EXACT PAIRS FOR THE IDEAL OF THE K-TRIVIAL SEQUENCES IN THE TURING DEGREES

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Abstract. The K-trivial sets form an ideal in the Turing degrees, which is generated by its computably enumerable (c.e.) members and has an exact pair below the degree of the halting problem. The question of whether it has an exact pair in the c.e. degrees was first raised in [MN06, Question 4.2] and later in [Nie09, Problem 5.5.8].

We give a negative answer to this question. In fact, we show the following stronger statement in the c.e. degrees. There exists a K-trivial degree $\mathbf{d}$ such that for all degrees $\mathbf{a}, \mathbf{b}$ which are not K-trivial and $\mathbf{a} > \mathbf{d}, \mathbf{b} > \mathbf{d}$ there exists a degree $\mathbf{v}$ which is not K-trivial and $\mathbf{a} > \mathbf{v}, \mathbf{b} > \mathbf{v}$. This work sheds light to the question of the definability of the K-trivial degrees in the c.e. degrees.

1. Introduction

The algebraic study of the Turing degrees has been a topic of considerable research in computability theory, ever since the establishment of degree theory as a research area in [KP54]. In this study, the ideals of this uppersemilattice are of particular interest. These are downward closed sets of degrees that also closed under the join operator. The recent study of algorithmic information theory by people in computability theory has brought forward a wealth of interactions between the two areas, including the discovery of a new ideal in the Turing degrees: the degrees of sequences with trivial initial segment complexity, the so-called K-trivial sequences. Since this discovery in [DHNS03, Nie05], the study of the K-trivial sequences and degrees have been established as a major area of research in the interface between computability theory and algorithmic information theory.

Issues of definability have been of special interest in the study of ideals in the Turing degrees. Such issues were already present in [KP54], where the notion of exact pairs of ideals was introduced. Two degrees $\mathbf{a}, \mathbf{b}$ form an *exact pair* of an ideal $\mathcal{C}$ in the Turing degrees if they are both upper bounds for the degrees in $\mathcal{C}$ and any degree below both $\mathbf{a}$ and $\mathbf{b}$ is in $\mathcal{C}$. By [KP54, Spe56] every ideal in the

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Turing degrees has an exact pair. By [Nie05] every K-trivial degree is bounded by a computably enumerable (c.e. for short) K-trivial degree. Hence for the purpose of finding exact pairs for this ideal it suffices to consider its restriction to the c.e. degrees. This turns out to be a Σ^0_3 ideal, in the sense that the index set of its members is Σ^0_3. Moreover by [BN11] it has a c.e. upper bound that is strictly below the degree $\mathbf{0}'$ of the halting problem (moreover, by [KS09] it has a low upper bound $\mathbf{b}$, which means that the halting problem relativized to $\mathbf{b}$ has degree $\mathbf{0}'$). By [Sho81], such ideals have an exact pair strictly below $\mathbf{0}'$. However it is well known that such an ideal may or may not have an exact pair in the c.e. degrees (this follows from the existence of branching and non-branching degrees that was established in [Lac66, Yat66]). Hence whether or not such an ideal have an exact pair in the c.e. degrees depends on the specific properties of it. The following question has come into focus.

Problem (Question 4.2 in [MN06] and Problem 5.5.8 in [Nie09])**.** *Is there an exact pair for the ideal of the K-trivial sequences in the c.e. degrees?*

The purpose of this paper is to give a negative answer to this question. In fact, our main result can be seen as a very strong negative answer to this question.

Theorem 1.1. *There exists a K-trivial c.e. degree $\mathbf{d}$ with the following property. For each pair of c.e. degrees $\mathbf{a}, \mathbf{b}$ which are not K-trivial, there exists a c.e. degree $\mathbf{v}$ which is not K-trivial and $\mathbf{v} < \mathbf{a} \cup \mathbf{d}, \mathbf{v} < \mathbf{b} \cup \mathbf{d}$.*

Here $\mathbf{a} \cup \mathbf{d}$ denotes the join (i.e. supremum) of the degrees $\mathbf{a}, \mathbf{d}$.

This theorem provides new and interesting information about the K-trivial sequences and their computational power. Moreover, as we elaborate in Section 2, it rests upon deeper information-theoretic properties that are specific to the K-trivial sequences, rather than some general property that this ideal happened to have. In contrast, the existence of a low bound of this ideal (another question from [MN06]) was obtained in [KS09] by observing that it satisfied a certain domination property, and proving that all ideals which share this property have a low bound.

We may obtain a negative answer to our problem by using some known properties of the K-trivial sequences.

Corollary 1.2. *The ideal of the K-trivial sequences does not have an exact pair of c.e. degrees.*

Proof. By [Nie02] there is no low c.e. upper bound for the K-trivial degrees. By [Nie05] every K-trivial degree is low. Therefore, if two c.e. degrees are an exact pair for the K-trivial degrees, then both of them are not K-trivial. The corollary now follows directly from Theorem 1.1. □

Note that the proof of Corollary 1.2 rests on the following weak (and nonuniform) version of Theorem 1.1: 'given a pair $\mathbf{a}, \mathbf{b}$ of c.e. degrees which are not K-trivial, there exists a K-trivial c.e. degree $\mathbf{d}$ and c.e. degree $\mathbf{v}$ which is not K-trivial such that $(\mathbf{d} \leq \mathbf{a} \wedge \mathbf{d} \leq \mathbf{b}) \to (\mathbf{v} \leq \mathbf{a} \wedge \mathbf{v} \leq \mathbf{b})$'.

