# GROUND STATES FOR A STATIONARY MEAN-FIELD MODEL FOR A NUCLEON

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ABSTRACT. In this paper we consider a variational problem related to a model for a nucleon interacting with the $\omega$ and $\sigma$ mesons in the atomic nucleus. The model is relativistic, and we study it in a nuclear physics nonrelativistic limit, which is of a very different nature than the nonrelativistic limit in the atomic physics. Ground states are shown to exist for a large class of values for the parameters of the problem, which are determined by the values of some physical constants.

## 1. INTRODUCTION

This article is concerned with the existence of minimizers for the energy functional

$$\mathcal{E}(\varphi) = \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx - \frac{a}{2}\int_{\mathbb{R}^3}|\varphi|^4\,dx \tag{1.1}$$

under the $L^2$-normalization constraint

$$\int_{\mathbb{R}^3}|\varphi|^2\,dx = 1. \tag{1.2}$$

More precisely, for a large class of values for the parameter $a$, we show the existence of solutions of the following minimization problem

$$I = \inf\left\{\mathcal{E}(\varphi)\,;\ \varphi\in X, \int_{\mathbb{R}^3}|\varphi|^2\,dx = 1\right\}\,, \tag{1.3}$$

where

$$X = \left\{\varphi\in L^2(\mathbb{R}^3,\mathbb{C}^2)\,;\ \int_{\mathbb{R}^3}\frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx < +\infty\right\}. \tag{1.4}$$

We remind that $\boldsymbol{\sigma}$ denotes the vector of Pauli matrices $(\sigma_1,\sigma_2,\sigma_3)$,

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The Euler-Lagrange equation of the energy functional $\mathcal{E}$ under the $L^2$-normalization constraint is given by the second order equation

$$-\boldsymbol{\sigma}\cdot\nabla\left(\frac{\boldsymbol{\sigma}\cdot\nabla\varphi}{(1-|\varphi|^2)_+}\right) + \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+^2}\varphi - a|\varphi|^2\varphi + b\varphi = 0\,, \tag{1.5}$$

where $b$ is the Lagrange multiplier associated with the $L^2$-constraint (1.2). Hence a solution of the minimization problem (1.3) is a solution of the equation (1.5).

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Moreover, Lemma 2.1 below proves that any $\varphi \in X$ satisfies $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$. So, a minimizer for (1.3) is actually a solution of

$$-\boldsymbol{\sigma} \cdot \nabla \left( \frac{\boldsymbol{\sigma} \cdot \nabla \varphi}{1-|\varphi|^2} \right) + \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi|^2}{(1-|\varphi|^2)^2} \varphi - a|\varphi|^2 \varphi + b\varphi = 0. \tag{1.6}$$

Solutions of (1.6) which are minimizers for $I$ are called ground states.

The equation (1.6) is a equivalent to the system

$$\begin{cases} i\boldsymbol{\sigma} \cdot \nabla \chi + |\chi|^2 \varphi - a|\varphi|^2 \varphi + b\varphi = 0, \\ -i\boldsymbol{\sigma} \cdot \nabla \varphi + \left(1-|\varphi|^2\right) \chi = 0 \,. \end{cases} \tag{1.7}$$

As we formally derived in a previous paper ([1]), this system is the nuclear physics nonrelativistic limit of the $\sigma$-$\omega$ relativistic mean-field model ([9, 10]) in the case of a single nucleon.

In [1], we proved the existence of square integrable solutions of (1.7) in the particular form

$$\begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix} = \begin{pmatrix} g(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ if(r) \begin{pmatrix} \cos\vartheta \\ \sin\vartheta e^{i\phi} \end{pmatrix} \end{pmatrix}, \tag{1.8}$$

where $f$ and $g$ are real valued radial functions. This ansatz corresponds to particles with minimal angular momentum, that is, $j = 1/2$ (for instance, see [8]). In this model, the equations for $f$ and $g$ read as follows:

$$\begin{cases} f' + \frac{2}{r} f = g(f^2 - ag^2 + b)\,, \\ \qquad g' = f(1-g^2)\,, \end{cases} \tag{1.9}$$

where we assumed $f(0) = 0$ in order to avoid solutions with singularities at the origin, and we showed that given $a, b > 0$ such that $a - 2b > 0$, there exists at least one nontrivial solution of (1.9) such that

$$(f(r), g(r)) \longrightarrow (0,0) \quad \text{as} \quad r \longrightarrow +\infty\,. \tag{1.10}$$

In this paper, we prove the existence of solutions of the above nuclear physics nonrelativistic limit of the $\sigma$-$\omega$ relativistic mean-field model without considering any particular ansatz for the nucleon's wave function.

Note that (1.6) is the Euler-Lagrange equation of the energy functional

$$\mathcal{F}(\varphi) = \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi|^2}{1-|\varphi|^2}\, dx - \frac{a}{2} \int_{\mathbb{R}^3} |\varphi|^4 \, dx \tag{1.11}$$

under the $L^2$ normalization constraint. In the Appendix, we prove that the energy functional $\mathcal{F}$ is not bounded from below. So, trying to find solutions of (1.6) which minimize the energy $\mathcal{F}$ is hopeless and the definition of ground states for (1.6) based on this functional is not clear.

In our previous work ([1]), we showed that for all the solutions of (1.9) which are square integrable, $g^2(r) < 1$ in $[0, +\infty)$. Hence, according to this result, we conjecture that a solution of (1.6) has to satisfy $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$. As we prove in the Appendix, this assumption is also justified when we consider the intermediate model

$$\varphi = \begin{pmatrix} u \\ 0 \end{pmatrix} \tag{1.12}$$

with $u : \mathbb{R}^3 \to \mathbb{R}$ and $a > b$. Moreover, in the physical literature finite nuclei are described via functions $\varphi$ such that, in the right units, $|\varphi|^2 \leq 1$ and $|\varphi|$ is rather flat near the center of the nucleus, and is equal to 0 outside it, see [5, 2].

