ON THE STOCHASTIC STRICHARTZ ESTIMATES AND THE STOCHASTIC NONLINEAR SCHRÖDINGER EQUATION ON A COMPACT RIEMANNIAN MANIFOLD

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Abstract. We prove the existence and the uniqueness of a solution to the stochastic NSLEs on a two-dimensional compact riemannian manifold. Thus we generalize (and improve) a recent work by Burq et all [11] and a series of papers by de Bouard and Debussche, see e.g. [16, 17] who have examined similar questions in the case of the flat euclidean space.
We prove the existence and the uniqueness of a local maximal solution to stochastic nonlinear Schrödinger equations with multiplicative noise on a compact d-dimensional riemannian manifold. Under more regularity on the noise, we prove that the solution is global when the nonlinearity is of defocusing or of focusing type, $d = 2$ and the initial data belongs to the finite energy space. Our proof is based on improved stochastic Strichartz inequalities.

1. Introduction

The aim is this paper is twofold. The first one is to generalise Theorem 2 from Burq et all [12] on the global existence of nonlinear Schrödinger equation (NLS) to a stochastic setting. The second one is to prove a general version of a Strichartz type inequality for stochastic convolutions. This inequality is the main technical improvement of our results as compared to papers by de Bouard and Debussche [16] and [17]. In the deterministic case our existence result is comparable with [12, Theorem 2] and our stochastic Strichartz inequality is comparable with [12, Corollary 2.10]. Some versions of the stochastic Strichartz inequalities were implicitly formulated in [16], see Lemma 3.1 and Corollary 3.2 or [17]. The proof in [16] and [17] is based on the dispersive estimates for the free Schrödinger group. However, see [12, Remark 2.6], such estimates are not valid in the case of a compact riemannian manifold; hence we had to rely on different methods. Thus, not only the proof but also the result differs from the corresponding result from [17] but for stochastic integrands our result is stronger than the previous ones. The local existence of the solution to the stochastic NLS equation can be obtained with some non linear diffusion coefficient. Another extension involves both the regularity of the noise and the form of the diffusion coefficient which ensure that blow-up does not occur in finite time.
As in the above cited papers by de Bouard and Debussche, the global existence result which is a consequence of the conservation of the $L^2(M)$ norm and the control of a certain Lyapounov function, requires our problem to be in the Stratonovich form. To provide a uniform treatment of equations in both the Itô and the Stratonovich forms, we assume that all vector spaces are over the field $\mathbb{R}$ of real numbers. In particular, the space $L^2(M, \mathbb{C})$ is considered as a real vector space of all (equivalence classes) from M to $\mathbb{R}^2$. As a consequence, by a linear/bilinear map we understand a linear/bilinear map over the field $\mathbb{R}$. Similarly, we speak only about $\mathbb{R}$-differentiability. In general, the spaces of all $\mathbb{R}$-linear, resp. bilinear,

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bounded maps from E, resp. $E \times E$, to X, where E and X are two real Banach spaces, will be denoted by $\mathcal{L}(E, X)$, resp. $\mathbb{L}_2(E, X)$.

Let us briefly describe the main results obtained in literature preceding our article. In [16], see Theorem 2.1, de Bouard and Debussche proved the existence of a continuous $L^2(\mathbb{R}^d)$-valued global solution for the NSLEs when the initial data was an $L^2(\mathbb{R}^d)$-valued random variable. In the subsequent paper [17], the same authours proved the existence of a continuous $H^{1,2}(\mathbb{R}^d)$-valued local and global solution (depending on the nonlinearity) when the initial data was an $H^{1,2}(\mathbb{R}^d)$-valued random variable.

Now let us briefly describe the content of the current article. In section 2 we study properties of the Nemytski operators. In particular we generalize the results from [6] which we proved various properties of the Nemytski operators in the Sobolev-Slobodetski spaces $W^{\theta,q}(D)$, where $D \subset \mathbb{R}^d$, $\theta > \frac{d}{q}$, to the spaces $W^{\theta,q}(D) \cap L^\infty(D)$, without any restriction on θ. This is quite important since later on we work with spaces $W^{\theta,2}(D)$, where $d = 2$ and $\theta \leq q$. Section 3 is devoted to the study of the stochastic Strichartz inequality. We recall the homogenous and inhomogeneous Strichartz estimates from [12]. Then we prove our main result from this section: a stochastic Strichartz inequality. This inequality is a generalization of the inhomogeneous Strichartz estimates from [12] and our argument is somehow motivated by the proof of the latter. However, we use the Burkholder inequality in the space $M^p(0, T; E)$, where E is a 2-smooth Banach space, and the Kahane-Khinchin inequality. Somehow related results have been obtained in very concrete setting by De Bouard and Debussche in [16, 17] for the stochastic Schrödinger equation and by Ondreját [31] for the stochastic wave equation. We believe that our result is new and that the approach has potential other applications. In the following section 4 we formulate an abstract result about the existence and uniqueness of the maximal local solution for stochastic evolution equations of NLS type, that is

$$idu(t) + \Delta u(t) = f(u(t))dt + g(u(t))dW(t).$$

As in [16, 17] we study the existence and uniqueness of solutions to appropriate approximated problems. The proof of the main result from this section, Theorem 5.4, which states the existence of a maximal solution is then given in section 5. Note that even if the lifetime τ_∞ of the solution is defined in terms of the sum of two norms, we prove that the $H^{1,2}$ norm of $u(t)$ explodes as $t \nearrow \tau_\infty < \infty$. In Section 6 we describe an abstract formulation of the NLS equation in Stratonovich form. As in [16, 17] this is needed to obtain a global solution since the L^2-norm of the solution is preserved in this formulation.

We then restrict the framework as follows. We consider a 2-dimensional compact riemannian manifold M, a regular function $\tilde{g} : \mathbb{R} \to \mathbb{R}$ and the diffusion coefficient of the form $g(u) = \tilde{g}(|u|^2)u$. In section 7 we establishe the local existence and uniqueness of the solution to the specific NLS equation in Stratonovich form

$$idu(t) + \Delta u(t) = f(u(t))\, dt + \tilde{g}(|u(t)|^2)u(t) \circ dW(t).$$

Note that unlike the usual parabolic case, see [6], the Stratonovich correction term does not contain the derivative of g or of $\tilde{g}$. This is somehow similar to the case of stochastic wave equation where the Stratonovich correction term is equal to 0; see for instance [27] and [8]. Finally, in section 8 we deal with the existence and uniqueness of a global solution for 2-dimensional manifold, an initial condition $u_0 \in H^{1,2}$, when the drift and the diffusion coefficients have the specific form $f(u) = \tilde{f}(|u|^2)u$ and $f(u) = \tilde{g}(|u|^2)u$ respectively (see Theorem 8.12). This is the natural framework to extend the deterministic well-posedeness proved in Burq et al in [12]. The function $\tilde{f}$ is either defocusing, that is a polynomial with a positive leading coefficient or $\tilde{f}(r) = Cr^\sigma$ with $C > 0$ and $\sigma \in [\frac{1}{2}, \infty)$, or $\tilde{f}$ is focusing, that is $\tilde{f}(r) = -Cr^\sigma$ with $\sigma \in [\frac{1}{2}, 1)$ and $C > 0$. The stochastic integral is defined in Stratonovich form and depending on the nonlinearity, some more conditions have to be imposed on the noise W. More precisely, W has to take values in a sub algebra $H^{1,2}(M) \cap H^{1,\alpha}(M)$ of the space $H^{1,2}(M) \cap W^{\hat{s},q}(M)$, with $\hat{s} = 1 - \frac{1}{p}$ and $2/p + 2/q = 1$, for some p chosen from the non linear term f. Note that this extends both the previous results of de Bouard and Debussche [17] who imposed $\tilde{g} = 1$ and some more smoothness on the noise, and also the deterministic global existence in [12].

Unlike [12] in the deterministic case, we did not try to shift the regularity of the initial condition and that of the solution. Note that several other problems were not considered here, such as the three dimensional global existence which was solved in the deterministic case for a cubic nonlinearity, and the finite time blow-up which was proven by de Bouard and Debussche [18] in the flat case of $\mathbb{R}^d$ with multiplicative noise. These will addressed in fortcoming papers.

Notation: Unless otherwise stated, all vector spaces are complex but treated as real vector spaces. To ease notations, we will denote C a generic constant which can change from one line to the next.

2. Nemytski operator

This section relies heavily on ideas from reference [6]. Assume that we are given two real numbers $q \in [1, \infty)$ and $\theta \in (0, 1)$. Let M be a compact d-dimensional riemannian manifold (without boundary). For $x_1, x_2 \in M$, $|x_2 - x_1|$ we will denote the riemannian distance between x_1 and x_2. By μ we will denote the riemannian volume measure on M but the integration with respect to μ we will denote by dx. By $L^q(M)$, $q \in [1, \infty]$ we will denote the classical real Banach space of all [equivalence classes of] $\mathbb{C} \cong \mathbb{R}^2$-valued q-integrable functions on M, endowed with the classical norm which will be denoted by $|\cdot|_q$ (or sometimes, in danger of ambiguity by $|\cdot|_{L^q}$).

Let us recall a definition of the classical Sobolev spaces $H^{\theta,q}(M)$ and of the Besov-Slobodetski space $W^{\theta,q}(M)$. The former space is defined as the complex interpolation space $[L^q(M), H^{k,q}(M)]_{\frac{\theta}{k}}$, where k is a natural number bigger that θ. It can be shown that $H^{\theta,q}(M) := D((-\Delta_q)^{\theta/2})$, where Δ_q is the Laplace-Beltrami operator on $L^q(M)$, i.e. the infinitesimal generator of the heat semigroup on the space $L^q(M)$. The latter space, defined by

$$W^{\theta,q}(M) := \left\{ f \in L^q(M) : \lvert f \rvert_{\theta,q}^q := \int_M \int_M \frac{|f(x_2) - f(x_1)|^q}{|x_2 - x_1|^{d+\theta q}} \, dx_1 \, dx_2 < \infty \right\}, \tag{2.1}$$

is endowed with the norm $\|f\|_{W^{\theta,q}(M)} := |f|_{L^q(M)} + \lvert f \rvert_{\theta,q}$. Note that $W^{\theta,2}(M) = H^{\theta,2}(M)$, whereas $H^{\theta,q}(M) \subset W^{\theta,q}(M)$ (or $H^{\theta,q}(M) \supset W^{\theta,q}(M)$) depending whether $q > 2$ (or $q < 2$). Let us recall that the Besov-Slobodetski spaces with fractional order $\theta \in (0, 1)$ are equal to the real interpolation spaces of order θ between the spaces $L^q(M)$ and $H^{1,q}(M) = W^{1,q}(M)$; see for instance [35]. We will also use the notation

$$\lvert f \rvert_{1,q}^q := \int_M |\nabla f|^q \, dx.$$

Thus, $f \in W^{1,q}(M)$ iff $f \in L^q(M)$ and $\lvert f \rvert_{1,q} < \infty$.

The space $\mathbb{R}^2$ can be replaced by any real separable Banach space Y for the L^q spaces and similarly, the Besov-Slobodetski $W^{\theta,q}$ spaces can naturally be defined for vector valued functions; see for instance [34]. Given a real Banach separable space Y, we denote by $L^q(M, Y)$ and $W^{\theta,q}(M, Y)$ the corresponding set of functions $f : M \to Y$. When no confusion arises and to ease notations, we will denote by $|f|_q$ and $\|f\|_{\theta,q}$ the corresponding $L^q(M, Y)$ and $W^{\theta,q}(M, Y)$ norms. However the Sobolev spaces $H^{\theta,q}$ can only be defined for functions with values in complex vector spaces and their treatments is a more delicate. Whenever we use these spaces in our paper we do this for $\mathbb{C} \cong \mathbb{R}^2$-valued functions.