The following fact is a direct consequence of the splitting theorem from [Bar11a, Section 5] and [Ste11, Chapter 2]. It shows that by replacing $\mathbf{v} < \mathbf{a} \cup \mathbf{d}, \mathbf{v} < \mathbf{b} \cup \mathbf{d}$ with $\mathbf{v} \leq \mathbf{a} \cup \mathbf{d}, \mathbf{v} \leq \mathbf{b} \cup \mathbf{d}$ in Theorem 1.1 we obtain an equivalent statement.

Proposition 1.3. *If $\mathbf{c}$ is a c.e. degree which is not K-trivial then there exist c.e. degrees $\mathbf{a} < \mathbf{c}$ and $\mathbf{b} < \mathbf{c}$ which are not K-trivial and $\mathbf{c} = \mathbf{a} \cup \mathbf{b}$.*

Since there exists a Δ_2^0 exact pair for the K-trivial degrees, the phenomenon described in Theorem 1.1 is specific to c.e. sets. The following observation contrasts Proposition 1.3 and confirms this intuition from a different angle.

Proposition 1.4. *There exists a degree* $\mathbf{x} < \mathbf{0}'$ *which is not* K*-trivial and for every* K*-trivial degree* $\mathbf{d}$*, the only c.e. degrees that are computable from* $\mathbf{x} \cup \mathbf{d}$ *are also computable from* $\mathbf{d}$*.*

Proof. A degree that is 1-generic relative to every K-trivial degree has the required properties, but is not necessarily below $\mathbf{0}'$. Moreover $\mathbf{0}'$ is not 1-generic. Hence it suffices to show that there exists a degree that is 1-generic relative to every K-trivial degree and is computable from the halting problem. This follows from the fact (see [KS09]) that there exists a function that is computable from the halting problem and dominates all partial computable functions relative to any K-trivial set. □

The proof of Theorem 1.1 rests on a few facts about K-trivial sequences and initial segment Kolmogorov complexity. We present these, along with their use in the proof, in Section 2. Some background on Kolmogorov complexity and K-trivial sequences that is directly relevant to our result is given in Section 2.1. For background material on computability theory we refer to [Odi89]. The main property of Kolmogorov complexity that is used in the proof of Theorem 1.1 is discussed in Section 2.2. It is a result from [Bar11b] which roughly says that any two c.e. sets of nontrivial initial segment complexity must have common lengths in their characteristic sequences where their complexity rises *simultaneously*. Our proof is essentially a derivation of Theorem 1.1 from this result. This route reduces the complexity of the main construction and results in a transparent presentation.

Two more tools from Kolmogorov complexity are used in order to reduce the calculations further and avoid the dynamic construction of machines in the main construction. The first is the use of Solovay functions to express K-triviality, which is based on [BD09, BMN11]. The second one is the standard computable invariance property that is intrinsic to most notions in Kolmogorov complexity. Both of these tools are discussed in Section 2.4. Section 2.5 provides the exact form of the result from [Bar11b] that will be used in the main argument, which is given in Section 3. These few preparatory steps (including the formulation of a sufficient set of requirements in Section 3.1) reduce the main argument to the simple construction and verification of Sections 3.3 and 3.4.

2. Preliminary facts

2.1. **Background on Kolmogorov complexity and K-trivial sequences.**

A standard measure of the complexity of a finite string was introduced by Kolmogorov in [Kol65] (an equivalent approach was due to Solomonoff [Sol64]). The basic idea behind this approach is that simple strings have short descriptions relative to their length while complex or random strings are hard to describe concisely. Kolmogorov (and Solomonoff) formalized this idea using the theory of computation. In this context, Turing machines play the role of our idealized computing devices, and we assume that there are Turing machines capable of simulating any mechanical process which proceeds in a precisely defined and algorithmic manner. Programs can be identified with binary strings.

A string τ is said to be a description of a string σ with respect to a Turing machine M if this machine halts when given program τ and outputs σ. Then the Kolmogorov complexity of σ with respect to M (denoted by $K_M(\sigma)$) is the length of its shortest description with respect to M. It can be shown that there exists an *optimal* machine V, i.e. a machine which gives optimal complexity for all strings, up to a certain constant number of bits. This means that for each Turing machine M there exists a constant c such that $K_V(\sigma) < K_M(\sigma) + c$ for all finite strings σ. Hence the choice of the underlying optimal machine does not change the complexity distribution significantly and the theory of Kolmogorov complexity can be developed without loss of generality, based on a fixed underlying optimal machine U.

When we come to consider the initial segment complexity of infinite strings, it becomes important to consider machines whose domain satisfies a certain condition; the machine M is called *prefix-free* if it has prefix-free domain (which means that no program for which the machine halts and gives output is an initial segment of another). Similarly to the case of ordinary Turing machines, there exists an *optimal* prefix-free machine U so that for each prefix-free machine M the complexity of any string with respect to U is up to a constant number of bits larger than the complexity of it with respect to M. We let K denote the prefix-free complexity with respect to a fixed optimal prefix-free machine.

