseteq [0, 1]^n$ be represented by (T, f) . Firstly, it is clear that every set $C_p = \bigcap_{i=0}^{\infty} \bigcup f(p|_i)$ is included in A for $p \in [T]$. We claim that also $\bigcup_{p \in [T]} C_p = A$. If $x \in A$, then for every i there is a unique $w_i \in T \cap \mathbb{N}^i$ such that $x \in \bigcup f(w_i)$. Since $w \sqsubseteq w_i$ and $w \neq w_i$ imply $\bigcup f(w_i) \subseteq \bigcup f(w)$, it follows that $T_x := \{w_i : i \in \mathbb{N}\}$ constitutes an infinite finitely branching tree and by Weak König's Lemma this tree has an infinite path p such that $x \in C_p$. Now we claim that $\bigcup_{p \in [T]} C_p$ is even a pairwise disjoint union. Let $x \in C_p \cap C_q$ for $p, q \in [T]$ with $p \neq q$. Then there is an $i \in \mathbb{N}$ such that $p|_i \neq q|_i$ and we have $x \in \bigcup f(p|_i) \cap \bigcup f(q|_i)$. This contradicts the fact that (T, f) is a tree of rational complexes. Hence, the union $\bigcup_{p \in [T]} C_p$ is a disjoint union. By definition of a tree of rational complexes, $\bigcup f(p|_i)$ is connected and compact for every $i \in \mathbb{N}$. It follows that C_p is connected, since the intersection of a sequence of continua is a continuum (i.e. connected and compact, see for instance Corollary 6.1.19 in [Eng89]). Altogether, this proves the claim. \square

As another interesting result we can deduce from Proposition 3.4 a classification of the operation that determines a connectedness component. We first define this operation. For short we denote by \mathcal{A}_n the subspace of non-empty closed subsets of $A_-([0, 1]^n)$.

Definition 3.7 (Connectedness components). By $\text{Con}_n : \mathcal{A}_n \rightrightarrows \mathcal{A}_n$ we denote the map with $\text{Con}_n(A) := \{C : C \text{ is a connectedness component of } A\}$ for every $n \geq 1$.

We note that the Weihrauch degree of Weak König's Lemma has been defined and studied in [GM09, BG11b, BG11a, BdBP12, BGM12]. Here we prove that the problem Con_n of finding a connectedness component of a closed set has the same strong Weihrauch degree as Weak König's Lemma for every dimension $n \geq 1$.

Theorem 3.8 (Connectedness components). $\text{Con}_n \equiv_{\text{sw}} \text{WKL}$ for $n \geq 1$.

Proof. Given a set $A \subseteq [0, 1]^n$ we can compute a tree (T, f) of rational complexes that represents A (together with a bound b of T). With the help of Weak König's

Lemma WKL we can find an infinite path $p \in [T]$ of T (since the bound b is available). Then $C = \bigcap_{i=0}^{\infty} \bigcup f(p|_i)$ is a connectedness component of A by Lemma 3.6 and given T, f, p we can actually compute $C \in \mathcal{A}_n$. This proves $\text{Con}_n \leq_W \text{WKL}$ and since WKL is a cylinder (see [BG11b]) this even implies $\text{Con}_n \leq_{sW} \text{WKL}$.

For the other direction $\text{WKL} \leq_{sW} \text{Con}_1$ we use a standard computable embedding $\iota : \{0, 1\}^{\mathbb{N}} \rightarrow [0, 1]$ of Cantor space into the unit interval with a computable right inverse. Given a tree T with infinite paths we can compute the set $A = [T] \in \mathcal{A}_-(\{0, 1\}^{\mathbb{N}})$ of infinite paths and hence we can also compute $\iota(A) \in \mathcal{A}_1$ (see [BG09]). Using Con_1 we obtain a connectedness component $C \in \mathcal{A}_1$ of $\iota(A)$. Since $\iota(\{0, 1\}^{\mathbb{N}})$ is totally disconnected, any connectedness component C of $\iota(A)$ is actually a singleton and hence we can compute x with $C = \{x\}$ (since $[0, 1]$ is compact). Hence $p = \iota^{-1}(x)$ is an infinite path in T . This proves $\text{WKL} \leq_{sW} \text{Con}_1$ and the higher dimensional case can be treated analogously (using the canonical embedding of $[0, 1]$ into $[0, 1]^n$) \square

In [LRZ08] le Roux and Ziegler have studied computability properties of closed sets and their connectedness components. For instance, they prove that any co-c.e. closed set with only finitely many connectedness components has only co-c.e. closed connectedness components and any co-c.e. closed set without co-c.e. closed connectedness components has continuum cardinality many connectedness components. This can easily be deduced from the previous theorem as well as many other properties of connectedness components. For instance, it is well known that there exists a computable tree with countably many infinite paths and a unique non-isolated infinite path which is not even limit computable (see Theorem 2.18 in [CDJS93]). This implies the following result, which resolves the Open Question 4.10 in [LRZ08].

Corollary 3.9. *There exists a non-empty co-c.e. closed set $A \subseteq [0, 1]$ with only countably many connectedness components one of which is not co-c.e. closed (and it is not even the set of accumulation points of a computable sequence).*

We mention that a closed set is the set of accumulation points of a computable sequence if and only if it has a limit computable name (i.e. if it is co-c.e. closed in the halting problem, see [LRZ08, BGM12]). Another consequence of Lemma 3.6 using the the Low Basis Theorem (see [Soa87]) is that every co-c.e. closed set has a low connectedness component in the sense that it is low in the space $\mathcal{A}_-([0, 1]^n)$. We describe this result in the special case of the representation of closed sets considered here.

Corollary 3.10. *Let $A \subseteq [0, 1]^n$ be co-c.e. closed. Then there is a computable sequence $(A_i)_{i \in \mathbb{N}}$ of bi-computable closed sets $A_i \subseteq [0, 1]^n$ and a low $p \in \mathbb{N}^{\mathbb{N}}$ such that $\bigcap_{i=0}^{\infty} A_{p(i)}$ is a connectedness component of A (which is then, in particular, the set of accumulation points of a computable sequence).*

We close this section by mentioning that one can use the representation of closed sets by trees or rational complexes in order to prove that the function $(A, x) \mapsto C$ that maps any non-empty closed set A together with a point $x \in A$ to the connectedness component C of A that contains x is computable. The point x guides the path in the tree of rational complexes that one has to take. This result was already proved in [LRZ08]. We formulate a non-uniform corollary here.

Corollary 3.11. *Every connectedness component of a co-c.e. closed set $A \subseteq [0, 1]^n$ that contains a computable point $x \in [0, 1]^n$ is itself co-c.e. closed.*

We note that in the one-dimensional case an inverse holds true: every non-empty connected co-c.e. closed set $A \subseteq [0, 1]$ contains a computable point. However,

the analogue statement is no longer true from dimension two upwards (see Corollary 5.6). Further interesting results on connected co-c.e. closed sets can be found in [Kih12].

4. BROUWER'S FIXED POINT THEOREM AND CONNECTED CHOICE

In this section we want to prove that the Brouwer Fixed Point Theorem is computably equivalent to connected choice for any fixed dimension. We first define these two operations. By $\mathcal{C}(X, Y)$ we denote the *set of continuous functions* $f : X \rightarrow Y$ and for short we write $\mathcal{C}_n := \mathcal{C}([0, 1]^n, [0, 1]^n)$.

Definition 4.1 (Brouwer Fixed Point Theorem). By $\text{BFT}_n : \mathcal{C}_n \rightrightarrows [0, 1]^n$ we denote the operation defined by $\text{BFT}_n(f) := \{x \in [0, 1]^n : f(x) = x\}$ for $n \in \mathbb{N}$.

We note that BFT_n is well-defined, i.e. $\text{BFT}_n(f)$ is non-empty for all f , since by the Brouwer Fixed Point Theorem every $f \in \mathcal{C}_n$ admits a fixed point x , i.e. with $f(x) = x$. We can also consider the infinite dimensional version of the Brouwer Fixed Point Theorem on the Hilbert cube $[0, 1]^{\mathbb{N}}$, which is represented by the map $\text{BFT}_\infty : \mathcal{C}([0, 1]^{\mathbb{N}}, [0, 1]^{\mathbb{N}}) \rightrightarrows [0, 1]^{\mathbb{N}}$ with $\text{BFT}_\infty(f) := \{x \in [0, 1]^{\mathbb{N}} : f(x) = x\}$. This can also be seen as a special case of the Schauder Fixed Point Theorem and hence BFT_∞ is well-defined too. We now define connected choice.

Definition 4.2 (Connected choice). By $\text{CC}_n : \subseteq \mathcal{A}_n \rightrightarrows [0, 1]^n$ we denote the operation defined by $\text{CC}_n(A) := A$ for all non-empty connected closed $A \subseteq [0, 1]^n$ and $n \in \mathbb{N}$. We call CC_n *connected choice (of dimension n)*.

