$G_{\delta\sigma}$-games

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Abstract

We elucidate the complexity of strategies for Σ_3^0 (also called $G_{\delta\sigma}$) games played on polish spaces of the form ${}^{\omega}X$. From previous work ([9]) it had been known that strong comprehension principles of the form Π_3^1 were sufficient, but Π_2^1 were not, to establish these amounts of determinacy. We characterise the first ordinal β_0 where such strategies are to be found in the constructible hierarchy for trees $T \subseteq {}^{<\omega}X$ for $X = 2$ or $\mathbb{N}$ (thus for Cantor or Baire space) in L_{β_0} as the first ordinal where L_γ admits certain kinds of end extensions. Secondly we give a conjecture for it to be characterised as a certain closure ordinal for a class of monotone inductive operators.[1]

1 Introduction

The work in the paper [9] was motivated by trying to see how the Σ_3^0-theory of *arithmetical quasi-inductive definitions* fits in with other subsystems of second order number theory. What had been left open was a more precise discussion of the location of strategies for Σ_3^0-games. We continue that discussion here.

To give this research a context we mention the results previously known in this area. The attempt to prove the determinacy of two person perfect inormation games (and the consequences of the existence of such winning strategies) has a long and fruitful history, starting with work of Banach and Mazur and continuing to the present.

In the next section we extract from [9] a criterion for where exactly the strategies appear in the constructible L_α hierarchy. Whilst we had this result for some while, the characterisation is somewhat unusual in that it is expressed in terms of the potential for such L_α to have ill-founded elelmentary end extensions, and is not so perspicaceous. Whilst waiting to discover something more standard we studied the work of Martin on non-monotone inductive definitions [5]. In that paper he concentrated on the inductive operators that were strictly in the *complement* of a Spector pointclass (these are defined in [7]). Now Spector pointclasses (such as Σ_1^0, and within the hierarchy of *projective sets*: Π_1^1, Σ_2^1 - and assuming Projective Determinacy, Σ_{2n}^1 and Π_{2n+1}^1 *etc.*) are very well behaved, comparatively well-understood and enjoy many amenable properties that pointclasses in the comple-

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mentary class do not. The theory of Γ-monotone inductive definitions is thus smooth for Γ a Spector pointclass. Martin's paper is remarkable for documenting properties of these co-Spector class operators. His work there is then applied to the present scenario where here we have the relevant pointclasses as $\Game\Sigma^0_3$ as the Spector pointclass, and its complimentary, or dual, class is (using the fact that Σ^0_3-games are determined) is the non-Spector pointclass $\Game\Pi^0_3$. The characterisation from Section 2 together with Martin's theorems allow us to conclude that the ordinal β_0 is in fact precisely the closure ordinal of $\Game\Pi^0_3$-non-monotone inductive definitions. (This sounds almost as if it could be trivially defining something in terms of itself, but it is not.)

We assume the reader has familiarity both with the constructible hierarchy of Gödel - for which see Devlin [3]. For the basic notions of descriptive set theory including the elementary theory of Gale-Stewart games, see Moschovakis [7]. Our notation is standard. Some of the results here relate to sub-systems of second order number, or analysis, and the basic theory of this is exposited in Simpson's monograph [8]. For models of admissible set theory, also called ''Kripke-Platek set theory'' see Barwise [1].

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We first extract from our earlier paper a criterion for the constructible rank of Π^0_3 games' strategies. (Note that we take our games as defined in L and using constructible game trees; the existence of a winning strategy for a particular Σ^0_3 (indeed arithemtic or Borel) game is a Σ^1_2 assertion about the countable tree T and the payoff set. As $T \in L$ the truth of such an assertion has the same truth value in the universe of sets or in L. We thus expect to find such strategies in L (since Davis in [2] proved such strategies exist in the universe V of sets). But where are they?

