LOWNESS FOR BOUNDED RANDOMNESS

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Abstract. In [3], Brodhead, Downey and Ng introduced some new variations of the notions of being Martin-Löf random where the tests are all clopen sets. We explore the lowness notions associated with these randomness notions. While these bounded notions seem far from classical notions with infinite tests like Martin-Löf and Demuth randomness, the lowness notions associated with bounded randomness turn out to be intertwined with the lowness notions for these two concepts. In fact, in one case, we get a new and likely very useful characterization of K-triviality

Keywords: Algorithmic randomness, K-triviality, low for random, finite randomness.

1. Introduction

The underlying idea behind algorithmic randomness is that *randomness* can only be understood through *computational* considerations. This interpretation of randomness means that the object in question *avoids* simpler algorithmic descriptions, either through effective betting strategies, effective regularities or effective compression.

This idea means that we avoid any idea that there is "true" randomness, and work with the idea that we only have calibrations of randomness according to the sensitivity of the measuring tool. That is, exactly what we mean here by "effective" delineates notions of algorithmic randomness. A major theme in the area of algorithmic randomness seeks to calibrate notions of randomness by varying the notion of effectivity. For example, classical Martin-Löf randomness[1] uses tests, shrinking connections of c.e. open sets whose measure have effective upper bounds, whereas Schnorr randomness is defined using tests of precise effective measure. We then see that Schnorr and Martin-Löf randomness are related but can have very different properties; for example outside the high degrees they coincide, but the lowness concepts are completely disjoint.

Given that we *define* randomness via computation it is natural to ascertain how randomness so defined relates to computational power. To do this we relate algorithmic randomness to measures of relative computability, such as the Turing degrees. The key question in this investigation is, if a string is random, can it have high computational power? A classic result is Stephan's theorem [18] that if a Martin-Löf real is random and has enough

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[1]We assume that the reader is familiar with the basic notions of algorithmic randomness as found in the early chapters of either Downey-Hirschfeldt [9] or Nies [17].

computational power to be able to compute a $\{0,1\}$-valued fixed-point-free function, then it must already be Turing complete.

The goal of the present paper is to study some new variations of algorithmic randomness introduced by the authors and Paul Brodhead in [3] exploring both of the themes above. In particular, we study what we call "bounded variations" of the notion of Martin-Löf randomness where the tests are all finite. These notions generalize the notion of Kurtz (or weak) randomness but are incomparable with both Schnorr and computable randomness. As more precisely defined in the next section, together with Brodhead we defined what we called computably bounded (CB) and finitely bounded (FB) notions of finite Martin-Löf randomness. The paper [3] showed that the bounded notions of randomness we shall encounter in the next section were strongly related to degree classes such as the totally ω-c.a. degrees, and notions of initial segment complexity.

The goal of the present paper is to explore the associated lowness notions. Aside from the intrinsic interest in this, one good reason for such study is to give insight into other studied lowness and randomness notions. Indeed our investigation reveals that this is indeed the case. In the case of CB-randomness we show that there are continuum many reals low for the concept, and discover that they are very closely related to the recently discovered reals low for Demuth randomness (as per Bienvenu, Downey, Greenberg, Nies and Turetsky [2]). We do this using an apparent extension of what is called in [2], BLR-traceability. It remains an open question if our lowness class coincides with lowness for Demuth randomness. In the case of FB-randomness (which is defined simply by considering only finite ML-tests) the lowness class coincides with K-triviality. In some sense, this last result is somewhat unexpected, and somehow says something deep about the nature of K-triviality. That is suggested by the fact that we can characterize lowness for ML-randomness via a much weaker notion of randomness, which is a very surprising discovery. In fact, we show that for characterizing K-triviality, it suffices to consider FB-randomness together with genericity.

We give the basic definitions in Section 2 and review some elementary properties from [3]. In Section 3 we include a proof of a basic result from [3], as its proof and statement are of importance for the present paper. In Section 4 we construct reals low for CB-randomness, and finally in Section 5 we show that lowness for FB-randomness and K-triviality coincide.

2. Notation

If W is a finite set then $\#W$ denotes the cardinality of W. $|\sigma|$ denotes the length of a finite string σ. We work in the Cantor space 2^ω with the usual clopen topology. The basic open sets are of the form $[\sigma]$ where σ is a finite string, and $[\sigma] = \{X \in 2^\omega \mid X \supset \sigma\}$. We fix some effective coding of the set of finite strings, and we freely identify finite strings with their code numbers. We denote $[W] = \cup\{[\sigma] : \sigma \in W\}$ as the Σ_1 open set associated with the c.e. set W. $\mu([W])$ denotes Lebesgue measure, and we write $\mu(W)$ instead of $\mu([W])$. We let $*$ be the string concatenation symbol. We let D_n be the n^{th} canonical finite set. If W is an open set and $\sigma \in 2^{<\omega}$ we let $\mu(W \mid [\sigma]) = \frac{\mu(W\cap[\sigma])}{\mu([\sigma])}$, i.e. the measure of W relative to $[\sigma]$.

Definition 2.1 (Brodhead, Downey and Ng [3]). (a) A Martin-Löf (ML) test is a uniform c.e. sequence $\{U_n\}_{n\in\omega}$ of sets U_n such that $\mu(U_n) < 2^{-n}$.

(b) A Martin-Löf test $\{U_n\}_{n\in\omega}$ is finitely bounded (FB) if $\#U_n < \infty$ for every n.

(c) A Martin-Löf test $\{U_n\}_{n\in\omega}$ is computably bounded (CB) if there is some total computable function f such that $\#U_n \leq f(n)$ for every n.

(d) A real $X \in 2^\omega$ passes a CB-test (FB-test) $\{U_n\}_{n\in\omega}$ if $X \notin \bigcap_n [U_n]$.

A real $X \in 2^\omega$ is *computably bounded random* if X passes every CB-test. X is *finitely bounded random* if it passes every FB-test. We always assume that in any FB-test $\{U_n\}$, each U_n contains only pairwise incomparable strings, since we can choose to enumerate long extensions of σ instead of σ itself. Note that for a CB-test we are unable to make such a convention.

It is not hard to see that in the definition of a CB-test, it is equivalent to require for a computable bound on the length of strings enumerated into the test. These two notions of randomness are weaker than Martin-Löf randomness, although they imply Kurtz randomness. The obvious implications are:

$$\begin{tikzcd}
ML\text{-random} \arrow[r] \arrow[drr] & FB\text{-random} \arrow[r] & CB\text{-random} \arrow[r] & \text{Kurtz random} \\
 & & \text{Schnorr random} \arrow[ur] &
\end{tikzcd}$$

In [3], with Brodhead we proved that no implications hold other than those stated in the diagram. We did this as follows. First, we showed that there is a Δ^0_3 1-generic real which is FB-random while no Schnorr random is weakly 1-generic. No incomplete c.e. degree can compute a FB-random (Proposition 3.1(i)). However, some incomplete c.e. degree bounds a CB-random since in [3] we prove the following.