Hence, connected choice is just the restriction of closed choice to connected sets. We also use the following notation for the set of fixed points of a functions $f \in \mathcal{C}_n$.

Definition 4.3 (Set of fixed points). By $\text{Fix}_n : \mathcal{C}_n \rightarrow \mathcal{A}_n$ we denote the function with $\text{Fix}_n(f) := \{x \in [0, 1]^n : f(x) = x\}$.

It is easy to see that Fix_n is computable, since $\text{Fix}_n(f) := (f - \text{id}|_{[0, 1]^n})^{-1}\{0\}$ and it is well-known that closed sets in \mathcal{A}_n can also be represented as zero sets of continuous functions (see [BW99, BP03]). We note that the Brouwer Fixed Point Theorem can be decomposed to $\text{BFT}_n \supseteq \text{CC}_n \circ \text{Con}_n \circ \text{Fix}_n$.

The main result of this section is that the Brouwer Fixed Point Theorem and connected choice are (strongly) equivalent for any fixed dimension n (see Theorem 4.9 below). An important tool for both directions of the proof is the representation of closed sets by trees of rational complexes. We start with the direction $\text{CC}_n \leq_{\text{sw}} \text{BFT}_n$ that can be seen as a uniformization of an earlier construction of Baigiger [Bai85] that is in turn built on results of Orevkov [Ore63].

We first formulate a stronger conclusion that we can derive from Proposition 3.4 in case of connected sets. In order to express these stronger conclusions we first recall the notion of effective pathwise connectedness as it was introduced in [Bra08]. Essentially, a set is called *effectively pathwise connected*, if for every two points in the set we can compute a path that connects these two points entirely within this set. We need a uniform such notion for sequences.

Definition 4.4 (Effectively pathwise connected). Let $(A_i)_{i \in \mathbb{N}}$ be a sequence of non-empty closed sets $A_i \subseteq \mathbb{R}^n$. Then $(A_i)_{i \in \mathbb{N}}$ is called *pathwise connected*, if there is a function $U : \subseteq \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathcal{C}([0, 1], \mathbb{R}^n)$, such that for every $p \in U(i, x, y)$ with $x, y \in A_i$ we obtain $p(0) = x$, $p(1) = y$ and $\text{range}(p) \subseteq A_i$. Such a U is called a *witness of pathwise connectedness*. If there is a computable such witness U , then $(A_i)_{i \in \mathbb{N}}$ is called *effectively pathwise connected*.

If a (name of a realizer of a) witness U of pathwise connectedness of $(A_i)_{i \in \mathbb{N}}$ can be computed from A , then we say that $(A_i)_{i \in \mathbb{N}}$ is pathwise connected uniformly in

A . We note that any rational complex $R \subseteq \mathbb{C}\mathbb{Q}^n$ is connected and also automatically pathwise connected, due to the simple structure of such complexes. It is easy to see that there is a computable map that maps any rational complex $R \in \mathbb{C}\mathbb{Q}^n$ to a witness of pathwise connectedness of $\bigcup R$. By $d(A, B) := \inf_{a \in A, b \in B} \|a - b\|$ we denote the *minimal distance* between sets $A, B \subseteq \mathbb{R}^n$. We note that $d(A, B^c) > 0$ is equivalent to $A \Subset B$ for non-empty compact $A, B \subseteq \mathbb{R}^n$.

Proposition 4.5 (Connected sets). *Given a non-empty connected closed set $A \subseteq [0, 1]^n$ we can compute a sequence of distance functions $(d_{A_i})_{i \in \mathbb{N}}$ and $(d_{A_i^c})_{i \in \mathbb{N}}$ of non-empty closed sets $A_i \subseteq [-1, 2]^n$ such that:*

- (1) $A = \bigcap_{i=0}^{\infty} A_i$,
- (2) $d(A_{i+1}, A_i^c) > 0$ for all $i \in \mathbb{N}$,
- (3) $(A_i)_{i \in \mathbb{N}}$ is pathwise connected uniformly in A .

Proof. Given a non-empty connected closed $A \subseteq [0, 1]^n$ we can compute an infinite tree of non-empty rational complexes (T, f) that represents A by Proposition 3.4. Since A is connected, A is its only connectedness component and by Lemma 3.6 there is exactly one infinite path $p \in [T]$. If we can find this path, then $A_i := \bigcup f(p|_i)$ is a sequence of closed sets $A_i \subseteq [-1, 2]^n$ with $A_{i+1} \Subset A_i$ for all i , which implies $d(A_{i+1}, A_i^c) > 0$ and $A = \bigcap_{i=0}^{\infty} A_i$. Since $f(p|_i)$ is a rational complex, it is straightforward how to determine d_{A_i} and $d_{A_i^c}$, given this complex and since $\bigcup f(p|_i)$ is connected it is also automatically pathwise connected and a witness U for pathwise connectedness can be easily computed.

It remains to show how we can compute the unique infinite path p in T . For each fixed i there are only finitely many words $w_0, \dots, w_k \in T \cap \mathbb{N}^i$ and due to connectedness of A and $\bigcup f(w_j)$ and the fact that all the $\bigcup f(w_j)$ are pairwise disjoint, it follows that there is exactly one such w_a with $A \subseteq \bigcup f(w_a)$. Due to compactness of the $\bigcup f(w_j)$ all the other w_j with $j \neq a$ will eventually be covered by negative information given as input for A and if this happens it can be recognized. Hence, one just needs to wait until all the $\bigcup f(w_j)$ except one are covered by negative information in order to identify w_a . Then $w_a \sqsubseteq p$ and by a repetition of this procedure for each i one can compute p . \square

Now we use Proposition 4.5 to prove that every non-empty connected closed set $A \subseteq [0, 1]^n$ can be effectively translated into a continuous function $f \in \mathcal{C}_n$ that has all its fixed points in A . The idea is to compute a compactly decreasing sequence $(A_i)_{i \in \mathbb{N}}$ of closed sets according to the previous proposition together with points $x_i \in A_i$ and paths p_i in A_i that connect x_{i+1} with x_i . In some sense we then use these paths like Ariadne's thread in order to construct a function f that is a modified identity with all fixed points shifted towards A along the given paths. By $\|f\| := \sup_{x \in [0, 1]^n} \|f(x)\|$ we denote the supremum norm for continuous functions $f : [0, 1]^n \rightarrow [0, 1]^n$.

Lemma 4.6. $\text{CC}_n \leq_{\text{SW}} \text{BFT}_n$ for all $n \geq 1$.

Proof. Given a non-empty closed and connected set $A \subseteq [0, 1]^n$ we will compute a function $f \in \mathcal{C}_n$ such that all fixed points of f are included in A . Firstly, we compute the sequences $(d_{A_i})_{i \in \mathbb{N}}$ and $(d_{A_i^c})_{i \in \mathbb{N}}$ according to Proposition 4.5.

Without loss of generality, we can assume that $A \subseteq [2^{-3}, 1 - 2^{-3}]^n$ and all $A_i \subseteq [2^{-4}, 1 - 2^{-4}]^n =: Q$. This can always be achieved using a suitable computable homeomorphism $T : [-1, 2]^n \rightarrow [2^{-4}, 1 - 2^{-4}]^n$ that is applied to all input data and afterwards the fixed point x that is found is transferred back by $T^{-1}(x)$.

Since we can compute the sequences of distance functions $(d_{A_i})_{i \in \mathbb{N}}$ we can also find a sequence of points $(x_i)_{i \in \mathbb{N}}$ with $x_i \in A_i$ for all $i \in \mathbb{N}$. Since $(A_i)_{i \in \mathbb{N}}$ is pathwise connected uniformly in A , we can also compute a sequence $(p_i)_{i \in \mathbb{N}}$ of continuous

functions $p_i : [0, 1]^n \rightarrow [0, 1]^n$ such that $p_i(0) = x_{i+1}$, $p_i(1) = x_i$ and $\text{range}(p_i) \subseteq A_i$. We can also uniformly compute a sequence $(D_i)_{i \in \mathbb{N}}$ of functions $D_i : [0, 1]^n \rightarrow [0, 1]$ defined by

$$D_i(x) := \frac{d(x, A_{i+1})}{d(x, A_{i+1}) + d(x, A_i^c)}$$

for all $x \in [0, 1]^n$ and $i \in \mathbb{N}$. Since $d(A_{i+1}, A_i^c) > 0$ for all $i \in \mathbb{N}$ it follows that the denominator is always non-zero and hence the functions D_i are well-defined. We obtain $D_i(x) = 0 \iff x \in A_{i+1}$ and $D_i(x) = 1 \iff x \in \overline{A_i^c}$.