Definition 1. *Let an m-*depth Σ_2-nesting *of an ordinal α be a sequence (ζ_n, σ_n) with (i) For $n < m$: $\zeta_n \leq \zeta_{n+1} < \alpha < \sigma_{n+1} < \sigma_n$; (ii) $L_{\zeta_n} \prec_{\Sigma_2} L_{\sigma_n}$.*

We shall want to consider non-standard admissible models (M, E) of KP

together with some other properties. We let $\mathrm{WFP}(M)$ be the wellfounded part of the model. By the so-called 'Truncation Lemma' it is well known that this well founded part must also be an admissible set. Usually the model will also be a countable one of ''$V = L$''. Let M be such and let $\alpha = \mathrm{On} \cap \mathrm{WFP}(M)$. By the above α is thus an 'admissible ordinal' and L_α will also be a KP model. An 'ω-depth' nesting cannot exist be the wellfoundedness of the ordinals. However an ill founded model M when viewed from the outside may have infinite descending chains of 'M-ordinals' in its ill founded part. These considerations motivate the following definition.

Theorem 1.

Definition 2. *An* infinite depth Σ_2-nesting of α based on M *is a sequence* (ζ_n, s_n) *with, for* $n < \omega$ *:*

(i) $\zeta_n \leq \zeta_{n+1} < \alpha \subset s_{n+1} \subset s_n$ *; (ii)* $s_n \in \mathrm{On}^M$; *(iii)* $(L_{\zeta_n} \prec_{\Sigma_2} L_{s_n})^M$.

Thus the s_n form an infinite descending E-chain through the illfounded part of the model M. In [9] we devised a game whereby one player produced an ω-model of a theory and the other player tried to find such infinite descending chains through M's ordinals. In this paper we shall switch the roles of the players, and have Player *II* produce the model and Player I attempt to find the chain. The game is then Σ^0_3. We shall assume the reader has a copy of this paper to hand and shall refer to it throughout for definitions and notation.

In order for there to exist a non-standard model with an infinite depth nesting (of the ordinal of its wellfounded part) then the wellfounded part will already be a relatively long countable initial segment of L (it is easy to see that if $\zeta = \sup_n \zeta_n$ then already $L_\zeta \models \Sigma_1$-Separation).

Example 1. (i) Let δ be least so that $L_\delta \models \Sigma_2$-Separation, and let (M, E) be an admissible non-wellfounded end extension of L_δ with L_δ as its wellfounded part. Then there is an infinite depth nesting of δ based on M.

(ii) By refining considerations of the last example, let γ_0 be least such that there is $\gamma_1 > \gamma_0$ with $L_{\gamma_0} \prec_{\Sigma_2} L_{\gamma_1} \models \mathrm{KP}$. Then again there is an infinite depth nesting of γ_1 based on some illfounded end extension M of L_{γ_1}.

Both of the above can be established by standard Barwise Compactness arguments. However both these δ and γ_0 we shall see are greater than the ordinal β_0 defined from this notion of nesting as follows.

Definition 3. *Let* β_0 *be the least ordinal* β *so that* L_β *has an admissible end-extension* (M, E) *based on which which there exists an infinite depth* Σ_2*-nesting of* β*.*

Definition 4. *Let* γ_0 *be the least ordinal so that for any game* $G(A, T)$ *with* $A \in \Sigma^0_3$*,* $T \in L_{\gamma_0}$ *a game tree, then there is a winning strategy for a player definable over* L_{γ_0}*.*

Theorem 2. $\gamma_0 = \beta_0$*. Moreover, any* Σ^0_3*-game for a tree* T*, with a strategy for Player I, has such a strategy an element of* L_{β_0}*. Any* Π^0_3*-game for such a tree has a strategy which may not be an element of* L_{β_0}*, but it is definable over* L_{β_0}*.*

Remark: The proof reveals more about the strategies for Σ^0_3-games: they in fact appear within a bounded initial segment of β_0.

Proof: We look at the construction of the proof of Theorem 5 of [9] in particular that of Lemma 3. There we used an assumption that there is a triple of ordinals $\gamma_0 < \gamma_1 < \gamma_2$ with (a) $L_{\gamma_0} \prec_{\Sigma_2} L_{\gamma_1}$ and (b) $L_{\gamma_0} \prec_{\Sigma_1} L_{\gamma_2}$ and (c) γ_2 was the second admissible ordinal beyond γ_1. One assumed that I did not have a winning strategy in $G(A;T)$. The Lemma 3 there ran as follows:

Lemma 1. *Let $B \subseteq A \subseteq \lceil T \rceil$ with $B \in \Pi^0_2$. If $(G(A;T)$ is not a win for $I)_{L_{\gamma_0}}$, then there is a quasi-strategy $T^* \in L_{\gamma_0}$ for II with the following properties:*
(i) $\lceil T^* \rceil \cap B = \varnothing$;
(ii) $(\, G(A;T^*)$ is not a win for $I\,)_{L_{\gamma_0}}$.