For two separable Banach spaces Y and Y_1, and a locally Lipschitz continuous function $f: Y \to Y_1$, we will denote by F the corresponding Nemytski map defined by

$$F(\gamma) := f \circ \gamma\,, \qquad \gamma \in W^{\theta,q}(M,Y). \tag{2.2}$$

We will at first study the regularity properties of the Nemytski operator F. Let us first note that in general F does not map the space $W^{\theta,p}(M,Y)$ into $W^{\theta,q}(M,Y_1)$. However, it does if either the function f is globally Lipschitz or if $\frac{d}{q} < \theta < 1$, see [6].
Given a real separable Banach space Y, $\theta \in [0,1]$ and $q \in [1,\infty)$, let

$$\mathcal{R}^{\theta,q}(Y) = W^{\theta,q}(M,Y) \cap L^\infty(M,Y), \tag{2.3}$$

endowed with the norm $\|u\|_{\mathcal{R}^{\theta,q}(Y)} = \|u\|_{W^{\theta,q}(M,Y)} + |u|_{L^\infty(M,Y)}$. Once more to ease notations, let $\mathcal{R}^{\theta,q} = \mathcal{R}^{\theta,q}(M,\mathbb{C})$ where $\mathbb{C} \equiv \mathbb{R}^2$.
Fix $\theta \in (0,1)$ and $q \in [1,\infty)$. We will show F is map from a Banach space $\mathcal{R}^{\theta,q}(Y)$ to $\mathcal{R}^{\theta,q}(Y_1)$ if $f: Y \to Y_1$ is locally Lipschitz. Note that if $\frac{d}{q} < \theta < 1$, then by the Sobolev embedding Theorem $W^{\theta,q}(M,Y) \subset L^\infty(M,Y)$ so that in that case the result from [6] is a special case of the one below. Let Y and Y_1 be separable Banach spaces, j be an integer larger than one and $f: Y \to Y_1$ be of class C^{j-1} such that $f^{(j-1)}$ is Lipschitz on balls or everywhere differentiable. Given $R > 0$ set

$$K_j(f,R) = \sup_{|x|,|y| \le R} \frac{|f^{(j-1)}(y) - f^{(j-1)}(x)|}{|y-x|}, \quad \tilde{K}_j(f,R) = \sup_{|x| \le R} |f^{(j)}(x)|. \tag{2.4}$$

Proposition 2.1. *Fix $\theta \in (0,1]$ and $q \in [1,\infty)$. Let Y and Y_1 be real separable Banach spaces and $f: Y \to Y_1$ be Lipschitz on balls when $\theta < 1$, or, everywhere differentiable when $\theta = 1$. Then the Nemytski map F corresponding to f (and defined in (2.2)) maps $\mathcal{R}^{\theta,q}(Y)$ to $\mathcal{R}^{\theta,q}(Y_1)$. More precisely, when $\theta < 1$, then for all $\gamma \in \mathcal{R}^{\theta,q}(Y)$,*

$$\|F(\gamma)\|_{\theta,q} \le |f(0)|\mathrm{vol}(M) + K_1(f,|\gamma|_\infty)\|\gamma\|_{\theta,q}. \tag{2.5}$$

When $\theta = 1$, inequality (2.5) holds with K_1 being replaced by $\tilde{K}_1$. In particular, F is of linear growth either if f is globally Lipschitz, i.e. $K_1(f) := \sup_{R>0} K_1(f,R)$ is finite when $\theta < 1$, or if f' is bounded, i.e. $\tilde{K}_1(f) := \sup_{R>0} \tilde{K}_1(f,R) < \infty$ when $\theta = 1$.
The above statements remain true if the space $\mathcal{R}^{\theta,q}(Y)$ is replaced by $\tilde{\mathcal{R}}^{\theta,q}_{s,p}(Y) = H^{1,2}(M) \cap W^{s,p}(M)$, provided that $1 > s > \frac{d}{p}$.

Proof. We at first check that we may assume that $f(0) = 0$. Indeed, the function $\tilde{f}: Y \to Y_1$ defined by $\tilde{f}(y) := f(y) - f(0)$, $y \in Y$, satisfies the same assumptions as f and the corresponding Nemytski operator $\tilde{F}$ is defined by $\tilde{F}(\gamma) = \tilde{f} \circ \gamma$, i.e. $\tilde{F}(\gamma) = F(\gamma) - f(0)1_M$. Since $\mathrm{vol}(M) < \infty$, 1_M belongs to $W^{\theta,q}(M)$ and $|F(\gamma)|_{\theta,q} = |\tilde{F}(\gamma)|_{\theta,q}$ for each $\gamma: M \to Y$. Hence the operator F maps $\mathcal{R}^{\theta,q}(Y)$ into $\mathcal{R}^{\theta,q}(Y_1)$ if and only if $\tilde{F}$ does.
Suppose that $f(0) = 0$ and $\theta < 1$. Let $\gamma \in \mathcal{R}^{\theta,q}(Y)$ and set $R := |\gamma|_\infty$. By assumption f is Lipschitz on the ball $B(0,R) := \{x \in Y : |x| \le R\}$ and

$$|f(y_1) - f(y_2)| \le K_1(f,R)|y_1 - y_2|, \quad y_1, y_2 \in B(0,R). \tag{2.6}$$

Thus, since $f(0) = 0$, $|f(y)| \le K_1(f,R)|y|$ for $|y| \le R$ and since $|\gamma(x)| \le |\gamma|_\infty$ for a.a. $x \in M$, we infer that $|F(\gamma)|_{L^q(M,Y_1)} \le K_1(f,|\gamma|_\infty)|\gamma|_{L^q(M,Y)}$. Similarly (2.6) yields

$$|f(\gamma(x_1)) - f(\gamma(x_2))| \le K_1(f,|\gamma|_\infty)|\gamma(x_2) - \gamma(x_1)|, \qquad \text{for a.a. } x_1, x_2 \in M,$$

which yields $|F(\gamma)|_{\theta,q} \le K_1(f,|\gamma|_\infty)|\gamma|_{\theta,q}$. Since the part of the result corresponding the L^∞ norm is obvious, the last inequality concludes the proof of (2.5) if $\theta < 1$.

If $\theta = 1$, $\gamma \in \mathcal{R}^{1,q}(Y)$, we have $|f(\gamma(x))| = \left|\int_0^1 f'(s\gamma(x))\gamma(x)ds\right| \le \tilde{K}_1(f,|\gamma|_\infty)|\gamma|_\infty$. Furthermore, $(f\circ\gamma)' = (f'\circ\gamma)\gamma'$, and since $|f'\circ\gamma| \le \tilde{K}_1(f,|\gamma|_\infty)$ we deduce that

$$\int_M |\nabla(f\circ\gamma)|^q dx \le \tilde{K}_1(f,|\gamma|_\infty)\int_M |\nabla\gamma(x)|^q dx,$$

which proves (2.5) when $\theta = 1$. □

Let us formulate the following important (but simple) consequence of Proposition 2.1.

Corollary 2.2. *Let $\theta \in (0,1]$ and $q \in [1,\infty)$. Then $\mathcal{R}^{\theta,q}$ is an algebra (with pointwise multiplication) and there exists a constant $C > 0$ such that for $\sigma,\gamma \in W^{\theta,q}(M)\cap L^\infty(M)$,*

$$|\sigma\gamma|_{\theta,q} \quad\le\quad \|\sigma\gamma\|_{\theta,q} \le |\sigma|_{L^\infty}\|\gamma\|_{\theta,q} + |\gamma|_{L^\infty}\|\sigma\|_{\theta,q} \le C'\|\sigma\|_{W^{\theta,q}\cap L^\infty}|\gamma|_{W^{\theta,q}\cap L^\infty}, \tag{2.7}$$

Now we will formulate the promised generalisation of Proposition 2.1.

Theorem 2.3. *Fix $\theta \in (0,1]$ and $q \in [1,\infty)$ and let Y, Y_1 be real separable Banach spaces. Assume that a function $f : Y \to Y_1$ is of $\mathcal{C}^1$ class and its Fréchet derivative $f' : Y \to \mathcal{L}(Y,Y_1)$ is Lipschitz on balls. Then the corresponding Nemytski map F defined in (2.2) is Lipschitz continuous on balls in $\mathcal{R}^{\theta,q}(Y)$. More precisely for any $K > 0$, and all $\gamma,\sigma \in \mathcal{R}^{\theta,q}(Y)$ with $\|\gamma\|_{\mathcal{R}^{\theta,q}(Y)} \vee \|\sigma\|_{\mathcal{R}^{\theta,q}(Y)} \le K$, we have:*

$$|F(\gamma)-F(\sigma)|_q \quad\le\quad K_1(f,|\gamma|_\infty\vee|\sigma|_\infty)\,|\gamma-\sigma|_q, \tag{2.8}$$

$$\begin{aligned} |F(\gamma)-F(\sigma)|_{\theta,q} \quad\le\quad & K_2(f,|\gamma|_\infty\vee|\sigma|_\infty)\,|\gamma-\sigma|_\infty\big[|\sigma|_{\theta,q} + \frac{1}{2}|\gamma-\sigma|_{\theta,q}\big] \\ & + K_1(f,|\gamma|_\infty\vee|\sigma|_\infty)\,|\gamma-\sigma|_{\theta,q}. \end{aligned}\tag{2.9}$$

The above statements remain true if the space $\mathcal{R}^{\theta,q}(Y)$ is replaced by $\tilde{\mathcal{R}}^{\theta,q}_{s,p}(Y) = H^{1,2}(M)\cap W^{s,p}(M)$, provided that $1 > s > \frac{d}{p}$.

Proof. We only consider the case $\theta < 1$ since the proof in the case $\theta = 1$ is easier.
Take $\gamma,\sigma \in \mathcal{R}^{\theta,q}(Y)$ and put $R = |\gamma|_\infty \vee |\sigma|_\infty$. For $x \in M$ the Taylor formula yields

$$f(\gamma(x)) - f(\sigma(x)) = \int_0^1 f'(y_s(x))(\gamma(x)-\sigma(x))\,ds, \tag{2.10}$$

where we let

$$y_s(x) := \sigma(x) + s[\gamma(x)-\sigma(x)], \quad s \in [0,1]. \tag{2.11}$$

As in the proof of (2.5), observe that since $|\gamma(x)|\vee|\sigma(x)| \le R$ for a.a. x in M,

$$|F(\gamma)-F(\sigma)|_q \le K_1(f,R)|\gamma-\sigma|_q, \quad |F(\gamma)-F(\sigma)|_\infty \le K_1(f,R)|\gamma-\sigma|_\infty.$$

Thus, it suffices to check (2.9).
Let $x_1,x_2 \in M$ be such that $|\gamma(x_1)|,|\gamma(x_2)|,|\sigma(x_1)|,|\sigma(x_2)| \le R$ and let $y_s(x_i)$ be defined in (2.11). Then $|y_s(x_i)| \le R$ for all $s \in [0,1]$ and

$$[f(\gamma(x_1)) - f(\sigma(x_1))] - [f(\gamma(x_2)) - f(\sigma(x_2))] = I_1(x_1,x_2) + I_2(x_1,x_2),$$

where

$$\begin{aligned} I_1(x_1,x_2) &= \int_0^1 \left[f'(y_s(x_1)) - f'(y_s(x_2))\right](\gamma(x_1)-\sigma(x_1))\,ds, \\ I_2(x_1,x_2) &= \int_0^1 f'(y_s(x_2))\left[\gamma(x_1)-\sigma(x_1)-(\gamma(x_2)-\sigma(x_2))\right]ds. \end{aligned}$$

Therefore, by the Minkowski inequality and (2.1) we infer that $|F(\gamma)-F(\sigma)|_{\theta,q} \le J_1+J_2$, where

$$J_1=\left\{\int_M\int_M \frac{|I_1(x_1,x_2)|^q}{|x_1-x_2|^{d+\theta q}}dx_1dx_2\right\}^{\frac1q},\quad J_2=\left\{\int_M\int_M \frac{|I_2(x_1,x_2)|^q}{|x_1-x_2|^{d+\theta q}}dx_1dx_2\right\}^{\frac1q}.$$

