

A Hierarchy of Ramified Theories below PRA

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For Helmut Schwichtenberg on his retirement, with friendship and respect.

Abstract

A two-sorted arithmetic EA(I;O) of elementary recursive strength, based on the Bellantoni–Cook variable separation, is first enriched by addition of quantifiers over “input” or “normal” variables, and then extended in two different ways to a hierarchy of theories whose provably computable functions coincide with the levels of the Grzegorzczuk hierarchy.

1 Introduction

The theory EA(I;O) of Ostrin–Wainer [5],[6] (see also [9]) is a stripped-down version of the ramified intrinsic theories of Leivant [4], designed to incorporate the “normal/safe” variable discipline of Bellantoni–Cook [2] in a two-sorted analogue of Peano arithmetic, but with a weaker, “pointwise” or “predicative” induction scheme:

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \rightarrow A(x)$$

where a is a safe variable and x is normal. We prefer to call them “output” variables and “input” variables respectively; hence the I;O notation. Input variables are not (at this stage) quantified, so they act as uninterpreted constants. The usual proof theoretic methods apply just as for PA (e.g. embedding and cut elimination in an infinitary arithmetic with ω -rule). But now, because the inductions are only “up to x ”, the natural bounding functions are supplied by the “slow growing” hierarchy rather than the “fast growing” one. Since the slow growing functions below ε_0 are the exponential polynomials, and those below ω^ω are just polynomials, it follows (as Leivant had already previously shown, but by different methods) that the provably computable functions of EA(I;O) are the elementary functions (Grzegorzczuk’s \mathcal{E}^3) and those provably computable in its Σ_1 -inductive fragment are the sub-elementary \mathcal{E}^2 functions, i.e. those Turing-machine computable in linear space. (By shifting to a binary, rather than our

unary, representation of numbers, one sees that the Σ_1 -inductive fragment then characterizes polytime.)

Though quite simple in its formulation, EA(I;O) is not very “user friendly”, as it does not permit quantification over inputs x, y, z , and therefore one cannot even show straightforwardly that the provably computable functions – as functions on inputs – are closed under composition. Of course it is true, and Wirz [10] supplies a variety of delicate proof theoretic analyses enabling the derivation of such results, but they also serve to highlight the awkwardness of the logic of EA(I;O). Here, we rectify this by extending the theory conservatively to a new theory EA(I;O)⁺ which allows quantification over inputs and incorporates also a certain “ Σ_1 Reflection Rule”. The induction however, continues to apply only to formulas of the base theory EA(I;O). One then sees that EA(I;O)⁺ forms just the first level of a ramified hierarchy of input/output theories, whose provably computable functions coincide, level-by-level, with the Grzegorzcyk hierarchy. A different approach to extending EA(I;O) was taken by the first author in his thesis [8], where the reflection rule and the quantifier rules on inputs were replaced by an “internalized” version of the ω -rule restricted to Σ_1 formulas. This “ Σ_1 Closure Rule” allows one to derive $\forall a A(a)$ from a proof of $A(x)$, and is a perfectly natural device given that in EA(I;O) inputs x act as uninterpreted constants. Thus EA(I;O)+ Σ_1 -Closure has the same computational strength as $\text{I}\Sigma_1$. This too is described in the final section.

2 Input-Output Arithmetic EA(I;O)

EA(I;O) has the language of arithmetic, with quantified “output” (or “safe”) variables a, b, c, \dots and unquantified “input” (or “normal”) variables x, y, z, \dots . For convenience other terms and defining axioms are added, for a pairing function $\pi(a, b)$ ($:= 1/2(a + b)(a + b + 1) + a + 1$) with inverses π_0, π_1 , from which sequence numbers can be constructed using $\pi(s, a)$ to append a to s , and deconstructed by functions $(s)_i$ extracting the i -th component. All of these initial functions are quadratically bounded. The induction axioms are:

$$A(0) \wedge \forall a(A(a) \rightarrow A(a + 1)) \rightarrow A(t)$$

where $t = t(\vec{x})$ is a *term on inputs only*, controlling induction-length. Note that if $A(a)$ is progressive then so is $\forall b \leq a. A(b) \equiv \forall b(b \leq a \rightarrow A(b))$, and so a more revealing instance of induction is

$$A(0) \wedge \forall a(A(a) \rightarrow A(a + 1)) \rightarrow \forall b \leq t. A(b).$$

In other words, EA(I;O) is, in a sense, a theory of bounded induction, the (implicit) bounds being terms $t(\vec{x})$ dependent on inputs \vec{x} which cannot be universally quantified and then later re-instantiated, as they can be in PA. Call this “input” or “predicative” induction. Note however that there is no restriction on the formula A .

Definition 2.1 A numerical function $f : N^k \rightarrow N$ is “provably computable” or “provably recursive” in $\text{EA}(\text{I};\text{O})$ if there is a Σ_1 formula $C_f(\vec{x}, a)$ (i.e. a bounded formula prefixed by unbounded existential quantifiers) such that $f(\vec{n}) = m$ if and only if $C_f(\vec{n}, m)$ is true, and $\text{EA}(\text{I};\text{O}) \vdash \exists! a C_f(\vec{x}, a)$, i.e. f is provably total on inputs. We shall occasionally use the shorthand $f(\vec{x}) \downarrow$ for the formula $\exists a C_f(\vec{x}, a)$.

Theorem 2.2 [Leivant, Ostrin-Wainer] *The provably computable functions of $\text{EA}(\text{I};\text{O})$ are exactly the Csillag-Kalmar elementary functions.*

By carefully restricting the witnessing terms in the existential introduction rule and in induction, to the so-called “basic” ones (i.e. those built out of unary term constructors only: successor, predecessor, π_i) one may also characterize the sub-elementary functions as those provably computable in the Σ_1 inductive fragment. Then increasing induction complexity in $\text{EA}(\text{I};\text{O})$ characterizes the successive levels of the Ritchie-Schwichtenberg hierarchy between sub-elementary and elementary. See [6].

3 Enriching $\text{EA}(\text{I};\text{O})$ to $\text{EA}(\text{I};\text{O})^+$

One sees immediately the deficiencies in the logic of $\text{EA}(\text{I};\text{O})$ if one tries to show, simply and directly, that the provably computable functions are closed under composition. For suppose one has proved that $f(x) \downarrow$ and $g(x) \downarrow$, i.e. $\exists a C_g(x, a)$. Then one needs first to reflect the value a of $g(x)$ as an input – *the guiding principle here, is that values a computable from inputs only may themselves be used as inputs*. Thus one obtains $\exists y C_g(x, y)$ and by generalizing over inputs, $\forall y (f(y) \downarrow)$. By logic,

$$\exists y C_g(x, y), \forall y (f(y) \downarrow) \vdash \exists a, b (C_g(x, b) \wedge C_f(b, a))$$

so by two cuts one immediately derives $\exists a, b (C_g(x, b) \wedge C_f(b, a))$ which is the desired $\exists a C_{f \circ g}(x, a)$.

We therefore now extend $\text{EA}(\text{I};\text{O})$ to the new theory $\text{EA}(\text{I};\text{O})^+$ by adding Input-quantifier rules (in Tait style):

$$\frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \quad \frac{\Gamma, A(y)}{\Gamma, \forall x A(x)}$$

(provided t contains only input variables and y is not free in Γ) and also the Σ_1 Reflection rule:

$$\frac{\Gamma(\vec{x}), \exists \vec{a} A(\vec{x}, \vec{a})}{\Gamma(\vec{x}), \exists \vec{y} A(\vec{x}, \vec{y})}$$

where $\Gamma(\vec{x}), \exists \vec{a} A(\vec{x}, \vec{a})$ is a set of Σ_1 formulas all of whose free variables are inputs.

The induction in $\text{EA}(\text{I};\text{O})^+$ is the same as that of $\text{EA}(\text{I};\text{O})$, it only applies to formulas without input quantifiers.

Our first task is to show that this extension of $\text{EA}(\text{I};\text{O})$ is conservative, in that no new provably recursive functions are produced.

Note 3.1 An alternative, simpler way of directly proving closure under composition is provided by the “Closure Rule”: from $f(x) \downarrow$ derive $\forall b(f(b) \downarrow)$. The proof of $\exists a, b(C_g(x, b) \wedge C_f(b, a))$ from $\exists b C_g(x, b)$ then follows in the same way as above, but without any further apparatus needed. As we shall later show, adding the Σ_1 Closure rule to EA(I;O) produces a version of primitive recursive arithmetic.

4 The Infinitary System EA(I;O) $^+_\infty$

Upper bounds on provable recursiveness in EA(I;O) $^+$ are obtained by the usual proof theoretic process – embedding into a suitable infinitary system which admits cut elimination.

The infinitary system EA(I;O) $^+_\infty$ derives Tait-style sequents

$$n:I; m:O \vdash^\alpha \Gamma$$

where Γ is a finite set of closed formulas, n bounds the input parameters, m declares a bound on any initial output values and the ordinal heights α are, for our purposes here, in fact “tree ordinals” (with assigned fundamental sequences) generated from 0 and $\omega = \sup_i(i + 1)$ by addition, multiplication and exponentiation. Thus, as set-theoretic ordinals, they all lie below ε_0 , and have standard fundamental sequences. For shorthand we write simply $n; m \vdash^\alpha \Gamma$ or $n; \vdash^\alpha \Gamma$ when $m = 0$.

Most of the rules are unsurprising and we don’t list the \vee, \wedge rules explicitly. The axioms are $n; m \vdash^\alpha \Gamma$ where the set Γ contains a true atom.