Note that if $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$, then $\mathcal{F}(\varphi) = \mathcal{E}(\varphi)$, and the ground states of (1.6) can be defined without further specification as the minimizers of $\mathcal{E}$.

The main result of our paper is the following

**Theorem 1.1.** *If $I < 0$ there exists a minimizer of (1.3). Moreover, $I < 0$ if and only if $a > a_0$ where $a_0$ is a strictly positive constant. In particular, $10.96 \approx \frac{2}{S^2} < a_0 < 48.06$, where $S$ the best constant in the Sobolev embedding of $H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$.*

**Remark 1.** *The upper estimate for $a_0$ is obtained by using a particular test function and is probably not optimal .*

The proof of the above theorem is an application of the concentration-compactness principle ([3, 4]) with some new ingredients. The main new difficulty is due to the presence of the term $\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx$ in the energy functional. As we will see below, to rule out the dichotomy case in the concentration-compactness lemma we have to choose *ad-hoc* cut-off functions allowing us to deal with possible singularities of the integrand. This is also necessary in order to show the localization properties of $\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx$.

In the next section, we will establish a concentration-compactness lemma in $X$ and then apply it to prove our main result. The Appendix contains some auxiliary results about various properties of the model problem that we consider here.

## 2. Proof of Theorem 1.1

To prove this theorem, we are going to apply a concentration-compactness lemma that we state below. The reader may refer to [3] and [4] for more details on this kind of approach. The particular shape of the energy functional, where the kinetic energy term is multiplied by a function which could present singularities as $|\varphi|$ gets close to 1 creates some complications in the use of concentration-compactness, that we deal with by using very particular cut-off functions.

Let us introduce

$$I_\nu = \inf\left\{\mathcal{E}(\varphi)\ ;\ \varphi \in X, \int_{\mathbb{R}^3} |\varphi|^2\,dx = \nu\right\} \tag{2.1}$$

where $\nu > 0$ and $I_1 = I$, and we make a few preliminary observations.

**Lemma 2.1** ([6])**.** *Let $\varphi \in X$. Then, $\varphi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ and $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$.*

*Proof.* First, by a straightforward calculation, we obtain

$$\int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx = \int_{\mathbb{R}^3} |\boldsymbol{\sigma}\cdot\nabla\varphi|^2\,dx \leq \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx < +\infty.$$

Hence, $\varphi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$. Next, let $n \in \mathbb{C}^2$ such that $|n| = 1$. Note that for $\varphi \in X$, $\mathbb{1}_{\mathrm{Re}\{n\cdot\varphi\}\geq 1}(\boldsymbol{\sigma}\cdot\nabla\varphi) = 0$, a.e. in $\mathbb{R}^3$. Define the functions $f = (\mathrm{Re}\{n\cdot\varphi\} - 1)_+$

and $\psi = fn$. (Note that for 2 complex vectors $A, B \in \mathbb{C}^2$, $A \cdot B$ denotes the scalar product $\Sigma_{i=1}^2 \overline{A}_i B$, where $\overline{z}$ stands for the complex conjugate of any complex number $z$).

We have $f \in H^1(\mathbb{R}^3, \mathbb{R})$ and $\psi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$. Moreover, for $k = 1, 2, 3$,

$$\partial_k \psi = \partial_k f\, n \quad \text{ and } \quad \partial_k f = \mathrm{Re}\,\{n \cdot \partial_k \varphi\}\, \mathbb{1}_{\mathrm{Re}\{n\cdot\varphi\}\geq 1} = n \cdot \partial_k \psi\,.$$

Hence, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla f|^2\, dx = \int_{\mathbb{R}^3} |\nabla \psi|^2\, dx &= \int_{\mathbb{R}^3} \sum_{k=1}^3 \mathrm{Re}\,\{\mathrm{Re}\,\{n \cdot \partial_k \varphi\}\, n \cdot \partial_k \psi\}\; dx \\
&= \int_{\mathbb{R}^3} \sum_{k=1}^3 \mathrm{Re}\,\{n \cdot \partial_k \varphi\}\, \mathrm{Re}\,\{n \cdot \partial_k \psi\}\; dx = \int_{\mathbb{R}^3} \sum_{k=1}^3 \mathrm{Re}\,\{\partial_k f\, n \cdot \partial_k \varphi\}\; dx \\
&= \int_{\mathbb{R}^3} \mathrm{Re}\,\{\nabla \psi \cdot \nabla \varphi\}\; dx = \int_{\mathbb{R}^3} \mathrm{Re}\,\{(\boldsymbol{\sigma} \cdot \nabla \psi) \cdot (\boldsymbol{\sigma} \cdot \nabla \varphi)\}\; dx \\
&= \int_{\mathbb{R}^3} \mathrm{Re}\,\left\{(\boldsymbol{\sigma} \cdot \nabla \psi) \cdot \mathbb{1}_{\mathrm{Re}\{n\cdot\varphi\}\geq 1} (\boldsymbol{\sigma} \cdot \nabla \varphi)\right\}\; dx = 0
\end{aligned}$$

As a consequence, $f = 0$ a.e. in $\mathbb{R}^3$ that means $\mathrm{Re}\{n \cdot \varphi\} \leq 1$ a.e. for all $n \in \mathbb{C}^2$ such that $|n| = 1$. This clearly implies that $|\varphi| \leq 1$ a.e. in $\mathbb{R}^3$. □

In what follows, we say that a sequence $\{\varphi_n\}_n$ is $X$-bounded if there exists a positive constant $C$ independent of $n$ such that

$$\|\varphi_n\|_{L^2}^2 + \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi_n|^2}{(1 - |\varphi_n|^2)_+}\, dx \leq C\,. \tag{2.2}$$

**Lemma 2.2.** *Let $\{\varphi_n\}_n$ be a minimizing sequence of (2.1), then $\{\varphi_n\}_n$ is $X$-bounded, bounded in $H^1(\mathbb{R}^3)$ and $I_\nu > -\infty$.*