We now compute a continuous function $f : [0, 1]^n \rightarrow [0, 1]^n$ with $\text{BFT}_n(f) \subseteq A$. The function f will be defined as $f := \text{id} + 2^{-4} \sum_{i=0}^{\infty} g_i$ using further continuous functions g_i . As an abbreviation we write $G_i := \sum_{j=0}^i g_j$ for the partial sums. We start with

$$g_0(x) := \begin{cases} 2^{-1} \frac{x_2 - x}{\|x_2 - x\|} d(x, A_1) & \text{if } x \notin A_2 \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in [0, 1]^n$. In the next step we define inductively

$$g_{i+1}(x) := \begin{cases} 2^{-i-2} \frac{G_i(x)}{\|G_i(x)\|} & \text{if } x \notin A_{i+1} \\ 2^{-i-2} \frac{p_{i+2}(D_{i+1}(x)) - x}{\|p_{i+2}(D_{i+1}(x)) - x\|} D_{i+1}(x) & \text{if } x \in A_{i+1} \setminus A_{i+2} \\ 0 & \text{if } x \in A_{i+2} \end{cases}$$

for all $x \in [0, 1]^n$ and $i \in \mathbb{N}$.

We first prove that all g_i and $\sum_{i=0}^{\infty} g_i(x)$ are well-defined and

$$(1) \quad x \in A = \bigcap_{i=0}^{\infty} A_i \iff \sum_{i=0}^{\infty} g_i(x) = 0 \iff f(x) = x.$$

The second equivalence follows immediately from the definition of f (once we know that the g_i and $\sum_{i=0}^{\infty} g_i$ are well defined). If $x \in \bigcap_{i=0}^{\infty} A_i$, then it follows immediately that $g_i(x) = 0$ for all i and hence $\sum_{i=0}^{\infty} g_i(x) = 0$. If $x \notin \bigcap_{i=0}^{\infty} A_i$, then there is a minimal $m \in \mathbb{N}$ with $x \notin A_m$, since $(A_i)_{i \in \mathbb{N}}$ is decreasing. If $m \in \{0, 1\}$, then $x \notin A_1$ and hence $x \notin A_2$. Since $x_2 \in A_2$ it follows that $\|x_2 - x\| \neq 0$. We also obtain $d(x, A_1) > 0$ and thus $g_0(x) \neq 0$. This implies

$$(2) \quad \sum_{i=0}^{\infty} g_i(x) = g_0(x) + \sum_{i=1}^{\infty} 2^{-i-1} \frac{g_0(x)}{\|g_0(x)\|} = 2^{-1} \frac{x_2 - x}{\|x_2 - x\|} (d(x, A_1) + 1) \neq 0.$$

If $m > 1$, then $x \in A_{m-1} \setminus A_m$ and it follows that $g_i(x) = 0$ for $i \leq m-2$. Since $\text{range}(p_m) \subseteq A_m$ and $x \notin A_m$ it follows that $\|p_m(D_{m-1}(x)) - x\| \neq 0$. We also have $D_{m-1}(x) \neq 0$ and hence $g_{m-1}(x) \neq 0$. This implies

$$\sum_{i=0}^{\infty} g_i(x) = g_{m-1}(x) + \sum_{i=m}^{\infty} g_i(x) = 2^{-m} \frac{p_m(D_{m-1}(x)) - x}{\|p_m(D_{m-1}(x)) - x\|} (D_{m-1}(x) + 1) \neq 0.$$

These two cases together prove the first equivalence in (1) together with the fact that all g_i and $\sum_{i=0}^{\infty} g_i(x)$ are well-defined. We can also conclude from Equation (1) that A is exactly the set of fixed point of f .

Next we want to show that by $f := \text{id} + 2^{-4} \sum_{i=0}^{\infty} g_i$ actually a continuous function of type $f : [0, 1]^n \rightarrow [0, 1]^n$ is defined. We show that $f([0, 1]^n) \subseteq [0, 1]^n$. If $x \in [0, 1]^n \setminus A_0$, then Equation (2) implies $f(x) = x + 2^{-5} \frac{x_2 - x}{\|x_2 - x\|} (d(x, A_1) + 1)$, which means that f moves x towards $x_2 \in A_0 \subseteq Q$ and, in particular, $f(x) \in [0, 1]^n$. If $x \in A_0 \subseteq Q$, then $f(x) = x + 2^{-4} \sum_{i=0}^{\infty} g_i(x) \in [0, 1]^n$ since $\|g_i\| = \sup_{x \in [0, 1]^n} \|g_i(x)\| \leq 2^{-i-1}$ and hence $\|2^{-4} \sum_{i=0}^{\infty} g_i(x)\| \leq 2^{-4}$. Now we prove that f is also continuous. First we show that each function g_i is continuous. We start

with g_0 . If x approaches ∂A_2 from the outside, then eventually $d(x, A_1) = 0$ and hence $g_0(x) = 0$. This means that g_0 is continuous. We now continue with g_{i+1} . If $x \in \partial A_{i+1} = \overline{\partial A_{i+1}^c}$, then $D_{i+1}(x) = 1$ and hence $p_{i+2}(D_{i+1}(x)) = x_{i+2}$ and we obtain

$$g_{i+1}(x) = 2^{-i-2} \frac{p_{i+2}(D_{i+1}(x)) - x}{\|p_{i+2}(D_{i+1}(x)) - x\|} D_{i+1}(x) = 2^{-i-2} \frac{x_{i+2} - x}{\|x_{i+2} - x\|}.$$

If, on the other hand, x approaches ∂A_{i+1} from the outside of A_{i+1} , then $D_i(x) \rightarrow 0$ and x is eventually in A_i and hence $g_j(x) = 0$ for $j \leq i-1$ and $G_i = g_i$. In case $i > 0$ we use $D_i(x) \rightarrow 0$ in order to conclude

$$g_{i+1}(x) = 2^{-i-2} \frac{G_i(x)}{\|G_i(x)\|} = 2^{-i-2} \frac{p_{i+1}(D_i(x)) - x}{\|p_{i+1}(D_i(x)) - x\|} \rightarrow 2^{-i-2} \frac{x_{i+2} - x}{\|x_{i+2} - x\|}.$$

In case of $i = 0$ we obtain

$$g_1(x) = 2^{-2} \frac{G_0(x)}{\|G_0(x)\|} = 2^{-2} \frac{x_2 - x}{\|x_2 - x\|}.$$

Finally, if x approaches ∂A_{i+2} from the outside, then $D_{i+1}(x) \rightarrow 0$ and x is eventually in A_{i+1} . Hence

$$g_{i+1}(x) = 2^{-i-2} \frac{p_{i+2}(D_{i+1}(x)) - x}{\|p_{i+2}(D_{i+1}(x)) - x\|} D_{i+1}(x) \rightarrow 0.$$

Altogether, this proves that the case distinction in the definition of g_{i+1} is continuous and hence all the functions g_i and f are continuous and can be uniformly computed in the input A .

We also obtain $\text{BFT}_n(f) = A$ by Equation (1), which proves $\text{CC}_n \leq_{\text{SW}} \text{BFT}_n$. \square

We note that the proof shows more than necessary. We only need that $\text{BFT}_n(f) \subseteq A$ and we even obtain equality.

For the other direction $\text{BFT}_n \leq_{\text{SW}} \text{CC}_n$ of the reduction we uniformize ideas of Joseph S. Miller [Mil02] and we use again the representation of closed sets by trees of rational complexes. We also exploit the fact that each rational complex can easily be converted into a simplicial complex. We recall that a *proper n -dimensional rational simplex* is the convex hull of $n+1$ geometrically independent rational points in $[0, 1]^n$ and a *proper rational simplicial complex* is a set of finitely many proper simplexes such that the interiors of distinct simplexes are disjoint. By SQ^n we denote the set of all such proper rational simplicial complexes and we assume that each such complex is represented by a specification of a list of $n+1$ geometrically independent rational points for each simplex in the complex. Hence, it is clear that there is a computable $f : \text{CQ}^n \rightarrow \text{SQ}^n$ with $\bigcup f(R) = \bigcup R$. That means that we can easily translate each tree of rational complexes into a corresponding tree of rational simplicial complexes (understood in the analogous way). Miller (see Section 2.3 in [Mil02]) has proved the following result.