Theorem 2.2 (Brodhead, Downey and Ng [3]). *Suppose A is a c.e. real. The following are equivalent.*

(i) *$deg_T(A)$ is not totally ω-c.a.,*

(ii) *$deg_T(A)$ contains a CB-random,*

(iii) *There is some c.e. real $B \leq_T A$ which is CB-random,*

(iv) *There is some $B \leq_T A$ which is CB-random.*

Here, a degree $\mathbf{a}$ is called *totally c.a.* iff every function $f \leq_T \mathbf{a}$ has a limit lemma approximation $f(x) = \lim_s g(x,s)$ where there is a computable h with $|\{s : g(x, s+1) \neq g(x,s)\}| < h(x)$ for all x. (See Downey, Greenberg, Weber [8], Downey and Greenberg [6, 7] and Barmpalias, Downey and Greenberg [1]). It is known that there are low c.e. degrees that are not totally ω-c.a..

Finally in [3], we showed that

Proposition 2.3 (Brodhead, Downey and Ng [3]). *Every CB-random is of effective packing dimension* 1.

Proof. For completeness we give a proof of Proposition 2.3 as it is quite short. Suppose $K(\alpha\restriction n) \leq cn$ for all $n \geq N$ for some $N \in \mathbb{N}$ and $c < 1$ i s rational. Fix a computable increasing sequence of natural numbers $\{n_i\}$ al l larger than N, such that $n_i > \frac{i}{1-c}$ for all i. Now define a CB- test $\{V_i\}$ by the following: $V_i := \{\sigma \in 2^{n_i} \mid K(\sigma) \leq cn_i\}$. Here we have $\#V_i \leq 2^{cn_i}$. □

Lathrop and Lutz [16] showed that there is a computably random set X such that for every order function g, $K(X \restriction n) \leq K(n) + g(n)$ for almost every n. Hence X cannot be CB-random, by Proposition 2.3. This gives the last separation for the diagram.

As we noted in [3], these finite notions of randomness turn out to have strong relationships with degrees classes hitherto unrelated to algorithmic randomness. We will show that FB-randomness and Martin-Löf -randomness coincide on the Δ^0_2 sets but are distinct on the Δ^0_3 sets (Theorem 3.1). There is one other known restriction on such reals.

Proposition 2.4 (Brodhead, Downey and Ng [3])**.** *No CB-random is c.e. traceable.*

Kurtz showed that every nonzero c.e. degree contains a Kurtz random real, but (by Theorem 2.2 above) the degrees containing CB-random reals is a subclass of the c.e. degrees : those that are not totally ω-c.a.. This is a class of c.e. degrees introduced by Downey, Greenberg and Weber [8] to explain certain "multiple permitting" phenomena in degree constructions such as "critical triples" in the c.e. degrees, and a number of other constructions as witnessed in the subsequent papers Barmpalias, Downey and Greenberg [1] and Downey and Greenberg [7]. This class extends the notion of array noncomputable reals, and correlates to the fact that all CB-random reals have effective packing dimension 1 (Theorem 2.3). Downey and Greenberg [6] have previously showed that the c.e. degrees containing reals of packing dimension 1 are exactly the array noncomputable reals. Brodhead, Downey and Ng [3] also show that if a c.e. degree **a** contains a CB-random then every (not necessarily c.e.) degree above **a** contains a CB-random as well. From all of this, we see that there remain a lot to understand for this class.

3. Basic results

One of the basic results shown in [3] is that the notion of FB-randomness and Martin-Löf -randomness coincide on the Δ^0_2 sets, and they differ on the Δ^0_3 sets. Since the proof is relevant to this paper, we include it here.

Proposition 3.1 (Brodhead, Downey and Ng [3])**.** (i) *Suppose $Z \leq_T \emptyset'$. Then Z is ML-random iff Z is FB-random.*
(ii) *There is some $Z \leq_T \emptyset''$ such that Z is 1-generic, FB-random and not ML-random.*

Proof. (i): Given an approximation Z_s of Z, and suppose $\{U_x\}$ is the universal ML-test where $Z \in \cap_x [U_x]$. Enumerate an FB-test $\{V_x\}$ by the following: at stage s, enumerate into V_x, the string $Z_s \restriction n$ for the least n such that $Z_s \restriction n \in U_x[s]$. Then, $\{V_x\}$ is uniformly c.e., where $\mu(V_x) \leq \mu(U_x) < 2^{-x}$ for all x. Clearly $Z \in [V_x]$ for all x. We know $Z \restriction n \in U_x$ for some least n, and let s be a stage such that $Z_s \restriction n$ is correct and $Z \restriction n$ has appeared in $U_x[s]$. Then, $Z \restriction n$ will be in V_x by stage s, and we will never enumerate again into V_x after stage s.

(ii): We build $Z = \cup_s \sigma_s$ by finite extension. Let $\{U_x\}$ be the universal ML-test, and $\{U_x^e\}_x$ be the e^{th} ML-test. Assume we have defined σ_s, where for all $e < s$, we have

- all infinite extensions of σ_s are in U_e,
- if $\#U_x^e < \infty$ for all x, then $\exists k$ such that no infinite extension of σ_s can be in U_k^e.

Now we define $\sigma_{s+1} \supset \sigma_s$. Firstly, find some $\tau \supseteq \sigma_s$ such that all infinite extensions of τ are in U_s; such τ exists because $\{U_e\}$ is universal. Let $k = |\tau|$. Next, ask if $\#U_k^s < \infty$. If not, let $\sigma_{s+1} = \tau * 0$ and we are done. If yes, then figure out exactly the strings ρ_i such that $[U_k^s] = \cup\{[\rho_1], [\rho_2], \cdots, [\rho_n]\}$. We cannot have $[U_k^s] \supseteq [\tau]$ since $\mu(U_k^s) < 2^{-k}$, so there has to be some $\sigma_{s+1} \supset \tau$ such that $[\sigma_{s+1}] \cap [U_k^s] = \emptyset$, by the finiteness of U_k^s. We can figure σ_{s+1} out effectively from $\rho_1, \rho_2, \cdots, \rho_n$. Next extend σ_{s+1} (if possible) to meet W_s. Clearly the properties above continue to hold for σ_{s+1}. All questions asked can be answered by the oracle $\emptyset''$. □

A set A is Δ_2^0*-jump dominated* if for every partial A-computable function Φ^A there is a $g \leq_T \emptyset'$ such that $g(x) > \Phi^A(x)$ for every x. This notion has also been called " weakly jump traceable", and implies that the set is GL_1. The proof of (i) above also shows that ML- and FB-randomness coincides over the reals which are Δ_2^0-jump dominated.