*Proof.* Indeed, since $\{\varphi_n\}_n$ is a minimizing sequence, there exists a constant $C$ such that

$$\begin{aligned}
C \geq \mathcal{E}(\varphi_n) \geq \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi_n|^2}{(1 - |\varphi_n|^2)_+}\, dx - \frac{a}{2}\nu &\geq \int_{\mathbb{R}^3} |\boldsymbol{\sigma} \cdot \nabla \varphi_n|^2\, dx - \frac{a}{2}\nu \\
= \int_{\mathbb{R}^3} |\nabla \varphi_n|^2\, dx - \frac{a}{2}\nu &\geq -\frac{a}{2}\nu.
\end{aligned}$$

As a conclusion, $\|\varphi_n\|_{H^1}$ is bounded independently of $n$ and $I_\nu$ is bounded from below. □

**Lemma 2.3.** *For all $\nu \in (0,1)$, $I_\nu \leq 0$. Moreover, the strict inequality $I < 0$ is equivalent to the strict concentration-compactness inequalities*

$$I < I_\nu + I_{1-\nu} \quad , \quad \forall \nu \in (0,1)\,. \tag{2.3}$$

*Proof.* Indeed, let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} |\varphi|^2 = \nu$ and $\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\, dx < +\infty$, and let $\varphi_\gamma(x) = \gamma^{-3/2} \varphi(\gamma^{-1} x)$ for $\gamma > 1$. Then

$$I_\nu \leq \mathcal{E}(\varphi_\gamma) = \frac{1}{\gamma^2} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi|^2}{\left(1 - \frac{1}{\gamma^3}|\varphi|^2\right)_+}\, dx - \frac{1}{\gamma^3} \frac{a}{2} \int_{\mathbb{R}^3} |\varphi|^4\, dx\,,$$

and letting $\gamma \to +\infty$, we prove $I_\nu \leq 0$.

By a scaling argument, we obtain

$$I_{\vartheta\nu} \le \inf\left\{\vartheta^{1/3}\int_{\mathbb{R}^3}\frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx - \frac{\vartheta\,a}{2}\int_{\mathbb{R}^3}|\varphi|^4\,dx|\varphi\in X,\int_{\mathbb{R}^3}|\varphi|^2\,dx=\nu\right\},$$

and, if $I_\nu < 0$, we may restrict the infimum $I_\nu$ to elements $\varphi$ satisfying

$$K(\varphi) = \int_{\mathbb{R}^3}\frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx \ge \delta > 0\,,$$

for some $\delta > 0$. Indeed, if there is a minimizing sequence $\{\varphi_n\}_n$ of $I_\nu$ such that $K(\varphi_n) \underset{n}{\to} 0$, then, by Sobolev embeddings, $\varphi_n \underset{n}{\to} 0$ in $L^p(\mathbb{R}^3)$ for $2 < p \le 6$ and $I_\nu \ge 0$. As a conclusion, if $I_\nu < 0$, then, for all $\vartheta > 1$ and for all $\nu > 0$,

$$I_{\vartheta\nu} < \vartheta\inf\left\{\mathcal{E}(\varphi)|\varphi\in X, K(\varphi)>0,\int_{\mathbb{R}^3}|\varphi|^2\,dx=\nu\right\} = \vartheta I_\nu\,. \tag{2.4}$$

Hence, a straightforward argument (see lemma II.1 of [3]) proves that (2.3) is equivalent to $I < 0$. □

In order to prove Theorem 1.1 we need to analyse the possible behaviour of minimizing sequences for $I$. This is done in the following lemma.

**Lemma 2.4.** *Let $\{\varphi_n\}_n$ be a $X$-bounded sequence such that $\int_{\mathbb{R}^3}|\varphi_n|^2\,dx = 1$ for all $n \ge 0$. Then there exists a subsequence that we still denote by $\{\varphi_n\}_n$ such that one of the following properties holds:*

(1) *Compactness up to a translation: there exists a sequence $\{y_n\}_n \subset \mathbb{R}^3$ such that, for every $\varepsilon > 0$, there exists $0 < R < \infty$ with*

$$\int_{B(y_n,R)}|\varphi_n|^2\,dx \ge 1-\varepsilon;$$

(2) *Vanishing: for all $0 < R < \infty$*

$$\sup_{y\in\mathbb{R}^3}\int_{B(y,R)}|\varphi_n|^2\,dx \underset{n}{\to} 0;$$

(3) *Dichotomy: there exist $\alpha\in(0,1)$ and $n_0 \ge 0$ such that there exist two $X$-bounded sequences, $\{\varphi_1^n\}_{n\ge n_0}$ and $\{\varphi_2^n\}_{n\ge n_0}$, satisfying the following properties:*

$$\|\varphi_n - (\varphi_1^n+\varphi_2^n)\|_{L^p} \underset{n}{\to} 0,\ \textit{for } 2\le p<6, \tag{2.5}$$

*and*

$$\int_{\mathbb{R}^3}|\varphi_1^n|^2\,dx \underset{n}{\to} \alpha \ \textit{and}\ \int_{\mathbb{R}^3}|\varphi_2^n|^2\,dx \underset{n}{\to} 1-\alpha, \tag{2.6}$$

$$\mathrm{dist}(\mathrm{supp}\,\varphi_1^n,\mathrm{supp}\,\varphi_2^n) \underset{n}{\to} +\infty. \tag{2.7}$$

*Moreover, in this case we have that*

$$\liminf_{n\to+\infty}\mathcal{E}(\varphi_n) - \mathcal{E}(\varphi_1^n) - \mathcal{E}(\varphi_2^n) \ge 0\,, \tag{2.8}$$

*which implies $I \ge I_\alpha + I_{1-\alpha}$.*

*Proof of Lemma 2.4.* Let $\{\varphi_n\}_n$ be a $X$-bounded sequence such that $\int_{\mathbb{R}^3}|\varphi_n|^2\,dx = \nu$ for all $n \geq 0$. We remind that $X$-bounded means that there exists $C > 0$ such that

$$\|\varphi_n\|^2_{L^2} + \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx \leq C.$$