Thus, the local Lipschitz property of f' implies

$$\begin{aligned}\|f'(y_s(x_1))-f'(y_s(x_2))\|_{\mathcal{L}(Y,Y_1)} &\le K_2(f,R)|y_s(x_1)-y_s(x_2)|\\ &\le K_2(f,R)\left\{|\sigma(x_1)-\sigma(x_2)|+s|\gamma(x_1)-\sigma(x_1)-(\gamma(x_2)-\sigma(x_2))|\right\},\end{aligned}$$

and hence

$$\begin{aligned}|I_1(x_1,x_2)| \;\le\;& K_2(f,R)|\sigma(x_1)-\sigma(x_2)|\,|\gamma(x_1)-\sigma(x_1)|\\ &+\frac12 K_2(f,R)|(\gamma-\sigma)(x_1)-(\gamma-\sigma)(x_2)|\,|\gamma(x_1)-\sigma(x_1)|.\end{aligned}$$

Since for $\phi\in W^{\theta,q}(M,Y_1)$ we have

$$\left\{\int_M\int_M \frac{|\phi(x_1)-\phi(x_2)|^q\,|\gamma(x_1)-\sigma(x_1)|^q}{|x_1-x_2|^{d+\theta q}}\,dx_1\,dx_2\right\}^{\frac1q}\le|\gamma-\sigma|_\infty|\phi|_{\theta,q},$$

we infer that

$$J_1\le K_2(f,R)|\gamma-\sigma|_\infty\Big[|\sigma|_{\theta,q}+\frac12|\gamma-\sigma|_{\theta,q}\Big]. \tag{2.12}$$

Let us observe that $|I_2(x_1,x_2)|\le K_1(f,R)|(\gamma-\sigma)(x_1)-(\gamma-\sigma)(x_2)|$. Therefore,

$$J_2\le K_1(f,R)|\gamma-\sigma|_{\theta,q}. \tag{2.13}$$

Summing up, the inequalities (2.12) and (2.13) yield (2.9), which completes the proof. □

Corollary 2.4. *Under the assumptions of Proposition 2.1 the map $F:\mathcal{R}^{\theta,q}(Y)\to\mathcal{R}^{\theta,q}(Y_1)$ is measurable.*

Proof. One can approximate f by a sequence of functions f_n which satisfy the assumptions of Theorem 2.3. Then each Nemytski map F_n associated with f_n is continuous and so Borel measurable. On the other hand, $F_n\to F$ pointwise on $\mathcal{R}^{\theta,q}(Y)$. □

Remark 2.5. Let $Y=\mathbb{C}\equiv\mathbb{R}^2$ and let $f:Y\to Y$ be defined by $f(z)=C\,|z|^{2\alpha}\,z$ for some real constants $\alpha\ge\frac12$ and C. Then f is of class $\mathcal{C}^1$ and both f and f' are Lipschitz on balls. Furthermore, given $\sigma\ge\frac32$, $\theta\in(0,1]$ and q such that $\theta d>q$, the map $\Phi:\in W^{\theta,q}(\mathbb{C})\to\mathbb{R}$ defined by

$$\Phi(u)=\int_M|u(x)|^{2\sigma}dx$$

is of class $\mathcal{C}^2$, and for $u,v_1,v_2\in W^{\theta,q}(\mathbb{C}$, we have

$$\begin{aligned}\Phi'(u)(v_1)&=\int_M 2\sigma|u(x)|^{2(\sigma-1)}Re(u(x)\overline{v_1(x)})dx\\ \Phi''(u)(v_1,v_2)&=\int_M\Big[4\sigma(\sigma-1)|u(x)|^{2(\alpha-2)}Re(u(x)\overline{v_1(x)})Re(u(x)\overline{v_2(x)})\\ &\quad+2\sigma|u(x)|^{2(\sigma-1)}Re(v_2(x)\overline{v_1(x)})\Big]dx.\end{aligned}$$

3. Stochastic Strichartz estimates

3.1. Deterministic Strichartz estimates.
We assume the following.

Assumption 3.1. (i) H_0 *is a separable Hilbert space and* E_0 *is a separable Banach space such that* $\mathrm{E}_0 \cap \mathrm{H}_0$ *is dense in both* E_0 *and* H_0*;*

(ii) *There exists a separable Hilbert space* $\mathcal{H}_0$ *such that* $\mathrm{H}_0 \subset \mathcal{H}_0$ *and a* $\mathcal{C}_0$ *unitary group* $\mathbf{U} = \big(U_t\big)_{t\in\mathbb{R}}$ *on* $\mathcal{H}_0$ *with the infinitesimal generator* iA*, where* A *is self-adjoint in* $\mathcal{H}_0$*. Assume that the restriction of* $-A$ *to* H_0*, denoted also by* $-A$*, is a positive operator in* H_0*.*

(iii) *There exists a positive linear operator* $-\tilde{A}$ *on the space* E_0 *such that* $D(A)\cap E_0 \subset D(\tilde{A})$*,* $D(\tilde{A}) \cap \mathrm{H}_0 \subset D(A)$ *and* $A = \tilde{A}$ *on* $D(A) \cap D(\tilde{A})$*. In what follows, unless in a danger of ambiguity, the operator* $\tilde{A}$ *will be denoted by* A*.*

(iv) *There exists a number* $p \in (2,\infty)$ *and a non decreasing function* $\tilde{C}_p : (0,\infty) \to (0,\infty)$ *such that* $\tilde{C}_p(0^+) = 0$ *and, for every* $v_0 \in \mathrm{H}_0$*, there exists a subset* $\Gamma(v_0)$ *of* $\mathbb{R}_+$*, of full Lebesgue measure, such that for each* $t \in \Gamma(v_0)$*,* $U_t(v_0) \in \mathrm{E}_0$*, and such that for* $T > 0$*, and every* $v_0 \in \mathrm{H}_0$*, one has:*

$$\Big(\int_0^T |U_t v_0|^p_{\mathrm{E}_0} dt\Big)^{\frac{1}{p}} \le \tilde{C}_p(T)|v_0|_{\mathrm{H}_0}. \tag{3.1}$$

Given $f \in L^1(0,T;H^0)$, set $U * f = \int_0^{\cdot} U_{\cdot - r} f(r)\, dr$; then we have the following inhomogeneous Strichartz inequalities.

Lemma 3.2. *Assume that Assumption 3.1 holds with* $p \in (2,\infty)$*. Then for* $T > 0$ *we have*

$$\begin{aligned} \Big| U * f \Big|_{L^p(0,T;\mathrm{E}_0)} &\le C\tilde{C}_p(T)\, |f|_{L^1(0,T;\mathrm{H}_0)},\ f \in L^1(0,T;\mathrm{H}_0), && (3.2)\\ \Big| U * f \Big|_{C([0,T];\mathrm{H}_0)} &\le C|f|_{L^1(0,T;\mathrm{H}_0)},\ f \in L^1(0,T;\mathrm{H}_0). && (3.3)\end{aligned}$$

where $C = \sup_{t\in[-T,T]} |U_t|_{\mathcal{L}(H_0)} \in (0,\infty)$

Proof. Since by assumptions $\mathcal{U} = \big(U_t\big)_{t\in\mathbb{R}}$ is a $\mathcal{C}_0$ group on H_0 we infer that for $t \in [0,T]$ we have:

$$[U * f](t) = U_t\Big(\int_0^t U_{-r} f(r)\, dr\Big).$$

Therefore, since the norms of U_t and $U_{-\tau}$ are bounded by C, the Jensen inequality implies (3.3). Following the argument in [12] it is easy to deduce (3.2); we sketch the argument for the sake of completeness. The Minkowski inequality and (3.1) imply that

$$|U{*}f|_{L^p(0,T;E_0)} \le \int_0^T |U_{\cdot - r} f(r)|_{L^p(0,T;E_0)} dr \le \tilde{C}_p(T)\int_0^T |U_{-r} f(r)|_{H_0} dr \le C\tilde{C}_p(T)|f|_{L^1(0,T;H_0)}.$$

□

In the next lemma we show that once Assumption 3.1 holds for a certain set of objects it holds for a much larger class of sets of objects.

Lemma 3.3. *Assume that the spaces* H_0 *and* E_0*, and the operator* A *satisfy Assumption3.1 with a number* p*. Assume that* $\hat{s} \ge 0$ *and put* $\mathrm{H} = D((-A)^{\frac{\hat{s}}{2}}))$*, a subspace of* H_0*. Assume also that* $\mathrm{E} \subset E_0$ *is a separable Banach space such that* $\mathrm{E} \supset D((-\tilde{A})^{\frac{\hat{s}}{2}})$*. Then the spaces* H *and* E*, and the restriction of the operator* A *to* H *and* E *satisfy Assumption 3.1 with the same number* p*.*

This yields the following:

Corollary 3.4. *In the framework of Lemma 3.3, there exists a non decreasing function* $C_p : (0,\infty) \to (0,\infty)$ *such that* $C_p(0^+) = 0$*, and such that for every* $u_0 \in L^p(\Omega, \mathrm{H})$ *and* $T > 0$ *the trajectories of the process* $(U_t u_0, t \in [0,T])$ *belong a.s. to* $C([0,T];\mathrm{E}) \cap L^p(0,T;\mathrm{H})$*; moreover*

$$\mathbb{E}\Big(\int_0^T |U_t u_0|_{\mathrm{E}}^p \, dt + \sup_{t\in[0,T]} |U_t u_0|_{\mathrm{H}}^p \Big) \;\le\; C\big(1 + C_p^p(T)\big) \mathbb{E}|u_0|_{\mathrm{H}}^p. \tag{3.4}$$

3.2. **Stochastic Strichartz estimates.** The formulation of the result from this section is motivated by an abstract approach to time in-homogenous Strichartz estimates proposed by [22] and [12]. To ease notations, let $L^q = L^q(M)$, $W^{\sigma,q} = W^{\sigma,q}(M)$ and $H^{\sigma,q} = H^{\sigma,q}(M)$ denote the spaces of complex-valued functions defined on M and which satisfy the corresponding integrability and regularity assumptions.

Let us recall that a Banach space E is of martingale type 2 if there exists a constant $L = L_2(E) > 0$ such that for every E-valued finite martingale $\{M_n\}_{n=0}^N$ the following holds:

$$\sup_n \mathbb{E}|M_n|^2 \le L \sum_{n=0}^N \mathbb{E}|M_n - M_{n-1}|^2, \tag{3.5}$$

where, as usually, we put $M_{-1} = 0$; see for instance [3, 4] and/or [6]. It is know that it is possible to extend the classical Hilbert-space valued Itô integral to the framework of martingale type 2 Banach spaces; see for instance [28] and [20].

We make the following assumption.

Assumption 3.5. *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*, where* $\mathbb{F} = \big(\mathcal{F}_t, t \ge 0\big)$*, is a filtered probability space satisfying the usual assumptions. We assume that* $W = \big(W(t), t \ge 0\big)$ *is an* K*-cylindrical Wiener process on some real separable Hilbert space* K*; see Definition 4.1 in* [10].

Let us recall the definition of an accessible stopping time.

Definition 3.6. *An* $\mathbb{F}$*-stopping time* τ *is called accessible if there exists an increasing sequence* $(\tau_n)_{n\in\mathbb{N}}$ *of* $\mathbb{F}$*-stopping times such that* $\tau_n < \tau$ *and* $\lim_{n\to\infty} \tau_n = \tau$ *a.s. Such a sequence* $(\tau_n)_{n\in\mathbb{N}}$ *will be called an approximating sequence for* τ.

Let us recall the following standard notation. Assume that X is a separable Banach space, $p \in [1,\infty)$ and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space. By $\mathcal{M}^p_{\mathrm{loc}}(\mathbb{R}_+, X)$ we denote the space of all $\mathbb{F}$-progressively measurable X-valued processes $\xi : \mathbb{R}_+ \times \Omega \to X$ for which there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of bounded stopping times such that $T_n \nearrow \infty$, $\mathbb{P}$-almost surely and $\mathbb{E}\int_0^{T_n} |\xi(t)|^p \, dt < \infty$ for every $n \in \mathbb{N}$.