Corollary 3.2. *Suppose that A is Δ_2^0-jump dominated. Then A is Martin-Löf random iff A is FB-random.*

There exists Martin-Löf randoms which are Δ_2^0-jump dominated. For example it is easy to see that each Demuth random is Δ_2^0-jump dominated.

4. CB-lowness and traceability

We investigate the lowness notions associated with FB- and CB-randomness. We call a real A *low for FB-randomness* if every FB-random real is FB-random relative to A. A is *low for FB-tests* if for every A-relative FB-test $\{U_x^A\}_{x\in\omega}$ there is an FB-test $\{E_x\}_{x\in\omega}$ such that $\cap_x[U_x^A] \subseteq \cap_x[E_x]$. We can make similar definitions for CB-randomness.

Recall that Cole and Simpson [4] defined a function $f : \omega \mapsto \omega$ to be $BLR(A)$ for an oracle A to mean that there exists $g \leq_T A$ and a computable function h such that for every x, $f(x) = \lim_{s\to\infty} g(x, s)$ and $chg_g(x) < h(x)$. Here we denote $chg_g(x) = \#\{s \mid g(x, s) \neq g(x, s + 1)\}$, the mind change function of g. Hence $BLR(A)$ is the class of all functions which are ω-c.e. relative to A, where the relativization is partial. Bienvenu, Downey, Greenberg, Nies and Turetsky [2] then called a set A BLR-traceable if there is a computable function h such that for every $f \in BLR(A)$ there is an ω-c.e. function k such that for every x, $\#W_{k(x)} < h(x)$ and $f(x) \in W_{k(x)}$.

BLR-traceability was shown in [2] to be crucial in understanding lowness for Demuth randomness. Being BLR-traceable allows us to obtain a c.e. trace (with few values) for the canonical index of a clopen Demuth test relative to A. We can then use this to build a Demuth test covering a given A-Demuth test. Unfortunately BLR-traceability appears too weak to imply lowness for CB-randomness, because in order to approximate a given

CB-test relative to A, we have to approximate the individual neighborhood enumerated into the test. This calls for a strengthening of BLR-traceability which we call *faithfully BLR-traceable.*

Definition 4.1. Given functions $f, g : \omega^2 \mapsto \omega$ such that $\lim_{s\to\infty} f(x,s)$ and $\lim_{s\to\infty} g(x,s)$ both exist, we say that g is a *faithful BLR witness* for f if for every x, $\lim_{s\to\infty} f(x,s) = \lim_{s\to\infty} g(x,s)$ and $\cup\{f(x,s) \mid s \in \omega\} \supseteq \cup\{g(x,s) \mid s \in \omega\}$. That is, g does not approximate f wastefully, and introduces no noise into the approximation.

We say that A is *faithfully BLR-traceable* if for every computable order h and every $BLR(A)$-approximation for a function $f \in BLR(A)$, there is an ω-c.e. function g and a function $n : \omega \mapsto \omega$ such that for every x, $\sum_{y<x} h(y) \leq n(x) < \sum_{y\leq x} h(y)$ and f has faithful BLR witness $g(n)$.

Note that there are no requirement on the complexity of n. It is easy to see that in the definition of faithful BLR-traceability we may replace "for every computable order h" with "for some computable order h".

Fact 4.2. *Faithfully BLR-traceable implies BLR-traceable.*

Being faithfully BLR-traceable means more than having an ω-c.e. trace for $f \in BLR(A)$ – it means that we can identify $h(x)$ many distinct attempts at approximating $f(x)$ faithfully.

Fact 4.3. *If A is faithfully BLR-traceable with respect to a constant function $h(x) = c$ then A is computable.*

Proof. Assume $c = 1$. Define the A-computable function $f(x,s) = A\restriction x$. Fix the ω-c.e. function g which is a faithful BLR witness for f. For every x, $g(x,0) = A\restriction x$ and hence A is computable. If $c > 1$ we use g to construct a Π^0_1 class containing A with at most c many infinite paths. □

Lemma 4.4. *If A is faithfully BLR-traceable and of hyperimmune-free degree then A is low for CB-tests.*

Proof. Fix a CB-test $\{U^A_x\}$ relative to A. We may assume that $\mu(U^A_x) < 2^{-2x}$ for each x. Let $f(x,s)$ approximate the canonical index for U^A_x. Since A is hyperimmune-free, $f \in BLR(A)$, with computable bound h on $\#U^A_x$. Let g and n be given such that g is ω-c.e. and $g(n)$ is a faithful BLR-witness for f with respect to the identity order function. For each x there are x many possibilities for $n(x)$ in $I_x = \{\sum_{y<x} y, \cdots, \sum_{y\leq x} y - 1\}$. Build T_x by copying, for each $z \in I_x$, the sets with canonical indices $g(z,0), g(z,1), \cdots$ until we find some s such that $\mu(\cup_{t\leq s}[D_{g(z,t)}]) \geq 2^{-2x}$, or $\# \cup_{t\leq s} D_{g(z,t)} > h(x)$. Then T_x is clearly a CB-test, and since $g(n)$ faithfully BLR-trace f, $[T_x] \supseteq [U^A_x]$. □

Proposition 4.5. *There is a perfect class of sets which are all low for CB-randomness.*

Proof. In [2] it was shown that there is a perfect Π^0_1 class of sets which are all BLR-traceable. It is easy to see that the same proof shows that there is a perfect Π^0_1 class of sets which are all faithfully BLR-traceable. We can then take a perfect subclass of reals which are all of hyperimmune-free degree. The proposition then follows from Lemma 4.4. □

Theorem 4.6. *If A is low for CB-randomness then A is of hyperimmune-free degree.*

Proof. Suppose A is of hyperimmune degree. Fix a strictly increasing total function $f \leq_T A$ such that f escapes domination by every computable function. We build $Z \leq_T A'$ such that Z is CB-random but not CB-random relative to A. The construction will be computable in A. We let $\{E^k_x\}_{x\in\omega}$ be the k^{th} possible CB-test with cardinality bound $g_k(x)$, where g_k is a partial computable function with domain an initial segment of ω. We assume that for every k, x, $\mu(E^k_x) < 2^{-x}$. We also assume that $E^k_x = \emptyset$ if $g_k(x)$ has not yet converged, and that $\#E^k_x < g_k(x)$ otherwise. The construction maintains global parameters $\{U_x\}_{x\in\omega}$, σ_i and k_i. $\{U_x\}_{x\in\omega}$ will be the CB-test relative to A which Z fails, and $\sigma_i \in 2^{<\omega}$ and $k_i \in \omega$ are the parameters of requirement P_i.