Moreover, thanks to Lemma 2.1, if $\{\varphi_n\}_n$ is a $X$-bounded sequence then $\{\varphi_n\}_n$ is bounded in $L^\infty$ (by the constant 1) and in $H^1(\mathbb{R}^3)$. Then, along the lines of [3], we introduce the so-called Lévy concentration functions

$$Q_n(R) = \sup_{y\in\mathbb{R}^3} \int_{|x-y|<R} |\varphi_n|^2\,dx, \tag{2.9}$$

$$K_n(R) = \sup_{y\in\mathbb{R}^3} \int_{|x-y|<R} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx \tag{2.10}$$

for $R > 0$. Note that $Q_n$ and $K_n$ are continuous non-decreasing functions on $[0,+\infty)$, such that for all $n \geq 0$ and for all $R > 0$

$$Q_n(R) + K_n(R) \leq C$$

since $\{\varphi_n\}_n$ is $X$-bounded. Then, up to a subsequence, we have for all $R > 0$

$$Q_n(R) \underset{n}{\to} Q(R), \tag{2.11}$$

$$K_n(R) \underset{n}{\to} K(R), \tag{2.12}$$

where $Q$ and $K$ are nonnegative, non-decreasing functions. Clearly, we have that

$$\alpha = \lim_{R\to+\infty} Q(R) \in [0,1],$$

and we denote $l = \lim_{R\to+\infty} K(R)$.

If $\alpha = 0$, then the situation (2) of the lemma arises as a direct consequence of Definition (2.9). If $\alpha = 1$, then (1) follows, see [3] for details. Assume that $\alpha \in (0,1)$, we have to show that (3) holds.

First of all, consider $\varepsilon > 0$, small, and $R_\varepsilon > 0$ such that $Q(R_\varepsilon) = \alpha - \varepsilon$ and $K(R_\varepsilon) \leq l - \varepsilon$. Then, for $n$ large enough,

$$Q_n(R_\varepsilon) - Q(R_\varepsilon) < 1/n, \quad K_n(R_\varepsilon) - K(R_\varepsilon) < 1/n\,,$$

and by definition of the Lévy functions $Q_n$, extracting subsequences if necessary, there exists $y_n \in \mathbb{R}^3$ such that

$$\left|\int_{|x-y_n|<R_\varepsilon} |\varphi_n|^2\,dx - Q_n(R_\varepsilon)\right| \leq \frac{1}{n},$$

$$\left|\int_{|x-y_n|<R_\varepsilon} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx - K_n(R_\varepsilon)\right| \leq \frac{1}{n}\,.$$

Next define $R_n > R_\varepsilon$ such that

$$\int_{R_\varepsilon<|x-y_n|<R_n} |\varphi_n|^2\,dx = \frac{3}{n} + \varepsilon\,.$$

Necessarily, $R_n \to +\infty$ as $n \to +\infty$. Indeed, if $R_n \leq M$ for some $M > 0$, then $Q(M) > \alpha$, which is impossible. We then deduce that for $n$ large enough,

$$\int_{\frac{R_n}{8}\leq|x-y_n|\leq R_n} |\varphi_n|^2\,dx \leq \frac{3}{n} + \varepsilon$$

Let $\xi$, $\zeta$ be cut-off functions: $\xi, \zeta \in \mathcal{D}(\mathbb{R}^3)$ such that

$$\xi(x) = \begin{cases} 1 & |x| \le 1 \\ 1 - \exp\left(1 - \dfrac{1}{1 - \exp\left(1 - \frac{1}{2-|x|}\right)}\right) & 1 < |x| < 2 \\ 0 & |x| \ge 2 \end{cases}$$

$$\zeta(x) = \begin{cases} 0 & |x| \le 1 \\ \exp\left(1 - \dfrac{1}{1 - \exp\left(1 - \frac{1}{2-|x|}\right)}\right) & 1 < |x| < 2 \;, \\ 1 & |x| \ge 2 \end{cases}$$

and let $\xi_\mu$, $\zeta_\mu$ denote $\xi\left(\frac{\cdot}{\mu}\right)$, $\zeta\left(\frac{\cdot}{\mu}\right)$. We define

$$\varphi_1^n(\cdot) = \xi_{\frac{R_n}{8}}(\cdot - y_n)\varphi_n(\cdot) = \xi_{\frac{R_n}{8}, y_n}(\cdot)\varphi_n(\cdot) \tag{2.13}$$

$$\varphi_2^n(\cdot) = \zeta_{\frac{R_n}{2}}(\cdot - y_n)\varphi_n(\cdot) = \zeta_{\frac{R_n}{2}, y_n}(\cdot)\varphi_n(\cdot) \tag{2.14}$$

with $R_n \to +\infty$. (2.7) follows easily from these definitions. Furthermore, (2.5) and (2.6) are obtained in the following way:

$$\begin{aligned} \lim_{n\to+\infty} \int_{\mathbb{R}^3} |\varphi_n - (\varphi_1^n + \varphi_2^n)|^2\, dx &= \lim_{n\to+\infty} \int_{\frac{R_n}{8} \le |x-y_n| \le R_n} |(1 - \xi_{\frac{R_n}{8}} - \zeta_{\frac{R_n}{2}})\varphi_n|^2\, dx \\ &\le \lim_{n\to+\infty} \int_{\frac{R_n}{8} \le |x-y_n| \le R_n} |\varphi_n|^2\, dx \le \varepsilon, \end{aligned}$$

Now by taking a sequence of $\varepsilon$ tending to 0, and by taking a diagonal sequence of the functions $\varphi_n$, and calling it by the same name, we find

$$\int_{\frac{R_n}{8} \le |x-y_n| \le R_n} |\varphi_n|^2\, dx \underset{n}{\to} 0,$$

and, since $\{\varphi_1^n\}_n$ and $\{\varphi_2^n\}_n$ are bounded in $H^1(\mathbb{R}^3)$, we also obtain

$$\lim_{n\to+\infty} \|\varphi_n - (\varphi_1^n + \varphi_2^n)\|_{L^p} \underset{n}{\to} 0,$$

for $2 \le p < 6$. Next, we have to prove that $\{\varphi_1^n\}_{n\ge n_0}$ and $\{\varphi_2^n\}_{n\ge n_0}$ are $X$-bounded. To this purpose, we show that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\, dx - \int_{\mathbb{R}^3} \frac{\xi^2_{\frac{R_n}{8}, y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\, dx = 0 \tag{2.15}$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_2^n|^2}{(1-|\varphi_2^n|^2)_+}\, dx - \int_{\mathbb{R}^3} \frac{\zeta^2_{\frac{R_n}{2}, y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_2^n|^2)_+}\, dx = 0 \tag{2.16}$$