The format of the lemma's proof involved showing that the $\Sigma_2^{L_{\gamma_0}}$ notion of 'goodness' embodied in (i) and (ii) held for $\varnothing$. To do this involved defining goodness in general. We first define T' as *II*'s *nonlosing quasi-strategy* for $(G(A;$ $T)$; this is Σ_1 definable over L_{γ_0} as the latter is a model KPI; in particular if we use the notation

Definition 5. $S^1_\gamma =_{\rm df} \{\delta < \gamma \mid L_\delta \prec_{\Sigma_1} L_\gamma\}$

then $T' \in \Pi_1^{L_{\zeta_0}}$, where $\zeta_0 =_{\rm df} \min S^1_{\gamma_0} \backslash \rho_L(T)$. More generally we define:

$p \in T'$ *good* if there is a quasi-strategy T^* for *II* in T'_p so that the following hold:
(i) $\lceil T^* \rceil \cap B = \varnothing$;
(ii) $G(A;T^*)$ is not a win for I.

Here T'_p is the subtree of T' below the node p. The point of requiring that the pair (γ_0, γ_1) have the Σ_2 reflecting property of (a) above, is that the class H of good p's of L_{γ_1} is the same as that of L_{γ_0} and so is a set in L_{γ_1} as it is thus definable over L_{γ_0}. The overall argument is a proof by contradiction, where we assume that $\varnothing$ is in fact not good, and proceeds to construct a strategy σ for Player I in the game $G(A;\, T')$, which is definable over L_{γ_1}, and is apparently winning in L_{γ_2}. (The requirement (c) that γ_2 be a couple of admissibles beyond γ_1 was only to allow for the strategy σ to be seen to be truly winning by going to the next admissible set, and verifying that there are no winning runs of play for *II*.) The contradiction arises since T' - which was defined as the subtree of T of *II*'s non-losing positions - is concluded still to be the same subtree of non-losing positions in L_{γ_2}. Being a non-losing position, p say, for *II* is a Π_1 property of p. This carries up from L_{γ_0} to L_{γ_2} as $L_{\gamma_0} \prec_{\Sigma_1} L_{\gamma_2}$, and this is the reason for the requirement (b). There is then no winning strategy for I in $G(A;\, T')$ definable over L_{γ_1}, contradicting the reasoning that σ is such.

This proves the Lemma: L_{γ_1} sees there is T^* a subtree of T' witnessing that $\varnothing$ is good. The existence of such a subtree is a Σ_2-sentence, and then again this reflects down to L_{γ_0}. We thus have such a T^* in L_{γ_0}.

The Theorem is proven by repeated applications of the Lemma, by using the argument for each Π^0_2 set B_n in turn where $A = \bigcup_n B_n$ and refining the trees using this procession from a tree to a subtree T^*. We thus repeat the argument with T^* replacing T. Because $T^* \in L_{\gamma_0}$ we have the same constellation of this triple of ordinals γ_i above the constructible rank of T^*, and can do this.

However we can get away with less. The definition of the subtree of non-losing positions of II now this time in the new T^* can be considered as taking place Π_1 over L_{δ_0} where δ_0 is the least element of $S^1_{\gamma_0}$ with $T^* \in L_{\delta_0}$. To get our contradiction we actually use that $L_{\delta_0} \prec_{\Sigma_1} L_{\gamma_2}$; we do not need that $L_{\gamma_0} \prec_{\Sigma_1} L_{\gamma_2}$. Notice that our argument that T^* exists is non-constructive: we simply say that the Σ_2-sentence of its existence reflects to L_{γ_0}: we do not have any control over its constructible rank below γ_0. Moreover any sufficiently large γ' greater than γ_1 would do for the upper ordinal, as long as it is a couple of admissibles larger than γ_1. Thus we could apply the Lemma repeatedly for different B_n if we have a guarantee that whenever a T^*_n-like subtree is defined there exists a $\zeta_n \in S^1_{\gamma_0}$ and a suitable upper ordinal $\gamma_n > \gamma_1$ with $T^*_n \in L_{\zeta_n} \prec_{\Sigma_1} L_{\gamma_n}$. Of course if there are arbitrarily large ζ_n below γ_0 with this extendability property, then this is tantamount to $L_{\gamma_0} \prec_{\Sigma_1} L_{\gamma'}$ for some suitable γ', and this shows why our original constellation of γ_i provides a sufficient condition.