The Cut rule, with cut formula C , is

$$\frac{n; m \vdash^{\beta_0} \Gamma, \neg C \quad n; m \vdash^{\beta_1} \Gamma, C}{n; m \vdash^\alpha \Gamma} .$$

There are two \exists -rules, one for $\exists a$ and one for $\exists x$, each with two premises:

$$\frac{n; m \Vdash^{\beta_0} m' \quad n; m \vdash^{\beta_1} A(m'), \Gamma}{n; m \vdash^\alpha \exists a A(a), \Gamma} \quad \frac{n; \Vdash^{\beta_0} m' \quad n; m \vdash^{\beta_1} A(m'), \Gamma}{n; m \vdash^\alpha \exists x A(x), \Gamma} .$$

Here the left-hand premise “computes” witness m' from m which we denote with a different proof gate \Vdash (not the forcing notation). We define such computations according to the following simple rules – The computational axiom is $n; m \Vdash^\alpha m'$ provided $m' \leq q(m)$, where q is a suitable quadratic function which bounds all the term-constructors. The computation rule (call-by-value) is:

$$\frac{n; m \Vdash^{\beta_0} m'' \quad n; m'' \Vdash^{\beta_1} m'}{n; m \Vdash^\alpha m'}$$

and this is also allowed to interact with the logic in the form:

$$\frac{n; m \Vdash^{\beta_0} m'' \quad n; m'' \vdash^{\beta_1} \Gamma}{n; m \vdash^\alpha \Gamma} .$$

The universal quantifiers $\forall a$ and $\forall x$ are introduced by versions of the ω -rule:

$$\frac{\{n; \max(m, i) \vdash^\beta A(i), \Gamma\}_{i \in \mathbb{N}}}{n; m \vdash^\alpha \forall a A(a), \Gamma} \quad \frac{\{\max(n, i); m \vdash^\beta A(i), \Gamma\}_{i \in \mathbb{N}}}{n; m \vdash^\alpha \forall x A(x), \Gamma}.$$

Note that here, the ordinal bound β on the premises does not vary with i . This is what keeps the theory “weak”.

4.1 The ordinal assignment

In all of the above rules the declared input n controls the ordinal assignment in the following way: if $n; m' \vdash^\beta \Gamma'$ is any premise of a rule with conclusion $n; m \vdash^\alpha \Gamma$ then the requirement is that $\beta \in \alpha[n]$ where $\alpha[n] = \emptyset$ if $\alpha = 0$, or $\beta[n] \cup \{\beta\}$ if $\alpha = \beta + 1$, or $\alpha_n[n]$ if $\alpha = \sup_i \alpha_i$ is a limit. Thus, while input n is fixed, derivations not containing the $\forall x$ rule are in fact of finite height because $n; m \vdash^\alpha \Gamma$ implies $n; m \vdash^{G_\alpha(n)} \Gamma$ where $G_\alpha(n) = |\alpha[n]|$ is the “slow growing” hierarchy. It is not difficult to check that for each fixed n , the map $\alpha \mapsto G_\alpha(n)$ preserves the arithmetic operations such as addition, multiplication and exponentiation. Thus by choosing $\omega = \sup(i+1)$ one sees that $G_\omega(n) = n+1$ and for each α , $G_\alpha(n)$ is the exponential polynomial which results by replacing ω by $n+1$ throughout. The following Bounding Lemma is easy by inductions on α ; recall that q is a fixed quadratic chosen to bound all the term-constructors, and q^k denotes its k -times iterate. (Note the Bellantoni-Cook-style variable separation.)

Lemma 4.1 $n; m \Vdash^\alpha m'$ if and only if $m' \leq q^k(m)$ where $k = 2^{G_\alpha(n)}$.

4.2 Cut elimination in $\text{EA}(\mathbf{I}; \mathbf{O})_\infty^+$

The standard methods apply here, just as for PA_∞ .

Theorem 4.2 (i) If $n; m \vdash^\gamma \Gamma, \neg C$ and $n; m \vdash^\alpha \Gamma, C$ are both derivable with cut formulas of “size” $\leq r$, where C is a formula of size $r+1$ with shape $\vee, \exists a$ or a true atom and $\alpha[n] \subseteq \gamma[n]$, then $n; m \vdash^{\gamma+\alpha} \Gamma$ is also derivable with cuts of size $\leq r$.

(ii) If $n; m \vdash^\alpha \Gamma$ is derivable with cut formulas of size at most $r+1$ with shape \vee or $\exists a$, then $n; m \vdash^{2^\alpha} \Gamma$ is derivable with cut formulas of size at most r .

Repeated application of (ii) eliminates cuts on formulas beginning \vee or $\exists a$, at the expense of an iterated exponential increase in the ordinal bound. (It does not eliminate cuts on formulas beginning with an input quantifier, but these can anyway be kept down to the Σ_1 level, as we shall see below.)

This gives an immediate glimpse of why the provably computable functions of $\text{EA}(\mathbf{I}; \mathbf{O})$ are elementary. For if an $\text{EA}(\mathbf{I}; \mathbf{O})$ proof of $f(x) \downarrow \equiv \exists a C_f(x, a)$ is embedded in the infinitary system, a cut-free derivation is obtained with ordinal bound $|\alpha| < \varepsilon_0$. Then for each $x := n$ a witness m may be read off such that $C_f(n, m)$ holds and $n; m \Vdash^\alpha m$. Therefore, by the bounding principle above, the elementary function $q^k(0)$, where $k = 2^{G_\alpha(n)}$, bounds the quantifiers in the Σ_1 defining formula of f , so f is an elementary function.

4.3 Embedding $\text{EA}(\mathbf{I};\mathbf{O})^+$ in $\text{EA}(\mathbf{I};\mathbf{O})_\infty^+$

Definition 4.3 By a “ $\Sigma(I)$ formula” (dually $\Pi(I)$) is meant a Σ_1 (resp. Π_1) formula beginning with at least one unbounded input quantifier $\exists x$ ($\forall x$). Write $n; m \vdash_{\Sigma(I)}^\alpha \Gamma$ to signify that there is a derivation of $n; m \vdash^\alpha \Gamma$ in which all cut formulas are $\Sigma(I)$ (dually $\Pi(I)$).

Theorem 4.4 *If $\text{EA}(\mathbf{I};\mathbf{O})^+ \vdash \Gamma(\vec{x}; \vec{a})$ then there is an α with $|\alpha| < \varepsilon_0$ such that for all $\vec{x} := \vec{n} = n_1, \dots, n_r \leq n$ and all $\vec{a} := \vec{m} = m_1, \dots, m_\ell \leq m$,*

$$n; m \vdash_{\Sigma(I)}^\alpha \Gamma(\vec{n}; \vec{m}).$$

Proof. First, as preparation, note that in $\text{EA}(\mathbf{I};\mathbf{O})^+$ we can directly eliminate all “free” cuts in which the cut formula C contains unbounded input quantifiers and is of greater logical complexity than $\Sigma(I)$ or $\Pi(I)$. Very roughly, the procedure goes as follows: if the premises of the cut are $\Gamma, \neg C$ and Γ, C where C begins, say, with an existential quantifier, then it cannot be the result of an induction or a reflection rule (since it’s neither an $\text{EA}(\mathbf{I};\mathbf{O})$ formula nor $\Sigma(I)$), and so could only arise by an \exists rule from a $\Gamma, D(t)$ where $C \equiv \exists a D(a)$ or $C \equiv \exists x D(x)$. Then by inverting the other premise to $\Gamma, \neg D(t)$ one sees that the original cut on C may now be replaced by a cut on its subformula D . (Similarly if $C \equiv D_1 \vee D_2$.) Repeating this process successively eliminates all such cuts from $\text{EA}(\mathbf{I};\mathbf{O})^+$ proofs, and we shall henceforth assume this to have been done.

The theorem is now proved by induction over the height of this (revised) proof of $\Gamma(\vec{x}; \vec{a})$ in $\text{EA}(\mathbf{I};\mathbf{O})^+$, with cases according to the last rule applied. The cut elimination of the last subsection ensures that cuts on $\text{EA}(\mathbf{I};\mathbf{O})$ formulas may continually be eliminated without increasing the ordinal bounds above the superexponential level. Thus only $\Sigma(I)$ cuts will remain.

The non-inductive axioms, the \vee, \wedge -rules and the cuts on $\Sigma(I)$ formulas all carry over easily to the infinitary setting, as do the $\forall a$ and $\forall x$ rules.

If $\Gamma, \exists a A(a)$ comes about by an \exists -rule from the premise $\Gamma, \exists a A(a), A(t(\vec{x}; \vec{a}))$ then we may inductively assume there is a β such that for all $\vec{n} \leq n, \vec{m} \leq m$, $n; m \vdash_{\Sigma(I)}^\beta \Gamma, \exists a A(a), A(t(\vec{n}; \vec{m}))$. Now the value m' of $t(\vec{n}; \vec{m})$ is polynomially bounded, and certainly $m' \leq q^k(m)$ where $k = 2^{G_{\omega+a}(n)}$ for some suitable d . Therefore $n; m \Vdash^{\omega+d} m'$ and, by weakening the ordinal bound β and replacing $t(\vec{n}; \vec{m})$ by its value, $n; m \vdash_{\Sigma(I)}^{\omega+d+\beta} \Gamma, \exists a A(a), A(m')$. An application of the $\exists a$ rule then gives $n; m \vdash_{\Sigma(I)}^\alpha \Gamma, \exists a A(a)$ as required, with $\alpha = \omega + d + \beta + 1$. The $\exists x$ rule is handled similarly but with $m = 0$.