Indeed, if (2.15) and (2.16) hold, we obtain that for all $\varepsilon > 0$, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\,dx &\leq \int_{\mathbb{R}^3} \frac{\xi^2_{\frac{R_n}{8},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx + o(1)_{n\to+\infty}\\ &\leq \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx + o(1)_{n\to+\infty} \leq C + o(1)_{n\to+\infty},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_2^n|^2}{(1-|\varphi_2^n|^2)_+}\,dx &\leq \int_{\mathbb{R}^3} \frac{\zeta^2_{\frac{R_n}{2},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_2^n|^2)_+}\,dx + o(1)_{n\to+\infty}\\ &\leq \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx + o(1)_{n\to+\infty} \leq C + o(1)_{n\to+\infty}.\end{aligned}$$

To prove (2.15) we proceed as follows. We remark that

$$\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\,dx - \int_{\mathbb{R}^3} \frac{\xi^2_{\frac{R_n}{8},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx = A_n + B_n\,,$$

where

$$\begin{aligned}A_n := \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot(\nabla\xi_{\frac{R_n}{8},y_n})\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx &= \int_{\frac{R_n}{8}\leq|x-y_n|\leq\frac{R_n}{4}} \frac{|\boldsymbol{\sigma}\cdot(\nabla\xi_{\frac{R_n}{8},y_n})\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx\\ &\leq \int_{\frac{R_n}{8}\leq|x-y_n|\leq\frac{R_n}{4}} \frac{|\boldsymbol{\sigma}\cdot(\nabla\xi_{\frac{R_n}{8},y_n})\varphi_n|^2}{1-\xi^2_{\frac{R_n}{8},y_n}}\,dx := C_n\,,\end{aligned}$$

and

$$|B_n| \leq 2\,(C_n)^{\frac12}\left(\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx\right)^{\frac12}.$$

Let us now prove that $C_n$ tends to 0 as $n$ goes to $+\infty$. Using spherical coordinates, we obtain

$$\begin{aligned}C_n &\leq \int_{\frac{R_n}{8}}^{\frac{R_n}{4}}\int_0^{\pi}\int_0^{2\pi} \frac{|(\sigma\cdot\boldsymbol{e}_r)\,\varphi_n(s,\theta,\phi)|^2\left(\xi'_{\frac{R_n}{8}}(s)\right)^2}{1-\xi^2_{\frac{R_n}{8}}(s)}\,s^2\,\sin\theta\,ds\,d\theta\,d\phi\\ &\leq \frac{64}{R_n^2}\int_{\frac{R_n}{8}}^{\frac{R_n}{4}}\int_0^{\pi}\int_0^{2\pi} \frac{|\varphi_n(s,\theta,\phi)|^2\left(\xi'\left(\frac{8}{R_n}s\right)\right)^2}{1-\xi^2\left(\frac{8}{R_n}s\right)}\,s^2\,\sin\theta\,ds\,d\theta\,d\phi\\ &\leq \frac{64}{R_n^2}\max_{1\leq r\leq 2}\frac{(\xi'(r))^2}{1-\xi^2(r)}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}|\varphi_n(s,\theta,\phi)|^2\,s^2\,\sin\theta\,ds\,d\theta\,d\phi = O\left(\frac{1}{R_n^2}\right)\end{aligned}$$

since $\max\limits_{1\leq r\leq 2}\frac{(\xi'(r))^2}{1-\xi^2(r)} \leq C$. Indeed, since $\xi^2(r) = 1$ if and only if $r = 1$, $\frac{(\xi'(r))^2}{1-\xi^2(r)}$ is a continuous function on $(1,2)$. Moreover, by a straightforward calculation, we obtain $\lim\limits_{r\to1^+}\frac{(\xi'(r))^2}{1-\xi^2(r)} = 0 = \lim\limits_{r\to2^-}\frac{(\xi'(r))^2}{1-\xi^2(r)}$. Hence, we can conclude, that $\frac{(\xi'(r))^2}{1-\xi^2(r)}$ is bounded in $[1,2]$. As a conclusion, since $R_n \to +\infty$, we obtain

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\,dx - \int_{\mathbb{R}^3} \frac{\xi^2_{\frac{R_n}{8},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx = 0.$$

With the same argument, we prove (2.16).

Finally, it remains to show that

$$\liminf_{n\to+\infty} \mathcal{E}(\varphi_n) - \mathcal{E}(\varphi_1^n) - \mathcal{E}(\varphi_2^n) \geq 0.$$

First of all, using the definitions (2.13) and (2.14), we obtain

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx \geq \lim_{n\to+\infty} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2-|\varphi_2^n|^2)_+}\,dx.$$

Next, we remark that

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2-|\varphi_2^n|^2)_+}\,dx - \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\,dx - \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_2^n|^2}{(1-|\varphi_2^n|^2)_+}\,dx& \\
= \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2-|\varphi_2^n|^2)_+}\,dx - \int_{\mathbb{R}^3} \frac{\xi^2_{\frac{R_n}{8},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx& \\
- \int_{\mathbb{R}^3} \frac{\zeta^2_{\frac{R_n}{2},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_2^n|^2)_+}\,dx + o(1)_{n\to\infty}& \\
= \int_{\mathbb{R}^3} \frac{\left(1-\xi^2_{\frac{R_n}{8},y_n}-\zeta^2_{\frac{R_n}{2},y_n}\right)|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2-|\varphi_2^n|^2)_+}\,dx + o(1)_{n\to\infty}& \\
\geq o(1)_{n\to\infty}.&
\end{aligned}$$