Proposition 4.7 (Topological index, Miller 2002). *There is a computable topological index function $\text{ind} : \subseteq \text{C}_n \times \text{SQ}^n \rightarrow \mathbb{Z}$ such that for all $f \in \text{C}_n$ and $S, S_1, S_2 \in \text{SQ}^n$ such that f has no fixed points on $\partial \bigcup S_1$ and $\partial \bigcup S_2$ the following holds:*

- (1) $\text{ind}(f, S)$ is defined if and only if $f(x) \neq x$ for all $x \in \partial \bigcup S$.
- (2) $\text{ind}(f, S) \neq 0$ implies that $f(x) = x$ for some $x \in \bigcup S$.
- (3) $\text{ind}(f, \{[0, 1]^n\}) \neq 0$.
- (4) If $\{x \in \bigcup S_1 : f(x) = x\} = \{x \in \bigcup S_2 : f(x) = x\}$, then one obtains $\text{ind}(f, S_1) = \text{ind}(f, S_2)$.
- (5) If $\bigcup S_1$ and $\bigcup S_2$ are disjoint, then $\text{ind}(f, S_1 \cup S_2) = \text{ind}(f, S_1) + \text{ind}(f, S_2)$.

Here we assume that f has no fixed points on $\partial \bigcup S_1$ and $\partial \bigcup S_2$.

The proof of this result uses simplicial homology theory. Computability aspects of homology have also been studied by in a discrete setting by Kaczynski et. al. [KMM04] and in the context of computable analysis by Collins [Col08, Col09]. We essentially use Miller's ideas to reduce the Brouwer Fixed Point Theorem uniformly to connected choice. First we prove that the map $\text{Con}_n \circ \text{Fix}_n$ is computable (which might be surprising in light of Theorem 3.8).

Proposition 4.8. $\text{Con}_n \circ \text{Fix}_n : \mathcal{C}_n \rightrightarrows \mathcal{A}_n$ is computable for all $n \in \mathbb{N}$.

Proof. Given a continuous function $f \in \mathcal{C}_n$ we can easily compute the set of fixed points $A := \{x \in [0, 1]^n : f(x) = x\} \in \mathcal{A}_n$. Using Proposition 3.4 we can compute a tree (T, f) of rational complexes that represents A . Using Proposition 4.7 we can now identify an infinite path p in T and hence by Lemma 3.6 a connectedness component C of A .

We start with the empty node ε in T . Given a node $w \in T$, we construct an extension $wi \in T$ that is part of an infinite path as follows. If there are several rational complexes $R_0 = f(w0), \dots, R_k = f(wk)$ to be considered, then we compute the corresponding simplicial complexes S_0, \dots, S_k . Since $A \subseteq \bigcup_{j=0}^k S_j$, it is clear that f cannot have any fixed point on any of the boundaries $\partial \bigcup S_j$ and hence we can compute the indexes $\text{ind}(f, S_0), \dots, \text{ind}(f, S_k)$ by Proposition 4.7 (1). One of them, say $\text{ind}(f, S_i)$, must be different from 0, as one can see inductively using Proposition 4.7 (3)-(5). By Proposition 4.7 (2) this means that f has a fixed point in S_i , which means that $A \cap \bigcup S_i \neq \emptyset$. We use this wi as an extension of w and we proceed inductively in the same manner.

Altogether, this algorithm produces an infinite path p of T and we can compute the connected component $C := \{\bigcap_{i=0}^{\infty} \bigcup f(p|i) : p \in [T]\} \in \mathcal{A}_n$ of A . This shows that $\text{Con}_n \circ \text{Fix}_n$ is computable. \square

Since $\text{BFT}_n \supseteq \text{CC}_n \circ \text{Con}_n \circ \text{Fix}_n$ we can directly conclude $\text{BFT}_n \leq_{\text{sW}} \text{CC}_n$ for all n . Together with Lemma 4.6 we obtain the following theorem.

Theorem 4.9 (Brouwer Fixed Point Theorem). $\text{BFT}_n \equiv_{\text{sW}} \text{CC}_n$ for all $n \in \mathbb{N}$.

It is easy to see that in general the Brouwer Fixed Point Theorem and connected choice are not independent of the dimension. In case of $n = 0$ the space $[0, 1]^n$ is the one-point space $\{0\}$ and hence $\text{BFT}_0 \equiv_{\text{sW}} \text{CC}_0$ are both computable. In case of $n = 1$ connected choice was already studied in [BG11a] and it was proved that it is equivalent to the Intermediate Value Theorem IVT (see Definition 6.1 and Theorem 6.2 in [BG11a]).

Corollary 4.10 (Intermediate Value Theorem). $\text{IVT} \equiv_{\text{sW}} \text{BFT}_1 \equiv_{\text{sW}} \text{CC}_1$.

It is also easy to see that the Brouwer Fixed Point Theorem BFT_2 in dimension two is more complicated than in dimension one. For instance, it is known that the Intermediate Value Theorem IVT always offers a computable function value for a computable input, whereas this is not the case for the Brouwer Fixed Point Theorem BFT_2 by Baigge's counterexample [Bai85]. We continue to discuss this topic in Section 5.

Here we point out that Proposition 4.8 implies that the fixed point set $\text{Fix}_n(f)$ of every computable function $f : [0, 1]^n \rightarrow [0, 1]^n$ has a co-c.e. closed connectedness component. Joseph S. Miller observed that also any co-c.e. closed superset of such a set is the fixed point set of some computable function and the following result is a uniform version of this observation. We denote by $(f, g) : \subseteq X \rightrightarrows Y \times Z$ the *juxtaposition* of two functions $f : \subseteq X \rightrightarrows Y$ and $g : \subseteq X \rightrightarrows Z$, defined by $(f, g)(x) = (f(x), g(x))$.

Theorem 4.11 (Fixability). *($\text{Fix}_n, \text{Con}_n \circ \text{Fix}_n$) is computable and has a multi-valued computable right inverse for all $n \in \mathbb{N}$.*

Proof. It follows directly from Proposition 4.8 and the fact that Fix_n is computable that $(\text{Fix}_n, \text{Con}_n \circ \text{Fix}_n)$ is computable for all $n \in \mathbb{N}$. We now describe how a right inverse $R : \subseteq \mathcal{A}_n \times \mathcal{A}_n \rightrightarrows \mathcal{C}_n$ can be computed. Firstly, given (A, C) such that $A \in \mathcal{A}_n$ and C is a connectedness component of A , we can find some $f \in \mathcal{C}_n$ such that $\text{Fix}_n(f) = C$ following the algorithm that is specified in the proof of Lemma 4.6. We can also find a continuous $g : [0, 1]^n \rightarrow [0, 1]$ such that $g^{-1}\{0\} = A$ (see [BW99]). Then we can also compute a continuous h with

$$h(x) := (1 - g(x))x + f(x)g(x)$$

and since this is a convex combination of id and f , it follows that h is actually a continuous function $h : [0, 1]^n \rightarrow [0, 1]^n$. Finally,

$$h(x) = x \iff (f(x) - x)g(x) = 0 \iff x \in C \cup A = A.$$

That is $\text{Fix}_n(h) = A$. That is the function R with $(A, C) \mapsto h$ is a suitable computable right inverse of $(\text{Fix}_n, \text{Con}_n \circ \text{Fix}_n)$. \square

Roughly speaking a closed set $A \in \mathcal{A}_n$ together with one of its connectedness components is as good as a continuous function $f \in \mathcal{C}_n$ with A as set of fixed points. As a non-uniform corollary we obtain immediately Miller's original result.

Corollary 4.12 (Fixable sets, Miller 2002). *A set $A \subseteq [0, 1]^n$ is the set of fixed points of a computable function $f : [0, 1]^n \rightarrow [0, 1]^n$ if and only if it is non-empty and co-c.e. closed and contains a co-c.e. closed connectedness component.*

We can also derive other interesting results from Theorem 4.11. For instance we can derive an upper bound on how complex a continuous functions needs to be that has an arbitrary given non-empty co-c.e. closed set as fixed point set.

Corollary 4.13. *Let $A \subseteq [0, 1]^n$ be a non-empty co-c.e. closed set. Then there is a continuous function $f : [0, 1]^n \rightarrow [0, 1]^n$ that is low as a point in \mathcal{C}_n and has A as fixed point set.*

This result follows from an application of the Uniform Low Basis Theorem (see [BdBP12]) since Fix_n has a weakly computable right inverse by Theorem 4.11 and Theorem 3.8. We mention that a function f that is low as a point in \mathcal{C}_n is not necessarily low as a function in the sense that $f \leq_{\text{sw}} \mathbb{L}$ (where $\mathbb{L} = \mathbb{J}^{-1} \circ \lim$ is the composition of the inverse of the Turing jump \mathbb{J} and the limit operation), but one only obtains $f \leq_{\text{w}} \mathbb{L}$ here (see [BdBP12, BGM12] for a discussion of low functions).

5. ASPECTS OF DIMENSION

In this section we want to discuss aspects of dimension of connected choice and the Brouwer Fixed Point Theorem. Our main result is that connected choice is computably universal or complete from dimension three upwards in the sense that it is strongly equivalent to Weak König's Lemma, which is one of the degrees of major importance. In order to prove this result, we use the following geometric construction.

