Base Invariance of Feasible Dimension

John M. Hitchcock* Elvira Mayordomo†

July 5, 2012

Abstract

Effective fractal dimension was defined by Lutz (2003) in order to quantitatively analyze the structure of complexity classes. Interesting connections of effective dimension with information theory were also found, implying that constructive dimension as well as polynomial-space dimension are closed under base-change while finite-state dimension is not.

We consider the intermediate case, polynomial-time dimension, and prove that it is indeed closed under base-change by a nontrivial argument which is quite different from the Kolmogorov complexity ones used in the other cases.

p-dimension can be characterized in terms of prediction-loss-rate, entropy, and compression algorithms. Our result implies that in an asymptotic way each of those concepts is invariant under base-change.

1 Introduction

The concept of randomness of a real number can be naturally defined from the randomness of the binary infinite sequence that represents this number. The choice of base two representation here is an arbitrary one–base three, base four, or any other base would work just as well–they all yield the same randomness notions. Surprisingly, when looking at effective versions

*Department of Computer Science, University of Wyoming, jhitchco@cs.uwyo.edu. This research was supported in part by NSF grants 9988483 and 0917417.

†Dept. de Informática e Ingeniería de Sistemas, Instituto de Investigación en Ingeniería de Aragón (I3A), Universidad de Zaragoza, Zaragoza, SPAIN. elvira@unizar.es. Part of this work was done during a visit to the Isaac Newton Institute for Mathematical Sciences at the University of Cambridge. This work was supported by the Spanish Ministry of Science and Innovation (Projects TIN2008-06582-C03-02, TIN2011-27479-C04-01)

of randomness one choice of base may not be equivalent to other base representations. For instance, in finite-state randomness and dimension [3] it is known that both randomness and dimension 1 sequences coincide with the normal sequences (consequence of [2, 14]), and therefore both finite-state randomness and dimension are not invariant under base change, since the existence of non-absolutely normal sequences is known [13]. On the other hand, Martin-Löf randomness [4] and constructive dimension [11] can be easily proven to be base-invariant by standard Kolmogorov complexity arguments, and for the same reason pspace-randomness [9] and pspace-dimension [10] are base invariant.

We study an intermediate case, polynomial-time resource-bounds, and prove p-dimension is invariant under base change. The proof is nontrivial since base change in not an honest function, in fact for infinitely many cases it is arbitrarily length decreasing. Consider for instance the process of changing the number 1/2 from base 3 to base 2. When given successively longer prefixes of the base 3 representation 0.1111... , there are always two possible candidates for finite prefixes of a base 2 representation. This makes any (time-bounded) randomness argument more complicated, while the Kolmogorov complexity of both candidates is very close.

Effective fractal dimension was defined by Lutz [10] in order to quantitatively analyze the structure of complexity classes, and later generalized to other resource-bounds such as constructive [11] or finite-state [3] computability. Important applications in computational complexity have been found including circuit-size complexity, polynomial-time degrees, the size of NP, zero-one laws, and oracle classes. See [12, 6, 4] for a summary of the main results.

p-dimension can be characterized in terms of prediction-loss-rate [5], entropy [7], and compression algorithms [8]. Our result implies that in an asymptotic way each of those concepts is invariant under base-change.

Another consequence of our main result is that p-dimension 1 numbers are absolutely normal, thus providing an interesting source of absolute normality.

Strong-p-dimension [1], a concept dual to p-dimension that corresponds to the effectivization of packing dimension from fractal geometry is also base-invariant, which can be proven with an argument similar to that used in the proof of our main theorem.

2 Preliminaries

2.1 p-dimension

For any natural number $k \geq 2$, we let $\Sigma_k = \{0, \ldots, k-1\}$ be a k-symbol alphabet. If $w \in \Sigma_k^*$ and $x \in \Sigma_k^* \cup \Sigma_k^\infty$, $w \sqsubseteq x$ means that w is a prefix of x.

For $0 \leq i \leq j$, we write $x[i \ldots j]$ for the string consisting of the i-th through the j-th symbols of x. We use λ for the empty string.

Definition. Let $s \in [0, \infty)$.

1. An *s-gale* on Σ_k is a function $d : \Sigma_k^* \to [0, \infty)$ satisfying

$$d(w) = |\Sigma_k|^{-s} \sum_{a \in \Sigma_k} d(wa)$$

for all $w \in \Sigma_k^*$.

2. A *martingale* is a 1-gale, that is, a function $d : \Sigma_k^* \to [0, \infty)$ satisfying

$$d(w) = \frac{\sum_{a \in \Sigma_k} d(wa)}{|\Sigma_k|}$$

for all $w \in \Sigma_k^*$.