As a conclusion,

$$\begin{aligned}
\lim_{n\to+\infty} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx \geq \lim_{n\to+\infty} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\,dx& \\
+ \lim_{n\to+\infty} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_2^n|^2}{(1-|\varphi_2^n|^2)_+}\,dx,&
\end{aligned}$$

and, using (2.5) and the localization properties of $\varphi_1^n$ and $\varphi_2^n$, we have

$$I = \lim_{n\to+\infty} \mathcal{E}(\varphi_n) \geq \liminf_{n\to+\infty} \mathcal{E}(\varphi_1^n) + \liminf_{n\to+\infty} \mathcal{E}(\varphi_2^n) \geq I_\alpha + I_{1-\alpha}\,.$$

□

*Proof of Theorem 1.1.* Assume that $I < 0$. By Lemma 2.2, any minimizing sequence $\{\varphi_n\}_n$ is $X$-bounded, and then we can use Lemma 2.4 to it. It is easy to rule out vanishing and dichotomy whenever $I < 0$.

Vanishing cannot occur. Indeed, If vanishing occurs, then, up to a subsequence, $\forall R < +\infty$ we have

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B(y,R)} |\varphi_n|^2 = 0. \tag{2.17}$$

This implies that $\varphi_n$ converges strongly in $L^p(\mathbb{R}^3)$ for $2 < p < 6$ and, as a consequence, $I \geq 0$. Clearly, this contradicts $I < 0$.

Moreover, if dichotomy occurs, we have

$$I = \lim_{n\to+\infty} \mathcal{E}(\varphi_n) \geq \liminf_{n\to+\infty} \mathcal{E}(\varphi_1^n) + \liminf_{n\to+\infty} \mathcal{E}(\varphi_2^n) \geq I_\alpha + I_{1-\alpha}$$

which contradicts Lemma 2.3, since $I < 0$.

Hence, for $n$ large enough, there exists $\{y_n\}_n \in \mathbb{R}^3$ such that $\forall\varepsilon > 0$, $\exists R < +\infty$,

$$\int_{B(y_n,R)} |\varphi_n|^2 \geq 1 - \varepsilon.$$

We denote by $\tilde{\varphi}_n(\cdot) = \varphi_n(\cdot + y_n)$. Since $\{\tilde{\varphi}_n\}_n$ is bounded in $H^1$, $\{\tilde{\varphi}_n\}_n$ converges weakly in $H^1$, almost everywhere on $\mathbb{R}^3$ and in $L^p_{loc}$ for $2 \le p < 6$ to some $\tilde{\varphi}$. In particular, as a consequence of weak convergence in $H^1$, $\boldsymbol{\sigma} \cdot \nabla \tilde{\varphi}_n$ converges weakly to $\boldsymbol{\sigma} \cdot \nabla \tilde{\varphi}$ in $L^2$. Moreover, thanks to the concentration-compactness argument, $\{\tilde{\varphi}_n\}_n$ converges strongly in $L^2$ and in $L^p$ for $2 \le p < 6$.

**Lemma 2.5.** *Let $\{f_n\}_n$ and $\{g_n\}_n$ be two sequences of functions such that $f_n : \mathbb{R}^3 \to \mathbb{R}_+$, $g_n : \mathbb{R}^3 \to \mathbb{C}^2$ , $f_n$ converges to $f$ a.e., $g_n$ converges weakly to $g$ in $L^2$ and there exists a constant $C$, that does not depend on $n$, such that $\int_{\mathbb{R}^3} f_n |g_n|^2\, dx \le C$. Then*

$$\int_{\mathbb{R}^3} f|g|^2\, dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^3} f_n|g_n|^2\, dx.$$

*Proof.* Given a function $h : \mathbb{R}^3 \to \mathbb{R}_+$, let $T_k$ be the function defined by

$$T_k(h)(x) = \begin{cases} h(x) & \text{if } h(x) \le k \\ k & \text{if } h(x) > k \end{cases}$$

for all $k \in [0, \infty)$. Hence, the following properties are satisfied for all $k \in [0, \infty)$:

$$T_k(f_n) \underset{n}{\to} T_k(f) \quad \text{a.e. in } \mathbb{R}^3, \tag{2.18}$$

$$T_k(f_n)|g|^2 \underset{n}{\to} T_k(f)|g|^2 \quad \text{in } L^1, \tag{2.19}$$

$$T_k(f_n)g \underset{n}{\rightharpoonup} T_k(f)g \quad \text{in } L^2, \tag{2.20}$$

$$\|T_k(f_n)g\|_{L^2} \underset{n}{\to} \|T_k(f)g\|_{L^2}, \tag{2.21}$$

where to obtain (2.19) and (2.21), we use Lebesgue's dominated convergence theorem. Moreover, as a consequence of (2.20) and (2.21), we have

$$T_k(f_n)g \underset{n}{\to} T_k(f)g \quad \text{in } L^2. \tag{2.22}$$

Next, we have

$$\begin{aligned} 0 \le & \liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g_n - g|^2\, dx = \liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g_n|^2\, dx \\ & + \liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g|^2\, dx - \liminf_{n\to+\infty} \left( \int_{\mathbb{R}^3} T_k(f_n)\, \overline{g}_n \cdot g\, dx + \int_{\mathbb{R}^3} T_k(f_n)\, g_n \cdot \overline{g}\, dx \right) \\ = & \liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g_n|^2\, dx + \int_{\mathbb{R}^3} T_k(f)|g|^2\, dx - 2 \int_{\mathbb{R}^3} T_k(f)|g|^2\, dx \end{aligned}$$

thanks to (2.19), (2.22) and the fact that $g_n$ converges weakly to $g$ in $L^2$. As a consequence,

$$\int_{\mathbb{R}^3} T_k(f)|g|^2\, dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g_n|^2\, dx \tag{2.23}$$

Since

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g_n|^2\, dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^3} f_n|g_n|^2\, dx \le C,$$

we can pass to the limit for $k$ that goes to $+\infty$ in (2.23) and we obtain

$$\int_{\mathbb{R}^3} f|g|^2\, dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^3} f_n|g_n|^2\, dx.$$