Proposition 5.1 (Twisted cube). *The function*

$$T : \subseteq \mathcal{A}_-[0, 1] \rightarrow \mathcal{A}_3, A \mapsto (A \times [0, 1] \times \{0\}) \cup (A \times A \times [0, 1]) \cup ([0, 1] \times A \times \{1\})$$

is computable and maps non-empty closed sets $A \subseteq [0, 1]$ to non-empty connected closed sets $T(A) \subseteq [0, 1]^3$.

Here tuples $(x_1, x_2, x_3) \in T(A)$ have the property that at least one of the first two components provide a solution $x_i \in A$, but the third component provides the additional information which one surely does. If x_3 is close to 1, then surely $x_2 \in A$ and if x_3 is close to 0, then surely $x_1 \in A$. If x_3 is neither close to 0 nor 1, then both $x_1, x_2 \in A$. Hence, there is a computable function H such that $C_{[0,1]} = H \circ CC_3 \circ T$, which proves $C_{[0,1]} \leq_{\text{sW}} CC_3$. Together with Theorem 4.9 we obtain the following conclusion.

Theorem 5.2 (Completeness of three dimensions). *For $n \geq 3$ we obtain*

$$CC_n \equiv_{\text{sW}} \text{BFT}_n \equiv_{\text{sW}} \text{BFT}_\infty \equiv_{\text{sW}} \text{WKL} \equiv_{\text{sW}} C_{[0,1]}.$$

Proof. We note that the reduction $CC_n \leq_{\text{sW}} C_{[0,1]^n}$ holds for all $n \in \mathbb{N}$, since connected choice is a just a restriction of closed choice and the equivalences

$$C_{[0,1]^n} \equiv_{\text{sW}} C_{[0,1]^n} \equiv_{\text{sW}} C_{[0,1]} \equiv_{\text{sW}} \text{WKL}$$

are known for all $n \geq 1$ (see [BdBP12]). The equivalence $CC_n \equiv_{\text{sW}} \text{BFT}_n$ has been proved in Theorem 4.9 for all $n \in \mathbb{N}$. We mention that $\text{BFT}_n \leq_{\text{sW}} \text{BFT}_\infty$ can be proved as follows. The function

$$K : \mathcal{C}([0, 1]^n, [0, 1]^n) \rightarrow \mathcal{C}([0, 1]^\mathbb{N}, [0, 1]^\mathbb{N}), f \mapsto ((x_i) \mapsto (f(x_1, \dots, x_n), 0, 0, 0, \dots))$$

is computable and together with the projection on the first n -coordinates this yields the reduction $\text{BFT}_n \leq_{\text{sW}} \text{BFT}_\infty$. Since

$$\mathcal{C}([0, 1]^\mathbb{N}, [0, 1]^\mathbb{N}) \rightarrow \mathcal{A}_-([0, 1]^\mathbb{N}), f \mapsto (f - \text{id}_{[0,1]^\mathbb{N}})^{-1}\{0\}$$

is computable too, it follows that $\text{BFT}_\infty \leq_{\text{sW}} C_{[0,1]^\mathbb{N}}$ holds. Finally, $C_{[0,1]} \leq_{\text{sW}} CC_n$ follows for $n \geq 3$ from Proposition 5.1. \square

In particular, we get the Baigger counterexample for dimension $n \geq 3$ as a consequence of Theorem 5.2. A superficial reading of the results of Orevkov [Ore63] and Baigger [Bai85] can lead to the wrong conclusion that they actually provide a reduction of Weak König's Lemma to the Brouwer Fixed Point Theorem BFT_n of any dimension $n \geq 2$. However, this is only correct in a non-uniform way and the corresponding uniform result is still open and does not follow from the known constructions. The Orevkov-Baigger result is built on the following fact.

Proposition 5.3 (Mixed cube). *The function*

$$M : \subseteq \mathcal{A}_-[0, 1] \rightarrow \mathcal{A}_2, A \mapsto (A \times [0, 1]) \cup ([0, 1] \times A)$$

is computable and maps non-empty closed sets $A \subseteq [0, 1]$ to non-empty connected closed sets $M(A) \subseteq [0, 1]^2$.

It follows straightforwardly from the definition that the pairs $(x, y) \in M(A)$ are such that one out of two components x, y is actually in A . In order to express the uniform content of this fact, we introduce the concept of a fraction.

Definition 5.4 (Fractions). Let $f : \subseteq X \rightrightarrows Y$ be a multi-valued function and $0 < n \leq m \in \mathbb{N}$. We define the *fraction* $\frac{n}{m}f : \subseteq X \rightrightarrows Y^m$ by

$$\frac{n}{m}f(x) := \{(y_1, \dots, y_m) \in \text{range}(f)^m : |\{i : y_i \in f(x)\}| \geq n\}$$

for all $x \in \text{dom}(\frac{n}{m}f) := \text{dom}(f)$.

The idea of a fraction $\frac{n}{m}f$ is that it provides m potential answers for f , at least $n \leq m$ of which have to be correct. The uniform content of the Orevkov-Baigger construction is then summarized in the following result.

Proposition 5.5 (Connected choice in dimension two). $\frac{1}{2}C_{[0,1]} \leq_{\text{sW}} CC_2 \leq_{\text{sW}} C_{[0,1]}$.

Proof. With Proposition 5.3 we obtain $\frac{1}{2}\mathbb{C}_{[0,1]} = \mathbb{C}\mathbb{C}_2 \circ M$ and hence $\frac{1}{2}\mathbb{C}_{[0,1]} \leq_{\text{sW}} \mathbb{C}\mathbb{C}_2$. The other reduction follows from $\mathbb{C}\mathbb{C}_2 \leq_{\text{sW}} \mathbb{C}_{[0,1]^2} \equiv_{\text{sW}} \mathbb{C}_{[0,1]}$. \square

That is, given a closed set $A \subseteq [0, 1]$ we can utilize connected choice $\mathbb{C}\mathbb{C}_2$ of dimension 2 in order to find a pair of points (x, y) one of which is in A . This result directly implies the counterexample of Baigger [Bai85] because the fact that there are non-empty co-c.e. closed sets $A \subseteq [0, 1]$ without computable point immediately implies that $\frac{1}{2}\mathbb{C}_{[0,1]}$ is not non-uniformly computable (i.e. there are computable inputs without computable outputs) and hence $\mathbb{C}\mathbb{C}_2$ is also not non-uniformly computable.

Corollary 5.6 (Orevkov 1963, Baigger 1985). *There exists a computable function $f : [0, 1]^2 \rightarrow [0, 1]^2$ that has no computable fixed point $x \in [0, 1]^2$. There exists a non-empty connected co-c.e. closed subset $A \subseteq [0, 1]^2$ without computable point.*

We mention that Proposition 5.5 does not directly imply $\mathbb{C}_{[0,1]} \equiv_{\text{sW}} \mathbb{C}\mathbb{C}_2$, since $\frac{1}{2}\mathbb{C}_{[0,1]} <_{\text{W}} \mathbb{C}\mathbb{C}_2$. In fact, we can prove an even stronger result which shows that $\frac{1}{2}\mathbb{C}_{[0,1]}$ computes almost nothing, not even choice for the two point space. This means that Proposition 5.5 has very little uniform content.

Proposition 5.7. $\mathbb{C}_{\{0,1\}} \not\leq_{\text{W}} \frac{1}{2}\mathbb{C}_{[0,1]}$.

Proof. We use $\mathbb{C}_{\{0,1\}} \equiv_{\text{sW}} \text{LLPO}$ and by ψ_- we denote the representation of \mathcal{A}_1 . We recall that $\text{LLPO} : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ is defined such that for $j \in \{0, 1\}$ and $p \in \{0, 1\}^{\mathbb{N}}$ it holds that

$$j \in \text{LLPO}(p) \iff (\forall i) p(2i + j) = 0,$$

where $\text{dom}(\text{LLPO})$ contains all sequences p such that $p(k) \neq 0$ for at most one k .

Let us now assume that $\text{LLPO} \leq_{\text{W}} \frac{1}{2}\mathbb{C}_{[0,1]}$ holds. Then there are continuous H, K such that $H(\text{id}, FK)$ realizes LLPO whenever F realizes $\frac{1}{2}\mathbb{C}_{[0,1]}$. We consider the inputs $p_{ji} := 0^{2i+j+1}10^{\mathbb{N}}$ and $p_{\infty} := 0^{\mathbb{N}}$ for LLPO . We obtain $\text{LLPO}(p_{ji}) = \{j\}$ for $j \in \{0, 1\}$ and $\text{LLPO}(p_{\infty}) = \{0, 1\}$. Now we let $K_{ji} := \psi_-(K(p_{ji}))$ and $K_{\infty} := \psi_-(K(p_{\infty}))$. These sets are all non-empty compact subsets of $[0, 1]$, hence there are $x_{ji} \in K_{ji}$ with names q_{ji} (with respect to the signed-digit representation of $[0, 1]$). Due to compactness for each $j \in \{0, 1\}$ there is some convergent subsequence (q_{ji_k}) of q_{ji} and we let $x_j := \lim_{k \rightarrow \infty} q_{ji_k}$ and $x_j := \lim_{k \rightarrow \infty} x_{ji_k}$.