Requirement P_i will ensure that Z passes $\{E^i_x\}_{x\in\omega}$ if the latter is a CB-test. If E is a c.e. open set, τ is a string and s is a stage we let $Survivor(E, \tau, s)$ be the lex-least string $\eta \supseteq \tau$ such that $|\eta| = s$ and $[\eta] \cap [E_s] = \emptyset$. We adopt the convention that E_s only contains strings of length $< s$, so $Survivor(E, \tau, s)$ is undefined iff $[E_s] \supseteq [\tau]$.

Construction of $\{U_x\}$: At stage $s = 0$ set $\sigma_0 = 00$ and $k_0 = 0$. At stage $s > 0$ we say that P_i requires attention if P_i has been started and there is some least $j_0 > 0$ such that $g_i(|\sigma_i| + j_0) \downarrow< f(k_i + j_0)$, and a new element has entered $E^i_{|\sigma_i|+j_0}$. If this is the first time we discover j_0 then we also say that P_i requires attention, regardless of the enumeration of $E^i_{|\sigma_i|+j_0}$.

At stage s pick the least $i < s$ such that P_i requires attention. If i exists we act for P_i by doing the following. For each $j < j_0$ enumerate $\sigma_i * 0^j$ into U_{k_i+j}. Initialize P_m for every $m > i$ (meaning that we cancel the values of σ_i, k_i). Set $\sigma_{i+1} = Survivor(E^i_{|\sigma_i|+j_0}, \sigma_i * 0^{j_0}, s) * 0^t$ where t is a fresh number. Set $k_{i+1} = k_i + j_0$.

If no P_i requires attention find the largest i such that P_i is not yet started. For this P_i, $\sigma_i, k_i \downarrow$. Declare the requirement to be started. Set $\sigma_{i+1} = \sigma_i * 0^t$ for a fresh number t and $k_{i+1} = k_i + 1$.

Finally for every P_i we put σ_i into U_{k_i} if these parameters are defined.

Verification: It is clear that every call for $Survivor$ during the construction returns a value. We verify that $\{U_x\}$ is an A-relative CB-test. The sequence $\{U_x\}$ is clearly uniformly c.e. in oracle f. Checking the construction reveals that at every stage $|\sigma_i| > k_i$ holds whenever these parameters are defined. In fact every time P_i is initialized $|\sigma_i|$ is picked to have a length (never seen before by the construction) larger than k_i. Now fix an x and consider U_x. At every stage there is at most one P_i contributing to U_x. In fact if P_i contributes to U_x then $k_i \leq x < k_{i+1}$ and $i \leq x$. Each requirement P_i can contribute at most one string of the form $\sigma_i * 0^j$ into U_{k_i+j} before it is initialized. Since $|\sigma_i| > k_i$ this means that $|\sigma_i * 0^j| > x$. If P_i gets initialized and later enumerates another string $\sigma'_i * 0^{j'}$ into U_x again then we must have $|\sigma'_i * 0^{j'}| > |\sigma'_i| > |\sigma_i * 0^j| > x$, as lengths are always chosen fresh. The total measure of P_i's contribution is at most 2^{-x}. Hence $\mu(U_x) \leq (x + 1)2^{-x}$.

Before we verify that $\#U_x$ is computably bounded in A, we need a technical lemma.

Lemma 4.7. *There is $F(i,x) \leq_T A$ such that for every x and every i, the number of stages where P_i is initialized when $k_i \leq x$ is at most $F(i,x)$.*

Proof. Fix x. We define $F(i,x)$ recursively in i. Clearly $F(0,x) = 0$. Assume $F(0,x), \cdots, F(i-1,x)$ has been defined. Fix $j < i$. If P_j initializes P_i when $k_i \leq x$ then $k_j < k_i \leq x$. If P_j does this then P_j has required attention which means that a new element has entered $E^j_{|\sigma_j|+z}$ where $k_j + z \leq k_i$ (unless this is the first time P_j is receiving attention). Hence P_j can do this at most $1 + g_j(|\sigma_j| + z) < 1 + f(k_j + z) \leq 1 + f(k_i) \leq 1 + f(x)$ times before P_j itself is initialized. When P_j is finally initialized $k_j < x$. So P_j initializes P_i at most $(1 + F(j,x))(1 + f(x))$ many times. We can use this to compute $F(i,x)$. □

Next we verify that $\#U_x \leq f(x)$. Now fix $i \leq x$. How many times can P_i contribute to U_x? Before P_i can contribute a second element to U_x it has to be first initialized by P_j, $j < i$ and at the point of initialization $k_i \leq x$. By Lemma 4.7 $\#U_x \leq \sum_{i \leq x} 1 + F(i,x)$, and so $\{U_x\}$ is a CB-test relative to A.

It is easy to check that every requirement is initialized finitely often, and that for every i, $\sigma_{i+1} \supset \sigma_i$ holds at every stage. Each σ_i must eventually get defined, so we let $\tilde{\sigma}_i$ be the limit value of σ_i. Let $Z = \cup_i \tilde{\sigma}_i \leq_T A'$. Clearly for each i, $\tilde{\sigma}_{i+1} \supset \tilde{\sigma}_i * 0^{\tilde{k}_{i+1} - \tilde{k}_i}$, and that $\tilde{\sigma}_i * 0^j \in U_{\tilde{k}_i + j}$ for every $j < \tilde{k}_{i+1} - \tilde{k}_i$. Thus $Z \in \cap_x [U_x]$. Finally fix i such that g_i is total. We argue that $Z \notin \left[E^i_{|\tilde{\sigma}_i| + \tilde{k}_{i+1} - \tilde{k}_i}\right]$. After P_i is never initialized, P_i must find j_0, because otherwise $g_i(|\tilde{\sigma}_i| + x) \geq f(\tilde{k}_i + x)$ holds for every x, contrary to the hyperimmunity of A. Once this $j_0 = \tilde{k}_{i+1} - \tilde{k}_i$ is found by P_i, it will ensure that $[\tilde{\sigma}_{i+1}] \cap [E^i_{|\tilde{\sigma}_i| + j_0}] = \emptyset$. Hence A is not low for CB-randomness. □

5. Lowness for FB-randomness and K-triviality coincide

We now turn to analyzing the class of reals which are low for FB-randomness. Since Ω is a Δ^0_2 real, by the relativized form of Theorem 3.1(i), each low for FB-randomness is low for Ω. We prove that in fact lowness for FB-randomness and several related lowness notions coincide with K-triviality.

The equivalence of (i) through (iv) below is proved easily by using the relativized form of Theorem 3.1(i). However this is somewhat unsatisfactory because it can be argued that we are exploiting the indistinguishability of ML- and FB-randomness at the Δ^0_3 level. We discover that if we consider the FB-random reals which are intrinsically not ML-random (i.e. generic), we still get coincidence with K-triviality. This is statement (v) and Theorem 5.3 below.