An induction axiom, in Tait style, is $\Gamma, \neg A(0), \exists a(A(a) \wedge \neg A(a+1)), A(t)$ with A an $\text{EA}(\mathbf{I};\mathbf{O})$ formula and t a term on input variables only. We suppress the other free variables \vec{x}, \vec{a} from Γ, A . As above, let m' be the value of $t(\vec{n})$ and choose $\gamma = \omega + d$ so that $n; m \Vdash^\gamma m'$. Next, note that for some fixed k depending on the size of the formula A , simple logic gives a cut-free derivation of $n; m' \vdash^k \Gamma, \neg A(0), \exists a(A(a) \wedge \neg A(a+1)), \neg A(i), A(i)$ for any i . Therefore if one assumes $n; m' \vdash^{k+2(i-1)} \Gamma, \neg A(0), \exists a(A(a) \wedge \neg A(a+1)), A(i-1)$ with $i \leq m'$, then by the \wedge -rule followed by the $\exists a$ rule (noting $n; m' \Vdash^0 i-1$) one immediately obtains

$n; m' \vdash^{k+2i} \Gamma, \neg A(0), \exists a(A(a) \wedge \neg A(a+1)), A(i)$. Hence by induction on i up to m' we have $n; m' \vdash^{k+2m'} \Gamma, \neg A(0), \exists a(A(a) \wedge \neg A(a+1)), A(m')$. But an easy tree ordinal computation shows $(k+2\omega)[m'] = \{0, \dots, k, k+1, \dots, k+2m'+1\}$ and so $n; m' \vdash^{k+2\omega} \Gamma, \neg A(0), \exists a(A(a) \wedge \neg A(a+1)), A(m')$. A computation rule with $n; m \Vdash^\gamma m'$ gives $n; m \vdash^{\gamma+k+2\omega+1} \Gamma, \neg A(0), \exists a(A(a) \wedge \neg A(a+1)), A(m')$. Since m' is the value of the term t , $\neg A(m')$, $A(t)$ is derivable with height k and then an eliminable cut on $A(m')$ yields $n; m \vdash_{\Sigma(I)}^\alpha \Gamma, \neg A(0), \exists a(A(a) \wedge \neg A(a+1)), A(t)$ as required.

Finally we must show that the Σ_1 reflection rule: from $\Gamma(\vec{x}), \exists \vec{a}A(\vec{x}, \vec{a})$ derive $\Gamma(\vec{x}), \exists \vec{y}A(\vec{x}, \vec{y})$, embeds into $\text{EA}(\text{I}; \text{O})_\infty^+$, where $\Gamma, \exists \vec{a}A(\vec{a})$ is a set of Σ_1 formulas with only the input variables \vec{x} free. Assume then, as induction hypothesis, that there is a β such that for all $\vec{n} \leq n$, $n; \vdash_{\Sigma(I)}^\beta \Gamma(\vec{n}), \exists \vec{a}A(\vec{n}, \vec{a})$. We need to prove, for a suitable α , $n; \vdash_{\Sigma(I)}^\alpha \Gamma(\vec{n}), \exists \vec{x}A(\vec{n}, \vec{x})$. This follows immediately from the following lemma with $m = 0$.

As a preliminary, note that only finitely many terms $t(\vec{x}; \vec{a})$ will be involved in the embedding of any $\text{EA}(\text{I}; \text{O})^+$ proof, and each one is polynomially bounded. So there will be a $\gamma = \omega + d$ for some fixed d , such that for all $\vec{n} \leq n, \vec{m} \leq m$ and every such term, $n; m \Vdash^\gamma \text{val}(t(\vec{n}; \vec{m}))$ or equivalently $\text{val}(t(\vec{n}; \vec{m})) \leq q^k(m)$ where $k = 2^{G_\gamma(n)}$. We then say that the derivation is “term controlled” by γ . \square

Lemma 4.5 *Suppose $n; m \vdash_{\Sigma(I)}^\beta \Gamma(\vec{n}; \vec{m})$ where Γ is a set of Σ_1 formulas. Suppose also that the derivation is term controlled by γ , and that m is such that $n; \Vdash^\gamma m$. Then for some fixed k we have $n; \vdash_{\Sigma(I)}^{\gamma+k^\beta} \Gamma'$ where Γ' results from Γ by replacing some (possibly all) unbounded output quantifiers $\exists a$ by input quantifiers $\exists x$.*

Proof. We proceed by induction on β with cases according to the last rule applied. The choice of k will become clear, and it is easy to show that if $\delta \in \beta[n]$ then $k^\delta \in k^\beta[n]$ so a derivation with ordinal bound β may be “weakened” to one with ordinal bound k^β .

If Γ is an axiom (i.e. contains a true atom) then so will be $n; m \vdash^\gamma \Gamma'$, and the computation rule with $n; \Vdash^\gamma m$ gives $n; \vdash^{\gamma+k^\beta} \Gamma'$ since $\gamma \in (\gamma + k^\beta)[n]$.

The \vee and \wedge rules are handled easily, by applying the induction hypothesis to each premise and then re-applying the rule.

If $n; m \vdash^\beta \Gamma$ comes about by a computation rule with premises $n; m \Vdash^{\beta_0} m'$ and $n; m' \vdash^{\beta_1} \Gamma$ then, first, set $\delta = \max(\beta_0, \beta_1) \in \beta[n]$. We have $n; \Vdash^{\gamma+k^\delta \cdot 2} m'$ by the computation rule, and so by the induction hypothesis, with γ replaced by $\gamma + k^\delta \cdot 2$, we obtain $n; \vdash^{\gamma+k^\delta \cdot 3} \Gamma'$. Since $\delta \in \beta[n]$, if one chooses $k \geq 3$ then either $\gamma + k^\delta \cdot 3 = \gamma + k^\beta$ or $\gamma + k^\delta \cdot 3 \in (\gamma + k^\beta)[n]$, so $n; \vdash^{\gamma+k^\beta} \Gamma'$ as required, and still with only $\Sigma(I)$ cuts.

Suppose $n; m \vdash^\beta \Gamma$ arises from premises $n; m \Vdash^{\beta_0} m'$ and $n; m \vdash^{\beta_1} \Gamma, A(m')$ by an $\exists a$ or $\exists x$ rule where (respectively) $\exists aA(a)$ or $\exists xA(x)$ belongs to Γ . Then, again with $\delta = \max(\beta_0, \beta_1) \in \beta[n]$, the computation rule and the induction

hypothesis yield $n; \Vdash^{\gamma+k^\delta} m'$ and $n; \vdash^{\gamma+k^\delta} \Gamma', A'(m')$. Since $k^\delta \in k^\beta[n]$ the appropriate $\exists a$ or $\exists x$ rule may then be applied to give $n; \vdash_{\Sigma(I)}^{\gamma+k^\beta} \Gamma'$.

If the last rule applied is a \forall -rule then it can only occur in a bounded context since the formulas are all Σ_1 . The premises will then have the form $\{n; \max(m, i) \vdash^\delta \Gamma, i \not\leq t \vee B(i)\}_{i \in \mathbb{N}}$ where $\delta \in \beta[n]$ and $\forall a \leq tB(a)$ belongs to Γ . Since the derivation is term controlled by γ we have $n; m \Vdash^\gamma \text{val}(t)$, so $n; \Vdash^{\gamma+k^\delta} \max(m, i)$ for each $i \leq \text{val}(t)$. For $i > \text{val}(t)$ the atom $i \not\leq t$ is true and derivable with any side formulas and any ordinal height. Therefore by the induction hypothesis, with γ replaced by $\gamma + k^\delta$ and m replaced by $\max(m, i)$, we obtain $n; i \vdash^{\gamma+k^\delta \cdot 2} \Gamma', i \not\leq t \vee B(i)$ for all i . Then (assuming $k \geq 3$ again) a final re-application of the \forall -rule gives $n; \vdash_{\Sigma(I)}^{\gamma+k^\beta} \Gamma'$.

Finally, suppose the last rule applied is a cut on a $\Sigma(I)$ formula, say $\exists \vec{c}B(\vec{c})$ where B contains only bounded quantifiers and the variables \vec{c} are either inputs or outputs. Then the premises are $n; m \vdash_{\Sigma(I)}^{\beta_0} \Gamma, \forall \vec{c} \neg B(\vec{c})$ and $n; m \vdash_{\Sigma(I)}^{\beta_1} \Gamma, \exists \vec{c}B(\vec{c})$. (Any “dummy” variables not in \vec{n}, \vec{m} are assumed to have been set to 0.) Again let $\delta = \max(\beta_0, \beta_1) \in \beta[n]$. Applying the induction hypothesis to the second premise yields $n; \vdash_{\Sigma(I)}^{\gamma+k^\delta} \Gamma', \exists \vec{y}B(\vec{y})$ with new input variables \vec{y} . By inverting the $\forall \vec{c}$ in the first premise, $\max(n, \vec{i}); \max(m, \vec{i}) \vdash^\delta \Gamma, \neg B(\vec{i})$ for all \vec{i} . Now the induction hypothesis can be applied to this, since $\max(n, \vec{i}); \Vdash^\gamma \max(m, \vec{i})$. Thus $\max(n, \vec{i}); \vdash^{\gamma+k^\delta} \Gamma', \neg B(\vec{i})$, and then by applying the $\forall y$ rule as many times as necessary, $n; \vdash_{\Sigma(I)}^{\gamma+k^\delta+r} \Gamma', \forall \vec{y} \neg B(\vec{y})$. We may now do a cut on $\exists \vec{y}B(\vec{y})$ to obtain $n; \vdash_{\Sigma(I)}^{\gamma+k^\delta+r+1} \Gamma'$ and the ordinal bound may be increased to $\gamma + k^\beta$ as required, provided we choose $k >$ the length r of any quantifier-prefix occurring in the embedded $\text{EA}(\text{I};\text{O})^+$ proof.

This completes the lemma and the proof of the theorem. \square

4.4 Extracting elementary bounds for $\text{EA}(\text{I};\text{O})^+$

The embedding of $\text{EA}(\text{I};\text{O})^+$ in $\text{EA}(\text{I};\text{O})_\infty^+$ preserves the $\Sigma(I)$ cuts, and since these are not allowed as induction formulas, it means that there must be a finite upper bound on the “depth of nesting” of such $\Sigma(I)$ cuts in any embedded $\text{EA}(\text{I};\text{O})_\infty^+$ derivation. We call this the “cut height” of the derivation and denote it h .

Definition 4.6 For α in the additive, multiplicative, exponential closure of $\{0, \omega\}$, let $B_\alpha(n, m)$ be the elementary function $q^k(m)$ where $k = 2^{G_\alpha(n)}$, so $n; m \Vdash^\alpha m'$ if and only if $m' \leq B_\alpha(n, m)$. Then define $B_\alpha^{(0)} = B_\alpha$ and, for any fixed positive h ,

$$B_\alpha^{(h)}(n, m) := B_\alpha^{(h-1)}(B_\alpha^{(h-1)}(n, m), B_\alpha^{(h-1)}(n, m)) .$$

Since $B_\alpha^{(h)}$ is a finite compositional term built up from B_α , it too is elementary.