Now we claim that $x_j \in K$ for both $j \in \{0, 1\}$ and by symmetry it suffices to prove this for $j = 0$. Let us assume that $x_0 \notin K_{\infty}$. Then by continuity of K there exists some open neighborhood U of x_0 and some $k \in \mathbb{N}$ such that $U \cap \psi_-(K(r)) = \emptyset$ for all $r \in \text{dom}(\psi_-K)$ with $0^k \sqsubseteq r$. Almost all p_{0i} satisfy this condition, which implies $U \cap K_{0i} = \emptyset$ for almost all i . This contradicts the construction of x_0 . Hence $x_0 \in K_{\infty}$ follows and analogously $x_1 \in K_{\infty}$.

Hence there is some realizer F_{∞} of $\frac{1}{2}\mathbb{C}_{[0,1]}$ with $F_{\infty}K(p_{\infty}) = \langle q_0, q_1 \rangle$. Without loss of generality we can assume $H\langle p_{\infty}, \langle q_0, q_1 \rangle \rangle = 0$. There are also realizers F_k of $\frac{1}{2}\mathbb{C}_{[0,1]}$ with $F_kK(p_{1i_k}) = \langle q_{0i_k}, q_{1i_k} \rangle$, since the second component contains a correct answer. Hence $H\langle p_{1i_k}, \langle q_{0i_k}, q_{1i_k} \rangle \rangle = 1$ has to hold. Continuity of H now implies $H\langle p_{\infty}, \langle q_0, q_1 \rangle \rangle = 1$, which is a contradiction. \square

In the following result we summarize the known relations for connected choice in dependency of the dimension.

Proposition 5.8. *We obtain $\mathbb{C}\mathbb{C}_0 <_{\text{W}} \mathbb{C}\mathbb{C}_1 <_{\text{W}} \mathbb{C}\mathbb{C}_2 \leq_{\text{W}} \mathbb{C}\mathbb{C}_n \equiv_{\text{W}} \mathbb{C}_{[0,1]}$ for all $n \geq 3$.*

Proof. It is clear that $\mathbb{C}\mathbb{C}_n \leq_{\text{sW}} \mathbb{C}\mathbb{C}_{n+1}$ holds for all $n \in \mathbb{N}$, since the computable map $A \mapsto A \times [0, 1]$ maps connected closed sets of dimension n to such sets of dimension $n + 1$. The reduction $\mathbb{C}\mathbb{C}_0 <_{\text{W}} \mathbb{C}\mathbb{C}_1$ is strict, since $\mathbb{C}\mathbb{C}_0$ is computable and $\mathbb{C}\mathbb{C}_1$ is not. The reduction $\mathbb{C}\mathbb{C}_1 <_{\text{W}} \mathbb{C}\mathbb{C}_2$ is strict, since $\mathbb{C}\mathbb{C}_1$ is non-uniformly

computable (since any non-empty connected co-c.e. closed set $A \subseteq [0, 1]$ is either a singleton and hence computable or it has non-empty interior and contains even a rational point) and CC_2 is not non-uniformly computable by Corollary 5.6. \square

Altogether, we are left with the major open problem whether $\text{C}_{[0,1]} \leq_{\text{W}} \text{CC}_2$ holds or not. We have a conjecture but currently no proof of it.

Conjecture 5.9 (Brouwer Fixed Point Theorem in dimension two). *We conjecture that $\text{CC}_2 <_{\text{W}} \text{C}_{[0,1]}$.*

We mention that this conjecture is equivalent to the property that CC_2 is not parallelizable, i.e. to the property that $\widehat{\text{CC}_2} \equiv_{\text{W}} \text{CC}_2$ does not hold. This is because $\widehat{\text{CC}_2} \equiv_{\text{W}} \text{C}_{[0,1]}$ follows from $\text{C}_{\{0,1\}} \leq_{\text{sW}} \text{CC}_2$ and $\widehat{\text{C}_{\{0,1\}}} \equiv_{\text{sW}} \text{C}_{[0,1]}$ and the fact that parallelization is a closure operator, which are known results (see [BG11a]). We will study a weaker property than parallelizability in Section 7.

We close this section with some further evidence that supports our Conjecture 5.9 and that indicates special properties of dimension two, which are not shared by higher dimensions. Firstly, one can extend Proposition 5.3 straightforwardly to higher dimensions (by choosing $A \mapsto (A \times A \times [0, 1]) \cup (A \times [0, 1] \times A) \cup ([0, 1] \times A \times A)$ in dimension three and so forth) and that leads to the following generalization of Proposition 5.5.

Proposition 5.10. $\frac{n-1}{n} \text{C}_{[0,1]} \leq_{\text{sW}} \text{CC}_n \leq_{\text{sW}} \text{C}_{[0,1]}$ for all $n \geq 2$.

On the other hand, one can use a majority voting strategy to obtain the following result.

Proposition 5.11 (Majority vote). $\frac{k}{n} \text{C}_{[0,1]} \equiv_{\text{sW}} \text{C}_{[0,1]}$ if $2k > n \geq k > 0$.

Proof. It is clear that $\frac{k}{n} \text{C}_{[0,1]} \leq_{\text{sW}} \text{C}_{[0,1]} \equiv_{\text{sW}} \widehat{\text{LLPO}}$ holds. Hence, we only need to prove $\widehat{\text{LLPO}} \leq_{\text{sW}} \frac{k}{n} \text{C}_{[0,1]}$. In the first step we show $\widehat{\text{LLPO}} \leq_{\text{sW}} \frac{k}{n} \widehat{\text{LLPO}}$. Given some answer $(p_1, \dots, p_n) \in \frac{k}{n} \widehat{\text{LLPO}}(q)$, a solution $p \in \widehat{\text{LLPO}}(q)$ can be obtained by bitwise majority voting: for any $i \in \mathbb{N}$, we let $p(i) := 1$ if and only if $|\{j : p_j(i) = 1\}| \geq k$ and $p(i) := 0$ otherwise. This guarantees majority since $2k > n$. To complete the proof it suffices to show $\frac{k}{n} \widehat{\text{LLPO}} \leq_{\text{sW}} \frac{k}{n} \text{C}_{[0,1]}$. We know that $\widehat{\text{LLPO}} \leq_{\text{sW}} \text{C}_{[0,1]}$ and hence there are computable H, K such that $HFK \vdash \widehat{\text{LLPO}}$ whenever $F \vdash \text{C}_{[0,1]}$ holds. Without loss of generality, we can assume that we use a total representation for $[0, 1]$ and hence H has to be total since $\text{C}_{[0,1]}$ is surjective. This implies that $H^n FK \vdash \frac{k}{n} \widehat{\text{LLPO}}$ whenever $F \vdash \frac{k}{n} \text{C}_{[0,1]}$, which completes the proof. \square

We note that $\frac{n-1}{n}$ satisfies $2(n-1) > n$ if and only if $n \geq 3$. This does constitute a second proof of Theorem 5.2. Moreover, Proposition 5.7 shows that the claim of Proposition 5.11 does not hold for $n = 2$ and $k = 1$. This illustrates from a combinatorial perspective why dimension two is special.

6. THE DISPLACEMENT PRINCIPLE

In this section we want to prove a displacement principle that provides some information on the power of binary choice $\text{C}_{\{0,1\}}$ on the left-hand side of a reduction. In order to prove our result we first need to study the convergence relation of $\mathcal{A}_-(X)$ induced by ψ_- . This convergence relation can be characterized in terms of *closed upper limits* as defined by Hausdorff. For a sequence (A_i) of closed subsets of a topological space X the *closed upper limit* of (A_i) is defined by

$$\text{Ls}(A_i) := \bigcap_{k=0}^{\infty} \bigcup_{i=k}^{\infty} A_i.$$

The common notation Ls is derived from the fact that this is also called the *topological limit superior* of (A_i) . The set $\text{Ls}(A_i)$ is always closed and possibly empty. If X is compact and all the A_i are non-empty, then $\text{Ls}(A_i)$ is also compact and non-empty by Cantor's Intersection Theorem. We mention the following known characterization of the topological limit superior by Choquet (see, for instance, Proposition 5.2.2 in [Bee93]).