Definition. Let $s \in [0, \infty)$ and d be an s-gale. We say that d *succeeds* on a sequence $S \in \Sigma_k^\infty$ if

$$\limsup_{n \to \infty} d(S[0 \ldots n]) = \infty.$$

The success set of d is

$$S^\infty[d] = \{S \in \Sigma_k^\infty \mid d \,\text{succeeds on}\, S\}.$$

Definition. We say that a function $d : \Sigma_k^* \to [0, \infty)$ is *p-computable* if there is a function $\hat{d} : \Sigma_k^* \times \mathbb{N} \to \mathbb{Q}$ such that $\hat{d}(w, r)$ is computable in time polynomial in $|w| + r$ and $|\hat{d}(w, r) - d(w)| \leq 2^{-r}$ holds for all w and r.

We say that a function $d : \Sigma_k^* \to [0, \infty) \cap \mathbb{Q}$ is *exactly p-computable* if $d(w)$ is computable in time polynomial in $|w|$.

Definition. Let $X \subseteq \Sigma_k^\infty$, The *p-dimension* of X is

$$\dim_{\mathrm{p}}(X) = \inf \left\{ s \in [0, \infty) \;\middle|\; \begin{array}{l} \text{there is a p-computable } s\text{-gale } d \text{ s.t.} \\ X \subseteq S^\infty[d] \end{array} \right\}$$

By the exact computation lemma in [10] p-computable and exactly p-computable gales are interchangeable in the definition above.

Theorem 2.1 *Let $X \subseteq \Sigma_k^\infty$,*

$$\dim_{\mathrm{p}}(X) = \inf \left\{ s \in [0,\infty) \;\middle|\; \begin{array}{l} \textit{there is an exactly p-computable s-gale d s.t.} \\ X \subseteq S^\infty[d] \end{array} \right\}.$$

We will use $\dim_{\mathrm{p}}^{(k)}(X)$ to refer to the p-dimension of $X \subseteq \Sigma_k^\infty$ when we want to stress that the underlying sequence space is Σ_k^∞.

We will briefly refer to n^r-dimension which corresponds to the use of gales computable in time n^r where n is the length of the input.

For a complete introduction and motivation of effective dimension see [12].

2.2 Representations of Reals

We will use infinite sequences over Σ_k to represent real numbers in [0,1). For this, we associate each string $w \in \Sigma_k^*$ with the half-open interval $[w]_k$ defined by

$$[w]_k = \left[\sum_{i=1}^{|w|} w[i-1]k^{-i}, k^{-|w|} + \sum_{i=1}^{|w|} w[i-1]k^{-i} \right).$$

Each real number $\alpha \in [0,1)$ is then represented by the unique sequence $S_k(\alpha) \in \Sigma_k^\infty$ satisfying

$$w \sqsubseteq S_k(\alpha) \iff \alpha \in [w]_k$$

for all $w \in \Sigma_k^*$. We have

$$\alpha = \sum_{i=1}^{\infty} S_k(\alpha)[i-1]k^{-i}$$

and the mapping $\alpha \mapsto S_k(\alpha)$ is a bijection from $[0,1)$ to Σ_k^∞ (notice that $[w]_k$ being half-open prevents double representations). If $x \in \Sigma_k^\infty$ then $real_k(x) = \alpha$ such that $x = S_k(\alpha)$. Therefore we always have that $real_k(S_k(\alpha)) = \alpha$ and $S_k(real_k(x)) = x$. A set of real numbers $A \subseteq [0,1)$ is represented by the set

$$X_k(A) = \{S_k(\alpha) \mid \alpha \in A\}$$

of sequences. If $X \subseteq \Sigma_k^\infty$ then

$$real_k(X) = \{real_k(x) \mid x \in X\}.$$

The question addressed in this paper is the following. Is the feasible dimension of a set $A \subseteq [0,1)$ invariant with respect to the base used for representation? That is, is the definition $\dim_{\mathrm{p}}(A) = \dim_{\mathrm{p}}^{(k)}(X_k(A))$ robust when k changes?

3 Main Theorem

Theorem 3.1 *Let $k, l \geq 2$. For any exactly p-computable s-gale d on Σ_l and rational $s' > s$, there is a p-computable s'-gale d' on Σ_k such that $real_l(S^\infty[d])) \subseteq real_k(S^\infty[d'])$.*

Proof.