Note 4.7 If $\beta \in \alpha[n]$ then $G_\beta(n) < G_\alpha(n)$ and so $B_\beta(n, B_\beta(n, m)) \leq B_\alpha(n, m)$ and similarly, $B_\beta^{(h)}(n, B_\beta(n, m)) \leq B_\alpha^{(h)}(n, m)$.

Definition 4.8 A closed Σ_1 formula $\exists \vec{c}A(\vec{c})$, with A a bounded formula and \vec{c} a mixture of input or output variables, is said to be “true at b ” if there are witnesses $\vec{m} \leq b$ such that $A(\vec{m})$ is true (in standard arithmetic). If there are no unbounded existential quantifiers then the bounded formula A is true at b if it is true. A set of Σ_1 formulas is true at b if at least one of them is.

Lemma 4.9 Let Γ be a set of Σ_1 formulas such that $n; m \vdash_{\Sigma(I)}^{\alpha} \Gamma$ with finite cut height $\leq h$. Assume that the derivation is term controlled by γ . Then Γ is true at $B_{\alpha'}^{(h)}(n, m)$ where $\alpha' = \gamma + \alpha$.

Proof. We use induction on h with sub-induction on α . To save having to decorate each ordinal with a “dash”, we may as well assume, by weakening, that γ has already been “added in” to each ordinal bound in the derivation. Then if $n; m \vdash^{\beta} \Gamma$ each closed term in Γ has value $\leq B_{\beta}(n, m)$. Let (*) denote the given derivation of $n; m \vdash_{\Sigma(I)}^{\alpha} \Gamma$.

If (*) is an axiom then Γ contains a true atom and is true at any b .

If (*) comes about by the \vee or \wedge rule then the premise(s) are of the form $n; m \vdash_{\Sigma(I)}^{\beta} \Gamma', D$ and $n; m \vdash_{\Sigma(I)}^{\beta} \Gamma', D'$ where D, D' are bounded formulas and of course, $\beta \in \alpha[n]$. By the induction hypothesis, either Γ' is true at $B_{\beta}^{(h)}(n, m)$ or, if not, D, D' are true. In this case $\Gamma = \Gamma', D \vee D'$ or $\Gamma = \Gamma', D \wedge D'$ is true at $B_{\beta}^{(h)}(n, m)$ and hence at $B_{\alpha}^{(h)}(n, m)$.

If (*) arises by the \forall rule then it can only occur in a bounded context, so the premises are $\{n; \max(m, i) \vdash^{\beta} \Gamma', i \not\leq t \vee B(i)\}_{i \in \mathbb{N}}$ where $\Gamma = \Gamma', \forall i \leq t B(i)$. Either $\forall i \leq t B(i)$ is true (and hence the result) or else there is an $i \leq \text{val}(t)$ such that Γ' is true at $B_{\beta}^{(h)}(n, \max(m, i))$. But by term control, $i \leq B_{\beta}(n, m)$ and so $B_{\beta}^{(h)}(n, \max(m, i)) \leq B_{\beta}^{(h)}(n, B_{\beta}(n, m)) \leq B_{\alpha}^{(h)}(n, m)$. Therefore Γ' and hence Γ is true at $B_{\alpha}^{(h)}(n, m)$.

If (*) arises by an \exists rule then there are two premises: $n; m \vdash^{\beta} \Gamma, A(m')$ and either $n; m \Vdash^{\beta} m'$ in case it's a $\exists a$ rule, or $n; \Vdash^{\beta} m'$ if it's a $\exists x$ rule. Here Γ contains $\exists cA(c)$ with the variable c being either an a or an x . By the induction hypothesis, either some formula in Γ is already true at $B_{\beta}^{(h)}(n, m)$ or else $A(m')$ is true at $B_{\beta}^{(h)}(n, m)$ and hence $\exists cA(c)$ is also true at $B_{\beta}^{(h)}(n, m)$ because the new witness m' is $\leq B_{\beta}(n, m) \leq B_{\beta}^{(h)}(n, m)$. Either way, Γ is true at $B_{\beta}^{(h)}(n, m)$ and hence at $B_{\alpha}^{(h)}(n, m)$.

If (*) comes about by a computation rule with premises $n; m' \vdash^{\beta} \Gamma$ and $n; m \Vdash^{\beta} m'$ then by the induction hypothesis Γ is true at $B_{\beta}^{(h)}(n, m')$, and also we have $m' \leq B_{\beta}(n, m)$. Therefore Γ is true at $B_{\beta}^{(h)}(n, B_{\beta}(n, m)) \leq B_{\alpha}^{(h)}(n, m)$ as required.

Finally suppose (*) arises by a cut from the premises $n; m \vdash^{\beta} \Gamma, \exists \vec{c}A(\vec{c})$ and $n; m \vdash^{\beta} \Gamma, \forall \vec{c} \neg A(\vec{c})$ with $A(\vec{c})$ a bounded formula and where \vec{c} is a sequence of input or output variables. Since Γ is derived with cut height $\leq h$, both of the premises must be derived with cut height $\leq h - 1$. By the induction hypothesis applied to the first premise, and letting $k = B_{\beta}^{(h-1)}(n, m)$, either

Γ is true at k or else there are $\vec{i} \leq k$ such that $A(\vec{i})$ is true. Now inverting the $\forall \vec{c}$ in the second premise, $k; k \vdash^\beta \Gamma, \neg A(\vec{i})$. The set $\Gamma, \neg A$ consists solely of Σ_1 formulas, so the induction hypothesis may be applied again, yielding $\Gamma, \neg A(\vec{i})$ true at $B_\beta^{(h-1)}(k, k)$. Since $\neg A(\vec{i})$ is false, we therefore have Γ true at $B_\beta^{(h-1)}(k, k) = B_\beta^{(h)}(n, m) \leq B_\alpha^{(h)}(n, m)$ and this completes the proof. \square

Theorem 4.10 *The provably computable functions of $EA(I; O)^+$ are exactly the elementary functions.*

Proof. Every elementary function is provably computable in $EA(I; O)$, and therefore in $EA(I; O)^+$. For the converse suppose $f(\vec{x})$ has the defining formula $\exists a C_f(\vec{x}, a)$ provable in $EA(I; O)^+$. Then by the embedding, $n; \vdash_{\Sigma(I)}^\alpha \exists a C_f(\vec{n}, a)$ with fixed cut height h , where $n = \max(\vec{n})$, and we may assume that this derivation is term controlled by some γ . Therefore by the Lemma, there are true witnesses for the existentially quantified variables prefixing $\exists a C_f(\vec{n}, a)$ and they are all bounded by the elementary function $B_{\gamma+\alpha}^{(h)}(n, 0)$. This holds uniformly for all inputs \vec{n} . Thus the graph of f is elementarily decidable and its value is elementarily bounded, so it is an elementary function. \square

5 A Hierarchy of Theories above $EA(I; O)^+$

We now introduce a new level of input variables, and a new tier of inductions over $EA(I; O)^+$ formulas. Henceforth denote I by I_1 , and call x, y, z the I_1 variables. Then the I_2 variables are new variables, denoted u, v, w . Add to $EA(I; O)^+$ these new I_2 variables and the new induction principle:

$$A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow A(t)$$

where A is an $EA(I; O)^+$ -formula, possibly with free I_2 parameters, and $t = t(\vec{u})$ is a term containing only I_2 variables. This theory is denoted $EA(I_2; I_1; O)$. Its extension $EA(I_2; I_1; O)^+$ is obtained by further adding I_2 -quantifier rules and a Σ_1 reflection rule at level 2:

$$\frac{\Gamma(\vec{u}), \exists \vec{x} A(\vec{u}, \vec{x})}{\Gamma(\vec{u}), \exists \vec{v} A(\vec{u}, \vec{v})}$$

where $\Gamma(\vec{u}), \exists \vec{x} A(\vec{u}, \vec{x})$ is a set of Σ_1 formulas all of whose free variables are I_2 inputs.

Definition 5.1 A function f is provably computable in $EA(I_2; I_1; O)^+$ if, on level-2 inputs \vec{u} , its defining formula $f(\vec{u}) \downarrow \equiv \exists a C_f(\vec{u}, a)$ is provable.

Note 5.2 Every function provably computable in $EA(I; O)^+$ is provably computable in $EA(I_2; I_1; O)$, for by trivial applications of the level-2 induction principle above, if $\exists a C_f(\vec{x}, a)$ is provable in $EA(I; O)^+$ then $\exists a C_f(\vec{u}, a)$ is provable in $EA(I_2; I_1; O)$.

Lemma 5.3 *The functions of Grzegorzczuk's \mathcal{E}^4 are all provably computable in $EA(I_2; I_1; O)^+$.*

Proof. It is only necessary to show that every function register-machine computable in a number of steps bounded by some finite iterate of the superexponential $2_u(u)$ is provably computable in $EA(I_2; I_1; O)^+$. Here, $2_u(v)$ is defined by $2_0(v) = v$ and $2_{u+1}(v) = 2^{2_u(v)}$.

First note that $2_u(u) \downarrow$ is provable in $EA(I_2; I_1; O)^+$. This is because $2^x \downarrow$ is already provable in $EA(I; O)$ and so $\exists y(2^x = y)$ and $\forall x \exists y(2^x = y)$ are provable in $EA(I; O)^+$ by the level-1 reflection rule and the \forall -rule. Therefore $\exists y(2_0(z) = y)$ and $\exists y(2_x(z) = y) \rightarrow \exists y(2_{x+1}(z) = y)$ are provable. Now the level-2 induction comes into play, yielding $\exists y(2_u(z) = y)$ and hence $2_u(u) \downarrow$. Arbitrary finite compositions of this are then provable in $EA(I_2; I_1; O)^+$ by the level-2 reflection and quantification rules, as done earlier at level-1.