Theorem 5.1. *Let A be a real. The following are equivalent.*

(i) *A is K-trivial.*
(ii) *A is low for FB-tests.*
(iii) *A is low for FB-randomness.*
(iv) *Every Martin-Löf random is FB-random relative to A.*
(v) *Every FB- not ML-random is FB-random relative to A.*

Proof. (i) $\Rightarrow$ (ii): Suppose A is K-trivial, hence A is low for K and $\Delta^0_2(A) = \Delta^0_2$. Fix an A-relative FB-test $\{U^A_x\}_{x\in\omega}$. We show how to build $\{V_x\}_{x\in\omega}$ covering $\{U^A_x\}$. Let $f(x) = \lim_s f_s(x)$ for a computable approximation $\{f_s\}$, such that $D_{f(x)} = U^A_x$ for every x. Effectively in A, for each σ and x, if we see $\sigma \in U^A_x$ we issue a description of $\langle\sigma, x\rangle$ of length $|\sigma|$. Since the total weight of all descriptions is less than 1, and since A is low for K, let d be a constant such that $K(\langle\sigma, x\rangle) \leq |\sigma| + d$ for every $\sigma \in U^A_x$.

Now for each x, enumerate σ into V_x if we find a stage s such that $\sigma \in D_{f(x,s)}$ and for every $y \leq x$ there is $\tau \subseteq \sigma$ such that $K(\langle\tau, y\rangle) \leq |\tau| + d$. Then $\#V_x$ is clearly finite. We argue that $\mu(V_x) < \frac{2^d}{x}$. For each $y \leq x$, let $E_y = \{\tau : K(\langle\tau, y\rangle) \leq |\tau| + d \text{ and } \exists\sigma \in V_x(\tau \subseteq \sigma)\}$. Suppose $\mu(V_x) \geq \frac{2^d}{x}$. Since each $\sigma \in V_x$ is an extension of some string in E_y, we have $\sum_{\tau\in E_y} 2^{-|\tau|} \geq \sum_{\sigma\in V_x} 2^{-|\sigma|} \geq \frac{2^d}{x}$. Each $\tau \in E_y$ corresponds to a description of length $|\tau| + d$. These descriptions are all for different numbers so the halting probability is at least $\sum_{y\leq x}\sum_{\tau\in E_y} 2^{-|\tau|-d} \geq \frac{2^d}{x}(x + 1)2^{-d} > 1$, a contradiction. It is easy to check that $\cap_x U^A_x \subseteq \cap_x V_x$. Hence $\{V_{2^{x+d}}\}$ is our required FB-test.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v) are obvious.

(iv) $\Rightarrow$ (i): Suppose every ML-random is FB-random relative to A. By the Kučera-Gács theorem there exists a ML-random real Z such that $A \leq_T Z \leq_T A \oplus \emptyset' \leq_T A'$. Hence Z is FB-random relative to A. By Theorem 3.1(i) relativized to A, Z is ML-random relative to A. Hence A is a base for ML-randomness.

(v) $\Rightarrow$ (i) : This follows from Theorem 5.3. □

Lemma 5.2. *The following are equivalent for a real A.*

(i) *A is not low for ML-randomness.*
(ii) *There is a uniform sequence of A-c.e. open sets $\{Q_p \mid p \in \omega\}$ such that for every p, $\mu(Q_p) < 2^{-p}$ and for each c.e. open set E such that $\mu(E) < 1$, we have $Q_p \not\subseteq E$.*
(iii) *There is a uniform sequence of A-c.e. open sets $\{S_{\sigma,p} \mid \sigma \in 2^{<\omega}, p \in \omega\}$ such that for every σ and p, $\mu(S_{\sigma,p} \mid [\sigma]) < 2^{-p}$, $S_{\sigma,p} \subseteq [\sigma]$ and for each c.e. open set E such that $\mu(E \mid [\sigma]) < 1$, we have $S_{\sigma,p} \not\subseteq E \cap [\sigma]$.*

Proof. We use the characterization of non-ML-randomness by Kjos-Hanssen [15]. Thus we replace (i) by "there exists an A-c.e. open set G such that $\mu(G) < 1$ and for each c.e. open set E where $\mu(E) < 1$, we have $G \not\subseteq E$". Clearly (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. (ii) $\Rightarrow$ (iii) is also trivial, by letting $S_{\sigma,p} = \sigma * Q_p$. We now prove (i) $\Rightarrow$ (ii). Fix G of measure at most $r \in \mathbb{Q} \cap (0, 1)$ where G is not covered by any c.e. open set of measure less than 1. We construct an A-c.e. open set Q where $\mu(Q) \leq \frac{r^2+r}{2}$ and which avoids being covered by c.e. open sets of measure less than 1. Intuitively Q is a slightly expanded version of $G * G$. This construction is effective. Since $r > \frac{r^2+r}{2} > \frac{\left(\frac{r^2+r}{2}\right)^2 + \left(\frac{r^2+r}{2}\right)}{2} > \cdots$ converges to 0, we can iterate this construction to get the sequence $\{Q_p\}$.

We let $\{E_i\}$ be the i^{th} c.e. open set. Clearly the predicate "$\mu(E_i) > q$" is uniformly c.e. in i, q. For each i, σ we let $Q^{i,\sigma} = [\sigma] - (E_{i,s} \cap [\sigma])$ if a stage s is found such that $\mu(E_{i,s} \mid [\sigma]) > 1 - \varepsilon 2^{-i-2|\sigma|}$, where $\varepsilon = \frac{r-r^2}{2}$. If no such s is found we let $Q^{i,\sigma} = \emptyset$. Let $Q = (G * G) \cup \{Q^{i,\sigma} \mid i \in \omega, \sigma \in G\}$. Since $\mu(Q^{i,\sigma}) \leq \varepsilon 2^{-i-2|\sigma|}$, this means that $\mu(Q) \leq r^2 + \varepsilon \sum_{i,\sigma} 2^{-i-2|\sigma|} \leq r^2 + \varepsilon = \frac{r+r^2}{2}$.