Recall that if K and E are separable Hilbert and respectively Banach spaces, then the space $R(\mathrm{K}, E)$ of γ-radonifying operators consists of all bounded operators $\Lambda : \mathrm{K} \to E$ such that the series $\sum_{k=1}^\infty \gamma_k \Lambda(e_k)$ converges in $L^2(\Omega, E)$ for some (or any) orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of K and some (or any) sequence $(\gamma_k)_{k\in\mathbb{N}}$ of i.i.d. $N(0,1)$ real random variables. We put

$$\|\Lambda\|_{\mathrm{R}(\mathrm{K},E)} = \Big(\tilde{\mathbb{E}} \big| \sum_{j\in\mathbb{N}} \beta_j \Lambda e_j \big|_E^2 \Big)^{\frac{1}{2}}.$$

By the Kahane-Khintchin inequality and the Itô-Nisio Theorem, for every Banach space E there exist a constant $C_p(E)$ such that for every linear map $\Lambda : \mathrm{K} \to E$,

$$C_p(E)^{-1} \|\Lambda\|_{\mathrm{R}(\mathrm{K},E)} \le \Big(\tilde{\mathbb{E}} \big| \sum_{j\in\mathbb{N}} \beta_j \Lambda e_j \big|_E^p \Big)^{\frac{1}{p}} \le C_p(E) \|\Lambda\|_{\mathrm{R}(\mathrm{K},E)}. \tag{3.6}$$

Hence the condition of convergence in $L^2(\Omega, E)$ can be replaced by a condition of convergence in $L^p(\Omega, E)$ for some (any) $p \in (1,\infty)$. The space $R(H, E)$ was introduced by

Neidhard in his PhD thesis [28] and was then used to study the existence and regularity of solutions to SPDEs in [4] and [10]. Recently it has been widely used; see for instance [30], [9], [29] and [2].
Furthermore, the Burkholder inequality holds in this framework; see [4, 19, 30]. If E is a martingale type 2 Banach space, for every $p \in (1,\infty)$ there exists a constant $B_p(E) > 0$ such that for each accessible stopping time $\tau > 0$ and $R(\mathrm{K},E)$-valued progressively measurable process ξ,

$$\mathbb{E} \sup_{0\le t\le\tau} \Big| \int_0^t \xi(s)\, dW(s)\Big|_E^p \le B_p(E)\,\mathbb{E}\Big(\int_0^\tau \|\xi(t)\|^2_{R(\mathrm{K},E)}\,dt\Big)^{p/2}. \tag{3.7}$$

Corollary 3.7. *Let E be a martingale type 2 Banach space and $p \in (1,\infty)$. Then there exists a constant $\hat{B}_p(E)$ depending on E such that for every $T \in (0,\infty]$ and every $L^p(0,T;E)$-valued progressively measurable process $(\Xi_s,\ s \in [0,T))$*

$$\mathbb{E}\Big|\int_0^T \Xi_s\, dW(s)\Big|^p_{L^p(0,T;E)} \le \hat{B}_p(E)\,\mathbb{E}\Big(\int_0^T \|\Xi_s\|^2_{R(\mathrm{K},L^p(0,T;E))}\,ds\Big)^{p/2}. \tag{3.8}$$

Moreover, for any $T > 0$, the above inequality (3.8) holds true also for the space $L^p(0,T;E)$ and the integral over interval $(0,T)$ with the same constant $\hat{B}_p(E)$.

Proof. Since the space $L^p(\mathbb{R}_+;E)$ is a martingale type 2, the above inequality holds with $T = \infty$. The second half is a consequence of the fact that $L^p(0,T;E)$ can be isometrically identified with a closed subspace of $L^p(\mathbb{R}_+;E)$. □

Before we state the main result in this section, let us prove an auxiliary result which has an interest in its own. Let us introduce the following notation. For an $R(\mathrm{K},H)$-valued process ξ and define a progressively measurable $R(\mathrm{K},L^p(0,T;E))$-valued process $(\Xi_r)_{r\in[0,T]}$ as follows:

$$\Xi_r := \big\{[0,T] \ni t \mapsto 1_{[r,T]}(t) U_{t-r}\xi(r)\big\},\quad r \in [0,T]. \tag{3.9}$$

Lemma 3.8. *Let Assumption 3.1 be satisfied. Then for each $r \in [0,T]$*

$$\|\Xi_r\|_{\mathrm{R}(\mathrm{K},L^p(0,T;\mathrm{E}_0))} \le C\tilde{C}_p(T) C_p(\mathrm{E}_0) C_p(\mathrm{H}_0)\|\xi(r)\|_{\mathrm{R}(\mathrm{K},\mathrm{H}_0)}. \tag{3.10}$$

Proof. Consider a sequence $\big(\beta_j, j \in \mathbb{N}\big)$ of i.i.d. $N(0,1)$ random variables defined on some auxiliary probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and a sequence $\big(e_j, j \in \mathbb{N}\big)$ which is an ONB of the Hilbert space K. Hence, by the Kahane-Khintchin inequality (3.6), the Fubini Theorem and the time-homogenous inequality (3.1), we infer that

$$\begin{aligned}
\|\Xi_r\|^p_{\mathrm{R}(\mathrm{K},L^p(0,T;\mathrm{E}_0))} &\le C_p^p(\mathrm{E}_0)\tilde{\mathbb{E}}\int_0^T \Big|\sum_{j\in\mathbb{N}} \beta_j \Xi_r(e_j)(t)\Big|^p_{\mathrm{E}_0}\,dt\\
&= C_p^p(\mathrm{E}_0)\tilde{\mathbb{E}}\int_r^T \Big|\sum_{j\in\mathbb{N}} \beta_j U_{t-r}\xi(r)(e_j)\Big|^p_{\mathrm{E}_0}\,dt = C_p^p(\mathrm{E}_0)\tilde{\mathbb{E}}\int_r^T \Big|U_{t-r}\Big(\sum_{j\in\mathbb{N}}\beta_j\xi(r)(e_j)\Big)\Big|^p_{\mathrm{E}_0}\,dt\\
&\le C_p^p(\mathrm{E}_0)\tilde{C}_p^p(T)\tilde{\mathbb{E}}\Big|U_{-r}\sum_{j\in\mathbb{N}}\beta_j\xi(r)(e_j)\Big|^p_{\mathrm{H}_0} \le CC_p^p(\mathrm{E}_0)\tilde{C}_p^p(T)\tilde{\mathbb{E}}\Big|\sum_j \beta_j\zeta(r)e_j\Big|^p_{\mathrm{H}_0}\\
&\le CC_p^p(\mathrm{E}_0)\tilde{C}_p^p(T)C_p^p(\mathrm{H}_0)\|\xi(r)\|^p_{R(\mathrm{K},\mathrm{H}_0)}.
\end{aligned}$$

The proof is complete. □

We define local p-integrable martingales starting from the random time T_0 as follows.

Definition 3.9. *Let T_0 be a finite accessible $\mathbb{F}$-stopping time. For $t \geq 0$ set $\mathcal{F}_t^{T_0} = \mathcal{F}_{T_0+t}$ and set $\mathbb{F}^{T_0} = \big(\mathcal{F}_t^{T_0}\big)_{t\geq 0}$. Let $\mathcal{M}^p_{loc}([0,\infty), \mathbb{F}^{T_0}, \mathrm{R}(\mathrm{K},\mathrm{H}_0))$ denote the set of $\mathbb{F}^{T_0}$-predictable, $\mathrm{R}(\mathrm{K},\mathrm{H}_0)$-valued processes $X = (X_t)_{t\geq 0}$ such that there exists a sequence (T_n) of finite accessible $\mathbb{F}^{T_0}$-stopping times such that $T_n \to \infty$ a.s. and $\mathbb{E}\int_0^{T_n} |X(t)|^p_{\mathrm{R}(\mathrm{K},\mathrm{H}_0)} dt < \infty$ for every integer n. To ease notation, set $\mathcal{M}^p_{loc}([0,\infty), \mathrm{R}(K,H)) = \mathcal{M}^p_{loc}([0,\infty), \mathbb{F}, \mathrm{R}(K,H))$.*

The following theorem, which is the main result in this section, extends (3.2) from the deterministic to the stochastic setting and will be called the stochastic Strichartz estimate. It is first stated on the time interval $[0,T]$ and generalizes the corresponding result of De Bouard and Debussche from [16, 17].
Informally, if $\zeta \in \mathcal{M}^p_{\mathrm{loc}}([0,\infty), \mathrm{R}(\mathrm{K},H))$, then for $t \geq 0$ the Itô integral $\int_0^t \zeta(r) dW(r)$ can be written as

$$\int_0^t \zeta(r) dW(r) = \sum_{k\geq 1} \int_0^t \zeta(r)[e_k]\, dW_k(r),$$

where $\{e_k\}_{k=1}^\infty$ is an ONB of K and $(W_k)_{k=1}^\infty$ is a sequence of i.i.d. real-valued Wiener processes defined for $k \in \mathbb{N}^*$ and $t \geq 0$ by $W_k(t) := W(t)[e_k]$

Theorem 3.10. *Assume that Assumption 3.1 is satisfied and E_0 is a martingale type 2 Banach space. For each $T > 0$ set $\hat{C}_p(T) := \tilde{C}_p^p(T) C_p^p(\mathrm{E}_0) C_p^p(\mathrm{H}_0) \hat{B}_p(\mathrm{E}_0)$, where $C_p(\mathrm{E}_0)$, $C_p(\mathrm{H}_0)$ and resp. $\hat{B}_p(\mathrm{E}_0)$ are defined in (3.6) and resp. (3.8). Then for every predictable process $\xi \in \mathcal{M}^p_{loc}([0,\infty), \mathrm{R}(\mathrm{K},\mathrm{H}_0))$ and every accessible stopping time τ satisfying $\tau \leq T$ and $\mathbb{E}\big(\int_0^\tau \|\xi(t)\|^2_{R(K,\mathrm{H}_0)}\, dt\big)^{\frac{p}{2}} < \infty$, one has*

$$\mathbb{E}\int_0^T |[J_{[0,\tau)}\xi](t)|^p_{\mathrm{E}_0}\, dt \leq \hat{C}_p(T)\mathbb{E}\Big(\int_0^\tau \|\xi(t)\|^2_{\mathrm{R}(\mathrm{K},\mathrm{H}_0)}\, dt\Big)^{p/2}, \tag{3.11}$$

where one puts

$$[J_{[0,\tau)}\xi](t) = \int_0^t 1_{[0,\tau)}(r) U_{t-r}\xi(r)\, dW(r), \quad t \geq 0.$$

Proof. We use the following crucial equalities for $t \in [0,T]$:

$$\int_0^t 1_{[0,\tau)}(r) U_{t-r}\xi(r)\, dW(r) = \int_0^T 1_{[0,\tau)}(r)\Xi_r(t)\, dW(r) = \Big(\int_0^T 1_{[0,\tau)}(r)\Xi_r\, dW(r)\Big)(t),$$

where Ξ_r is defined by (3.9). Hence, with $u = J_{[0,\tau)}\xi$, we have

$$\int_0^T |u(t)|^p_{\mathrm{E}_0}\, dt = \int_0^T \Big|\Big(\int_0^T 1_{[0,\tau)}(r)\Xi_r\, dW(r)\Big)(t)\Big|^p_{\mathrm{E}_0} dt = \Big|\int_0^T 1_{[0,\tau)}(r)\Xi_r\, dW(r)\Big|^p_{L^p(0,T;\mathrm{E}_0)}.$$

Next by Corollary 3.7 and Lemma 3.8, we deduce

$$\begin{aligned}
\mathbb{E}\int_0^T |J_{[0,\tau)}\xi(t)|^p_{E_0}\, dt &\leq \hat{B}_p(E_0)\mathbb{E}\Big|\int_0^T 1_{[0,\tau)}(r)\|\Xi_r\|^2_{R(\mathrm{K};L^p(0,T;E_0))}\, dr\Big|^{\frac{p}{2}} \\
&\leq \hat{B}_p(E_0)\tilde{C}_p^p(T) C_p^p(E_0) C_p^p(\mathrm{H}_0)\mathbb{E}\Big[\int_0^\tau \|\xi(r)\|^2_{R(\mathrm{K};\mathrm{H}_0)}\, dr\Big]^{\frac{p}{2}}.
\end{aligned}$$

This concludes the proof. □

Next we formulate a result which is related with (3.11) as the inequality (3.3) is to (3.2).