Actually as the final paragraph of the Theorem 5 there shows, we are doing slightly more than this: we are, each time, applying the Lemma infinitely often to each possible subtree of of T^* below some node p_2 of it which is of length 2, to define our strategy τ applied to moves of length 4. We then move on to the next Π^0_2 set. Although we are applying the Lemma infinitely many times to each such p_2, and thus infinitely many new Σ_2-sentences, or trees, have to be instantiated, we had that L_{γ_0} is a Σ_2-admissible set, and as the class of such p_2 is just a set of L_{γ_0}, Σ_2-admissibility works for us to find a bound for the ranks of the witnessing trees, as some $\delta < \gamma_0$. We thus can claim that our final τ is an element of L_{γ_0} even after ω-many iterations of this process.

($\beta_0 \geq \gamma_0$) We argue for this. Let (M, E) be a non-standard model of KP with an infinite nesting (ζ_n, s_n) about β_0 as described. Note that $S^1_{\beta_0}$ must be unbounded in β_0 (so that $L_{\beta_0} \models \Sigma_1$-Separation), and each ζ_n is a limit point of $S^1_{\beta_0}$. We do not assume that β_0 is Σ_2-admissible (which in fact it is not as the proof shows). Let $T \in L_{\beta_0}$ be a game tree. By omitting finitely much of the outer nesting we assume $T \in L_{\zeta_0}$. We assume that Player I has no winning strategy for $G(A; T)$ in L_{β_0} (for otherwise we are done). Note that in M we have that L_{s_0} also has no winning strategy for this game (otherwise the existence of such would reflect into L_{β_0}. We show that II has a winning strategy definable over L_{β_0}. Let $A = \bigcup B_n$ with each $B_n \in \Pi^0_2$. For $n = 0$ we apply the argument of the Lemma using the pair (ζ_1, s_1) in the role of (γ_0, γ_1) from before, with (ζ_0, s_0) in the role of (δ_0, γ_2) described above, *i.e.* we use only that $T \in L_{\zeta_0}$ and that $L_{\zeta_0} \prec_{\Sigma_1} L_{s_0}$.

The Lemma then asserts the existence of a quasi-strategy for *II* definable using the pair (ζ_1, s_1): $T^*(\varnothing)$. By Σ_2-reflection the L-least such lies in L_{ζ_1}, and we shall assume that $T^*(\varnothing)$ refers to it.

Claim: For any pair (ζ_n, s_n) for $n \geq 1$ the same tree $T^*(\varnothing)$ would have resulted using this pair.

Proof: Note that we can define such a tree like $T^*(\varnothing)$ using such pairs, since for all of them we have that $(\zeta_0, s_0) \supset (\zeta_1, s_1) \supset (\zeta_m, s_m)$ for $m > 1$. As $T^*(\varnothing) \in L_{\zeta_1}$ and satisfies a Σ_2 defining condition there, and since we also have $\zeta_1 \in S^1_{\zeta_m}$, it thus satisfies the same Σ_2 condition in L_{ζ_m}. Q.E.D. *Claim*

For any position $p_1 \in T$ with $\mathrm{lh}(p_1) = 1$, let $\tau(p_1)$ be some arbitrary but fixed move in $T'(\varnothing)$, this now *II*'s non-losing quasi-strategy for the game $G(A, T^*(\varnothing))$ as defined in L_{ζ_2}. The relation "$p \in T'(\varnothing)$" is $\Pi_1^{L_{\zeta_1}}(\{T^*(\varnothing)\})$ or equivalently $\Pi_1^{L_{\zeta_2}}(\{T^*(\varnothing)\})$, or indeed $\Pi_1^{L_\delta}(\{T^*(\varnothing)\})$ where δ is least in $S^1_{\zeta_1}$ above $\rho_L(T^*(\varnothing))$. Hence "$y = T'(\varnothing)$" $\in \Delta_2^{L_\delta}(\{T^*(\varnothing)\})$ and thus $T'(\varnothing)$ also lies in L_{ζ_1}. For definiteness we let $\tau(p_1)$ be the numerically least move.