The original motivation behind Kolmogorov complexity was a mathematical definition of random infinite sequences. Kolmogorov's idea was that these should be infinite sequences with very complex initial segments. Based on this intuition, Levin [Lev73] and Chaitin [Cha75] gave a robust definition of randomness for infinite binary sequences. They called X random if $\exists c \forall n, K(X \upharpoonright_n) \geq n - c$. In other words, X is random if its initial segments cannot be 'compressed' (i.e. be described more concisely) by more than a constant number of bits. For a thorough presentation of this theory we refer to the monographs [Nie09, DH10], while [LV97] is a standard reference for the more general theory of Kolmogorov complexity.

In this paper we are concerned with the other end of the spectrum: sequences with trivial initial segment complexity. These are sequences whose initial segments are very highly compressible, in the sense that they have very short descriptions.

Definition 2.1 (K-trivial sequences)**.** *An infinite binary sequence X is called K-trivial if* $\exists c \forall n,\ K(X \upharpoonright_n) \leq K(n) + c$.

Here $K(n)$ denotes the complexity of the number n (which may be seen as a name for the sequence 0^n). Hence the first n bits of a K-trivial sequence have the same complexity as the sequence 0^n. By identifying subsets of $\mathbb{N}$ with their characteristic sequence we can also talk about K-trivial sets of numbers. Chaitin drew some attention to K-trivial sets by noticing that they are computable from the halting problem and by asking whether they are all computable. Solovay [Sol75] produced the first example of a noncomputable K-trivial set. The work in [DHNS03] signaled a renewed interest on this notion and initiated a deeper study of K-triviality which revealed surprising connections between initial segment complexity and classical computability. For example, Hirschfeldt and Nies showed in [Nie05] that K-triviality is downward closed under Turing computation. Moreover the K-trivial sets form an ideal in the Turing degrees, which is generated by its c.e. members (in the sense that every K-trivial set is computable by a c.e. K-trivial set). In the following we focus on aspects of K-triviality that are directly relevant with the proof of

Theorem 1.1. For a more thorough presentation of this area of complexity-theoretic weakness we refer to [DH10, Chapter 11].

2.2. **Common complexity in pairs of c.e. sets of nontrivial complexity.** Much of the excitement about the K-trivial sequences comes from the fact that they provide an ideal platform for the study of the interaction between the information that can be coded into an infinite binary sequence and the complexity of its initial segments. The latter has been a primary focus of research in the interface between computability theory and Kolmogorov complexity. The fact that there are noncomputable K-trivial sequences showed that one can code nontrivial information into a sequence without increasing the complexity of its initial segments. A limitation to this phenomenon was revealed in [DHNS03] where it was shown that K-trivial sequences cannot compute the halting problem (in other words, they are not Turing complete). In contrast, there are Turing complete sequences of arbitrarily low nontrivial prefix-free initial segment complexity. More precisely, in [Bar11b] it was shown that for every c.e. set A which is not K-trivial, there exists a Turing complete c.e. set V of lower complexity, i.e. such that $\exists c \forall n,\ K(V\upharpoonright_n) \leq K(A\upharpoonright_n) + c$. This was also generalized for the case of any finite collection $A_i, i < k$ of c.e. sets which are not K-trivial, producing a Turing complete c.e. set V such that $\exists c \forall n \forall i < k,\ K(V\upharpoonright_n) \leq K(A_i\upharpoonright_n) + c$. One of the many consequences of this result (see [Bar11b, Corollary 1.7]) is that any two c.e. sets of nontrivial initial segment prefix-free complexity exhibit common lengths of nontrivial prefix-free complexity.

$$\text{Let } A, B \text{ be c.e. sets which are not } K\text{-trivial.}\qquad \text{Then } \forall c \exists n\, [K(A\upharpoonright_n) > K(n)+c \text{ and } K(B\upharpoonright_n) > K(n)+c]. \tag{2.1}$$

This fact is the crux of the proof of Theorem 1.1. Moreover it is just one of a series of results which indicate that any two c.e. sets of non-trivial initial segment complexity have some kind of common complexity, or even information. In view of the existence of minimal pairs in the c.e. Turing degrees (a classic result from [Lac66]), such information is not common in the terms of the Turing reducibility but in terms of weaker measures of relative complexity. See [Bar11b, Theorem 1.2] and [Bar10, Theorem 1.3].

There are alternative roots to the proof of Theorem 1.1. We have chosen to derive it as a consequence of (2.1), with the use of an additional device that is known as 'Solovay functions' (see Section 2.4). This root reduces the bulk of the proof to the rather simple construction and verification of Section 3.

2.3. **Construction of prefix-free machines.** A (rather simple) direct construction of a prefix-free machine will be used in Section 2.5. There are certain notions and tools associated with such constructions, which are standard in the arguments employed in algorithmic randomness and also relate to the main argument of Section 3. We briefly discuss them. The *weight* of a prefix-free set S of strings, denoted $\texttt{wgt}(S)$, is defined to be the sum $\sum_{\sigma\in S} 2^{-|\sigma|}$. The *weight* of a prefix-free machine M is defined to be the weight of its domain and is denoted $\texttt{wgt}(M)$. Prefix-free machines are most often built in terms of *request sets*. A request set L is a set of pairs $\langle \rho, \ell\rangle$ where ρ is a string and ℓ is a positive integer. A 'request' $\langle \rho, \ell\rangle$ represents the intention of describing ρ with a string of length ℓ. We define the *weight of the request* $\langle \rho, \ell\rangle$ to be $2^{-\ell}$. We say that L is a *bounded request set* if the sum of the weights of the requests in L is less than 1. This sum is the *weight of the request set* L and is denoted by $\texttt{wgt}(L)$. The Kraft-Chaitin theorem (see e.g.