Proposition 3.11. *Asumme that the assumptions of Theorem 3.10 are satisfied; then there exists a constant $C_p > 0$ such that for ξ and τ as in Theorem 3.10 we have:*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\Big|[J_{[0,\tau)}\xi](t)\Big|^p_{\mathrm{H}_0}\Big) \le C_p\mathbb{E}\Big[\int_0^\tau \|\xi(t)\|^2_{R(K,\mathrm{H}_0)}dt\Big]^{\frac{p}{2}}. \tag{3.12}$$

Proof. The proof of inequality (3.12) is classical; we include it for the sake of completeness. Since $(U_t, t\in\mathbb{R})$ is a $\mathcal{C}_0$ group on H_0 with bounded norms on $[-T,T]$, the Burkholder inequality yields

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t 1_{[0,\tau)}(s)U_{t-s}\xi(s)dW(s)\Big|^p_{\mathrm{H}_0}\Big) \le C\mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t 1_{[0,\tau)}(s)U_{-s}\xi(s)dW(s)\Big|^p_{\mathrm{H}_0}\Big)$$
$$\le CB_p(H_0)\mathbb{E}\Big[\int_0^\tau \|U_{-s}\xi(s)\|^2_{R(K,\mathrm{H}_0)}ds\Big]^{\frac{p}{2}} = C_p\mathbb{E}\Big[\int_0^\tau \|\xi(s)\|^2_{R(K,\mathrm{H}_0)}\,ds\Big]^{\frac{p}{2}}.$$

□

We next extend the above results replacing the starting time 0 by a random one. Let T_0 be a finite accessible $\mathbb{F}$-stopping time, $\xi\in\mathcal{M}^p_{\mathrm{loc}}([0,\infty),\mathbb{F}^{T_0},\mathrm{R}(\mathrm{K},\mathrm{H}_0))$, $T>0$ and τ be a finite accessible $\mathbb{F}^{T_0}$ stopping time bounded from above by T and such that $\mathbb{E}\big(\int_0^\tau \|\xi(t)\|^2_{R(K,H_0)}dt\big)^{p/2} < \infty$. Then since the process W^{T_0} defined by $W^{T_0}(t) := W(T_0+t)-W(T_0)$, $t\ge 0$ is a $\mathbb{F}^{T_0}$ Wiener process, the operator $J^{T_0}_{[0,\tau)}\xi$ defined by

$$[J^{T_0}_{[0,\tau)}\xi](t) = \int_0^t 1_{[0,\tau)}(r)U_{t-r}\xi(r)\,dW^{T_0}(r), \quad t\ge 0, \tag{3.13}$$

satisfies the inequality (3.11). Informally, if one lets $u(T_0+t)=\xi(t)$ and $[J_{[T_0,T_0+\tau)}u](t) = [J^{T_0}_{[0,\tau)}\xi](t)$ so that $[J_{[T_0,T_0+\tau)}u](t) = \int_{T_0}^{T_0+t} 1_{[T_0,T_0+\tau)}(s)U_{T_0+t-s}u(s)\,dW(s)$, Theorem 3.10 yields for $\hat{C}_p(T) := \tilde{C}^p_p(T)C^p_p(E_0)C^p_p(\mathrm{H}_0)\hat{B}_p(E_0)$

$$\mathbb{E}\int_0^T |J_{[T_0,T_0+\tau)}u(t)|^p_{E_0}\,dt \le \hat{C}_p(T)\mathbb{E}\Big(\int_{T_0}^{T_0+\tau}\|u(t)\|^2_{\mathrm{R}(\mathrm{K},\mathrm{H}_0)}\,dt\Big)^{\frac{p}{2}}. \tag{3.14}$$

Thus we have the following version of Theorem 3.10 using Lemma 3.3.

Corollary 3.12. *Assume that the assumptions of Lemma 3.3 are satisfied. Then for each $T>0$ there exists a constant $\hat{C}_p(T)$ such that:*

(i) $\lim_{T\to 0}\hat{C}_p(T) = 0$,

(ii) *For every finite accessible $\mathbb{F}$-stopping time T_0, every $\mathbb{F}^{T_0}$ stopping time τ bounded by T and every process $\xi\in\mathcal{M}^p_{loc}([0,\infty),\mathbb{F}^{T_0},\mathrm{R}(\mathrm{K},\mathrm{H}))$ such that $\mathbb{E}\big(\int_0^\tau \|\xi(t)\|^2_{\mathrm{R}(\mathrm{K},H)}dt\big)^{p/2} < \infty$ one has*

$$\mathbb{E}\int_0^T |J^{T_0}_{[0,\tau)}\xi(t)|^p_E\,dt \le \hat{C}_p(T)\mathbb{E}\Big(\int_0^\tau \|\xi(t)\|^2_{\mathrm{R}(\mathrm{K},H)}\,dt\Big)^{\frac{p}{2}}. \tag{3.15}$$

where $J^{T_0}_{[0,\tau)}$ be defined by (3.13).

3.3. **Examples of the deterministic and the stochastic Strichartz estimates.** Let M be a compact Riemannian manifold M of dimension $d\ge 2$. According to Burq et all [12], Assumption 3.1 is satisfied by the Hilbert space $\mathcal{H}_0 = L^2(M)$, the $\mathcal{C}_0$-group of unitary operators $(U_t, t\in\mathbb{R})$ with infinitesimal generator iA, where $A:=\Delta$ is the Laplace-Beltrami operator on M, and the spaces $\mathrm{H}_0 = H^{\frac{1}{p},2}(M)$ and $\mathrm{E}_0 = L^q(M)$, provided the parameters, $p \in [2,\infty)$ and $q \in (2,\infty)$ satisfy the so called scaling admissible condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \tag{3.16}$$

Indeed, on $H_0 = H^{\frac{1}{p},2}(M)$ we may consider either the norm $\|.\|_{H^{1/p,2}(M)}$ or, since $H_0 = D((-\Delta)^{1/(2p)})$ the equivalent norm $|\Delta^{1/(2p)}.|_{L^2(M)}$ for which $(U_t, t \in \mathbb{R})$ is a group of isometries.

It is well known that when M is replaced by $\mathbb{R}^d$, then Assumption 3.1 is satisfied by the $\mathcal{C}_0$ unitary group generated by the operator $i\Delta$, and the spaces $\mathcal{H}_0 = \mathrm{H}_0 = L^2(\mathbb{R}^d)$ and $\mathrm{E}_0 = L^q(\mathbb{R}^d)$ provided $p \in [2,\infty)$ and $q \in (2,\infty)$ satisfy the scaling admissible condition (3.16). In this setting, the identity (3.16) is optimal with these spaces for (3.1) to hold true.

It is shown in [12, Theorem 4] that when $M = S^2$ is the two-dimensional sphere, then Assumption 3.1 is satisfied by the $\mathcal{C}_0$-group of unitary operators generated by the operator $i\Delta$ with the following choice parameters: $p = 4$, $\mathrm{E}_0 = L^4(M)$ and $\mathrm{H}_0 = H^{s,2}(M)$ for $s > s_0(2) = \frac{1}{8}$, which proves that (3.2) is not optimal for $\mathrm{H}_0 = H^{\frac{1}{4},2}(M)$ and $\mathrm{E}_0 = L^4(M)$. Note also that (3.2) does not hold when $s < s_0(2)$.

The following result proves that on compact manifolds, the homogenous Strichartz estimates (3.1) and Lemma 3.2 hold for the following spaces: $\mathrm{H} = H^{\sigma+\frac{1}{p},2}(M)$ and $\mathrm{E} = W^{\sigma,q}(M)$, for $\sigma \geq 0$.

Proposition 3.13. *Let M be compact Riemanian manifold, $\big(U(t) = e^{it\Delta}, t \in \mathbb{R}\big)$, (p,q) satisfy the scaling admissible condition (3.16). Then for each $\sigma \geq 0$ and $T > 0$ there exists a constant $\bar{C}_q(T) > 0$ such that $\lim_{T\searrow 0} \bar{C}_q(T) = 0$ and*

(i) *For every $v_0 \in H^{\sigma+\frac{1}{p},2}(M)$,*

$$\Big(\int_0^T \|U(t)v_0\|^p_{W^{\sigma,q}} dt\Big)^{1/p} \leq \bar{C}_q(T)\|v_0\|_{H^{\sigma+\frac{1}{p},2}}. \tag{3.17}$$

(ii) *For every $g \in L^1(0,T; H^{\sigma+\frac{1}{p},2}(M))$,*

$$\Big(\int_0^T \|(U * g)(t)\|^p_{W^{\sigma,q}} dt\Big)^{1/p} \leq \bar{C}_q(T)\int_0^T \|g(t)\|_{H^{\sigma+\frac{1}{p},2}} dt. \tag{3.18}$$

Proof. This result follows Lemma 3.2 and Lemma 3.3. Set $\mathcal{H}_0 = L^2(M)$. First let us notice that for $\sigma = 0$ and $\mathrm{H}_0 = H^{\frac{1}{p},2}(M)$ and $\mathrm{E}_0 = L^q(M)$ the above inequalities are satisfied by Lemma 3.2 since in view of [12] Assumption 3.1 is satisfied for this choice of spaces. Let us denote by A_r the version of the operator A on the space $L^r(M)$, $r \in [1,\infty)$. Then the space $\mathrm{H} = H^{\frac{1}{p}+\sigma,2}(M)$ is equal to $D(A_2^{\frac{1}{2p}+\frac{\sigma}{2}})$. Moreover, $H^{\sigma,q}(M) = D(A_q^{\frac{\sigma}{2}})$; since $q \in (2,\infty)$ we have $H^{\sigma,q}(M) \subset W^{\sigma,q}(M) =: \mathrm{E}$. The proof is complete. $\square$

Similarly, Corollary 3.4 has the following particular formulation.

Corollary 3.14. *In the framework of Proposition 3.13, if $\mathrm{H} = H^{\sigma+\frac{1}{p},2}(M)$ and either $\mathrm{E} = H^{\sigma,q}(M)$ or $\mathrm{E} = W^{\sigma,q}(M)$, then for every $u_0 \in L^p(\Omega,\mathrm{H})$ and every $T > 0$ the trajectories of the process $(U_t u_0, t \in [0,T])$, belong a.s. to $C([0,T];\mathrm{E}) \cap L^p(0,T;\mathrm{H})$ and moreover*

$$\mathbb{E}\Big(\int_0^T |U_t u_0|^p_{\mathrm{E}}\, dt + \sup_{t\in[0,T]} |U_t u_0|^p_{\mathrm{H}}\Big) \;\leq\; \big(1 + \bar{C}^q_p(T)\big)\mathbb{E}|u_0|^p_{\mathrm{H}} \tag{3.19}$$

Finally note that Corollary 3.12 holds for $\mathrm{H} = H^{\sigma+\frac{1}{p},2}(M)$ and $\mathrm{E} = W^{\sigma,q}(M)$ for any $\sigma \geq 0$, and for $p \in [2,\infty)$, $q \in (2,\infty)$ satisfying the scaling admissible condition $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$. .

4. Stochastic NSEs: abstract local existence result

The aim of this section is to prove an abstract local existence result that will be used subsequently to prove the local existence for certain nonlinear Schrödinger equations. This section is divided into five subsections.

4.1. **Asumptions and truncated equation.** We begin with a description of the main assumptions. The first one of them is just Assumption 3.1.

Assumption 4.1.(i) *Assume that the spaces* H_0 *and* E_0 *and the operator* A *satisfy Assumption 3.1 with some number* $p \in (2,\infty)$. *Let* $\hat{s} \geq 0$ *and put* $\mathrm{H} = D((-A)^{\frac{\hat{s}}{2}})) \subset \mathrm{H}_0$. *Assume also that* $\mathrm{E} \subset E_0$ *is a separable Banach space such that* $\mathrm{E} \supset D((-\tilde{A})^{\frac{\hat{s}}{2}})$.