Fact 6.1 (Choquet). *Let X be a Hausdorff space and let \mathcal{N}_x denote the set of open neighborhoods of $x \in X$. For each sequence (A_i) of closed sets $A_i \subseteq X$ one has*

$$\text{Ls}(A_i) = \{x \in X : (\forall U \in \mathcal{N}_x)(\forall k)(\exists i \geq k) U \cap A_i \neq \emptyset\}.$$

It is well-known that the topological limit superior (and the related topological limit inferior) are used to define Kuratowski-Painlevé convergence, which is closely related to convergence with respect to the Fell topology (see Chapter 5 in [Bee93]). Here we characterize the convergence relation of $\mathcal{A}_-(X)$ in terms of the topological limit superior. For a sequence (A_i) and a set A in $\mathcal{A}_-(X)$ we write $A_i \rightarrow A$ if there are p_i and p such that $\psi_-(p_i) = A_i$, $\psi_-(p) = A$ and $p_i \rightarrow p$. We note that this convergence relation on $\mathcal{A}_-(X)$ is not unique in general, i.e. one sequence (A_i) can have many different limits. The following result gives an exact characterization.

Lemma 6.2 (Closed upper limit). *Let X be a computable metric space and let $A_i, A \in \mathcal{A}_-(X)$ for all $i \in \mathbb{N}$. Then $A_i \rightarrow A$ if and only if $\text{Ls}(A_i) \subseteq A$.*

Proof. Let p_i and p be such that $\psi_-(p_i) = A_i$, $\psi_-(p) = A$.

We now assume $p_i \rightarrow p$. Let $x \notin A$. Then there is some basic open neighborhood B_m of x that is eventually listed in position j of p . Since the p_i converge to p , there is a $k \in \mathbb{N}$ such that B_m is also listed in position j of p_i for all $i \geq k$. According to Fact 6.1 this means that $x \notin \text{Ls}(A_i)$. Hence, we have proved $\text{Ls}(A_i) \subseteq A$.

Let us now assume that $\text{Ls}(A_i) \subseteq A$. It suffices to find q_i with $\psi_-(q_i) = A_i$ and $q_i \rightarrow p$. We choose $q_i := p|_{m_i} p_i$, where $p|_{m_i}$ is the prefix of p of suitable length m_i . It is clear that $q_i \rightarrow p$ follows if the m_i are strictly increasing and we need to prove that we can choose such m_i with $\psi_-(q_i) = \psi_-(p_i)$. We note that for each n the set $U = B_{p(n)}$ does not intersect A , i.e. $U \cap A = \emptyset$ and hence there is some k such that for all $i \geq k$ we have $U \cap A_i = \emptyset$ by Fact 6.1. That means that we can add the ball $B_{p(n)}$ to the negative information of A_i without changing A_i . This guarantees the existence of a suitable strictly increasing sequence m_i . \square

This result implies that the convergence relation on $\mathcal{A}_-(X)$ induced by ψ_- is the convergence relation of the upper Fell topology and hence ψ_- is admissible with respect to this topology (which was already known, see [Sch02]). We introduce some further terminology. If $\mathcal{S} \subseteq \mathcal{A}_-(X)$, then we denote by

$$\overline{\mathcal{S}} := \{A \in \mathcal{A}_-(X) : (\exists (A_i) \in \mathcal{S}^{\mathbb{N}}) \text{Ls}(A_i) \subseteq A\}$$

the *sequential closure* of \mathcal{S} in $\mathcal{A}_-(X)$ and by

$$2\mathcal{S} := \{A \in \mathcal{S} : (\exists A_1, A_2 \in \mathcal{S})(A_1 \cap A_2 = \emptyset \text{ and } A_1 \cup A_2 \subseteq A)\}$$

the set of those sets in \mathcal{S} that have two disjoint subsets in \mathcal{S} . By $\mathcal{C}_X|_{\mathcal{S}}$ we denote the restriction of \mathcal{C}_X to \mathcal{S} .

Theorem 6.3 (Displacement Principle). *Let f be a multi-valued function on represented spaces, let X be a computable metric space and let $\mathcal{S} \subseteq \mathcal{A}_-(X)$. Then*

$$f \times \mathcal{C}_{\{0,1\}} \leq_{\text{w}} \mathcal{C}_X|_{\mathcal{S}} \implies f \leq_{\text{w}} \mathcal{C}_X|_{\mathcal{S} \cap 2\overline{\mathcal{S}}}.$$

An analogous statement holds with \leq_{w} replaced by \leq_{sw} in both instances.

Proof. We use the computable metric space (X, δ_X) and represented spaces (Y, δ_Y) and (Z, δ_Z) . We assume that f is of type $f : \subseteq Y \rightrightarrows Z$ and we use $\mathbb{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\text{sW}} \text{LLPO}$ (see [BdBP12]). Let $H, K : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ be computable functions that witness the reduction $f \times \text{LLPO} \leq_{\text{W}} \mathbb{C}_X|_{\mathcal{S}}$, i.e. $H(\text{id}, GK) \vdash f \times \text{LLPO}$ whenever $G \vdash \mathbb{C}_X|_{\mathcal{S}}$.

We recall that $\text{LLPO} : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ is defined such that for $j \in \{0, 1\}$ and $p \in \{0, 1\}^{\mathbb{N}}$ it holds that $j \in \text{LLPO}(p) \iff (\forall i) p(2i + j) = 0$, where $\text{dom}(\text{LLPO})$ contains all sequences p such that $p(k) \neq 0$ for at most one k . We consider the inputs $p_{j,i} := 0^{2i+j+1}10^{\mathbb{N}}$ and $p_{\infty} := 0^{\mathbb{N}}$ for LLPO . We obtain $\text{LLPO}(p_{j,i}) = \{j\}$ for $j \in \{0, 1\}$ and $\text{LLPO}(p_{\infty}) = \{0, 1\}$. For every $p \in \text{dom}(f\delta_Y)$, $i \in \mathbb{N}$ and $j \in \{0, 1\}$ we now define

$$A_{j,i}^p := \psi_{-}K\langle p, p_{j,i} \rangle \text{ and } A_{\infty}^p := \psi_{-}K\langle p, p_{\infty} \rangle.$$

Since $p_{j,i} \rightarrow p_{\infty}$ for $i \rightarrow \infty$, continuity of K implies $\text{Ls}(A_{j,i}^p) \subseteq A_{\infty}^p$ for $j \in \{0, 1\}$ by Lemma 6.2. Now we consider the corresponding subsets of $\text{dom}(H)$:

$$B_j^p := \bigcup_{i=0}^{\infty} \langle \{p, p_{j,i}\} \times \delta_X^{-1}(A_{j,i}^p) \rangle, B_{\infty}^p := \langle \{p, p_{\infty}\} \times \delta_X^{-1}(A_{\infty}^p) \rangle.$$

By π_j we denote the projection on the j -th component of a tuple in Baire space. Then $h := \delta_{\mathbb{N}}\pi_2 H : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$ is a computable function such that $h|_{B_j^p}$ is constant with value j for $j \in \{0, 1\}$. We claim that due to continuity of h this implies $\text{Ls}(A_{0,i}^p) \cap \text{Ls}(A_{1,i}^p) = \emptyset$. Let us assume that q is such that $\delta_X(q) \in \text{Ls}(A_{0,i}^p) \cap \text{Ls}(A_{1,i}^p)$. Then $\delta_X(q) \in A_{\infty}^p$ and hence $r := \langle \langle p, p_{\infty} \rangle, q \rangle \in B_{\infty}^p \subseteq \text{dom}(h)$. Let now U be a neighborhood of r and let $j \in \{0, 1\}$. By Fact 6.1 the point $\delta_X(q)$ is a cluster point of the sequence $(A_{j,i}^p)_i$ and hence there is a sequence (q_i) with $\delta_X(q_i) \in A_{j,i}^p$ for all i with a subsequence that converges to q . Hence, for some sufficiently large i we obtain $\langle \langle p, p_{j,i} \rangle, q_i \rangle \in B_j^p \cap U$, which means $B_j^p \cap U \neq \emptyset$ for $j \in \{0, 1\}$. Hence $h|_U$ has to take both values 0 and 1 on any neighborhood U of r , which contradicts continuity of h . This proves the claim $\text{Ls}(A_{0,i}^p) \cap \text{Ls}(A_{1,i}^p) = \emptyset$.

Altogether, we have proved $A_{\infty}^p \in 2\bar{\mathcal{S}}$ and $A_{\infty}^p \in \mathcal{S}$ is clear. We now define computable functions $H', K' : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ by

$$H'\langle p, q \rangle := \pi_1 H\langle \langle p, p_{\infty} \rangle, q \rangle \text{ and } K'(p) = \pi_1 K\langle p, p_{\infty} \rangle.$$

Then $H'(\text{id}, GK') \vdash f$ whenever $G \vdash \mathbb{C}_X|_{\mathcal{S} \cap 2\bar{\mathcal{S}}}$, i.e. $f \leq_{\text{W}} \mathbb{C}_X|_{\mathcal{S} \cap 2\bar{\mathcal{S}}}$. If H does not depend on the first component, then $H' = \pi_1 H$ also does not depend on the first component. Hence the claim also holds for strong reducibility \leq_{sW} in place of \leq_{W} . \square

If \mathcal{S} only contains non-empty closed sets $A \subseteq X$ and X is compact, then $\bar{\mathcal{S}}$ also contains only non-empty sets and $2\bar{\mathcal{S}}$ contains only sets that have at least two points. Hence we obtain the following corollary, where $\mathcal{U}(X) := \{\{x\} : x \in X\}$ denotes the set of singleton subsets of X .