Let d be an exactly p-computable s-gale on Σ_l, without loss of generality we assume that $d(\lambda) = 1$. For any $n \in \mathbb{N}$, we define a function $D_n : \Sigma_k^* \to [0, \infty)$ as follows. Let $m = \lfloor n \log_k l \rfloor$. For any $y \in \Sigma_k^*$, we define

$$D_n(y) = \begin{cases} k^{s'|y|}\left(\displaystyle\sum_{\substack{x \in \Sigma_l^n \\ [x]_l \subseteq [y]_k}} d(x) + \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y]_k \\ [x]_l \cap [y]_k \neq \emptyset}} \frac{1}{2} d(x) \right) & \text{if } |y| \leq m \\ k^{(s'-1)*(|y|-m)} D_n(y[0..m-1]) & \text{otherwise.} \end{cases}$$

The desired s'-gale d' on Σ_k is then defined by

$$d'(y) = \sum_{n=0}^{\infty} l^{-s'n} D_n(y).$$

The intuition in the definition of D_n and d' is that $d'(y)$ takes the full value of $d(x)$ for those x for which y is (the beginning of) the base k representation of x, while it only takes $1/2d(x)$ for those x for which we still don't know if (an extension of) y will be the base k representation of (an extension of) x.

Claim 1 d' is an s'-gale on Σ_k.

Let $y \in \Sigma_k^{<m}$. For any $x \in \Sigma_l^n$, we have

$$\begin{aligned} [x]_l \subseteq [y]_k \quad \Longleftrightarrow \quad & (\exists a \in \Sigma_k)\ [x]_l \subseteq [ya]_k \\ & \text{or } (\exists a \in \Sigma_k - \{(k-1)\})\ [x]_l \subseteq [ya]_k \cup [y(a+1)]_k \end{aligned}$$

because $[x]_l$ can intersect at most two of the intervals $[ya]_k$ for $a \in \Sigma_k$. (This is because $|[ya]_k| = k^{-|ya|} \geq k^{-m} \geq k^{-n \log_k l} = l^{-n} = |[x]_l|$.) For the same reason, we also have

$$[x]_l \not\subseteq [y]_k \text{ and } [x]_l \cap [y]_k \neq \emptyset \iff \begin{array}{l} ([x]_l \cap [y0]_k \neq \emptyset \text{ and } [x]_l \cap [y1]_k = \emptyset) \\ \text{or } ([x]_l \cap [y(k-1)]_k \neq \emptyset \text{ and } [x]_l \cap [y(k-2)]_k = \emptyset) \end{array}$$

for any $x \in \Sigma_l^n$. By these relationships, we have

$$\begin{aligned} \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \subseteq [y]_k}} d(x) &= \sum_{a \in \Sigma_k} \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \subseteq [ya]_k}} d(x) + \sum_{a \in \Sigma_k - \{(k-1)\}} \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \subseteq [ya]_k \cup [y(a+1)]_k \\ [x]_l \not\subseteq [ya]_k \\ [x]_l \not\subseteq [y(a+1)]_k}} d(x) \\ &= \sum_{a \in \Sigma_k} \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \subseteq [ya]_k}} d(x) + \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \cap [y0]_k \neq \emptyset \\ [x]_l \cap [y1]_k \neq \emptyset}} \frac{1}{2} d(x) \\ &\quad + \sum_{a \in \Sigma_k - \{0, (k-1)\}} \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [ya]_k \\ [x]_l \cap [ya]_k \neq \emptyset}} \frac{1}{2} d(x) + \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \cap [y(k-1)]_k \neq \emptyset \\ [x]_l \cap [y(k-2)]_k \neq \emptyset}} \frac{1}{2} d(x) \end{aligned}$$

and

$$\sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y]_k \\ [x]_l \cap [y]_k \neq \emptyset}} \frac{1}{2} d(x) = \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y0]_k \\ [x]_l \cap [y0]_k \neq \emptyset \\ [x]_l \cap [y1]_k = \emptyset}} \frac{1}{2} d(x) + \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y(k-1)]_k \\ [x]_l \cap [y(k-1)]_k \neq \emptyset \\ [x]_l \cap [y(k-2)]_k = \emptyset}} \frac{1}{2} d(x).$$

Combining these two sums establishes that D_n is an s'-gale on Σ_k.

Claim 2 d' is p-computable.

We first show how to efficiently compute $D_n(y)$. For this, we iteratively define a sequence of sets $B_i^n(y)$ for $i = 0, \ldots, n$ by

$$B_i^n(y) = \left\{ x \in \Sigma_l^i \;\middle|\; [x]_l \subseteq [y]_k \text{ and } x \text{ has no prefix in } \bigcup_{j=0}^{i-1} B_j^n(y) \right\}.$$

That is, $B_i^n(y)$ is the strings x of length i that represent maximal intervals included in $[y]_k$. We can now represent $D_n(y)$ for $y \in \Sigma_k^{\leq m}$ in the following form.

$$D_n(y) = k^{s'|y|}\left(\sum_{i=0}^{n}\sum_{x\in B_i^n(y)} l^{s(n-i)}d(x) + \sum_{\substack{x\in\Sigma_l^n \\ [x]_l\not\subseteq[y]_k \\ [x]_l\cap[y]_k\neq\emptyset}} \frac{1}{2}d(x)\right)$$

This is equivalent to the original definition of D_n because d is an s-gale. Each $B_i^n(y)$ will have at most $2(l-1)$ strings, and these are easily computable. There are two strings to consider for the second sum.