(ii) *Assume that* F *is a locally Lipschitz map from* $\mathrm{H}\cap\mathrm{E}$ *to* H *in the following sense. There exists positive constants* C *and* $\beta \in [1,p)$ *such that for all* $u, v \in \mathrm{H}\cap\mathrm{E}$

$$|F(u)|_{\mathrm{H}} \leq C\left[\left(1+|u|_{\mathrm{E}}^{\beta}\right) + \left(1+|u|_{\mathrm{E}}^{\beta-1}\right)|u|_{\mathrm{H}}\right], \tag{4.1}$$

$$\begin{aligned}\left|F(u)-F(v)\right|_{\mathrm{H}} \leq{}& C\left[1+|u|_{\mathrm{E}}^{(\beta-2)^+} + |v|_{\mathrm{E}}^{(\beta-2)^+}\right]\left[1+|u|_{\mathrm{H}}+|v|_{\mathrm{H}}\right]|u-v|_{\mathrm{E}} \\ &+ C\left[1+|u|_{\mathrm{E}}^{\beta-1}+|v|_{\mathrm{E}}^{\beta-1}\right]|u-v|_{\mathrm{H}}.\end{aligned} \tag{4.2}$$

(iii) *Assume that* G *is a locally Lipschitz map from* $\mathrm{H}\cap\mathrm{E}$ *to* $R(K,\mathrm{H})$ *in the following sense. there exist positive constants* C *and* $a \in [1,p/2)$ *such that for all* $u, v \in \mathrm{H}\cap\mathrm{E}$

$$|G(u)|_{R(K,\mathrm{H})} \leq C\left[\left(1+|u|_{\mathrm{E}}^{a}\right) + \left(1+|u|_{\mathrm{E}}^{a-1}\right)|u|_{\mathrm{H}}\right], \tag{4.3}$$

$$\begin{aligned}|G(u)-G(v)|_{R(K,\mathrm{H})} \leq{}& C\left(1+|u|_{\mathrm{E}}^{a-1}+|v|_{\mathrm{E}}^{a-1}\right)|u-v|_{\mathrm{H}} \\ &+ C\left(1+|u|_{\mathrm{E}}^{(a-2)^+}+|v|_{\mathrm{E}}^{(a-2)^+}\right)\left(1+|u|_{\mathrm{H}}+|v|_{\mathrm{H}}\right)|u-v|_{\mathrm{E}}.\end{aligned} \tag{4.4}$$

We use the convention $x^0 = 1$. Lemma 3.3 implies that the spaces H and E satisfy the Assumption 3.1. Although the above growth and local Lipschitz continuity conditions are a bit unusual one can easily see that, as in the more typical situations, (4.2) implies (4.1) and (4.4) implies (4.3) if $\beta, a \geq 2$.
In this section we will consider the following stochastic Itô nonlinear Schrödinger Equation of the following form:

$$idu(t) + Au(t)\,dt = F(u)\,dt + G(u)\,dW(t), \quad u(0) = u_0, \tag{4.5}$$

where the initial data u_0 belongs to the Hilbert space H. For $d \geq c \geq 0$, let us denote

$$Y_{[c,d]} := \mathcal{C}([c,d];\mathrm{H}) \cap L^p(c,d;\mathrm{E}), \tag{4.6}$$

Obviously, $Y_{[c,d]}$ is a Banach space with norm defined by:

$$|u|_{Y_{[c,d]}}^p := \sup_{r\in[c,d]} |u(r)|_{\mathrm{H}}^p + \int_c^d |u(r)|_{\mathrm{E}}^p\,ds.$$

Note that the $(Y_{[c,t]})_{t\geq c}$ is an increasing family of Banach spaces. More precisely, if $t > \tau > c$ and $u \in Y_{[c,t]}$, then $u_{|[c,\tau]} \in Y_{[c,\tau]}$ and $|u_{|[c,\tau]}|_{Y_{[c,\tau]}} \leq |u|_{Y_{[c,t]}}$. To ease notation, we will simply write $Y_t = Y_{[0,t]}$.

Let $\mathbb{M}^p(Y_{T_1},\mathbb{F}^{T_0}) := \mathbb{M}^p(Y_{[0,T_1]},\mathbb{F}^{T_0})$ denote the Banach space of continuous H-valued $\mathbb{F}^{T_0}$-adapted local processes $(X_t, t\in[0,T_1])$ which satisfy

$$\|X\|^p_{\mathbb{M}^p(Y_{T_1},\mathbb{F}^{T_0})} = \mathbb{E}\Big(\sup_{r\in[0,T_1]}|X(t)|^p_{\mathrm{H}} + \int_0^{T_1}\|X(r)\|^p_{\mathrm{E}}dr\Big) < \infty. \tag{4.7}$$

Similarly, let $\mathbb{M}^p_{\mathrm{loc}}(Y_{[0,T_1)},\mathbb{F}^{T_0})$ denote the set of all H-valued $\mathbb{F}^{T_0}$-adapted and continuous local processes $(X_t, t\in[0,T_1))$ such that $X\in\mathbb{M}^p(Y_{\tau_n},\mathbb{F}^{T_0})$ for any sequence of $\mathbb{F}^{T_0}$ stopping times (τ_n) approximating T_1.
Now we will introduce definitions of local and maximal local solutions; they are modifications of definitions used earlier, such as in [4], [6] and [7].

Definition 4.2. *Assume that T_0 is a finite accessible $\mathbb{F}$-stopping time and u_0 is a* H*-valued $\mathcal{F}_{T_0}$-measurable random variable. A **local mild solution** to equation* (4.5) *with initial condition u_0 at time T_0 is a process u defined as $u(T_0+t)=X(t)$, $t\in[0,T_1)$, where*
(i) T_1 is an accessible $\mathbb{F}^{T_0}$ stopping time,
(ii) $X=(X(t),t\in[0,T_1))$ belongs to $\mathbb{M}^p_{loc}(Y_{[0,T_1)},\mathbb{F}^{T_0})$,
(iii) for some approximating sequence (τ_n) of $\mathbb{F}^{T_0}$ stopping times for T_1, one has,

$$X(t\wedge\tau_n) = U_{t\wedge\tau_n}u_0 + \int_0^{t\wedge\tau_n} U_{t\wedge\tau_n - r}F(X(r))\,dr + I_{\tau_n}(G(X))(t), \tag{4.8}$$

for every every $n=1,2,\cdots$ and $t\geq 0$, where $I_{\tau_n}(G(X))$ is the process defined by

$$I_{\tau_n}(G(X))(t) = \int_0^\infty 1_{[0,t\wedge\tau_n]}(r)U_{t-r}G(X(r))\,dW^{T_0}(r). \tag{4.9}$$

A local mild solution $u=\big(u(T_0+t)\,,\,0\leq t<T_1\big)$ to problem (4.5) *is **pathwise unique** if for any other local mild solution $\tilde u=\big(\tilde u(T_0+t)\,,\,0\leq t<\tilde T_1\big)$ for this problem, $u(T_0+t,\omega)=\tilde u(T_0+t,\omega)$ for almost every $(t,\omega)\in[0,T_1\wedge\tilde T_1)\times\Omega$.*
*A local mild solution $u=(u(T_0+t)\,,\,t\in[0,T_1))$ is called **maximal** if for any other local mild solution $\tilde u=(\tilde u(T_0+t)\,,\,t\in[0,\tilde T_1))$ satisfying $\tilde T_1\geq T_1$ a.s. and $\tilde u|_{[T_0,T_0+T_1)\times\Omega}\sim u$, one has $T_1=\tilde T_1$ a.s. The $\mathbb{F}$-stopping time T_0+T_1 will be called the life span of the maximal local mild solution u. Furthermore, a maximal local mild solution $(u(T_0+t),\ t\in[0,T_1))$ is called global if its lifespan is equal to ∞ a.s., i.e. $T_1=\infty$ a.s.*

The existence and uniqueness of a local maximal solution to (4.5) will be proved in section 5. We at first prove the existence and the uniqueness of the solution when its norm is truncated. Thus let $\theta:\mathbb{R}_+\to[0,1]$ be a $\mathcal{C}^\infty_0$ non increasing function such that

$$\inf_{x\in\mathbb{R}_+}\theta'(x)\geq -1,\quad \theta(x)=1 \text{ iff } x\in[0,1] \quad\text{and } \theta(x)=0 \text{ iff } x\in[2,\infty). \tag{4.10}$$

and for $n\geq 1$ set $\theta_n(\cdot)=\theta(\frac{\cdot}{n})$. Let us fix some finite accessible $\mathbb{F}$-stopping time T_0, some constant $T>0$ and some accessible $\mathbb{F}^{T_0}$-stopping time T_1 such that $T_1\leq T$. The rest of this section is devoted to prove existence and uniqueness of the solution X^n to the following evolution equation for $t\in[0,T_1]$:

$$\begin{aligned} X^n(t) =& U_t u^n(T_0) + \int_0^t U_{t-r}\big[\theta_n(|X^n|_{Y_r})F(X^n(r))\big]\,dr \\ &+ \int_0^t U_{t-r}\big[\theta_n(|X^n|_{Y_r})G(X^n(r))\big]\,dW^{T_0}(r). \end{aligned} \tag{4.11}$$

It is similar to that introduced by de Bouard and Debussche [16] [17]. The first step consists in showing that for small T_1 the right handside of (4.11) is a strict contraction.

Norm estimates of the corresponding three terms are studied in separate subsections. We will often use the following straightforward inequalities.

Lemma 4.3. *If $h:\mathbb{R}_+\to\mathbb{R}_+$ is a non decreasing function, then for every $x,y\in\mathbb{R}$,*

$$\theta_n(x)h(x)\le h(2n),\quad |\theta_n(x)-\theta_n(y)|\le \frac{1}{n}|x-y|. \tag{4.12}$$

4.2. **Estimates for the deterministic term.** The following results are modified and extended versions of the argument from [12]. Since $p>\beta$ by Assumption 4.1 (i), we have

$$\gamma:=1-\frac{\beta}{p}>0. \tag{4.13}$$

Let us recall that the space $Y_{[T_0,T_0+T]}$ has been defined in (4.6). For $X\in Y_T:=Y_{[0,T]}$ put

$$[\Phi^n_T(X)](t) \;=\; \int_0^t U_{t-r}\big[\theta_n(|X|_{Y_r})F(X(r)\big]\,dr,\ t\in[0,T]. \tag{4.14}$$

The following two results are formulated and proved for $T_0=0$ but their generalization to any T_0 is straightforward since the integrals are deterministic.

Lemma 4.4. *Assume that Assumption 3.1 is satisfied and that the map F satisfies Assumption 4.1(ii). Let $n>0$ and $T>0$. Then the map Φ^n_T defined by (4.14) maps the space Y_T into itself. Moreover, there exists a generic constant $C>0$ such that for each $X\in Y_T$,*

$$\|\Phi^n_T(X)\|_{C([0,T];\mathrm{H})}\le C\Big[T+\big(T^\gamma+T^{\gamma+\frac{1}{p}}\big)(2n)^\beta\Big], \tag{4.15}$$

$$\|\Phi^n_T(X)\|_{L^p(0,T;E)}\le C\tilde{C}_p(T)\Big[T+\big(T^\gamma+T^{\gamma+\frac{1}{p}}\big)(2n)^\beta\Big], \tag{4.16}$$

where $\tilde{C}_p(T)$ is the constant from Assumption 3.1 with H and E instead of H_0 and E_0 respectively.