□

By applying Lemma (2.5) to $f_n = \frac{1}{(1-|\tilde{\varphi}_n|^2)}_+$ and $g_n = |\boldsymbol{\sigma}\cdot\nabla\tilde{\varphi}_n|$, we obtain

$$\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\tilde{\varphi}|^2}{(1-|\tilde{\varphi}|^2)_+}\,dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\tilde{\varphi}_n|^2}{(1-|\tilde{\varphi}_n|^2)_+}\,dx.$$

Hence, $\tilde{\varphi}\in X$, $\int_{\mathbb{R}^3}|\tilde{\varphi}|^2\,dx = 1$, and

$$\mathcal{E}(\tilde{\varphi}) \le \liminf_{n\to+\infty} \mathcal{E}(\tilde{\varphi}_n) \le \mathcal{E}(\tilde{\varphi}).$$

As a conclusion, the minimum of $I$ is achieved by $\tilde{\varphi}$.

Finally, it remains to prove that there exists $a_0 > 0$ such that for all $a > a_0$ we have $I < 0$.

It is clear that $I < 0$ for $a$ large enough. Since $I$ is non-increasing with respect to $a$, we may denote by $a_0$ the least positive constant such that $I < 0$ for $a > a_0$. We have to prove that $a_0 > 0$ or in other words $I = 0$ for $a$ small enough. Using Sobolev and Hölder inequalities, we find, for $\varphi\in X$ such that $\int_{\mathbb{R}^3}|\varphi|^2\,dx = 1$,

$$\mathcal{E}(\varphi) \ge \frac{1}{S^2}\left(\int_{\mathbb{R}^3}|\varphi|^6\,dx\right)^{1/3} - \frac{a}{2}\left(\int_{\mathbb{R}^3}|\varphi|^6\,dx\right)^{1/3}.$$

Hence, if $a \le \frac{2}{S^2}$, $I = 0$. This implies $a_0 > \frac{2}{S^2}$. According to [7] the best constant for the Sobolev inequality

$$\|u\|_{L^q(\mathbb{R}^m)} \le C\|\nabla u\|_{L^p(\mathbb{R}^m)}$$

with $1 < p < m$ and $q = \frac{mp}{(m-p)}$ is given by

$$C = \pi^{-1/2}m^{-1/p}\left(\frac{p-1}{m-p}\right)^{1-1/p}\left(\frac{\Gamma(1+m/2)\Gamma(m)}{\Gamma(m/p)\Gamma(1+m-m/p)}\right)^{1/m}.$$

In particular,

$$S = \frac{1}{\sqrt{3\pi}}\left(\frac{4}{\sqrt{\pi}}\right)^{1/3},$$

and

$$\frac{2}{S^2} = \frac{3\pi^{4/3}}{2^{1/3}} \approx 10.96.$$

To obtain an upper estimate for $a_0$, we consider the following test function

$$\bar{\varphi}(x) = \begin{pmatrix} \bar{f}_R(|x|) \\ 0 \end{pmatrix}$$

where $\bar{f}_R(|x|) = \bar{f}\left(\frac{|x|}{R}\right)$,

$$\bar{f}(|x|) = \begin{cases} \cos(|x|) & |x| \le \dfrac{\pi}{2} \\ 0 & |x| > \dfrac{\pi}{2} \end{cases}$$

and $R\in(0,1)$ is such that $\int|\bar{f}_R|^2\,dx = 1$. This implies

$$R = \left(\frac{2}{\pi}\right)^{2/3}\left(\frac{3}{\pi^2-6}\right)^{1/3}.$$

Next, we denote by $\bar{a}$ the positive constant such that $\mathcal{E}(\varphi) = 0$. By definition,

$$\bar{a} = \frac{2\int_{\mathbb{R}^3}\frac{|\boldsymbol{\sigma}\cdot\nabla\bar{\varphi}|^2}{(1-|\bar{\varphi}|^2)_+}\,dx}{\int|\bar{\varphi}|^4} = \frac{2\int_{\mathbb{R}^3}\frac{|\nabla\bar{f}_R|^2}{1-|f_R|^2}\,dx}{\int|\bar{f}_R|^4},$$

and, by a straightforward calculation, we obtain

$$\int_{\mathbb{R}^3} \frac{|\nabla \bar{f}_R|^2}{1-|\bar{f}_R|^2}\,dx = \frac{\pi^4}{6}R = \frac{\pi^{10/3}}{3^{2/3}(2(\pi^2-6))^{1/3}},$$

$$\int |\bar{f}_R|^4 = \frac{\pi^2(2\pi^2-15)}{32}R^3 = \frac{3(2\pi^2-15)}{8(\pi^2-6)}.$$

As a consequence,

$$\bar{a} = \frac{8\pi^{10/3}\left(\frac{2}{3}(\pi^2-6)\right)^{2/3}}{3(2\pi^2-15)} \approx 48.06$$

Since the energy functional $\mathcal{E}$ is decreasing in $a$, if $a > \bar{a}$ then $I \leq \mathcal{E}(\bar{\varphi}) < 0$. As a conclusion, $a_0 \leq \bar{a} + \varepsilon$ for all $\varepsilon > 0$.