Suppose that $Q \subseteq E_i$ where $\mu(E_i) < 1$. If $\sigma \in G$ then $\mu(E \mid [\sigma]) = 1$, because otherwise $\{\tau \mid \sigma * \tau \in E\}$ is a c.e. open set of measure less than 1 covering G. Hence for all $\sigma \in G$ and all i, $Q^{i,\sigma} = 2^\omega - E_{i,s}$ for some s. Since $E_i \supseteq Q \supseteq Q^{i,\sigma}$ this means that $[\sigma] \subseteq [E_i]$. Hence G is covered by E_i, a contradiction. □

Theorem 5.3. *Suppose that A is not low for ML-randomness. Then there is a FB-random real Z which is 1-generic relative to A and not FB-random relative to A.*

Proof. Assume that A is not low for ML-randomness. By Lemma 5.2(iii) fix the sequence $\{S_{\sigma,p}\}$. Since $[S_{\sigma,p}] \subseteq [\sigma]$, for ease of notation, we consider $\{\tau \mid \sigma * \tau \in S_{\sigma,p}\}$ instead. Henceforth $S_{\sigma,p}$ refers to this set of truncated strings. We first describe a construction C which we will later use as a black box to build Z. C takes in parameters $\eta \in 2^{<\omega}, I, \varepsilon \in \omega$. It effectively outputs a FB-test relative to A, $\{U_x\}$, where $\mu(U_x) < 2^{-(x+1)\varepsilon - x}$, and an A-computable sequence $\{\sigma_s\}$ of finite strings. The construction $C(\eta, I, \varepsilon)$ ensures that if $\{E^I_x\}$ is a FB-test then $\sigma = \lim_s \sigma_s \supset \eta$ exists, $[E^I_x] \cap [\sigma] = \emptyset$ for some x and $[\sigma] \subseteq \cap_{k \leq y}[U_k]$, where y is the least stage such that σ_y is stable. Here $\{E^I_x\}$ is the I^{th} ML-test. Intuitively construction C searches, relative to the input parameters, for a safe spot for lower priority requirements to act.

Description of strategy: The construction C is an effective version of the proof of Proposition 3.1(ii). η is the environment in which C is called to work in, and is handed to us by higher priority requirements. For simplicity we assume that $\eta = \emptyset$, and that we are trying to avoid some c.e. open set $E = E^i_x$ of measure less than 1. We build the approximation σ_s which attempts to locate a neighbourhood from which E^i_x is disjoint. We know this exists if E is finite, but since we have to diagonalize against every test, we have to deal with the possibility of E being infinite.

Since σ_s will be an initial segment of our real Z (provided E is finite), we have to enumerate σ_s into a FB-test $\{U_x\}$ relative to A. The main difficulty here is that we have to keep each U_x finite even when E is infinite. We use the fact that A is not low for random. Let U_0 copy $S_{\emptyset,p}$ for some sufficiently large p. More specifically we let σ_s be the first string enumerated in $S_{\emptyset,p}$. We enumerate σ_s in U_0. We wait until $[E] \supseteq [\sigma_s]$, and we move on to the second string enumerated in $S_{\emptyset,p}$, and so on. Clearly if U_0 is infinite then $[E] \supseteq [S_{\emptyset,p}]$, which by the properties of S implies that $\mu(E) = 1$, a contradiction. However this naive plan does not work well for us because E can cover extensions of each $\sigma \in S_{\emptyset,p}$ without covering σ itself. We need to find a safe spot $\lim_s \sigma_s$ for lower priority genericity requirements to work in, and thus $[\lim_s \sigma_s]$ has to be disjoint from $[E]$.

Thus we need to iterate the above strategy. While we are waiting for $[E]$ to cover the current $[\sigma^0]$ in $S_{\emptyset,p}$, we need to consider S_{σ^0,p^2}. For simplicity we drop p^2 from the notation, with the understanding that each iteration of S is called with a sufficiently large p^n. We take the first string $\sigma^1 \in S_{\sigma^0}$, enumerate σ^1 in U_1 and wait for $[E]$ to cover $[\sigma^1]$. In the meantime we call on S_{σ^1} and so on. If $[E]$ does end up covering $[\sigma^n]$ then by compactness we see this at a finite stage, and we can set σ^n to be the next element enumerated in $S_{\sigma^{n-1}}$. We let σ_s be σ^n for the largest n where σ^n is defined at stage s. Clearly if E is finite then $[\lim_s \sigma^s]$ can be chosen to be disjoint from $[E]$. The problem with this approach is when $\#E = \infty$. In this case we may return to σ^n infinitely often, i.e. $[E]$ covers $[\sigma^n]$ for every value of σ^n we pick from $S_{\sigma^{n-1}}$. In this case $\mu(E \mid [\sigma^{n-1}])$ must be 1, but if $[E] \not\supseteq [\sigma^{n-1}]$ we will never abandon σ^{n-1} during the construction. This means that we will end up copying the infinite set $S_{\sigma^{n-1}}$ in U_n. However we are committed to making $\{U_x\}$ an FB-test relative to A, even when E is infinite.

The way around the problem above is to *force* $[E]$ to cover $[\sigma^{n-1}]$. Above each string σ^{n-1} that we are currently guessing is an initial segment of Z, we pick σ^n from the set $S_{\sigma^{n-1}}$ until $\mu(E_s \mid [\sigma^{n-1}])$ is very close to 1 (say larger than $1-\varepsilon$). When this happens, say at stage s, we switch and pick σ^n from the clopen set $[\sigma^{n-1}] - [E_s]$. This forces us to add a small amount of additional measure to U_n not accounted for by $\mu(S_{\sigma^{n-1}})$, but we can choose the threshold ε to be as small as we like to keep the measure of U_n small. Additionally we will be able to ensure that we consider only finitely many different strings σ^n above each value for σ^{n-1}.

Clearly if $\#E < \infty$ then $\lim_s \sigma_s$ exists and we will be able to choose $[\lim_s \sigma_s]$ disjoint from $[E]$. We then allow a lower priority requirement to act above $[\lim_s \sigma_s]$, by calling C above $[\lim_s \sigma_s]$. On the other hand if $\#E = \infty$ then $\liminf\{n : \sigma^n \text{ is abandoned}\}$ must be ∞. To see this suppose that the $\liminf$ is some number n. Then $\mu(E \mid [\sigma^{n-1}]) = 1$ and so we must switch to pick σ^n from $[\sigma^{n-1}] - [E_s]$. Hence $[E] \supseteq [\sigma^{n-1}]$ and so σ^{n-1} must be abandoned infinitely often, a contradiction. Since $\liminf\{n : \sigma^n$ is abandoned$\} = \infty$, this particular run of C contributes only finitely many elements to each U_x. At each stage where the output σ_s changes we allow the next requirement to call C above the node 0. Then $Z \supset 0$ and may possibly be in $[E]$, but it does not matter as $\#E = \infty$.

The formal proof is organized as follow. We first specify the working of the basic module $C(\eta, I, \varepsilon)$. The actual construction of Z is then carried out on a priority tree. Each node on the priority tree is allowed to call C with certain parameters. Z can then be read off the true path of the construction.

Construction $C(\eta, I, \varepsilon)$: Let $E = E^I_{|\eta|}$. We define, for each σ, p, the set $\hat{S}_{\sigma,p}$ to copy $S_{\sigma,p}$ until the first stage s is found such that $\mu(E_s \mid [\sigma]) > 1 - 2^{-\varepsilon-p}$. If this s is found we say that $\hat{S}_{\sigma,p}$ has *switched*, and we let $\hat{S}_{\sigma,p} = S_{\sigma,p,s} \cup ([\sigma] - E_s)$. Otherwise we say that $\hat{S}_{\sigma,p}$ remains *unswitched* and in this case $\hat{S}_{\sigma,p} = S_{\sigma,p}$.