[DH10, Section 2.6]) says that for every bounded request set L which is c.e., there exists a prefix-free machine M with the property that for each $\langle \rho, \ell \rangle \in L$ there exists a string τ of length ℓ such that $M(\tau) = \rho$. Hence the dynamic construction of a prefix-free machine can be reduced to a mere description of a corresponding c.e. bounded request set.

For each prefix-free machine N and string σ, let $P_N(\sigma)$ be the weight of all the strings τ such that $N(\tau) = \sigma$ (if this set of strings is empty, let this weight be 0). A basic result in Kolmogorov complexity from [Sol64, Lev74, Cha75] (also see [Nie09, Theorem 2.2.25] or [DH10, Theorem 3.9.4] for a modern presentation) is called the *coding theorem*. This says that for each prefix-free machine N, there is a constant c such that $2^{-K(\sigma)} > 2^{-c} \cdot P_N(\sigma)$ for each string σ.

2.4. **Solovay functions and computable invariance.** Building on work from [Sol75], the following characterization of K-trivial sets was given in [BD09].

(2.2) There exists a computable function $g : \mathbb{N} \to \mathbb{N}$ such that

$$(\star) \quad X \text{ is } K\text{-trivial} \iff \exists c \forall n \; (K(X \upharpoonright_n) \le g(n) + c)$$

for all sets X and also $\sum_n 2^{-g(n)}$ is a random real.

Here by a random real we mean a real number in $(0, 1)$ whose binary expansion is a random sequence. Later it was demonstrated in [BMN11] that the functions g of (2.2) are exactly the *computable tight upper bounds* of the Kolmogorov function $K(n)$, in the sense for some constant c we have $K(n) \le g(n) + c$ for all n and $g(t) \le K(t) + c$ for infinitely many t. These functions were called *Solovay functions*.

Note that (2.2) replaces a non-computable component in the definition of K-triviality (namely $K(n)$) with a computable function. In certain situations this allows for a simplification of the calculations involved in arguments about the K-trivial sets. This is the case with the proof of Theorem 1.1. In the following sections, we fix a computable function g as in (2.2) and use $(\star)$ as a definition of K-triviality.

Another device that we will use in the proof of Theorem 1.1 is a certain computable invariance that is common in many notions related to Kolmogorov complexity.

Proposition 2.2 (Computable invariance of Solovay functions). *Let f be a Solovay function and let $m_i : \mathbb{N} \to \mathbb{N}$ be a computable increasing sequence. Then $i \mapsto f(m_i)$ is a Solovay function.*

Proof. Since (m_i) is increasing and computable, $\exists c \forall i, \; K(i) \le K(m_i) + c$. Hence $i \mapsto f(m_i)$ is a computable upper bound of $K(i)$. Also $\exists b \forall i, \; K(m_i) \le K(i) + b$. So $i \mapsto f(m_i)$ is a computable tight upper bound of $K(i)$ and the proposition follows from the characterization of Solovay functions from [BMN11]. □

The following observation is a consequence of $\exists b \forall i, \; K(X \upharpoonright_i) \le K(X \upharpoonright_{m_i}) + b$ and Proposition 2.2.

Proposition 2.3 (K-triviality in terms of Solovay functions). *Let f be a Solovay function and let $m_i : \mathbb{N} \to \mathbb{N}$ be a computable increasing sequence. A set X is K-trivial if and only if $\exists c \forall i, \; K(X \upharpoonright_{m_i}) \le f(m_i) + c$.*

The following observation is a direct consequence of the fact that $\sum_t 2^{-f(t)}$ is noncomputable when f is a Solovay function.

Proposition 2.4 (Accumulation of weight in Solovay functions)**.** *If (m_i) is a computable increasing sequence and f is a Solovay function then for every k there exist infinitely many n such that $\sum_{t>m_n} 2^{-f(t)} > \frac{1}{n-k-1}$.*

Finally we give (2.1) in terms of an arbitrary monotone computable injection.

(2.3) Let A, B be c.e. sets which are not K-trivial and (m_i) a computable increasing sequence.
Then $\forall c \exists i\ [K(A\restriction_{m_i}) > g(m_i) + c \text{ and } K(B\restriction_{m_i}) > g(m_i) + c]$.

Indeed, by [Bar11b] there exists a Turing complete c.e. set X and a constant x such that $\forall n,\ K(X\restriction_n) \le K(A\restriction_n) + x$ and $\forall n,\ K(X\restriction_n) \le K(B\restriction_n) + x$. By [DHNS03], X is not K-trivial so (2.3) follows from Proposition 2.3.

2.5. **Modulus functions of c.e. sets and K-triviality.** We use the following standard notion of 'modulus of convergence' which is associated with the approximation to a function or a set.