## Appendix A

**A.1.** We begin this section by proving that if $(\varphi, \chi)$ a solution of (1.7) with $\varphi \in H^1(\mathbb{R}^3)$ of the form (1.12), then $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$. As we saw before, $\varphi$ is a solution of

$$-\boldsymbol{\sigma}\cdot\nabla\left(\frac{\boldsymbol{\sigma}\cdot\nabla\varphi}{1-|\varphi|^2}\right) + \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)^2}\varphi - a|\varphi|^2\varphi + b\varphi = 0, \tag{A.1}$$

or equivalently,

$$\frac{\Delta\varphi}{|\varphi|^2-1} - \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(|\varphi|^2-1)^2}\varphi - a|\varphi|^2\varphi + b\varphi = 0\,,$$

or still,

$$\Delta\varphi - \frac{|\nabla\varphi|^2}{|\varphi|^2-1}\varphi - (a|\varphi|^2\varphi - b\varphi)(|\varphi|^2-1) = 0\,,$$

because for functions $\varphi$ of the form (1.12),

$$|\sigma\cdot\nabla\varphi|^2 = |\nabla\varphi|^2 \quad \text{and} \quad \sigma\cdot(\nabla\varphi\wedge\nabla\varphi) = 0 \quad \text{a. e.}$$

For any $K > 1$, we define the truncation function $T_K(s)$ by $T_K(s) = s$ if $1 < s < K$, and $T_K(s) = 0$ otherwise. Multiplying the above equation by $\varphi\, T_K(|\varphi|^2) \in L^2(\mathbb{R}^3)$, we obtain

$$-\int_{\mathbb{R}^3} |\nabla\varphi|^2\, T_K(|\varphi|^2) - \int_{\mathbb{R}^3} (\nabla\varphi\cdot\varphi)\,\nabla T_K(|\varphi|^2) - \int_{\mathbb{R}^3} \frac{|\nabla\varphi|^2}{|\varphi|^2-1}|\varphi|^2\, T_K(|\varphi|^2)$$
$$- \int_{\mathbb{R}^3} (a|\varphi|^2-b)(|\varphi|^2-1)|\varphi|^2\, T_K(|\varphi|^2) = 0. \tag{A.2}$$

Moreover, for all $K > 1$,

$$\nabla T_K(|\varphi|^2) = \begin{cases} 2\varphi\cdot\nabla\varphi & 1 < |\varphi|^2 < K \\ 0 & |\varphi|^2 \leq 1 \ \text{ or } \ |\varphi|^2 \geq K \end{cases}.$$

Therefore, if $a - b > 0$ the l.h.s of (A.2) is negative and this implies that either $|\varphi|^2 \leq 1$ or $|\varphi|^2 \geq K$ a.e. As a conclusion, taking the limit $K \to +\infty$, if $a - b > 0$ then any solution $\varphi$ of (A.1) of the form (1.12) satisfies $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$, and in the equation (A.1) we can replace the term $(1-|\varphi|^2)$ by $(1-|\varphi|^2)_+$ without changing its solution set. The same happens for solutions of the form (1.8).

**A.2.** Let us next prove that the functional $\mathcal{F}$, defined by (1.11), is not bounded from below. Consider the function $\xi$ introduced in the proof of Lemma 2.4. Let us denote $A := \int_{\mathbb{R}^3} |\xi(x)|^2\, dx$.

Then, let us define the radially symmetric function

$$f(r) = \begin{cases} e^{(r-\sqrt{\ln 2})^2}, & 0 \le r < \sqrt{\ln 2}\,, \\ \bar{\xi}(r+1-\sqrt{\ln 2}), & r \ge \sqrt{\ln 2}\,, \end{cases}$$

where $\bar{\xi}(|x|) = \xi(x)$ for all $x$, and take $a := \int_{\mathbb{R}^3} f(|x|)^2\, dx$. Note that $\mathrm{supp}(f) \subset [0, 1+\sqrt{\ln 2}]$ and $\max\limits_{0\le r\le 1+\sqrt{\ln 2}} \frac{(f'(r))^2}{1-f^2(r)} \le C$, for some constant $C > 0$.

Next, for all integers $n > 0$, define the rescaled functions $\xi_n(x) := n^{3/2}\xi(nx)$. This change of variables leaves invariant the $L^2(\mathbb{R}^3)$ norm. Then for $n$ large, consider the function

$$g_n(x) := \max_{\mathbb{R}^3}\{\xi_n, f\}\,.$$

Note that the measure of the set $\{x \in \mathbb{R}^3 \ ;\ g_n = \xi_n\}$ tends to 0 as $n$ goes to $+\infty$. This function satisfies $\int_{\mathbb{R}^3} |g_n(x)|^2\, dx = A + a + o(1)$, as $n$ goes to $+\infty$. In order to normalize it in the $L^2$ norm, let us finally define the rescaled function $g_n^R(x) := g_n\left(\frac{x}{R}\right)$, $R > 0$ and choose $R_n$ such that $\int_{\mathbb{R}^3} |g_n^{R_n}(x)|^2\, dx = 1$. As $n$ goes to $+\infty$, $R_n \to \bar{R} := (A+a)^{-1/3} > 0$. We compute now the energy $\mathcal{F}$ of the vector function $\varphi_n^{R_n}$ defined by

$$\varphi_n^{R_n}(x) = \begin{pmatrix} g_n^{R_n}(x) \\ 0 \end{pmatrix}.$$

We find

$$\begin{aligned}
\mathcal{F}(\varphi_n^{R_n}) = &\int_{\xi_n^{R_n} \ge f^{R_n}} \frac{\left(\left(\xi_n^{R_n}\right)'(r)\right)^2}{1-(\xi_n^{R_n}(r))^2}\, dx - \frac{a\, n^3\, R_n^3}{2} \int_{\xi_n^{R_n} \ge f^{R_n}} |\xi|^4\, dx \\
&+ R_n \int_{\xi_n^{R_n} \le f^{R_n}} \frac{(f'(r))^2}{1-f^2(r)}\, dx - \frac{a\, R_n^3}{2} \int_{\xi_n^{R_n} \le f^{R_n}} f(x)^4\, dx \\
\le -\frac{a\, n^3\, R_n^3}{2} \int_{\mathbb{R}^3} |\xi|^4\, dx &+ R_n \int_{\mathbb{R}^3} \frac{(f'(r))^2}{1-f^2(r)}\, dx - \frac{a\, R_n^3}{2} \int_{\mathbb{R}^3} f(x)^4\, dx + o(n^3)\,,
\end{aligned}$$

because whenever $\xi_n^{R_n} \ge f^{R_n}$, $\left(\xi_n^{R_n}\right)^2 > 1$ and because the sequence $\{R_n\}_n$ is bounded. This clearly shows that $\mathcal{F}$ is unbounded from below.

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