For any play, p_2 say, of length 2 consistent with the above definition of τ so far, we apply the lemma again with $B = A_1$ replacing $B = A_0$ and with $(T^*(\varnothing))_{p_2}$ replacing T. We use the nested pair (ζ_2, s_2) to define quasi-strategies for *II*, call them $T^*(p_2)$, one for each of the countably many p_2. These are each definable in a Σ_2 way over L_{ζ_2}, in the parameter $(T^*(\varnothing))_{p_2}$. This argument uses that $(T^*(\varnothing))_{p_2} \in L_{\zeta_1} \prec_{\Sigma_1} L_{s_1}$. Let $T'(p_2) \in L_{\zeta_2}$ be *II*'s non-losing quasi-strategy for $G(A, T^*(p_2))$, this time with "$y = T'(p_2)$" $\in \Delta_2^{L_{\zeta_2}}(\{T^*(p_2)\})$. (Again these will satisfy the same definitions as over L_{ζ_m} for any $m \geq 2$.) Note that we may assume that the countably many trees $T'(p_2)$ appear boundedly below ζ_2 (using the Σ_2-admissibility of ζ_2). Again for $p_3 \in T^*(p_2)$ any position of length 3, let $\tau(p_3)$ be some arbitrary but fixed move in $T'(p_2)$. Now we consider appropriate moves p_4 of length 4, and reapply the lemma with $B = A_2$ and $(T^*(p_2))_{p_4}$. Continuing in this way we obtain a strategy τ for *II*, so that $\tau \restriction {}^{[1,2k+2)}\omega$, for $k < \omega$, is defined by a length k-recursion that is $\Sigma_2^{L_{\zeta_k}}(\{T\})$.

As the argument continues more and more of the strategy τ is defined using successive (ζ_m, s_m) to justify the existence of the relevant trees in L_{ζ_m}. *Knowing* that the trees are there for the asking, we see that τ can actually be defined by a Σ_2-recursion over L_{β_0} in the parameter T in precisely the manner given above.

If x is any play consistent with τ, then for every n, by the defining properties of $T^*(p_{2n})$ given by the relevant application of the lemma, $x \in \lceil T^*(x \restriction 2n) \rceil \subseteq \neg A_n$. Hence $x \notin A$, and τ is a winning strategy for *II* as required. Thus $\beta_0 \geq \gamma_0$ is demonstrated.

For $\beta_0 \leq \gamma_0$: suppose then $\beta_0 > \gamma_0$. Then the existence of such a γ_0 would be part of the Σ_1-Theory of L_{β_0}, and thus $\gamma_0 < \bar{\alpha}$ where $\bar{\alpha}$ is least with $T^1_{\bar{\alpha}} = T^1_{\beta_0}$ (and thus $L_{\bar{\alpha}} \prec_{\Sigma_1} L_{\beta_0}$). We may now run the argument of Theorem 4 with Player *II* constructing an ω-model of T + ''*There is no transitive set model of* T'' where T is the theory: $\mathrm{KP} + V = L + \psi$ where ψ says: ''γ_0 *exists*''. This defines a Σ^0_3-game, which I must win. For if the model that *II* constructs is illfounded below β_0, I, who is trying to find a descending chain, will be able to detect one, because the argument of Theorem 4's proof depended precisely on there being no infinite nested sequence based on the wellfounded part of *II*'s model. But the wellfounded part of the model *II* is building cannot be larger than α_ψ. Contradiction. Hence $\beta_0 \leq \gamma_0$. Q.E.D.

Let T^n_δ denote the Σ_n theory of L_δ. Recall that a set $X \subseteq \mathbb{N}$ ($\mathbb{N}^\mathbb{N}$) is said to be in $\Game\Gamma$ for some adequate pointclass Γ if there is a set $Y \subseteq \mathbb{N} \times \mathbb{N}^\mathbb{N}$ ($\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$) so that $X = \{x \mid$ *Player I has a winning strategy in* $G(Y_x, {}^{<\mathbb{N}}\mathbb{N})\}$ where $Y_x = \{y \mid \langle x, y\rangle \in Y\}$.