Definition 2.5 (Modulus functions of c.e. sets)**.** *Let A be a c.e. set with a computable enumeration $(A[s])$. The modulus function $n \mapsto a(n)$ of A maps each n to the least stage $s > n$ such that $A[s]\restriction_n \subset A$.*

Note that the modulus function of a c.e. set A always refers to a particular computable enumeration $(A[s])$ of it. In this paper all c.e. sets will be given via a certain computable enumeration of them. Hence we may talk about *the* modulus function of a given c.e. set (suppressing the corresponding computable enumeration) without causing confusion. Modulus functions and K-triviality are related.

Proposition 2.6. *Let A, B be c.e. sets which are not K-trivial, let $a(n), b(n)$ be their modulus functions and let $d(n) := \min\{a(n), b(n)\}$. If g is a Solovay function, then for each c there exists n such that $\sum_{i>d(n)} 2^{-g(i)} < 2^{-c} \cdot \sum_{i>n} 2^{-g(i)}$.*

For the proof of Theorem 1.1 we will use a more explicit (stronger) version of Proposition 2.6. This is based on the following fact.

Lemma 2.7 (Modulus and Solovay functions)**.** *Let A be a c.e. set which is not K-trivial and let a be its modulus function. If g is a Solovay function, there exists a constant q such that*

$$K(A\restriction_n) > g(n) + c + q \Rightarrow \sum_{i \ge a(n)} 2^{-g(i)} < 2^{-c} \cdot \sum_{i>n} 2^{-g(i)}$$

for all numbers n, c.

Proof. By the coding theorem (see Section 2.3) it suffices to construct a prefix-free machine N such that for all numbers n, c,

$$2^{-c} \cdot \sum_{i>n} 2^{-g(i)} \le \sum_{i \ge a(n)} 2^{-g(i)} \Rightarrow P_N(A\restriction_n) \ge 2^{-g(n)-c}.$$

Let $\Omega_n = \sum_{0 \le i < n} 2^{-g(n)}$. Hence it suffices to construct a prefix-free machine N such that

$$P_N(A\restriction_n) \ge \Big(\sum_{i \ge a(n)} 2^{-g(i)} \Big) \cdot \frac{2^{-g(n)}}{1 - \Omega_n} \qquad \text{for all } n. \tag{2.4}$$

We define the approximation to the modulus function of A. Let $a_e(n)[s]$ be n if $s \leq n$ and the least stage $t > n$ with $t \leq s$ such that $A[t] \restriction_n \subset A[s]$ otherwise. We construct N by enumerating a suitable set of requests. The set $V_n[s]$ contains the requests that have been enumerated for $A[s] \restriction_n$ by the end of stage s. At stage $s+1$, for each $n < s$ let

$$p_n[s] = 2^{-g(n)} \cdot \frac{\sum_{a(n)[s] \leq i < s} 2^{-g(i)}}{1 - \Omega_n} - \mathtt{wgt}(V_n[s])$$

and if $p_n[s] > 0$ enumerate a set of requests for $A[s] \restriction_n$ in N of total weight $p_n[s]$ (formally, if $\sum_{i<k} t_i \cdot 2^{-i}$ is the unique binary representation of $p_n[s]$, for each $i < k$ such that $t_i \neq 0$ enumerate request $\langle A[s] \restriction_n, i\rangle$ into N).

We verify that the request set is bounded by 1. Indeed, fix n and let $s_i, i < t$ be the successive stages where $a(n)[s]$ changes value (and $s_0 = 0$). Note that $a(n)[s_0] = n$ and $a(n)[s_i] < a(n)[s_{i+1}]$. The weight of requests that we enumerate for the current approximation to $A \restriction_n$ in the interval $[s_i, s_{i+1})$ is bounded by

$$\frac{2^{-g(n)}}{1-\Omega_n} \cdot \Big(\sum_{i \in J} 2^{-g(i)}\Big), \quad \text{where } J = [a(n)[s_i], a(n)[s_{i+1}])$$

(and $s_t := \infty$). Since $a(n)[s_0] = n$, the total weight of requests that we enumerate for the various approximations to $A \restriction_n$ is bounded by

$$\frac{2^{-g(n)}}{1-\Omega_n} \cdot \Big(\sum_{i \geq n} 2^{-g(i)}\Big) = 2^{-g(n)}.$$

Hence the total weight of N is bounded by $\sum_n 2^{-g(n)} < 1$ and N is a prefix-free machine. Finally, (2.4) is an explicit feature of the construction of N. □

Finally, we may derive the statement that we actually need in Section 3.

Corollary 2.8 (Convergence of two sets and weight of Solovay functions)**.** *Let A, B be c.e. sets which are not K-trivial and let $a(n), b(n)$ be their modulus functions. If $d(n) := \min\{a(n), b(n)\}$ and g is a Solovay function, there exists q such that*

$$\left.\begin{array}{l} K(A \restriction_n) > g(n) + c + q \\ K(B \restriction_n) > g(n) + c + q \end{array}\right\} \Rightarrow \sum_{i > d(n)} 2^{-g(i)} < 2^{-c} \cdot \sum_{i > n} 2^{-g(i)}$$

for all numbers n, c.

This is a direct consequence of Lemma 2.7.

3. Proof of Theorem 1.1

We formulate a sufficient set of requirements in Section 3.1 and give the specifics of the construction in Section 3.2. We conclude with the formal construction in Section 3.3 and the verification of the requirements in Section 3.4.