Proof. It is sufficient to prove inequalities (4.15-4.16). For this aim let us fix $X\in Y_T$.
Step 1. We at first prove (4.15). The inequality (3.3) from Lemma 3.2 yields

$$\sup_{t\in[0,T]}|[\Phi^n_T(X)](t)|_{\mathrm{H}} \;\le\; \int_0^T \theta_n(|X|_{Y_t})|F(X(t))|_{\mathrm{H}}\,dt. \tag{4.17}$$

Thus, it is enough to estimate the $L^1(0,T;\mathrm{H})$-norm of $\theta_n(|X|_{Y_\cdot})F(u(\cdot))$. Let us define $T^*:=\inf\{t\ge 0: |X|_{Y_t}\ge 2n\}\wedge T$ and note that $\theta_n(|X|_{Y_t})=0$ for $|X|_{Y_t}\ge 2n$. Then since $\tau\to|X|_{Y_t}$ is non decreasing on $[0,T]$, $\theta_n(|X|_{Y_t})=0$ for $t\ge T^*$. Using Assumption (4.1) and Hölder's inequality we infer that for some $C>0$:

$$\begin{aligned}
\int_0^T\theta_n(|X|_{Y_t})|F(X(t))|_{\mathrm{H}}\,dt &\le C\Big[T^*+\int_0^{T^*}\big[|X(t)|_E^\beta+|X(t)|_E^{\beta-1}|X(t)|_{\mathrm{H}}\big]\,dt\Big]\\
&\le C\Big[T+\int_0^{T^*}|X(t)|_E^\beta dt+\sup_{t\in[0,T^*]}|X(t)|_{\mathrm{H}}\int_0^{T^*}|X(t)|_E^{\beta-1}dt\Big]\\
&\le C\Big[T+T^\gamma\Big(\int_0^{T^*}|X(t)|_E^p dt\Big)^{\frac{\beta}{p}}+T^{\gamma+\frac{1}{p}}\sup_{t\in[0,T^*]}|X(t)|_{\mathrm{H}}\Big(\int_0^{T^*}|X(t)|_E^p dt\Big)^{\frac{\beta-1}{p}}\Big]\\
&\le C\Big[T+\big(T^\gamma+T^{\gamma+\frac{1}{p}}\big)(2n)^\beta\Big].
\end{aligned} \tag{4.18}$$

The inequalities (4.17) and (4.18) conclude the proof of (4.15).
Step 2. We turn to the proof of (4.16); Assumption 4.1(i) implies $\hat{s}+\frac{1}{p}=s$ and, by (4.14), $\Phi^n_T(X)=U*\big[\theta_n(|X|_{Y_\cdot})F(X(\cdot))\big]$. Hence Corollary 3.4 shows that it is enough to

upper estimate $\int_0^T \theta_n(|X|_{Y_t})|F(X(t))|_{\rm H}\,dt$, which has been done in (4.18). This completes the proof. $\square$

The next result establishes the Lipschitz properties of Φ^n_T as a map acting on Y_T with some explicit bound of the Lipschitz constant. This the main result of this subsection.

Proposition 4.5. *Assume that Assumption 3.1 is satisfied and that F satisfies Assumption 4.1(ii). Then the map Φ^n_T defined in* (4.14) *is Lipschitz from the space Y_T into itself. More precisely, for some generic constant $C>0$ and all $X_1, X_2 \in Y_T$ we have*

$$\|\Phi^n_T(X_2)-\Phi^n_T(X_1)\|_{C([0,T];{\rm H})} \leq C\,\big[T+nT^{\frac{p-1}{p}}+n^\beta(T^\gamma+T^{\gamma+1/p})\big]|X_1-X_2|_{Y_T},$$

$$\|\Phi^n_T(X_2)-\Phi^n_T(X_1)\|_{L^p(0,T;E)} \leq C\,\tilde{C}_p(T)\big[T+nT^{\frac{p-1}{p}}+n^\beta(T^\gamma+T^{\gamma+1/p})\big]|X_1-X_2|_{Y_T},$$

where $\tilde{C}_p(T)$ is the constant from Assumption 3.1. Furthermore, given any $T>0$ there exists a positive constant $L_n(T)$ such that $L_n(.)$ is non decreasing, $\lim_{T\to 0} L_n(T)=0$ locally uniformly in n, and such that

$$|\Phi^n_T(X_2)-\Phi^n_T(X_1)|_{Y_T} \leq L_n(T)|X_1-X_2|_{Y_T}. \tag{4.19}$$

Proof. Let $X_1, X_2 \in Y_T$. We at first upper estimate the $C([0,T],{\rm H})$ and $L^p(0,T;E)$-norms of the difference $\Phi^n_T(X_2)-\Phi^n_T(X_1)$ in terms of A_T defined by

$$A_T = \int_0^T \big|\theta_n(|X_2|_{Y_t})F(X_2(t))-\theta_n(|X_1|_{Y_t})F(X_1(t))\big|_{\rm H}\,dt.$$

Indeed, arguing as in the proof of Lemma 4.4, we have

$$\|\Phi^n_T(X_2)-\Phi^n_T(X_1)\|_{C([0,T];{\rm H})} \leq A_T, \quad |\Phi^n_T(X_2)-\Phi^n_T(X_1)|_{L^p(0,T;E)} \leq C\tilde{C}_p(T)A_T, \tag{4.20}$$

where $\tilde{C}_p(T)$ is the constant introduced in (3.1). For $i=1,2$ set $T_i := \inf\{t\geq 0 : |X_i|_{Y_t} \geq 2n\}\wedge T$; then for $i=1,2$ we have

$$\sup_{t\in[0,T_i]} |X_i(t)|^p_{\rm H} + \int_0^{T_i} |X_i(t)|^p_E dt \leq (2n)^p.$$

Without loss of generality we may assume that $T_1 \leq T_2$. Using once more the fact that the functions $[0,T]\ni t\mapsto |X_i|_{Y_t}$ are non decreasing, we deduce that $\theta_n(|X_i|_{Y_\tau})=0$ for $\tau\geq T_i$, $i=1,2$, and hence

$$A_T \leq \int_0^{T_2}\big|\theta_n(|X_1|_{Y_t})-\theta_n(|X_2|_{Y_t})\big||F(X_2(t))|_{\rm H}dt + \int_0^{T_1}\theta_n(|X_1|_{Y_t})\big|F(X_2(t))-F(X_1(t))\big|_{\rm H}dt.$$

Therefore, in view of the conditions (4.1) and (4.2), since $2\leq\beta<p$, Hölder's inequality yields the existence of a constant $C>0$ such that, for γ defined by (4.13), we have

$$\begin{aligned}
A_T \leq & \int_0^{T_2}\frac{C}{n}\big||X_1|_{Y_t}-|X_2|_{Y_t}\big|\big[1+|X_2(t)|^\beta_E+\big(1+|X_2(t)|^{\beta-1}_E\big)|X_2(t)|_{\rm H}\big]dt\\
&+C\int_0^{T_1}\big(1+|X_1(t)|^{(\beta-2)^+}_E+|X_2(t)|^{(\beta-2)^+}_E\big)|X_1(t)-X_2(t)|_{\rm E}\big[1+|X_1(t)|_{\rm H}+|X_2(t))|_{\rm H}\big]dt\\
&+C\int_0^{T_1}\big(1+|X_1(t)|^{\beta-1}_{\rm E}+|X_2(t)|^{\beta-1}_E\big)|X_1(t))-X_2(t)|_H dt\\
\leq & \frac{C}{n}|X_1-X_2|_{Y_T}\Big\{T+T^\gamma\Big(\int_0^{T_2}|X_2(t)|^p_E dt\Big)^{\frac{\beta}{p}}\\
&\qquad+\sup_{t\in[0,T_2]}|X_2(t)|_H\,\Big[T+T^{\gamma+1/p}\,\Big(\int_0^{T_2}|X_2(t)|^p_E dt\Big)^{\frac{\beta-1}{p}}\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+C\sup_{t\in[0,T_1]}\big(1+\big|X_1(t)\big|_H+\big|X_2(t)\big|_{\rm H}\big)\\
&\qquad\times\Big(\int_0^{T_1}\Big[1+\sum_{i=1,2}|X_i(t)|_E^{\frac{(\beta-2)^+p}{p-1}}\Big]dt\Big)^{\frac{p-1}{p}}\Big(\int_0^{T_1}|X_1(t)-X_2(t)|_E^p dt\Big)^{\frac1p}\\
&+C\sup_{t\in[0,T_1]}\big|X_1(t)-X_2(t)\big|_{\rm H}\Big[T+T^\gamma\Big(\int_0^{T_1}\sum_{i=1,2}|X_i(t)|_E^p dt\Big)^{\frac{\beta-1}{p}}\Big].
\end{aligned}
$$

Recall that $p>\beta$; thus $p>\frac{p}{p-1}(\beta-2)$. Hence, since $T_1\le T_2$, Hölder's inequality yields

$$
\begin{aligned}
A_T\;&\le\;\frac{C}{n}|X_1-X_2|_{Y_T}\big[T+T^\gamma(2n)^\beta+(2n)\big\{T+T^{\gamma+1/p}(2n)^{\beta-1}\big\}\big]\\
&\quad+C(1+4n)|X_1-X_2|_{L^p(0,T;E)}\big(T^{\frac{p-1}{p}}+T^{\gamma+1/p}(2n)^{(\beta-2)^+}\big)\\
&\quad+C\sup_{t\in[0,T_1]}\big|X_1(t)-X_2(t)\big|_{\rm H}\big(T+T^\gamma(2n)^{\beta-1}\big)\\
&\le\;C\,|X_1-X_2|_{Y_T}\big[T+nT^{\frac{p-1}{p}}+n^{\beta-1}T^\gamma+n^\beta T^{\gamma+1/p}\big].
\end{aligned}
\tag{4.21}
$$

The inequalities (4.20)-(4.21) conclude the proof of the two first upper estimates of the difference $\Phi_T^n(X_2)-\Phi_T^n(X_1)$ in the proposition. Finally, let

$$
L_n(T)=C\big(1+\tilde C_p(T)\big)\big[T+nT^{\frac{p-1}{p}}+n^\beta(T^\gamma+T^{\gamma+1/p})\big].
$$

Then since $\lim_{T\to0}\tilde C_p(T)=0$, it remains bounded for $T\in(0,1]$ and we deduce that $\lim_{T\to0}L_n(T)=0$ locally uniformly in n. This completes the proof of the proposition. $\square$

4.3. **Estimates for the stochastic term.** Let us recall that according to Assumption 3.5, we assume that K is a separable Hilbert space and $W=(W(t),t\ge0)$ is a K-valued cylindrical Brownian motion on a filtered probability space $(\Omega,\mathcal F,\mathbb F,\mathbb P)$. We fix a finite accessible $\mathbb F$-stopping time T_0 and keep the notation introduced in Definition 3.9. Recall that $\mathcal F_t^{T_0}=\mathcal F_{T_0+t}$ for $t\ge0$ and $(W^{T_0}(t),t\ge0)$ is the $\mathbb F^{T_0}$ Brownian motion defined by $W^{T_0}(t)=W(T_0+t)-W(T_0)$. Finally T will be some positive constant and T_1 a $\mathbb F^{T_0}$ accessible stopping time such that $T_1\le T$. Recall that the space $\mathbb M^p(Y_{T_1},\mathbb F^{T_0})$ has been defined in (4.7). As usual, we let $a^0=1$ for any $a\ge0$.
Let $X\in\mathbb M^p(Y_T,\mathbb F^{T_0})$. Then for every $n\ge1$ we set

$$
\xi^n(t)=\theta_n(|X|_{Y_t})G(X(t)),\ t\in[0,T],
$$

and put $\Psi_T^{T_0,n}(X)=J^{T_0}\xi^n$ with J^{T_0} defined by

$$
\big[J^{T_0}\xi^n\big](t)=\int_0^t U_{t-r}\xi^n(r)\,dW^{T_0}(r),\quad t\in[0,T].
\tag{4.22}
$$

We at first prove that $\Psi_T^{T_0,n}$ maps $\mathbb M^p(Y_T,\mathbb F^{T_0})$ into itself. More precisely, we have the following result.

Lemma 4.6. *Assume that Assumption 3.1 is satisfied,* E *is a martingale type* 2 *Banach space and that the map G satisfies Assumption 4.1(iii). Then $\Psi_T^{T_0,n}$ maps $\mathbb M^p(Y_T,\mathbb F^{T_0})$ into itself. Moreover, one can find a constant $C_p>0$ such that for every $X\in\mathbb M^p(Y_T,\mathbb F^{T_0})$ and the constant $\hat C_p(T)$ defined in* (3.15)*, we have:*

$$
\mathbb E\int_0^T|\Psi_T^{T_0,n}(X)(t)|_E^p dt\le C_p\,\hat C_p(T)\Big[T^{p/2}+T^{p/2-a}\big(1+T\big)n^{pa}+T^{p/2}n^p\Big],
\tag{4.23}
$$

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|\Psi_T^{T_0,n}(X)(t)|_{\mathrm{H}}^p\Big)\le C_p\Big[T^{p/2}+T^{p/2-a}\big(1+T\big)n^{pa}+T^{p/2}n^p\Big]. \tag{4.24}$$

To ease notation, in the proof below, as well as in the other proofs in this section, we will omit the subscript T_0. For instance we will simply write $\mathbb{F}$ and J instead of $\mathbb{F}^{T_0}$ and J^{T_0}.