We fix a 1-1 enumeration of the infinite set $S_{\sigma,p}$. This gives rise in the obvious way to a 1-1 (possibly finite) enumeration of $\hat{S}_{\sigma,p}$. We define $\hat{S}^i$ inductively on i: Set $\hat{S}^0 = \hat{S}_{\eta,\varepsilon+1}$, and $\hat{S}^{i+1} = \{\sigma * \tau \mid \sigma \in \hat{S}^i, \tau \in \hat{S}_{\sigma,\varepsilon+i+2}\}$.

For each $\sigma \in \hat{S}^i$, and $\tau \in \hat{S}_{\sigma,\varepsilon+i+2}$ we define $\alpha(\sigma * \tau) = \alpha(\sigma) * i$ where τ is the $i+1^{th}$ element to be enumerated in $\hat{S}_{\sigma,\varepsilon+i+2}$. In this way we associate each $\sigma \in \cup_i \hat{S}^i$ with $\alpha(\sigma)$. We order $\omega^{<\omega}$ first lexicographically then by length ($\alpha \subset \beta$ means $\alpha < \beta$).

We now define the sequence $\alpha_0 \leq \alpha_1 \leq \cdots$. At stage $s = 0$ set $\alpha_s = 0$. At stage $s+1$ we see if E has a new element. If so we set α_{s+1} to be the $<$-least string of length $s+3$ such that $\forall i \leq s$ $([\alpha_{s+1} \upharpoonright i+1] \not\subseteq [E_{s+1}])$. Otherwise let $\alpha_{s+1} = \alpha_s$.

This produces an A-computable sequence of finite strings $\{\alpha_s\}$. It is easy to verify that $\alpha_s \leq \alpha_{s+1}$ for every s.

Claim 5.4. *Each time we need to pick α_{s+1} we can do so.*

Proof. Suppose we are unable to pick α_{s+1} at some stage $s+1$. This means there is some least $i+1$ and some β of length $i+1$ such that $[\beta] \not\subseteq [E_{s+1}]$ but for every j, $[\beta * j] \subseteq [E_{s+1}]$. (If $i = -1$ then β is associated with the string η, where it is clear that $[E] = [E^I_{|\eta|}] \not\supseteq [\eta]$).

If $\hat{S}_{\beta,\varepsilon+i+2}$ is never switched, then we must be able to avoid the finite set E_{s+1}, since $S_{\beta,\varepsilon+i+2}$ must be dense above β. Therefore $\hat{S}_{\beta,\varepsilon+i+2}$ must switched at some stage t. If $t > s+1$ then $[\beta] - [E_t]$ must be disjoint from $[E_{s+1}]$ and so we can pick $\beta * j$ appropriately. On the other hand if $t \leq s+1$ and if $[E_{s+1}] \supseteq [\hat{S}_{\beta,\varepsilon+i+2}]$ then this means that $[E_{s+1}] \supseteq [\beta]$, a contradiction. $\square$

Claim 5.5. *If $\#E = \infty$ then for each i, $\lim_s \alpha_s(i)$ exists.*

Proof. Suppose that $\beta = \langle \alpha_s(j) \rangle_{j<i+1}$ is constant for all large enough s. Hence by compactness $[E] \not\supseteq [\beta]$. If $\hat{S}_{\beta,\varepsilon+i+2}$ is ever switched then we are done, because $\lim_s \alpha_s(i+1)$ is one of finitely many choices. Suppose $\hat{S}_{\beta,\varepsilon+i+2}$ is never switched. This means that $\mu(E \mid [\beta]) < 1$ and so $[\beta * \hat{S}_{\beta,\varepsilon+i+2}] = [\beta * S_{\beta,\varepsilon+i+2}] \not\subseteq [E]$. Thus there exists some $\tau \in \hat{S}_{\beta,\varepsilon+i+2}$ such that $[\beta * \tau] \not\subseteq [E]$. $\square$

Claim 5.6. *For each i, $\mu(\hat{S}^i) < 2^{-\varepsilon i - i}$.*

Proof. If $i = 0$ then $\mu(\hat{S}^0) = \mu(\hat{S}_{\eta,\varepsilon+1}) \leq \mu(S_{\eta,\varepsilon+1}) + 2^{-2\varepsilon-1} < 1$. If $i+1 > 0$ then similarly

$$
\begin{aligned}
& \mu(\hat{S}^{i+1}) \\
= & \sum \left\{ 2^{-|\sigma|-|\tau|} \mid \sigma \in \hat{S}^i, \tau \in \hat{S}_{\sigma,\varepsilon+i+2} \right\} \\
= & \sum_{\sigma \in \hat{S}^i} 2^{-|\sigma|} \sum_{\tau \in \hat{S}_{\sigma,\varepsilon+i+2}} 2^{-|\tau|} \\
< & \sum_{\sigma \in \hat{S}^i} 2^{-|\sigma|} \left(\mu(S_{\sigma,\varepsilon+i+2}) + \text{conditional measure added due to a switch} \right) \\
\leq & \sum_{\sigma \in \hat{S}^i} 2^{-|\sigma|} (2^{-\varepsilon-i-2} + 2^{-\varepsilon-i-2}) \leq 2^{-\varepsilon(i+1)-(i+1)}. \quad \square
\end{aligned}
$$

Finally let $U_k = \{\alpha_s \upharpoonright k+2 \mid s \in \omega\}$. If $\#E < \infty$ then α_s is eventually stable, and so $\#U_k < \infty$. If $\#E = \infty$ then by Claim 5.5 $\#U_k < \infty$. Since

$[U_k] \subseteq [\hat{S}^{k+1}]$, by Claim 5.6 this means that $\mu(U_k) \leq \mu(\hat{S}^{k+1}) < 2^{-(k+1)\varepsilon - k}$. Set σ_s = string coded by α_s. If $\#E < \infty$ then clearly $\eta \subset \lim_s \sigma_s$ exists. Let $\alpha = \lim_s \alpha_s$. Then for each $2 \leq i \leq |\alpha|$, $\alpha \restriction i \in U_{i-2}$, where $|\alpha| - 2$ is the least stage where α_s is stable. Clearly $[E] \cap [\alpha] = \emptyset$, since for every t, E_t contains no string of length longer than t.

Now we use C as a procedure to build Z. We define the priority tree to be the full binary tree, with labels ∞ instead of 0 and f in place of 1. $\eta <_L \gamma$ denotes the usual left to right lexicographic ordering. If $\eta \subset \gamma$ then we say $\eta <_L \gamma$. Each node is assigned parameters $head(\eta) \in 2^{<\omega}$ and $m(\eta) \in \omega$. As usual all parameters retain their assigned values until they are initialized or reassigned. A node η of length i will attempt to diagonalize against $\{E^i_x\}$ and meet the i^{th} genericity requirement.