3.1. **Requirements for the construction of D.** Let U be the universal prefix-free machine which underlies the prefix-free Kolmogorov complexity function, i.e. such that $K = K_U$. We may assume that $\mathtt{wgt}(U) < 2^{-4}$. Also let (A_e, B_e) be an effective list of all pairs of c.e. sets. Note that the sets A_e, B_e are given via specific computable enumerations that are provided by a fixed universal Turing machine. The sets A_e, B_e correspond to guesses about representatives of the degrees $\mathbf{a}, \mathbf{b}$ of Theorem 1.1. For each pair (A_e, B_e) let a_e, b_e denote the corresponding modulus functions. Moreover let $a_e[s], b_e[s]$ denote their approximations at stage s. In particular, $a_e(n)[s]$ is n if $s \le n$ and the least stage $t > n$ with $t \le s$ such that $A[t] \restriction_n \subset A[s]$ otherwise; similarly for $b_e(n)[s]$. Let $(i,j) \mapsto \langle i,j\rangle$ be a standard computable increasing (in both arguments) pairing function and define $\mathbb{N}^{[k]} = \{\langle k,n\rangle \mid n \in \mathbb{N}\}$. We define a version of the parameter $\min\{a_e(n), b_e(n)\}$ which can be treated dynamically (at any stage of the construction) as a number that is eligible for enumeration into the set D that will be constructed. Define $d_e(n)[s]$ to be the least number in $\mathbb{N}^{[\langle e,n\rangle]} - D[s]$ which is larger than $\min\{a_e(n)[s], b_e(n)[s]\}$. Moreover let $d_e(n) = \lim_s d_e(n)[s]$. The parameters $a_e(n)[s], b_e(n)[s], d_e(n)[s]$ can be seen as movable markers on $\mathbb{N}$. Moreover a direct consequence of their definition is that they always move monotonically, i.e. $a_e(n)[s] \le a_e(n)[s+1]$ and similarly for $b_e(n)[s], d_e(n)[s]$.

We will define a K-trivial c.e. set D and a sequence of c.e. sets (V_e) such that the following conditions are met.

$$R_e : V_e \le_T A_e \oplus D \ \wedge \ V_e \le_T B_e \oplus D.$$

We will also ensure the following condition on V_e.

$$P_e : \text{ If } A_e, B_e \text{ are not } K\text{-trivial then } V_e \text{ is not } K\text{-trivial.}$$

These conditions on $D, (V_e)$ are sufficient for the proof of Theorem 1.1. Let g be a fixed Solovay function, i.e. a function satisfying (2.2), for the duration of this proof. Without loss of generality we may assume that $\sum_i 2^{-g(i)} < 2^{-4}$. We may split each condition P_e into more elementary conditions P^*_{ekt}. Let $(e,k,i) \mapsto n_{ek}(i)$ be a computable function such that $n_{ek}(i) < n_{ek}(i+1)$. In Section 3.2 we will define a specific such function, but at this point we may express P^*_{ekt} in terms of any fixed such choice. We may write n_{ekt} to denote $n_{ek}(t)$ in the interest of space.

$$P^*_{ekt} : \Big(\sum_{i > d_e(n_{ekt})} 2^{-g(i)} < 2^{-e-k} \cdot \sum_{i > n_{ekt}} 2^{-g(i)} \Big) \Rightarrow K(V_e \restriction_{n_{ekt}}) > g(n_{ekt}) + k.$$

We let P^*_{ek} denote the conjunction of all P^*_{ekt} , $t \in \mathbb{N}$. We verify that the satisfaction of P_e may be reduced to the satisfaction of P^*_{ek}, $k \in \mathbb{N}$.

Lemma 3.1 ($(\forall k\ P^*_{ek}) \to P_e$)**.** *Fix a computable function $(e,k,i) \mapsto n_{ek}(i)$ which is increasing on i. For each e, the conjunction of P^*_{ek}, $k \in \mathbb{N}$ implies P_e.*

Proof. Assume that A_e, B_e are not K-trivial and P^*_{ekt} are met for all k,t. It suffices to show that for each y there exists some n such that $K(V_e \restriction_n) > g(n) + y$. Let q_e be the constant q of Corollary 2.8 for $A = A_e$ and $B = B_e$. Since A_e, B_e are not K-trivial, by (2.3) there exists some $t > x_0$ such that $K(A_e \restriction_{n_{ekj}}) > g(n_{ekj}) + y + e + q_e$ and $K(B_e \restriction_{n_{ekj}}) > g(n_{ekj}) + y + e + q_e$. By the choice of q_e, the fact that $\forall n, s,\ d_e(n)[s] \ge \min\{a_e(n)[s], b_e(n)[s]\}$ and Corollary 2.8, the left hand side of the implication in P^*_{ekt} is met for $k = y$ and $t = j$. Therefore $K(V_e \restriction_n) > g(n) + y$ for $n = n_{eyj}$. □

The requirement that D is K-trivial can be expressed as

$$\exists c \forall n,\ K(D \restriction_n) \le g(n) + c. \tag{3.1}$$

The cost associated with the enumeration of a number n in D at stage $s+1$ of the construction in view of (3.1) is given by

$$c(n,s) = \sum_{n \le i \le s} 2^{-g(i)}. \tag{3.2}$$

The satisfaction of (3.1) will be achieved by ensuring that the total cost of the enumerations into D is bounded, in other words

$$\sum_{(n,s)\in I_D} c(n,s) < 1 \quad \text{where } I_D = \big\{(n,s) \mid n = \min\{x \mid x \in D[s+1] - D[s]\}\big\}. \tag{3.3}$$

The fact that (3.3) implies (3.1) was established in [DHNS03] when g is replaced by the Kolmogorov function $K(n)$ (also see [DH10, Section 11.1] and [Nie09, Section 5.3] for elaborate presentations of this method). The same argument shows that this implication holds when $K(n)$ is replaced by any right-c.e function (i.e. a function with a computable approximation 'from above') f such that $\sum_i 2^{-f(i)} < 1$.