For the p-computation of d', let b be such that for every r, $\sum_{n=br+1}^{\infty} l^{-n(s'-s)/2} \leq 2^{-r}$. Let c be such that $k^{s'} < l^{(s'-s)c/2}$. Let $f(y,r) = \sum_{n=0}^{br+c|y|} l^{-s'n}D_n(y)$. Then f is clearly computable in polynomial time on $|y|$ and r and we have that

$$\begin{aligned}
|d'(y) - f(y,r)| &= \sum_{n=br+c|y|+1}^{\infty} l^{-s'n}D_n(y) \\
&\leq \sum_{n=br+c|y|+1}^{\infty} l^{-s'n}k^{s'|y|}D_n(\lambda) \\
&= \sum_{n=br+c|y|+1}^{\infty} l^{-s'n}l^{sn}d(\lambda)k^{s'|y|} \\
&\leq \sum_{n=br+c|y|+1}^{\infty} l^{-(s'-s)n}l^{|y|(s'-s)c/2} \\
&\leq \sum_{n=br+c|y|+1}^{\infty} l^{-(s'-s)n}l^{n(s'-s)/2} \\
&= \sum_{n=br+c|y|+1}^{\infty} l^{-(s'-s)n/2} \\
&\leq 2^{-r}.
\end{aligned}$$

The first three inequalities come form the fact that both d and D_n are gales, and $d(\lambda) = 1$. The remaining inequalities come from the choice of constants b and c.

Claim 3 $real_l(S^\infty[d])) \subseteq real_k(S^\infty[d'])$.

Let $\alpha \in [0,1)$. Letting $x_n = S_l(\alpha)[0..n-1]$ and $y_n = S_k(\alpha)[0..m-1]$, we have $[y_n]_k \cap [x_n]_l \neq \emptyset$ and by definition $D_n(y_n) \geq k^{s'|y_n|}\frac{1}{2}d(x_n)$.

Therefore if $\alpha \in real_l(S^\infty[d]))$ then $S_l(\alpha) \in S^\infty[d]$ and since for every n,

$$d'(y_n) \geq l^{-s'n} D_n(y_n) \geq l^{-s'n} k^{s'|y_n|} \frac{1}{2} d(x_n) \geq \frac{1}{2} d(x_n),$$

we have that $S_k(\alpha) \in S^\infty[d']$. □

We now have our main theorem.

Theorem 3.2 *For any $A \subseteq [0,1)$ and $k, l \geq 2$, $\dim_{\mathrm{p}}^{(k)}(X_k(A)) = \dim_{\mathrm{p}}^{(l)}(X_l(A))$.*

Proof. Let $s > \dim_{\mathrm{p}}^{(l)}(X_l(A))$. Then there is an s-gale d on Σ_l such that $X_l(A) \subseteq S^\infty[d]$ thus $A \subseteq real_l(S^\infty[d])$. For each $s' > s$ the previous theorem gives a p-computable s'-gale d' on Σ_k such that $real_l(S^\infty[d])) \subseteq real_k(S^\infty[d'])$. Therefore $X_k(A) \subseteq S^\infty[d']$, and $\dim_{\mathrm{p}}^{(k)}(X_k(A)) \leq s'$. As $s > \dim_{\mathrm{p}}^{(l)}(X_l(A))$ and $s' > s$ were arbitrary, this establishes $\dim_{\mathrm{p}}^{(k)}(X_k(A)) \leq \dim_{\mathrm{p}}^{(l)}(X_l(A))$. The converse inequality follows by a symmetric argument. □

This contrasts with the fact that Finite-State dimension [3] is not closed under base change (consequence of [2, 14]). For instance Finite-State dimension 1 coincides with normality [2, 14], and normality is not closed under base change. Thus we prove that any real number for which polynomial-time dimension is 1 in a certain base is absolutely normal.

Corollary 3.3 *For any $x \in [0,1)$ and $k \geq 2$, if $\dim_{\mathrm{p}}^{(k)}(\{X_k(x)\}) = 1$ then x is absolutely normal.*

Proof. It is simple to see that p-dimension 1 implies Finite State dimension 1, therefore if $\dim_{\mathrm{p}}^{(k)}(\{X_k(x)\}) = 1$ then $X_k(x)$ has Finite State dimension 1 and therefore x is normal in base k. By our main theorem $\dim_{\mathrm{p}}^{(l)}(\{X_l(x)\}) = 1$ holds for every $l \geq 2$ and x is normal in every base l. □

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