At stage $s = 0$ we set $head(\emptyset) = \emptyset$, and do nothing else. At stage s we define δ_s of length s, the stage s approximation to the true path. Assume that $\eta = \delta_s \restriction i$ and $head(\eta)$ have been defined. We now act for η. If $m(\eta) \uparrow$ or if $head(\eta)$ has changed since the last visit to η, we pick a fresh number for $m(\eta)$. Run construction $C(head(\eta), |\eta|, m(\eta))$ for one more step, say step t. If the output of this construction, σ_t, does not change (i.e. $\sigma_t = \sigma_{t-1}$) then let $\delta_s(i) = f$, otherwise let $\delta_s(i) = \infty$. We now update $head(\eta * \delta_s(i))$. If $\delta_s(i) = \infty$ let $head(\eta * \infty)$ be the first string $\tau \supseteq head(\eta) * 0^{m(\eta)}$ found such that $\tau \in W^A_{|\eta|,s}$. If no such τ exists let $head(\eta * \infty) = head(\eta) * 0^{m(\eta)}$. If $\delta_s(i) = f$ let $head(\eta * f)$ be the first string $\tau \supseteq \sigma_t$ found such that $\tau \in W^A_{|\eta|,s}$. If no such τ exists let $head(\eta * f) = \sigma_t$. Finally initialize every node to the right of $\eta * \delta_s(i)$.

Now let $TP = \liminf_s \delta_s$ be the true path of construction. Clearly if $i < j < s$ then $head(\delta_s \restriction i) \subset head(\delta_s \restriction j)$. It is easy to verify that the following claim holds.

Claim 5.7. *For every i, $head(TP \restriction i)$ and $m(TP \restriction i)$ are eventually stable.*

Now let $Z = \cup_i \lim_s head(TP \restriction i)$.

Claim 5.8. *Z is FB-random.*

Proof. Fix i such that $\{E^i_x\}$ is a FB-test, and let $\eta = TP \restriction i$. Let h and m be the final values of $head(\eta)$ and $m(\eta)$ respectively. At almost every visit to η we will run $C(h, i, m)$. By properties of C we have $\lim_t \sigma_t$ exists, so $\eta * f \subset TP$. This means that $Z \supset \lim_s head(\eta * f) \supset \lim_t \sigma_t$ and so $Z \notin \cap_x [E^i_x]$. □

Claim 5.9. *Z is 1-generic relative to A.*

Proof. Fix i and look at $\eta = TP \restriction i + 1$. We have $Z \supset \lim_s head(\eta)$. By the construction either $head(\eta) \in W^A_{|\eta|-1}$ or else no extension of $head(\eta)$ is in $W^A_{|\eta|-1}$. □

Claim 5.10. *Z fails to be FB-random relative to A.*

Proof. The construction is effective in oracle A. Let $t(\emptyset, s) = 0$. For each node η and each stage s we let $t(\eta, s) \uparrow$ if $\delta_s \not\supset \eta$. Otherwise η is visited at stage s. If $\eta(|\eta| - 1) = \infty$ let $t(\eta, s) = t(\eta^-, s) + 1$. Otherwise $\eta(|\eta| - 1) = f$ and we let $t(\eta, s) = t(\eta^-, s) + 1+$ the largest $w \leq t$ such that the output σ_w

of the construction C called by η^- (with the current parameters) at stage s has changed. Here t is the number of steps C (with the current parameters) has been run by η^-. For instance each time $C(head(\emptyset), 0, 1)$ changes its output σ_w we increase $t(f, s)$ to match w.

We define the following FB-test $\{V_x\}$ relative to A. For every node η and every s in which η is visited, we look at the output $\{U_x\}$ of C run by η at s. For each $x \in \omega$ add U_x to $V_{x+t(\eta,s)}$. If $\eta * \infty$ is visited at s add $head(\eta * \infty)$ to $V_{t(\eta,s)}$. We verify that $\{V_x\}$ is an FB-test relative to A.

First observe that $t(\eta, s) < t(\gamma, s)$ for every $\eta \subset \gamma$. It is also easy to see that $m(\eta) > t(\eta)$ whenever η is visited. Fix an x. Only a node η of length $\leq x$ can contribute to V_x. Each time the parameters of η is changed or $t(\eta)$ is changed we will pick a fresh value for $m(\eta)$. At each visit to η we add U_y to $V_{y+t(\eta,s)}$, and possibly a string of length at least $m(\eta)$ to $V_{t(\eta,s)}$. Since $\mu(U_y) < 2^{-(y+1)m(\eta)-y} \leq 2^{-m(\eta)}2^{-t(\eta,s)-y}$, it follows that the sum total of all contributions is at most 2^{-x}. Hence $\mu(V_x) < 2^{-x+2}$.

Now let us argue that η enumerates only finitely many elements into V_x. If η is to the left of TP then η is visited only finitely often, so we only add finitely many different versions of its output $\{U_y\}$. If η is on TP then the parameters of η and $t(\eta)$ are eventually stable, so η too, adds finitely many different versions of its output $\{U_y\}$. Assume $\eta >_L TP$. Let $\gamma = TP \cap \eta$. It is easy to see that $\lim_s t(\gamma * f, s) = \infty$. Since $t(\eta) \geq t(\gamma * f)$ at every stage, this means that η will eventually stop adding elements to V_x. Hence $\{V_x\}$ is an FB-test relative to A.

Finally we show that $Z \in [V_x]$ for every x. Fix η on TP. If η has true outcome f, let t_0 be the last step in C where the output σ_{t_0} changes. Then $Z \supset \lim_s head(\eta * f) \supseteq \sigma_{t_0}$ and by the properties of C, $[\sigma_{t_0}] \subseteq \cap_{w \leq t_0}[U_w]$. Since $\lim_s t(\eta * f, s) = \lim_s t(\eta, s) + t_0 + 1$, we copy $\{U_w\}$ in $V_{w+\lim_s t(\eta,s)}$, it follows that $Z \in \left[V_{\lim_s t(\eta,s)}\right] \cap \cdots \cap \left[V_{\lim_s t(\eta * f,s)-1}\right]$.

On the other hand if η has true outcome ∞ then $Z \supset \lim_s head(\eta * \infty)$. Since $\lim_s t(\eta * \infty, s) = \lim_s t(\eta, s) + 1$ and $\lim_s head(\eta * \infty) \in V_{\lim_s t(\eta,s)}$, it follows similarly that $Z \in \left[V_{\lim_s t(\eta,s)}\right] \cap \cdots \cap \left[V_{\lim_s t(\eta * f,s)-1}\right]$. □

This ends the proof of Theorem 5.3. □

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