Theorem 3. *Let* $\bar{\alpha}$ *be least with* $T^1_{\bar{\alpha}} = T^1_{\beta_0}$.

(i) $T^1_{\bar{\alpha}}$ *is a complete* $\Game\Sigma^0_3$ *set of integers.*

(ii) Hence for any $\alpha \leq \bar{\alpha}$, T^1_α *is a* $\Game\Sigma^0_3$ *set of integers and the reals of* $L_{\bar{\alpha}}$ *are all* $\Game\Sigma^0_3$ *set of integers.*

Proof of Theorem 3. The argument is really close to that of the Corollary 2 of [9]. Indeed there we showed that the $T^1_{\alpha_\psi}$ were $\Game\Sigma^0_3$ sets. Some details of this are repeated. First remark that we need only show that $T^1_{\bar{\alpha}}$ is $\Game\Sigma^0_3$ since the other T^1_α for $\alpha < \bar{\alpha}$ are all recursive in $T^1_{\bar{\alpha}}$ and $\Game\Sigma^0_3$, being a Spector class, is closed under recursive substitution. For the same reason each real $a \in L_{\bar{\alpha}}$ is $\Game\Sigma^0_3$ as a set of integers.

We define a game G^*.

Rules for II.

In this game *II*'s moves in x must be a set of Gödel numbers for the complete Σ_1-theory of an ω-model of $\mathrm{KP} + V = L + \mathrm{Det}(\Game\Sigma^0_3) + \neg\varphi$.

Everything else remains the same *mutatis mutandis*: *I*'s Rules remain the same and his task is to find an infinite descending chain through the ordinals of *II*'s model. Note that if $\varphi \in T^1_{\bar{\alpha}}$, *I* now has a winning strategy: for if *II* obeys her rules, and x codes an ω-model M of this theory, then M is not wellfounded, and has $\mathrm{WFP}(M) \cap \mathrm{On} < \rho(\varphi)$ where $\rho(\varphi)$ is defined as the least ρ such that $\varphi \in T^1_{\rho+1}$. However *I* playing (just as *II* did in the main Theorem 4) can find a descending chain and win. For we have $\mathrm{WFP}(M) \cap \mathrm{On} < \beta_0$ and so the argument goes through, as there are no infinite depth nestings there. On the other hand if $\varphi \notin T^1_{\alpha_\psi}$, *II* may just play a code for the true wellfounded $L_{\beta_0^+}$ with β_0^+ the least admissible above $\beta_0 + 1$, and so win. This shows that $T^1_{\bar{\alpha}}$ is a complete $\Game\Sigma^0_3$ set of integers.

Now suppose $a \in \Game\Sigma^0_3$. Then we have some Σ^0_3 set $A \subseteq \omega \times {}^\omega\omega$ with $n \in a \longleftrightarrow$ I has a winning strategy to play into $A_a = \{y \in {}^\omega\omega \mid (a, y) \in A\}$. Then a is $\Sigma_1^{L_{\bar{\alpha}}}$, and thus is recursive in $T^1_{\bar{\alpha}}$. Hence $T^1_{\bar{\alpha}}$ is a complete $\Game\Sigma^0_3$ set of integers. Q.E.D.

In conclusion: we saw above that $\bar{\alpha}$ was the least α with $T^1_\alpha = T^1_{\beta_0}$. Phrased in other terms, by elementary constructible hierarchy considerations, this is saying that $\bar{\alpha}$ is the minimum of $S^1_{\beta_0}$. Hence $L_{\bar{\alpha}} \prec_{\Sigma_1} L_{\beta_0}$ but for no smaller δ is $L_\delta \prec_{\Sigma_1} L_{\beta_0}$. Since the statement "There is a winning strategy for Player I in $G(A, T)$" is equivalent in KPI to a Σ_1-assertion, if true in L_{β_0} it is true in $L_{\bar{\alpha}}$. In short for those Σ^0_3-games that are wins for I on trees $T \in L_{\bar{\alpha}}$, there are strategies for such also within $L_{\bar{\alpha}}$ itself. For those that are wins for Player *II* these may be defined over L_{β_0} at the end of the interval $[\bar{\alpha}, \beta_0)$ or else may be found also in $L_{\bar{\alpha}}$. This somewhat asymmetrical picture reflects the earlier theorems cited above. The theorems of the next section harmonise perfectly with this.