Corollary 6.4. *Let f be a multi-valued function on represented spaces, let X be a compact computable metric space and let $\mathcal{S} \subseteq \mathcal{A}_{-}(X) \setminus \{\emptyset\}$. Then*

$$f \times \mathbb{C}_{\{0,1\}} \leq_{\text{W}} \mathbb{C}_X|_{\mathcal{S}} \implies f \leq_{\text{W}} \mathbb{C}_X|_{\mathcal{S} \setminus \mathcal{U}(X)}.$$

An analogous statement holds with \leq_{W} replaced by \leq_{sW} in both instances.

7. IDEMPOTENCY OF CONNECTED CHOICE

The goal of this section is to prove that connected choice CC_1 of dimension one is not idempotent, i.e. $\text{CC}_1 \not\equiv_{\text{W}} \text{CC}_1 \times \text{CC}_1$. For this purpose we use CC_1^{-} , which is just the restriction of CC_1 to such connected sets that are not singletons. In [BG11a] it was proved that $\text{CC}_1^{-} \leq_{\text{W}} \mathbb{C}_{\mathbb{N}}$, which follows since one can just guess a rational number in a non-degenerate interval and with finitely many mind changes

one can find a correct number. Using the Displacement Principle we can prove the following result.

Proposition 7.1. $CC_1 <_W CC_1 \times C_{\{0,1\}}$.

Proof. It is clear that $CC_1 \leq_W CC_1 \times C_{\{0,1\}}$. Let us assume that also $CC_1 \times C_{\{0,1\}} \leq_W CC_1$. Then by Corollary 6.4 we obtain $CC_1 \leq_W CC_1^-$. Since $CC_1^- \leq_W C_{\mathbb{N}}$ (by Proposition 3.8 in [BG11a]), we obtain $CC_1 \leq_W C_{\mathbb{N}}$, which is a contradiction (to Lemma 4.9 in [BG11a]). \square

While this result shows that binary choice $C_{\{0,1\}}$ enhances the power of connected choice CC_1 if multiplied with it, products of binary choice with itself are not that powerful, as the next result shows.

Proposition 7.2. $C_{\{0,1\}}^* \leq_{sW} CC_1^-$.

Proof. Given a pair $\langle n, p \rangle$ as input to $C_{\{0,1\}}^*$ we need to construct a non-degenerate connected closed set $A \subseteq [0, 1]$ any point of which allows us to find a point in $C_{\{0,1\}}^n$ for input p . The input p describes a product $A_1 \times \dots \times A_n$ of non-empty sets $A_i \subseteq \{0, 1\}$ by an enumeration of the complement.

In order to construct A we use an auxiliary tree of rational complexes with branching degree $2n$ in which each complex exists exactly of one rational interval $[a, b]$ with $a < b$. More precisely, we start with the root $[\frac{1}{2n+2}, \frac{1}{2n+1}]$ and on each successor node of the tree we use $2n$ canonical pairwise disjoint subintervals of the previous interval, sorted in the natural order.

We now describe how we use this tree to construct A . Given p we start to produce the root interval $[\frac{1}{2n+2}, \frac{1}{2n+1}]$ as long as no negative information on any of the sets A_1, \dots, A_n is available. If A_k is the first of these sets that is determined by p , then we proceed with child node number $2k - 1$ or $2k$ depending on whether $A_k = \{0\}$ or $A_k = \{1\}$. We then produce a description of the interval associated with this child node until further information on one of the sets A_{k+1}, \dots, A_n becomes available, in which case we proceed inductively as described above.

Altogether, this procedure produces an interval I that is somewhere between the root level (in case that all the sets A_i remain undetermined) and level n below the root level of the tree (in case that all the sets A_i are eventually determined). Given a point $x \in I$, we can find one of the (at most two) intervals J on level n that are closest to x and included in I . Given J , we can reconstruct all decisions in the above algorithm and in this way we can produce a point $(x_1, \dots, x_n) \in A_1 \times \dots \times A_n$. \square

We mention that one can use the level (as introduced by Hertling [Her96]) to prove that $C_{\{0,1\}}^* <_{sW} CC_1^-$. One can show that CC_1^- has no level, whereas the level of $C_{\{0,1\}}^*$ is ω_0 . Since the level is preserved downwards by reducibility, it follows that the reduction must be strict. Altogether, we arrive at the main result of this section.

Theorem 7.3 (Idempotency). $CC_1 <_W CC_1 \times CC_1 <_W CC_2$.

Proof. Firstly, it is clear that $CC_1 <_W CC_1 \times C_{\{0,1\}} \leq_W CC_1 \times CC_1$ holds by Propositions 7.1 and 7.2. Secondly, it is also clear that $CC_1 \times CC_1 \leq_W CC_2$, since the product map $(A, B) \mapsto A \times B$ is computable on closed sets and the product of two connected sets is connected. Finally, $CC_1 \times CC_1$ is non-uniformly computable, whereas CC_2 is not by Proposition 5.5 and hence $CC_1 \times CC_1 <_W CC_2$. \square

In particular, CC_1 is not idempotent and the same reasoning that was used in the proof shows that $CC_2 \not\leq_W CC_1^*$ holds for the idempotent closure CC_1^* . That means that not even an arbitrary finite number of copies of CC_1 in parallel is powerful

enough to compute connected choice in dimension two. With Corollary 4.10 we obtain the following conclusion of Theorem 7.3.

Corollary 7.4. *The Brouwer Fixed Point Theorem BFT_1 of dimension one and the Intermediate Value Theorem IVT are both not idempotent.*

This means that two realizations of the Intermediate Value Theorem in parallel are more powerful than just one. Finally, we can strengthen our Conjecture 5.9 in the following way.

Conjecture 7.5 (Idempotency). *We conjecture that CC_2 is not idempotent.*

It is clear that this conjecture, if correct, implies Conjecture 5.9, since $\text{C}_{[0,1]}$ and hence CC_n for $n \geq 3$ are all idempotent (by Corollary 4.7 and Theorem 8.5 in [BG11b] and Theorem 5.2).

A problem related to idempotency is whether CC_n is a cylinder. Again it is clear that CC_n is a cylinder for $n \geq 3$, which follows from Theorem 5.2 and the fact that $\text{C}_{[0,1]}$ is a cylinder. We can use the techniques of this section to prove that CC_1 is not a cylinder.

Theorem 7.6 (Cylinder). *CC_1 is not a cylinder.*

Proof. Let us assume that $\text{id} \times \text{CC}_1 \leq_{\text{sW}} \text{CC}_1$. Then $\text{id} \times \text{C}_{\{0,1\}} \leq_{\text{sW}} \text{CC}_1$ follows by Proposition 7.2 and hence $\text{id} \leq_{\text{sW}} \text{CC}_1^-$ by Corollary 6.4. Since CC_1^- has a realizer that always selects a rational number, we obtain $\text{CC}_1^- \leq_{\text{sW}} \text{CC}_1^-|_{\mathbb{Q} \cap [0,1]}$, where $\text{CC}_1^-|_{\mathbb{Q} \cap [0,1]}$ denotes the restriction of CC_1^- to $\mathbb{Q} \cap [0,1]$ in the image. By Proposition 13.2 in [BGM12] this implies that $\text{range}(\text{id})$ is countable, which is a contradiction! \square

Once again, the only unresolved case is the case of dimension two.

Conjecture 7.7 (Cylinder). *We conjecture that CC_2 is not a cylinder.*

8. CONCLUSIONS

We have systematically studied the uniform computational content of the Brouwer Fixed Point Theorem for any fixed dimension and we have obtained a systematic classification that leaves only the status of the two-dimensional case unresolved. Besides a solution of this open problem, one can proceed into several different direction.

For one, one could study generalizations of the Brouwer Fixed Point Theorem, such as the Schauder Fixed Point Theorem or the Kakutani Fixed Point Theorem. On the other hand, one could study results that are based on the Brouwer Fixed Point Theorem, such as equilibrium existence theorems in computable economics (see for instance [RW99]). Nash equilibria existence theorems have been studied in [Pau10a] and they can be seen to be strictly simpler than the general Brouwer Fixed Point Theorem (in fact they can be considered as linear version of it). In this context the question arises of how the Brouwer Fixed Point Theorem can be classified for other subclasses of continuous functions, such as Lipschitz continuous functions?

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