We close this section by providing a condition which implies R_e and shows explicitly the required Turing reductions. By the definition of $a_e[s]$ it follows that $a_e(n)$ (the final position of $a(n)[s]$) is computable from A_e. Similarly, $b_e(n)$ is computable from B_e. Hence $A_e \oplus D$ computes an upper bound of $n \mapsto d_e(n)$ (provided that $\mathbb{N}^{[\langle e,n\rangle]} - D$ is finite) and the same is true of $B_e \oplus D$. The following condition expresses a *weak coding of* V_e *into* D.

$$R_e^* : \ \forall n, s \ \Big(n \in V_e[s+1] - V_e[s] \Rightarrow d_e(n)[s] \in D[s+1] - D[s]\Big).$$

Condition R_e^* implies condition R_e. Indeed, suppose that R_e^* holds. Then to determine if $n \in V_e$ it suffices compute $a_e(n)$ (using A_e) and then (using $A_e \oplus D$) find a stage where the approximation to $D \restriction_{d_e(n)+1}$ has reached a limit. Assuming that $\mathbb{N}^{[\langle e,n\rangle]} - D$ is finite, $d_e(n)[s]$ reaches a limit as $s \to \infty$ and such a stage will be found. At such a stage, $d_e(n)[s]$ has reached a limit. Hence by R_e^* the approximation to $V_e(n)$ has also reached a limit. The same procedure can be performed via $B_e \oplus D$-computations, with the use of $b_e(n)$ which can also reveal an upper bound for $d_e(n)$ (with the help of D).

We have established that a construction of $D, (V_e)$ which meets conditions (3.3) and R_e^*, P_{ek} for $e, k \in \mathbb{N}$ (and at at least one choice of a computable function $(e,k,i) \mapsto n_{ek}(i)$ which is increasing on i) is sufficient for the proof of Theorem 1.1. An underlying assumption is that for each e, k the set $\mathbb{N}^{[\langle e,n\rangle]} - D$ is finite. The latter will be an immediate feature of the construction.

3.2. **Strategy and witnesses for conditions P_{ek}^*.** Recall that P_{ek}^* denotes the conjunction of the conditions P_{ekt}^* of Section 3.1 (which depend on the choice of $(e,k,i) \mapsto n_{ek}(i)$). The construction of Section 3.3 is driven by actions (enumerations into D, V_e) for the satisfaction of P_{ek}^*. Here we define some parameters that are used in these actions. For each e, k we define an increasing sequence $(n_{ek}(i))$ of numbers. Recall the definitions of $(i,j) \mapsto \langle i,j\rangle$ and $\mathbb{N}^{[k]}$ from Section 3.1. Define

$$J_{ek}(\langle k,x\rangle) = \Big\{\langle k,m\rangle \mid m > x+1 \ \wedge \sum_{t>\langle k,m\rangle} 2^{-g(t)} > \frac{1}{m-x-1}\Big\}.$$

The sets J_{ek} are uniformly c.e. and by Proposition 2.4 they are all infinite. Hence we may choose a uniformly computable family of sets J_{ek}^* such that $J_{ek}^* \subseteq J_{ek}$ for each e, k. Define $(n_{ek}(i))$ recursively as follows.

$$\begin{aligned} n_{ek}(-1) &= \min \mathbb{N}^{[k]} \\ n_{ek}(i) &= \min J_{ek}^*\big(n_{ek}(i-1)\big) \end{aligned}$$

Note that the function $(e,k,i) \mapsto n_{ek}(i)$ is computable. Moreover

$$\textstyle\sum_{i>n_{ek}(t)} 2^{-g(i)} > 1/\big|\big(n_{ek}(t-1), n_{ek}(t)\big) \cap \mathbb{N}^{[k]}\big|. \tag{3.4}$$

From this point on, P^*_{ekt} refers to this choice of $(e,k,i) \mapsto n_{ek}(i)$. We say that P^*_{ek} *requires attention at stage* $s+1$ if there is some $t < s$ such that

$$\sum_{d_e(n_{ek}(t))[s]<i\le s} 2^{-g(i)} < 2^{-e-k} \cdot \sum_{n_{ek}(t)<i\le s} 2^{-g(i)} \tag{3.5}$$

and

$$\forall i \le p_{ek}[s],\ K(V_e \restriction_i)[s] \le g(i) + k \tag{3.6}$$

where $p_{ek}[s]$ is the largest stage $\le s$ where P^*_{ek} required attention (and $p_{ek}[s] = 0$ if such a stage does not exist). In this case we say that P^*_{ek} *requires attention for* t at stage $s+1$.