Remark: (i) Since $\Game\Sigma^0_3$ is a Spector class, one will have a $\Game\Sigma^0_3$-prewellorderings of $T^1_{\bar{\alpha}}$ as a $\Game\Sigma^0_3$ set of integers, of maximal length, here $\bar{\alpha}$.

We write down one on $T = T^1_{\bar{\alpha}}$. Abbreviate $\Gamma = \Game\Sigma^0_3$ and $\check{\Gamma} = \Game\Pi^0_3$. We need to provide relations $\leq_\Gamma$ and $\leq_{\check{\Gamma}}$ in Γ and $\check{\Gamma}$ respectively, so that the following hold:

$$T(y) \Longrightarrow \forall x\{[T(x) \wedge \rho(x) \leq \rho(y)] \Longleftrightarrow x \leq_\Gamma y \Longleftrightarrow x \leq_{\check{\Gamma}} y\}.$$

For the relation $x \leq_\Gamma y$, we define the game where *II* produces a model M^{II} of $T(y) \wedge (\neg T(x) \vee \rho(x) \nleq \rho(y))$ and I tries to demonstrate that it is wellfounded. Assume then $T(y)$. If $T(x) \wedge \rho(x) \leq \rho(y)$ then *either* $(\neg T(x))^{M^{II}}$ and thus M^{II} is illfounded with $\mathrm{WFP}(M^{II}) \cap \mathrm{On} < \rho(x)$ and hence I can win as in this region there are no ω-nested sequences. *Or*: $(\rho(x) \nleq \rho(y))^{M^{II}}$. Thus $(\rho(x) > \rho(y))^{M^{II}}$ and again this implies $\mathrm{WFP}(M^{II}) \cap \mathrm{On} < \rho(x)$ with I winning.

Conversely suppose $x \leq_\Gamma y$. Since $T(y)$ is assumed, if $\neg T(x)$, then *II* can play a wellfounded model with $(y \wedge \neg x)^{M^{II}}$ and win. If $\rho(x) > \rho(y)$ then again the same can be done. This proves the first equivalence above. The second is similar, with now I producing a model M^I of $T(x) \wedge \rho(x) \leq \rho(y)$ and *II* finding descending chains. We leave the details to the reader.

(ii) One may also write out directly the theories T^1_α for $\alpha < \bar{\alpha}$ in a $\Game\Pi^0_3$ form. This should not be surprising: a $\Game\Sigma^0_3$ norm as above should have 'good' $\Delta(\Game\Sigma^0_3)$ initial segments.

3 A non-monotone inductive closure ordinal

We consider here a very different possible characterisation of β_0. Let $\Phi\colon \mathcal{P}(\omega) \longrightarrow \mathcal{P}(\omega)$ be any map. We think of Φ as an inductive definition by means of the following: we 'iterate' Φ and define $\Phi^\alpha \subseteq \omega$ as follows: assume Φ^β is defined for $\beta < \alpha$, then $\Phi^{<\alpha} = \bigcup_{\beta<\alpha} \Phi^\beta$. Now set $\Phi^\alpha = \Phi^{<\alpha} \cup \Phi(\Phi^{<\alpha})$. Then Φ iterated in this way is a *progressive operator* and for some countable ordinal γ we shall have a fixed point $\Phi^\gamma = \Phi^{<\gamma}$. We shall write this γ as $o(\Phi)$. We shall further write Φ^∞ for $\Phi^{o(\Phi)}$.

Definition 6. Φ *is* monotone *if:* $A \subseteq B \longrightarrow \Phi(A) \subseteq \Phi(B)$.

Then for monotone Φ the Φ^∞ defined above is the *smallest fixed point* of Φ, *i.e.* the smallest set X with $\Phi(X) = X$.

Definition 7. *If* Γ *is a pointclass of relations on* $\mathcal{P}(\omega) \times \mathcal{P}(\omega)$ *then*
$o(\Phi\text{-mon}) =_{\mathrm{df}} \sup\{o(\Phi) \mid \Phi \in \Gamma,\ \Phi\ \textit{monotone}\}$.