Letting $f(u)$ be any such finite iteration of $2_u(u)$, suppose $g(\vec{v})$ is register-machine computable in a number of steps bounded by $f(\max \vec{v})$. Associate with the register machine program a bounded formula $C_g(s, \vec{v}, i, r_1, \dots, r_k)$ representing the well-definedness of all successive internal configurations of the machine up to step s . Thus i stores the sequence of next-to-be-obeyed program-instructions starting at step 0, and each r_j is a sequence-number recording the numerical content of the j -th working register at each step up to s . The basic instructions either update a register by a successor or predecessor, or jump to another instruction according to the value in one register being zero or nonzero. The initial configuration on input \vec{v} is $(1, 0, \dots, 0)$. It is therefore easy to prove, in $EA(I_2; I_1; O)$,

$$\exists i \exists \vec{r} C_g(0, \vec{v}, i, \vec{r}) \wedge \forall s (\exists i \exists \vec{r} C_g(s, \vec{v}, i, \vec{r}) \rightarrow \exists i \exists \vec{r} C_g(s+1, \vec{v}, i, \vec{r})) .$$

Hence $\exists i \exists \vec{r} C_g(u, \vec{v}, i, \vec{r})$ is provable in $EA(I_2; I_1; O)$ and $\forall u \exists i \exists \vec{r} C_g(u, \vec{v}, i, \vec{r})$ is provable in $EA(I_2; I_1; O)^+$. Therefore so is $\exists i \exists \vec{r} C_g(f(\max \vec{v}), \vec{v}, i, \vec{r})$, using level-2 reflection and quantification. But this is the termination condition for g , so g is provably computable in $EA(I_2; I_1; O)^+$. \square

5.1 The infinitary system $EA(I_2; I_1; O)_\infty^+$

As before, upper bounds on provable recursiveness in $EA(I_2; I_1; O)^+$ are obtained by embedding into a suitable infinitary system which admits cut elimination.

The infinitary system $EA(I_2; I_1; O)_\infty^+$ derives Tait-style sequents

$$n_2 : I_2 ; n_1 : I_1 ; m : O \vdash^{\alpha, \gamma} \Gamma$$

where Γ is a finite set of closed formulas, n_2, n_1 bound level-2 and level-1 inputs respectively, and m declares a bound on the output values. The tree-ordinals α, γ are still exponential forms to the base ω , i.e. generated by addition, multiplication and exponentiation from $0, \omega$. As before, we use the shorthand $n_2; n_1; m \vdash^{\alpha, \gamma} \Gamma$.

Only the α and n_2 control the derivations in the sense that if β, γ are the bounds on a premise with conclusion $n_2; n_1; m \vdash^{\alpha, \gamma} \Gamma$ then $\beta \in \alpha[n_2]$. The γ remains fixed throughout, and only plays a role in the following axiom which layers the new system on top of the old one $\text{EA}(\text{I}; \text{O})_\infty^+$:

$$n_2; n_1; m \vdash^{\alpha, \gamma} \Gamma \text{ if } n_1; m \vdash^\gamma \Gamma' \text{ where } \Gamma' \subseteq \Gamma$$

with α and n_2 arbitrary.

The rules of $\text{EA}(\text{I}_2; \text{I}_1; \text{O})_\infty^+$ are those of $\text{EA}(\text{I}; \text{O})_\infty^+$ appropriately redecorated with n_2 and α , together with new level-2 rules:

The $\exists u$ -rule is

$$\frac{n_2; 0 \Vdash^{\beta_0, \gamma} m' \quad n_2; n_1; m \vdash^{\beta_1, \gamma} A(m'), \Gamma}{n_2; n_1; m \vdash^{\alpha, \gamma} \exists u A(u), \Gamma}.$$

and the $\forall u$ -rule is

$$\frac{\{ \max(n_2, i); n_1; m \vdash^{\beta, \gamma} A(i), \Gamma \}_{i \in \mathbb{N}}}{n_2; n_1; m \vdash^{\alpha, \gamma} \forall u A(u), \Gamma}.$$

The new computation axiom is:

$$n_2; n_1; m \Vdash^{\alpha, \gamma} m' \text{ if } n_1; m \vdash^\gamma m' \text{ in } \text{EA}(\text{I}; \text{O})_\infty^+.$$

The new computation rule is:

$$\frac{n_2; n_1; 0 \Vdash^{\beta_0, \gamma} n'_1 \quad n_2; n'_1; m \Vdash^{\beta_1, \gamma} m'}{n_2; n_1; m \Vdash^{\alpha, \gamma} m'}$$

and this can interact with the logic in the form:

$$\frac{n_2; n_1; 0 \Vdash^{\beta_0, \gamma} n'_1 \quad n_2; n'_1; m \vdash^{\beta_1, \gamma} \Gamma}{n_2; n_1; m \vdash^{\alpha, \gamma} \Gamma}.$$

Definition 5.4 (*Bounding Functions*)

$$B_{\alpha, \gamma}(n_2, n_1, m) = \begin{cases} B_\gamma(n_1, m) & \text{if } \alpha = 0, \\ B_{\alpha-1, \gamma}(n_2, B_{\alpha-1, \gamma}(n_2, n_1, 0), m) & \text{if } \alpha \text{ is a successor,} \\ B_{\alpha_{n_2}, \gamma}(n_2, n_1, m) & \text{if } \alpha \text{ is a limit.} \end{cases}$$

Lemma 5.5 *For each fixed pair α, γ , the function $B_{\alpha, \gamma}$ lies in \mathcal{E}^4 .*

Proof. This is because, by an easy induction on α ,

$$B_{\alpha, \gamma}(n_2, n_1, m) = B_\gamma(f^k(n_1), m) \text{ where } f(n) = B_\gamma(n, 0) \text{ and } k = 2^{G_\alpha(n_2)} - 1.$$

Since the binary B_γ is elementary, f is elementary, and so its iterate function $f^k(n)$ lies in \mathcal{E}^4 , because one iteration jumps up to the next level of the Grzegorzczek hierarchy. But k is also an elementary function of n_2 . Therefore $B_{\alpha, \gamma} \in \mathcal{E}^4$. \square

Lemma 5.6 (*Bounding Lemma*)

$$n_2; n_1; m \Vdash^{\alpha, \gamma} m' \text{ if and only if } m' \leq B_{\alpha, \gamma}(n_2, n_1, m)$$

provided (in the “only if” part) that γ is at least ω .

Proof. For the “if” suppose $m' \leq B_{\alpha, \gamma}(n_2, n_1, m)$ and proceed by induction on α . If $\alpha = 0$ then $m' \leq B_\gamma(n_1, m)$, so $n_1; m \Vdash^\gamma m'$ and hence $n_2; n_1; m \Vdash^{\alpha, \gamma} m'$ by the computation axiom. If α is a successor then $m' \leq B_{\alpha-1, \gamma}(n_2, n_1, m)$ where $n_1' = B_{\alpha-1, \gamma}(n_2, n_1, 0)$. Therefore by the induction hypothesis, $n_2; n_1; 0 \Vdash^{\alpha-1, \gamma} n_1'$ and $n_2; n_1'; m \Vdash^{\alpha-1, \gamma} m'$. One application of the computation rule then yields $n_2; n_1; m \Vdash^{\alpha, \gamma} m'$. If $\alpha = \sup_i \alpha_i$ is a limit the result follows immediately from the induction hypothesis at α_{n_2} .

For the “only if” assume $n_2; n_1; m \Vdash^{\alpha, \gamma} m'$ and call this sequent (*). If it comes about by a computation axiom then $n_1; m \Vdash^\gamma m'$ and hence $m' \leq B_\gamma(n_1, m) = B_{0, \gamma}(n_2; n_1; m) \leq B_{\alpha, \gamma}(n_2, n_1, m)$. If (*) arises from the level-2 computation rule the premises are $n_2; n_1; 0 \Vdash^{\beta_0, \gamma} n_1'$ and $n_2; n_1'; m \Vdash^{\beta_1, \gamma} m'$. Inductively, we may therefore assume $m' \leq B_{\beta_1, \gamma}(n_2, n_1', m)$ where $n_1' \leq B_{\beta_0, \gamma}(n_2, n_1, 0)$. Since $\beta_0, \beta_1 \in \alpha[n_2]$ it then follows from the definition of $B_{\alpha, \gamma}$ that $m' \leq B_{\alpha, \gamma}(n_2, n_1, m)$. Finally suppose (*) arises from an application of the level-1 computation rule, with premises $n_2; n_1; m'' \Vdash^{\beta_1, \gamma} m'$ and $n_2; n_1; m \Vdash^{\beta_0, \gamma} m''$. Then $m' \leq B_{\beta_1, \gamma}(n_2, n_1, m'')$ where $m'' \leq B_{\beta_0, \gamma}(n_2, n_1, m)$. Letting $\beta = \max(\beta_0, \beta_1) \in \alpha[n_2]$, the desired result will follow immediately from

$$B_{\beta, \gamma}(n_2, n_1, B_{\beta, \gamma}(n_2, n_1, m)) \leq B_{\beta, \gamma}(n_2, B_{\beta, \gamma}(n_2, n_1, 0), m)$$

since the latter is $\leq B_{\alpha, \gamma}(n_2, n_1, m)$. But this is checked by a careful induction on β . If $\beta = 0$ the left hand side of the inequality is $B_{\gamma+1}(n_1, m) = q^k(m)$ where $k = 2^{G_\gamma(n_1)+1}$. The right hand side is $q^{k'}(m)$ with $k' = 2^{G_\gamma(B_\gamma(n_1, 0))}$. Then $k \leq k'$ provided γ is at least ω . If β is a limit the result is immediate by applying the induction hypothesis on β_{n_2} . Now suppose β is a successor. Then, unravelling the left hand side, one obtains

$$B_{\beta-1, \gamma}(n_2, B_{\beta-1, \gamma}(n_2, n_1, 0), B_{\beta-1, \gamma}(n_2, B_{\beta-1, \gamma}(n_2, n_1, 0), m))$$

and, by the induction hypothesis, one sees that this is less than or equal to

$$B_{\beta-1, \gamma}(n_2, B_{\beta-1, \gamma}(n_2, B_{\beta-1, \gamma}(n_2, n_1, 0), 0), m) = B_{\beta-1, \gamma}(n_2, B_{\beta, \gamma}(n_2, n_1, 0), m)$$

which is $\leq B_{\beta, \gamma}(n_2, B_{\beta, \gamma}(n_2, n_1, 0), m)$ as required. \square