The intuition for the main action of the construction is that if (3.5) holds, by enumerating $d_e(n_{ek}(t))[s]$ into D and changing the approximation to $V_e \restriction_{n_{ek}(t)}$ *the cost of the opponent for maintaining* (3.6) *is a large multiple of our cost for maintaining* (3.3). Our choice of the sequence $(n_{ek}(i))$ ensures that such attacks are sufficient in order to drive the opponent out of the available descriptions that are needed for maintaining (3.6). Moreover recall that by the analysis of Section 3.1, (3.5) has to hold for infinitely many t if A_e, B_e are indeed not K-trivial.

3.3. **Construction of the sets D, V_e.** At stage $s+1$ check if there is some $\langle e,k\rangle < s$ such that P^*_{ek} requires attention. If there is such a number, let $\langle e,k\rangle$ be the least one and let t be the least number such that (3.5) and (3.6) hold. Enumerate $d_e(n_{ek}(t))[s]$ into D and enumerate the largest number of

$$\mathbb{N}^{[k]} \cap \big(n_{ek}(t-1), n_{ek}(t)\big) - V_e[s] \tag{3.7}$$

into V_e.

3.4. **Verification of the requirements.** At every stage $s+1$ where P^*_{ek} requires attention for t and $\langle e,k\rangle < s$, a change in $V_e \restriction_{n_{ek}(t)}$ is caused by an enumeration of a number of the set in (3.7) into V_e (provided that the set in (3.7) is nonempty). There are

$$\big|\big(n_{ek}(t-1), n_{ek}(t)\big) \cap \mathbb{N}^{[k]}\big|$$

many such enumerations that can be performed. Because of (3.4) and (3.6), each time that P^*_{ek} requires attention after such an enumeration, we can count an additional weight of

$$1/\big|\big(n_{ek}(t-1), n_{ek}(t)\big) \cap \mathbb{N}^{[k]}\big|$$

in the underlying universal prefix-free machine U. Consequently, since $\mathtt{wgt}(U) < 2^{-2}$,

$$P^*_{ek} \text{ requires attention less than } 2^{n_{ek}(t)} \text{ times for } t. \tag{3.8}$$

Marker $d_e(t)$ moves at stage $s+1$ only if one of the following events occur:

(a) $A \restriction_i [s] \not\subset A[s+1]$ or $B \restriction_i [s] \not\subset B[s+1]$;
(b) $d_e(i)[s] \in V_e[s+1] - V_e[s]$.

Clearly (a) can only occur at most finitely many times. Moreover (b) only occurs if $i = n_{ek}(t)[s]$ for some t such that P^*_{ek} requires attention for t at stage $s+1$. By (3.8), case (b) only occurs at most finitely many times. Consequently,

$$\lim_s d_e(i)[s] \text{ exists for each } e. \tag{3.9}$$

In other words $\mathbb{N}^{[\langle e,i\rangle]} - D$ is finite, which was an underlying assumption for the requirements of Section 3.1.

Lemma 3.2. *For each e, condition R_e is met.*

Proof. Fix e. The construction clearly meets condition R^*_e. By (3.9) and the analysis in Section 3 it follows that R_e is met. □

Lemma 3.3. *For each e, condition P_e is met.*

Proof. Fix $e > 3$. By Lemma 3.1, it suffices to show that P^*_{ekt} is met for each k, t. Fix k, t and assume that the left hand side of the implication in P^*_{ekt} holds. Then according to the construction, (3.8) implies that $K(V_e \restriction_{n_{ekt}}) > g(n_{ekt}) + k$. □

Lemma 3.4. *The set D is K-trivial.*

Proof. By the analysis in Section 3.1 it suffices to show (3.3). Let

$$I_D(e,k) = \Big\{ \big(d_e(n_{ek}(t)[s-1], s\big) \in I_D \mid s, t > 0 \Big\}.$$

Note that $I_D(e,k)$ contains the pairs in I_D that correspond to actions for P^*_{ek}. In particular, $I_D = \bigcup_{e,k} I_D(e,k)$ and it suffices to show that

$$\sum_{(n,s)\in I_D(e,k)} c(n,s) < 2^{-e-k-3} \tag{3.10}$$

for each e, k. Fix e, k and let (n_i, s_i) be a monotone enumeration of $I_D(e,k)$, in the sense that $s_i < s_{i+1}$ for each i. Let us say that at stage s_{i+1} the ith cycle of P_{ek} is completed. Note that the sequence (n_i, s_i) is possibly infinite. However upon the completion of the ith cycle of P_{ek} we may count an additional set of descriptions of the universal machine U (describing current values of V_e) of weight at least $2^{e+k} \cdot c(n_i, s_i)$. This is a consequence of (3.5) and (3.6). For the case that (n_i, s_i) is finite (so the last cycle is never completed) note that $c(n_i, s_i) < 2^{-e-k-4}$ for all i due to (3.5). Since $\mathtt{wgt}(U) < 2^{-4}$ we obtain $\sum_i c(n_i, s_i) < 2^{-e-k-3}$, i.e. (3.10). □

According to the analysis of Section 3.1, this concludes the proof of Theorem 3.

4. Conclusion

Our result shows that a certain simple definition of the ideal of the K-trivial degrees with parameters is not possible in the c.e. degrees. The question of parameter definability of this ideal in the c.e. degrees remains open.

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