Definition 8. *For* $\Gamma \subseteq \mathcal{P}\mathcal{P}(\omega)$ *we define the* pointclass dual to Γ *as the pointclass* $\{\mathcal{P}(\omega) \backslash X \mid X \in \Gamma\}$ *and is denoted* $\check{\Gamma}$.

Thus $\check{\Sigma}^1_1 = \Pi^1_1$; $\check{\Sigma}^1_2 = \Pi^1_2$ *etc.* In the latter case it is Σ^1_2 that is an example of a *Spector pointclass*. The latter is defined in [7]; we shall not need to go into the defintions or properties of Spector classes that much, but note that a Spector class of pointsets is closed under union, intersection, number quantification, contains Σ^0_1, is ω*-parametrized* (which implies that it has a universal set), and importantly has the *prewellordering property*.

Martin points out that it is not always the case that inductive definitions lead from simple sets via iteration of an operator in a particular pointclass to a complicated set: he shows that the fixed point Φ^∞ of a monotone Π^1_2 operator, in fact is still a Π^1_2 set. It is a one line argument: suppose Φ is such, then the following is also Π^1_2:
$n \in \Phi^\infty \leftrightarrow \forall X(\Phi(X) \subseteq X \longrightarrow n \in X) \leftrightarrow \forall X(\exists m(m \in \Phi(X) \vee n \in X)$.

He wishes to study $o(\check{\Gamma}\text{-mon})$ for Γ a Spector pointclass, and he takes Π^1_2 as the typical example of such. For this paper however the Spector pointclass of iterest is $\Game\Sigma^0_3$ and we are interested in $o(\widecheck{\Game\Sigma^0_3}\text{-mon})$. As remarked above by $\mathrm{Det}(\Sigma^0_3)$, $\widecheck{\Game\Sigma^0_3} = \Game\check{\Sigma}^0_3 = \Game\Pi^0_3$. The relevant ordinal for us will then be $\pi_0 =_{\mathrm{df}} o(\Game\Pi^0_3\text{-mon})$.

He shows:

Theorem 4. *(Theorem D [5]) Let* Γ *be a Spector pointclass. Suppose that for every* $X \subseteq \omega$*, and every* $\check{\Gamma}(X)$ *monotone* Φ*, that* $\Phi^\infty \in \check{\Gamma}(X)$*, then* $o(\check{\Gamma}\text{-mon})$ *is non-projectible, that is* $S^1_{o(\check{\Gamma}\text{-mon})}$ *is unbounded in* $o(\check{\Gamma}\text{-mon})$.

It is remarked that it is unknown in general if $o(\check{\Gamma}\text{-mon})$ is admissible, but of those of the kind in the theorem not only is $L_{o(\check{\Gamma}\text{-mon})}$ admissible, it is a model of Σ_1-Separation (which is another way of saying that it is non-projectible). We should like to apply the theorem for $\Gamma = \Game\Sigma^0_3$, and then we might conclude that π_0 is non-projectible. The required supposition stated in the last theorem needed to apply this, we obtain from the following of Martin's theorems:

Theorem 5. *(Theorem E [5]) Suppose* Γ *is closed under union, intersection, recursive pre-images and existential number quantification and contains* Σ^0_3*. Suppose* $\mathrm{Det}(\Gamma)$ *holds, and that* $\Game\Gamma$ *has the prewellordering property. If* Φ *is then* $\Game\check{\Gamma}$ *monotone, then* $\Phi^\infty \in \Game\check{\Gamma}$.

In fact we apply the theorem with $\Gamma = \Sigma^0_3$ itself. All the assumptions are met ($\Game\Sigma^0_3$ is a Spector class and thus has the prewellordering property). The theorem then relativizes uniformly in any $X \subseteq \omega$, to conclude that such $\Phi^\infty \in \Game\Pi^0_3$.

Corollary 1. $\pi_0 = o(\Game\Pi^0_3\text{-mon})$ *is non-projectible.*

Theorem 6. $\pi_0 = \beta_0$.

Clearly $\bar{\alpha} < \pi_0 \leq \beta_0$. By the last clause of Theorem 4, $S^1_{\pi_0}$ is unbounded in π_0; and thus $\bar{\alpha}$=min $S^1_{\pi_0}$. Thus the only question left is whether $\pi_0 < \beta_0$ is conceivable.

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