5.2 \mathcal{E}^4 bounds for provable Σ_1 formulas in $\text{EA}(\mathbf{I}_2; \mathbf{I}_1; \mathbf{O})^+$

The procedure for extracting numerical bounds now runs along the same lines as before for $\text{EA}(\mathbf{I}; \mathbf{O})^+$. Suppose $\text{EA}(\mathbf{I}_2; \mathbf{I}_1; \mathbf{O})^+ \vdash \exists \vec{a} C(\vec{u}, \vec{a})$ with C a bounded formula. First, remove all cuts on formulas of complexity greater than Σ_1 or Π_1 containing at least one unbounded level-2 quantifier (recall that these cannot be induction formulas, nor the results of reflection rules, so they are “free cuts”

which can be eliminated straightforwardly within $EA(I_2; I_1; O)^+$. Next, by the same methods as in Theorem 4.4, embed this proof into $EA(I_2; I_1; O)_\infty^+$, simultaneously eliminating all cuts on $EA(I_2; I_1; O)$ formulas (cut elimination works for the new extended infinitary system just as it did for the old one, and the ordinal bounds remain exponential forms to the base ω). The result is a derivation of

$$k; 0; 0 \vdash_{\Sigma(I)}^{\alpha, \gamma} \exists \vec{a} C(\vec{k}, \vec{a})$$

for some fixed α, γ and all $\vec{k} = k_1, \dots, k_r \leq k$. The subscript $\Sigma(I)$ now indicates that the only cuts remaining are on Σ_1 (dually Π_1) formulas which contain a level-2 quantifier. A bounding result similar to that of Lemma 4.9 then shows that $\exists \vec{a} C(\vec{k}, \vec{a})$ is true at $f(\max \vec{k})$ where f is some finite iterate of $B_{\alpha, \gamma}$.

Theorem 5.7 *The provably computable functions of $EA(I_2; I_1; O)^+$ are exactly the \mathcal{E}^4 functions.*

5.3 Extending the hierarchy upwards

The theory $EA(I_j; \dots; I_2; I_1; O)^+$ is formed from $EA(I_{j-1}; \dots; I_2; I_1; O)^+$ at each stage j , by adding new level- j variables, inductions “up to” level- j terms on $EA(I_{j-1}; \dots; I_2; I_1; O)^+$ formulas, and then adding level- j quantifiers and a level- j reflection rule. It is then embedded into an infinitary system:

$$n_j; n_{j-1}; \dots; n_2; n_1; m \vdash^{\alpha, \vec{\gamma}} \Gamma$$

whose computation rules determine bounding functions $B_{\alpha, \vec{\gamma}}$ defined by one further iteration from $B_{\vec{\gamma}}$. Since this latter is in \mathcal{E}^{j+1} the new bounding functions $B_{\alpha, \vec{\gamma}}$, and their finite compositions, will therefore lie in \mathcal{E}^{j+2} . The methods, at each stage, are essentially those already described.

Theorem 5.8 *The provably computable functions of $EA(I_j; \dots; I_2; I_1; O)^+$ are exactly the \mathcal{E}^{j+2} functions.*

6 Extending $EA(I; O)$ with a Closure Rule

The theory $EA(I; O)^*$ extends $EA(I; O)$ with the Σ_1 Closure rule:

$$\frac{\Delta(\vec{x}; \vec{b})}{\Gamma, \Delta(\vec{a}, \vec{b})}$$

where Δ is a set of Σ_1 formulas, Γ is an arbitrary set of formulas and the variables \vec{a} are free for \vec{x} in Δ .

The rule replaces the uninterpreted (arbitrary) input constants \vec{x} in the premise by fresh output variables which may then be universally quantified (note how the semi-colon in Δ is dropped from premise to conclusion). Thus it resembles a formalized ω -rule for Σ_1 formulas in $EA(I; O)^*$. This causes a

collapse of the variable separation and thus strengthens EA(I;O) considerably. For a given EA(I;O)* derivation of a Σ_1 formula $A(\vec{x})$ (such as that defining a provably computable function) we may now deduce $\forall \vec{a}A(\vec{a})$. The universal quantifiers $\forall \vec{a}$ could be regarded as quantifiers *with computational content* in the sense of Schwichtenberg [7] (also Berger [3]).

Definition 6.1 In EA(I;O)*, provably computable functions are now defined on output variables. That is, $f : N^k \rightarrow N$ is “provably computable” if there is a Σ_1 formula $C_f(\vec{a}, b)$ such that $f(\vec{n}) = m$ if and only if $C_f(\vec{n}, m)$ is true, and $\text{EA(I;O)*} \vdash \forall \vec{a} \exists ! b C_f(\vec{a}, b)$.

Theorem 6.2 *The primitive recursive functions are provably computable in EA(I;O)*.*

Proof. Clearly if a function is provably computable in EA(I;O) it is also provably computable in EA(I;O)* using the closure rule and universal quantification. Furthermore, closure under composition comes immediately from the logic. Hence we need only show that the provably computable functions of EA(I;O)* are closed under primitive recursion.

Without loss of generality assume that the function f is defined by the primitive recursion $f(0, b) = g(b)$, $f(a + 1, b) = h(f(a, b))$ where g and h are already provably computable. Then we have derivations of $\forall b \exists d C_g(b, d)$ and $\forall c \exists d C_h(c, d)$. It is straightforward to define a computational formula $C_f(a, b, d)$ for f such that we may prove $\exists d C_f(0, b, d)$ and $\exists d C_f(a, b, d) \rightarrow \exists d C_f(a + 1, b, d)$. Applying predicative induction yields $\exists d C_f(x; b, d)$ from which the closure rule and universal quantifications leave $\forall a, b \exists d C_f(a, b, d)$. Thus f is provably computable in EA(I;O)*. \square

6.1 Refinements of EA(I;O)*

We may refine the previous result by defining a hierarchy of theories below EA(I;O)* which carefully control the applications of the closure and predicative induction rules.

Definition 6.3 Let EA(I;O) be denoted EA⁰(I;O). Then for any natural number $k > 0$ the theories EA^k and EA^k(I;O) are generated inductively. EA^k is a theory with just one type of variable, outputs, and the usual rules of inference and axioms. EA^k has no induction rule but we do add one non-logical axiom schema: the Σ_1 closure axiom

$$\text{EA}^k \vdash \Gamma, \Delta(\vec{a}, \vec{b}) \quad \text{if} \quad \text{EA}^{k-1}(I;O) \vdash \Delta(\vec{x}; \vec{b})$$

where Δ is a set of Σ_1 formulas, Γ is an arbitrary set of formulas and the variables \vec{a} are free for \vec{x} in Δ .

EA^k(I;O) is then defined as a two sorted extension of EA^k. We add an infinite supply of input constants x, y, x_0, x_1, \dots as symbols of the language along with the predicative induction rule

$$\frac{\Gamma, A(0) \quad \Gamma, \neg A(a), A(a+1)}{\Gamma, A(x)}$$

where Γ is an arbitrary set of formulas and a is not free in $\Gamma, A(0)$.

Note that if a function is provably computable in $EA(I;O)^*$ and the derivation contains at most k nested applications of the closure rule then this derivation may be replicated in $EA^k(I;O)$.

Theorem 6.4 *For each natural number $k > 0$, the functions in Grzegorzcyk's class \mathcal{E}^{k+2} are provably computable in EA^k*

Proof. We use induction over k . The basis of the induction follows immediately since the elementary functions are provably computable in $EA(I;O)$ and thus also EA^1 . Now assume the result holds for k . By a result of Axt [1], for $i \geq 3$, the Grzegorzcyk class \mathcal{E}^{i+1} may be characterized as the smallest class of functions containing \mathcal{E}^i which is closed under composition and closed under a single primitive recursion. Thus if $f \in \mathcal{E}^{k+3}$ we need only consider three cases for its definition and we use a sub-induction according to these cases.

i. If $f \in \mathcal{E}^{k+2}$ then by the induction hypothesis for k we know f is provably computable in EA^k and hence also in EA^{k+1} .

ii. If f is definable by composition where the auxiliary functions g_i are in \mathcal{E}^{k+3} then by the sub-induction hypothesis g_i is provably computable EA^{k+1} . As in theorem 6.2, closure under composition of provably computable functions in EA^{k+1} is straightforward from the logic without any appeal to predicative induction. Hence, f is provably computable in EA^{k+1} .

iii. Finally it may be the case that $f \in \mathcal{E}^{k+3}$ is defined by a primitive recursion using auxiliary functions $g_i \in \mathcal{E}^{k+2}$. Using the main induction hypothesis for k we have g_i provably computable in EA^k . Following the proof of closure under primitive recursion in 6.2 we may, with a single use of predicative induction, show f is provably computable in $EA^k(I;O)$ and thus also provably computable in EA^{k+1} . \square

6.2 The Infinitary System $EA(I;O)_\infty^*$

We now show that the provably recursive functions of $EA(I;O)^*$ are *at most* the primitive recursive functions using the infinitary system $EA(I;O)_\infty^*$. Its Tait-style sequents take the form

$$n; m \vdash^{\alpha, \vec{\gamma}} \Gamma$$

where Γ is a finite set of closed formulas, n declares a bound on input parameters and m declares a bound on output values. The tree ordinals $\alpha, \vec{\gamma}$ are again exponential forms to the base ω . Here $\vec{\gamma}$ represents a finite (possibly empty) sequence $\gamma_k, \dots, \gamma_1$.

With one key exception, the rules of $EA(I;O)_\infty^*$ are controlled by α and n with $\vec{\gamma}$ remaining fixed. That is to say from premise(s) with ordinal bounds $\beta, \vec{\gamma}$ the conclusion takes the bound $\alpha, \vec{\gamma}$ where $\beta \in \alpha[n]$. We have an axiom rule in which $\alpha, \vec{\gamma}$ are arbitrary and the usual conjunction, disjunction and cut rules. Quantification rules only apply to outputs, hence there is one \exists -rule

$$\frac{n; m \Vdash^{\beta_0, \vec{\gamma}} m' \quad n; m \vdash^{\beta_1, \vec{\gamma}} \Gamma, A(m')}{n; m \vdash^{\alpha, \vec{\gamma}} \Gamma, \exists a A(a)}$$

and one \forall -rule

$$\frac{\{n; \max(m, i) \vdash^{\beta, \vec{\gamma}} \Gamma, A(i)\}_{i \in \mathbb{N}}}{n; m \vdash^{\alpha, \vec{\gamma}} \Gamma, \forall a A(a)}$$

where again β does not vary with i . The exception arises in the addition of the closure rule

$$\frac{n'; m' \vdash^{\vec{\gamma}} \Delta}{n; \max(n', m') \vdash^{\alpha, \vec{\gamma}} \Gamma}$$

where Δ is Σ_1 , $\Delta \subseteq \Gamma$, α and n are arbitrary and $\vec{\gamma}$ is non-empty. In this rule a new ordinal α is prefixed to the existing sequence $\vec{\gamma}$ such that subsequent applications of the other rules are now controlled by this new α and n .

We have three rules governing computations. The computational axiom is $n; m \Vdash^{\alpha, \vec{\gamma}} l$ where $l \leq q(m)$ for some suitable quadratic function which bounds all the term-constructors. The computational cut rule is

$$\frac{n; m \Vdash^{\beta_0, \vec{\gamma}} m' \quad n; m' \Vdash^{\beta_1, \vec{\gamma}} l}{n; m \Vdash^{\alpha, \vec{\gamma}} l}$$

which interacts with the logic in the form

$$\frac{n; m \Vdash^{\beta_0, \vec{\gamma}} m' \quad n; m' \vdash^{\beta_1, \vec{\gamma}} \Gamma}{n; m \vdash^{\alpha, \vec{\gamma}} \Gamma}.$$

Finally we have a computational closure rule

$$\frac{n'; m' \Vdash^{\vec{\gamma}} l}{n; \max(n', m') \Vdash^{\alpha, \vec{\gamma}} l}$$

where α and n are arbitrary and $\vec{\gamma}$ is non-empty.

Remark 6.5 The closure rule is analogous to the closure rule in $\text{EA}(\text{I}; \text{O})^*$ in that input declarations in the premise become output declarations in the conclusion. Again note that the semi-colon in Δ is dropped from premise to conclusion signifying the change. As the declarations have now shifted we also require a corresponding computational closure rule. Thus the infinitary theory here is similar to one for $\text{EA}(\text{I}_j; \dots; \text{I}_2; \text{I}_1; \text{O})^+$ in which multiple levels of declarations $n_j; n_{j-1}; \dots; n_1; m$ are reduced to just two: n_j and $\max(n_{j-1}; \dots; n_1; m)$.

We now broadly follow the usual methods for extracting numerical bounds on derivations of Σ_1 sets beginning by defining a suitable bounding function.

Definition 6.6 (*Bounding Functions*)

$$B_{\alpha, \vec{\gamma}}(n; m) = \begin{cases} q(m) & \text{if } \alpha, \vec{\gamma} = 0, \\ B_{\vec{\gamma}}(m; m) & \text{if } \alpha = 0, \\ B_{\alpha-1, \vec{\gamma}}(n; B_{\alpha-1, \vec{\gamma}}(n; m)) & \text{if } \alpha \text{ is a successor,} \\ B_{\alpha_n, \vec{\gamma}}(n; m) & \text{if } \alpha \text{ is a limit.} \end{cases}$$

Lemma 6.7 *For each fixed $\alpha, \vec{\gamma}$ and for a fixed $d \in \mathbb{N}$, $B_{d, \vec{\gamma}}$ lies in \mathcal{E}^{k+2} and $B_{\alpha, \vec{\gamma}}$ lies in \mathcal{E}^{k+3} .*

Proof. We use induction over the length of the sequence $\vec{\gamma}$. When the sequence is empty $k = 0$ and $B_d(n; m)$ is bounded by a polynomial. Thus it is contained in \mathcal{E}^2 . B_α lies in \mathcal{E}^3 since $B_\alpha(n; m) = q^d(m)$ where $q := 2^{G_\alpha(n)}$ and $G_\alpha(n)$ is elementary. Now assume the result holds for $\vec{\gamma} := \alpha', \gamma_{k-1}, \dots, \gamma_1$. Then $B_{\vec{\gamma}} \in \mathcal{E}^{k+2}$. For a fixed $d \in \mathbf{N}$, $B_{d, \vec{\gamma}} \in \mathcal{E}^{k+2}$ since it is defined by composition from $B_{\vec{\gamma}}$. An easy induction on α shows $B_{\alpha, \vec{\gamma}}(n; m)$ is equal to $B_{d, \vec{\gamma}}(n; m)$ where $d := G_\alpha(n)$. Hence we may define $B_{\alpha, \vec{\gamma}}$ by a primitive recursion whose auxiliary functions lie in \mathcal{E}^{k+2} and we conclude $B_{\alpha, \vec{\gamma}} \in \mathcal{E}^{k+3}$. \square

Lemma 6.8 (*Bounding Lemma*)

$$n; m \Vdash^{\alpha, \vec{\gamma}} l \text{ if and only if } l \leq B_{\alpha, \vec{\gamma}}(n; m).$$

Proof. Tackling the “if” part first we use induction on the length of $\alpha, \vec{\gamma}$ with a sub-induction on α . Assume that $\vec{\gamma}$ is empty. Then if $\alpha = 0$ we have $l \leq B_0(n; m) = q(m)$ and the result follows immediately by the computational axiom. If α is a successor $\beta + 1$ then $l \leq B_{\beta+1}(n; m) = B_\beta(n; m')$ where $m' = B_\beta(n; m)$. Using the induction hypothesis for α we have the derivations $n; m \Vdash^\beta m'$ and $n; m' \Vdash^\beta l$ and the result follows by a computational cut rule. Now assume α is a limit $\sup_n(\lambda_n)$ so $l \leq B_\lambda(n; m) = B_{\lambda_n}(n; m)$. Then for every n we may apply the induction hypothesis for α to give $n; m \Vdash^{\lambda_n} l$. We must use a sub-induction on this derivation to change the proof height from λ_n to λ . The axiom case is self-evident. Now assume an application of a computational cut was the last rule of inference from premises of heights β_i . Then since $\beta_i \in \lambda_n[n] = \lambda[n]$ we may have in each case taken the new height to be λ . Hence $n; m \Vdash^\lambda l$.

Now, assuming the result for $\vec{\gamma}$, if $\alpha = 0$ then $l \leq B_{0, \vec{\gamma}}(n; m) = B_{\vec{\gamma}}(n; m)$ and the main induction hypothesis gives $m; m \Vdash^{\vec{\gamma}} l$. Applying the computational closure rule we get $n; m \Vdash^{0, \vec{\gamma}} l$ as required. The cases where α is a successor or a limit follow just as above.

For the “only if” we use induction over the derivation of $n; m \Vdash^{\alpha, \vec{\gamma}} l$ with a case distinction according to the final rule of inference applied. We rely heavily upon certain majorization properties of the functions $B_{\alpha, \vec{\gamma}}(n; m)$, namely that they are increasing in n, m and $\alpha, \vec{\gamma}$. These properties follow by simple inductions on $\alpha, \vec{\gamma}$. If the derivation is a computational axiom then $l \leq q(m) \leq B_{\alpha, \vec{\gamma}}(n; m)$. If the last rule of inference were a computational cut from premises $n; m \Vdash^{\beta_0, \vec{\gamma}} m'$ and $n; m' \Vdash^{\beta_1, \vec{\gamma}} l$ then inductively $m' \leq B_{\beta_0, \vec{\gamma}}(n; m)$ and $l \leq B_{\beta_1, \vec{\gamma}}(n; m')$. Taking β to be the maximum of β_0, β_1 , since $\beta \in \alpha[n]$ we find

$$l \leq B_{\beta, \vec{\gamma}}(n; B_{\beta, \vec{\gamma}}(n; m)) = B_{\beta+1, \vec{\gamma}}(n; m) \leq B_{\alpha, \vec{\gamma}}(n; m).$$

The only other possibility is that the derivation results from the computational closure rule. Using the induction hypothesis we have $l \leq B_{\vec{\gamma}}(n'; m')$ for some $n', m' \leq m$. Hence $l \leq B_{\vec{\gamma}}(m; m) \leq B_{\alpha, \vec{\gamma}}(n; m)$. \square

6.3 Extracting bounds for EA(I;O)*

The standard methods of cut-elimination apply in $EA(I;O)_\infty^*$ except that the cut-rank reduction becomes stuck in the presence of the closure rule. However, since this rule only applies where the formulas in the premise are at worst Σ_1 , we are able to reduce cuts to the Σ_1 level. It is straightforward to show that $EA(I;O)_\infty^*$ admits weakening, conjunction inversion and universal quantifier inversion as well as the following:

Theorem 6.9 (i) *If $n; m \vdash^{\alpha, \vec{\gamma}} \Gamma, -C$ and $n; m \vdash^{\alpha', \vec{\gamma}} \Gamma, C$ are both derivable with cut formulas of size $\leq r$, where C is a formula of size $r + 1$ which is either a true atom, or has shape \vee or $\exists a$ but is not Σ_1 , then $n; m \vdash^{\alpha + \alpha', \vec{\gamma}} \Gamma$ is also derivable with cuts of size $\leq r$ provided $\alpha'[n] \subseteq \alpha[n]$.*

(ii) *Let $\vec{2}^{\vec{\gamma}} := 2^{\gamma_1}, \dots, 2^{\gamma_k}$. If $n; m \vdash^{\alpha, \vec{\gamma}} \Gamma$ is derivable with cut formulas of size at most $r + 1$ then $n; m \vdash^{2^\alpha, \vec{2}^{\vec{\gamma}}} \Gamma$ is derivable with cut formulas either of size at most r or Σ_1 .*

Repeated applications of (ii) will therefore eliminate cuts down to the Σ_1 level at the cost of iterated exponential increases in each of the ordinals $\alpha, \vec{\gamma}$.

Theorem 6.10 (Embedding) *If $EA(I;O)^* \vdash \Gamma(\vec{x}; \vec{a})$ then this derivation determines some $d \in \mathbf{N}$ such that for all $\vec{x} := \vec{n} = n_1, \dots, n_r \leq n$ and all $\vec{a} := \vec{m} = m_1, \dots, m_\ell \leq m$, EA_∞^* derives*

$$n; m \vdash^{\alpha, \vec{\gamma}} \Gamma(\vec{n}; \vec{m})$$

where each of $\alpha, \vec{\gamma}$ is of the form $\omega \cdot d_i$ for some $d_i \leq d$.

Proof. We proceed by induction on the height of the proof of $\Gamma(\vec{x}; \vec{a})$ in $EA(I;O)^*$ with a case distinction according to the final rule applied. The axioms, rules for \vee, \wedge and $\forall a$ as well as cuts all carry over easily using the corresponding infinitary rules. The cases for $\exists a$ and predicative induction follow the reasoning in the proof of 4.4 except that we need not eliminate cuts as we go thus restricting the heights to ordinals of the form $\omega \cdot d$.

The only case remaining is Σ_1 Closure. Assume $\Gamma(\vec{x}; \vec{a})$ came about via such a rule and without loss of generality, for clarity, let $\vec{a} := a, b, c$ and $\vec{x} = x$ so that $\Gamma(\vec{x}; \vec{a}) := \Gamma'(x; c), \Delta(a, b)$ where Δ is a Σ_1 set of formulas. Then the premise is of the form $\Delta(y; b)$ where y is an input which changes to the output a in the conclusion. Appealing to the induction hypothesis we obtain

$$i, m' \vdash^{\alpha', \vec{\gamma}} \Delta(i; m').$$

The result follows immediately since the closure rule of EA_∞^* and weakening yield

$$n; \max(i, m', m) \vdash^{\omega \cdot 0, \vec{\gamma}} \Gamma'(n; m), \Delta(i, m')$$

where i has become an output declaration and an additional ordinal $\omega \cdot 0$ has been introduced in front of $\vec{\gamma} := \alpha', \vec{\gamma}$. \square

Lemma 6.11 *Let Γ be a set of Σ_1 formulas such that $n; m \vdash^{\alpha, \vec{\gamma}} \Gamma$ using only Σ_1 cuts and assume that the derivation is term controlled by δ . Then Γ is true at $B_{\alpha', \vec{\gamma}}(n, m)$ where $\alpha' = \delta + \alpha$ and each $\gamma'_i = \delta + \gamma_i$.*

Proof. We use induction over the derivation of $n; m \vdash^{\alpha, \vec{\gamma}} \Gamma$ with a case distinction according to the last rule applied. We follow closely the proof of 4.9 but simplified in this context – we do not have a $\forall x$ rule so there are no substitutions on inputs in the cut case. We shall only expand on the cases for the cut and closure rules where, as before, we dispense with the use of “dashes” on ordinals by assuming each ordinal bound in the derivation has been weakened to “add in” δ . Hence $n; m \vdash^{\beta, \vec{\gamma}} \Gamma$ implies that any closed term in Γ is bounded by $B_{\beta, \vec{\gamma}}(n; m)$. We also assume that ordinal bounds in premises have been matched by weakening.

Assume that the derivation comes from a Σ_1 cut with premises of the form $n; m \vdash^{\beta, \vec{\gamma}} \Gamma, \exists \vec{c}A(\vec{c})$ and $n; m \vdash^{\beta, \vec{\gamma}} \Gamma, \forall \vec{c}\neg A(\vec{c})$ where A is a bounded formula. Let $k = B_{\beta, \vec{\gamma}}(n; m)$. The induction hypothesis applies to the first premise to reveal $\Gamma, \exists \vec{c}A(\vec{c})$ is true at k . Either Γ is true at k and we are done since $k \leq B_{\alpha, \vec{\gamma}}(n; m)$, or else there are $\vec{i} \leq k$ such that $A(\vec{i})$ is true. Inverting the universal quantifiers $\forall \vec{c}$ in the second premise gives $n; \max(m, k) \vdash^{\beta, \vec{\gamma}} \Gamma, \neg A(\vec{i})$. This allows an application of the induction hypothesis so that $\Gamma, \neg A(\vec{i})$ is true at $B_{\beta, \vec{\gamma}}(n; k)$. Therefore, as $\neg A(\vec{i})$ is false, Γ is true at $B_{\beta, \vec{\gamma}}(n; k) = B_{\beta, \vec{\gamma}}(n; B_{\beta, \vec{\gamma}}(n; m)) \leq B_{\alpha, \vec{\gamma}}(n; m)$.

Now assume we have an application of closure from the premise $n'; m' \vdash^{\alpha, \vec{\gamma}} \Gamma'$ where $n', m' \leq m$. By the induction hypothesis Γ' , and hence Γ , is true at $B_{\vec{\gamma}}(n'; m') \leq B_{\vec{\gamma}}(m; m) \leq B_{\alpha, \vec{\gamma}}(n; m)$. \square

Theorem 6.12 (i) *The provably computable functions of $EA(I; O)^*$ are exactly the primitive recursive functions.*

(ii) *The provably computable functions of EA^k , for each $k > 0$, are exactly Grzegorzczuk’s class \mathcal{E}^{k+2} .*

Proof. (i) We have shown that the primitive recursive functions are provably computable in $EA(I; O)^*$ in theorem 6.2. For the converse, if $f(\vec{a})$ is provably computable in $EA(I; O)^*$ then we must have a derivation of $\forall \vec{a} \exists b C_f(\vec{a}, b)$ where C_f is a defining formula for f . Successively applying the embedding theorem, universal inversion of $\forall \vec{a}$ and then cut-reduction we find that $EA(I; O)^*_\infty$ will prove $0; m \vdash^{\alpha, \vec{\gamma}} \exists b C_f(\vec{m}, b)$ with only Σ_1 cuts where $m := \max(\vec{m})$ and $\alpha, \vec{\gamma}$ is some finite sequence of tree-ordinals with $|\alpha|, |\gamma_i| < \varepsilon_0$ for each i . We may assume the derivation is term controlled using weakening so that the witnessing result above tell us $\exists b C_f(\vec{m}, b)$ is true with existential witnesses bounded by $B_{\alpha, \vec{\gamma}}(0; m)$ (a primitive recursive function by lemma 6.7). Hence the graph of f is decidable using primitive recursive functions and its value is bounded by a primitive recursive function, so f is itself a primitive recursive function.

(ii) Theorem 6.4 shows that any \mathcal{E}^{k+2} function is provably computable in EA^k . For the converse we begin by proving the following:

(*) For $k > 0$, EA^k embeds into $EA(I;O)_\infty^*$ with ordinal bound $d, \gamma_k, \dots, \gamma_1$ where $d \in \mathbf{N}$ and each $|\gamma_i| < \varepsilon_0$ whilst $EA^k(I;O)$ embeds into $EA(I;O)_\infty^*$ with ordinal bound $\gamma_{k+1}, \dots, \gamma_1$.

Proof of (*): We use induction on k . First we note that $EA^0(I;O)$ is just $EA(I;O)$ and, following the proof of 6.10, it will embed into the infinitary theory with only one ordinal γ_1 since the closure rule never applies. Now assume $k = 1$. EA^1 contains no induction rule but it does have the closure axiom which reads: $EA^1 \vdash \Gamma, \Delta(\vec{a}, \vec{b})$ if $EA^0(I;O) \vdash \Delta(\vec{x}; \vec{b})$. When embedding EA^1 into $EA(I;O)_\infty^*$ we deal with this axiom by appealing to the previously mentioned embedding of $EA^0(I;O)$ with ordinal height γ_1 from which the closure rule will give a derivation of height d, γ_1 for any d . All the other rules of EA^1 are just logic and will embed with a finite measure d in front of the (now fixed) γ_1 . Furthermore, without any inputs present in EA^1 , the input declaration n may be set to 0.

$EA^1(I;O)$ is formed from EA^1 by the addition of the predicative induction rule up to input constants x . We must therefore extend the embedding of EA^1 to include this case. The γ_1 will remain fixed whilst embedding inductions will force the finite measure d to become infinite. Hence embedding $EA^1(I;O)$ into $EA(I;O)_\infty^*$ yields ordinal bounds γ_2, γ_1 .

The induction step is straightforward using the same argument as the base case. Assume the result holds for k so that $EA^k(I;O)$ embeds with ordinals $\gamma_{k+1}, \dots, \gamma_1$. Then using this result to embed any use of the closure axiom in EA^{k+1} we obtain ordinal bounds $d, \gamma_{k+1}, \dots, \gamma_1$ for $d \in \mathbf{N}$. From here, an embedding of $EA^{k+1}(I;O)$ will require the finite d to become an infinite γ_{k+2} . This completes the proof of (*).

The result now follows by the usual argument. Let $\vec{\gamma}$ denote $\gamma_k, \dots, \gamma_1$. For a provably computable function f , we embed the EA^k proof of $\forall \vec{a} \exists b C_f(\vec{a}, b)$ into $EA(I;O)_\infty^*$ which by (*), inversions and cut reduction gives $0; m \vdash^{d, \vec{\gamma}} \exists b C_f(\vec{m}, b)$ for some $|\gamma_i| < \varepsilon_0$. This derivation may be term controlled by weakening the d to some $d' \in \mathbf{N}$ and each γ_i to some $\delta + \gamma_i$. The witnessing result now shows that $\exists b C_f(\vec{m}, b)$ is true at $B_{d, \vec{\gamma}}(0; m)$ which by lemma 6.7 is a function in \mathcal{E}^{k+2} . Hence f itself is in \mathcal{E}^{k+2} . \square

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