Proof. Let us take $X\in\mathbb{M}^p(Y_T,\mathbb{F})$; first we will prove (4.23). Using inequality (3.15) from Corollary 3.12 with $\tau=T$ we deduce that

$$\mathbb{E}\int_0^T|J\xi^n(t)|_E^p dt\le\hat{C}_p(T)\mathbb{E}\Big(\int_0^T\|\xi^n(t)\|^2_{R(K,\mathrm{H})}\,dt\Big)^{p/2}.$$

Let $T^*:=\inf\{t\ge 0:|X|_{Y_t}\ge 2n\}\wedge T$. Then $\theta_n(|X|_{Y_t})=0$ for $t\in[T^*,T]$ and

$$\sup_{t\in[0,T^*]}|X(t)|_{\mathrm{H}}^p+\int_0^{T^*}|X(t)|_E^p dt\le(2n)^p.$$

Hence the growth condition (4.3) on G and Hölder's inequality imply that

$$\begin{aligned}
&\int_0^T\|\xi^n(t)\|^2_{R(K,\mathrm{H})}\,dt\le C\int_0^{T^*}\Big[\big(1+|X(t)|_E^{2a}\big)+\big(1+|X(t)|_E^{2a-2}\big)|X(t)|_{\mathrm{H}}^2\Big]\,dt\\
&\le C\Big[T+T^{1-\frac{2a}{p}}\Big(\int_0^{T^*}|X(t)|_E^p\,dt\Big)^{\frac{2a}{p}}+\sup_{t\in[0,T^*]}|X(t)|_{\mathrm{H}}^2\Big\{T+T^{1-\frac{2a-2}{p}}\Big(\int_0^{T^*}|X(t)|_E^p\,dt\Big)^{\frac{2a-2}{p}}\Big\}\Big]\\
&\le C\Big[T+T^{1-\frac{2a}{p}}(2n)^{2a}+(2n)^2\Big(T+T^{1-\frac{2a-2}{p}}(2n)^{2a-2}\Big)\Big].
\end{aligned}\tag{4.25}$$

This completes the proof of (4.23). To prove (4.24) we apply inequality (3.12) to get

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t U_{t-r}\xi^n(r)dW(r)\Big|_{\mathrm{H}}^p\Big)\le C_p\mathbb{E}\Big[\int_0^T\|\xi^n(r)\|^2_{R(K,\mathrm{H})}dr\Big]^{\frac{p}{2}}.$$

Combining the above with inequality (4.25) we deduce (4.24). This completes the proof of the proposition. □

We next result proves that the map $\Psi_T^{T_0,n}$ is Lipschitz on $\mathbb{M}^p(Y_T,\mathbb{F}^{T_0})$ and gives an upper bound of its Lipschitz constant.

Proposition 4.7. *Assume that Assumption 3.1 is satisfied and that the map G satisfies Assumption 4.1(iii). Then for every $T>0$ there exists a constant $\hat{L}_n(T)>0$ such that $\hat{L}_n(.)$ is non decreasing, $\lim_{T\to 0}\hat{L}_n(T)=0$ locally uniformly in n, and for $X_1,X_2\in\mathbb{M}^p(Y_T,\mathbb{F}^{T_0})$ and the constant $\hat{C}_p(T)$ defined in* (3.15),

$$\|\Psi_T^{T_0,n}(X_2)-\Psi_T^{T_0,n}(X_1)\|_{\mathbb{M}^p(Y_T,\mathbb{F}^{T_0})}\le\big(1+\hat{C}_p(T)\big)\hat{L}_n(T)\,\|X_1-X_2\|_{\mathbb{M}^p(Y_T,\mathbb{F}^{T_0})}. \tag{4.26}$$

Proof. For $i=1,2$ set $\xi_i^n(t)=\theta_n(|X_i|_{Y_t})G(X_i(t))$. Using once again inequality (3.15) from Corollary 3.12, we deduce that

$$\mathbb{E}\int_0^T\Big|\int_0^t U_{t-r}\big[J\xi_1^n(r)-J\xi_2^n(r)\big]dr\Big|_E^p dt\le\hat{C}_p(T)B_T, \tag{4.27}$$

where $B_T=\mathbb{E}\big(\int_0^T|\xi_2^n(t)-\xi_1^n(t)|^2_{R(K,\mathrm{H})}dt\big)^{\frac{p}{2}}$. Furthermore, the Burkholder-Davis-Gundy inequality (3.8) yields

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|J\xi_1^n(t)-J\xi_2^n(t)|_{\mathrm{H}}^p\Big)\le\hat{B}_p(H)B_T. \tag{4.28}$$

For $i=1,2$ let $T_i=\inf\{t\geq 0: |X_i|_{Y_t}\geq 2n\}\wedge T$. Then one has

$$B_T \;\leq\; C_p \sum_{\{i,j\}=\{1,2\}} \mathbb{E}\Big(\int_0^T \big|\theta_n(|X_i|_{Y_t})-\theta_n(|X_j|_{Y_t})\big|^2 \;\|G(X_i(t))\|^p_{R(K,\mathrm{H})}\, 1_{\{T_j\leq T_i\}}\; dt\Big)^{\frac{p}{2}}$$
$$+C_p \sum_{\{i,j\}=\{1,2\}} \mathbb{E}\Big(\int_0^T \theta_n^2(|X_j|_{Y_t})\|G(X_i(t))-G(X_j(t))\|^2_{R(K,\mathrm{H})}\, 1_{\{T_j\leq T_i\}}\; dt\Big)^{\frac{p}{2}}.$$

Furthermore, $\theta_n(|X_i|_{Y_t})=0$ provided that $t\geq T_i$, $i=1,2$. Therefore, using the property (4.12) on θ, the growth and Lipschitz conditions (4.3) and (4.4) on G, we deduce that for some constant $C>0$

$$B_T\leq C\sum_{\{i,j\}=\{1,2\}}\sum_{l=1,2,3}\big(B^l_T(i,j)\big)^{p/2}, \tag{4.29}$$

where

$$B^1_T(i,j)=\frac{1}{n^2}\mathbb{E}\,\Big(\int_0^{T_i} 1_{\{T_j\leq T_i\}}\big||X_i|_{Y_t}-|X_j|_{Y_t}\big|^2$$
$$\times\big[\big(1+|X_i(t)|_E^{2a}\big)+\big(1+|X_i(t)|_E^{2a-2}\big)|X_i(t)|^2_{\mathrm{H}}\big]dt\Big),$$
$$B^2_T(i,j)=\mathbb{E}\,\Big(\int_0^{T_i\wedge T_j}\big(1+|X_i(t)|_E^{2a-2}+|X_j(t)|_E^{2a-2}\big)|X_i(t)-X_j(t)|^2_{\mathrm{H}}dt\Big),$$
$$B^3_T(i,j)=\mathbb{E}\,\Big(\int_0^{T_i\wedge T_j}\big(1+|X_i(t)|_E^{2(a-2)^+}+|X_j(t)|_E^{2(a-2)^+}\big)\big(1+|X_i(t)|^2_{\mathrm{H}}+|X_j(t)|^2_{\mathrm{H}}\big)$$
$$\times|X_i(t)-X_j(t)|^2_E dt\Big).$$

Using Hölder's inequality since $a<p/2$ we obtain

$$n^2B^1_T(i,j)\leq\big|X_1-X_2\big|^2_{Y_T}\int_0^{T_i}\big[\big(1+|X_i(t)|_E^{2a}\big)+\big(1+|X_i(t)|_E^{2a-2}\big)|X_i(t)|^2_{\mathrm{H}}\big]dt$$
$$\leq C\big|X_1-X_2\big|^2_{Y_T}\Big[T+T^{1-\frac{2a}{p}}n^{2a}+n^2\Big(T+T^{1-\frac{2a-2}{p}}n^{2a-2}\Big)\Big].$$

Using once more Hölder's inequality we deduce

$$B^2_T(i,j)\leq\sup_{t\in[0,T]}|X_i(t)-X_j(t)|^2_{\mathrm{H}}\int_0^{T_i\wedge T_j}\big(1+|X_i(t)|_E^{2a-2}+|X_j(t)|_E^{2a-2}\big)dt$$
$$\leq\sup_{t\in[0,T]}|X_i(t)-X_j(t)|^2_{\mathrm{H}}\Big(T+2T^{1-\frac{2a-2}{p}}n^{2a-2}\Big),$$

and

$$B^3_T(i,j)\leq C\sup_{t\in[0,T_i\wedge T_j]}\big(1+|X_i(t)|^2_{\mathrm{H}}+|X_j(t)|^2_{\mathrm{H}}\big)\Big[\int_0^T|X_i(t)-X_j(t)|^p_E\,dt\Big]^{\frac{2}{p}}$$
$$\times\Big[T^{1-\frac{2}{p}}+T^{1-\frac{a}{p}}\Big(\int_0^{T_i\wedge T_j}\big(|X_i(t)|^p_E+|X_j(t)|^p_E\big)\,dt\Big)^{\frac{(a-2)^+}{p}}\Big]$$
$$\leq\Big(1+8n^2\Big)\Big(T^{1-\frac{2}{p}}+2T^{1-\frac{2(a-1)}{p}}n^{2(a-2)^+}\Big)\Big[\int_0^T|X_i(t)-X_j(t)|^p_E\,dt\Big]^{\frac{2}{p}}.$$

Therefore, $B_T\leq\hat{L}_n(T)\|X_1-X_2\|^p_{\mathbb{M}^p(Y_T,\mathbb{F}^{T_0})}$ if one lets

$$\hat{L}_n(T):=C\big[T^{p/2}+T^{p/2-1}n^p+T^{p/2-a}(1+T)n^{(a-1)p}+n^{(a\vee 2)p/2}T^{p/2-a+1}\big].$$

Since $a < p/2$ and $p > 2$ we have $\lim_{T\to 0}\hat{L}_n(T) = 0$ locally uniformly in n; thus the inequalities (4.27) and (4.28) conclude the proof. $\square$

4.4. **The estimates on the free term.** In order to prove the existence and uniqueness of a solution to the approximating equation (4.11) we have also to show that the first term on the RHS of that equation belongs to the space $\mathbb{M}^p(Y_T,\mathbb{F}^{T_0})$. This in fact follows from Corollary 3.4 and Lemma 3.3. Thus we have.

Lemma 4.8. *Let T_0 be a finite accessible $\mathbb{F}$-stopping time and $u(T_0)\in L^p(\Omega,\mathcal{F}_{T_0},\mathrm{H})$ and $d > c > 0$. Define a process $\mathcal{U}_{[c,d]}(u_0)$ by $\big([\mathcal{U}_{[c,d]}(u(T_0))](t) = U(t-c)(u(T_0))\,,\, t\in[c,d]\big)$. Then the process X defined by $X(t) = \mathcal{U}_{[c,d]}(u(T_0))(t+c)$, $t\in[0,d-c]$, belongs to $\mathbb{M}^p(Y_{d-c},\mathbb{F}^{T_0})$.*

4.5. **Existence and uniqueness of a global solutions to approximating equations.** A direct consequence of all the results proved in sections 4.2-4.4 (which follows by applying the Banach-Cacciopoli Fixed Point Theorem) is presented below.

Theorem 4.9. *Assume that the Assumptions 3.5, 3.1 and 4.1 satisfied with $\beta\in[2,p)$ and $a\in[1,\frac{p}{2})$. Assume also that* E *is a martingale type* 2 *Banach space. Let T_0 be a finite and accessible $\mathbb{F}$-stopping time T_0 and $u(T_0)\in L^p(\Omega,\mathcal{F}_{T_0},\mathrm{H})$. Then for every positive integer n, there exists a unique process $u^n = (u^n(t)\,, t\in[T_0,\infty))$, such that $u^n(t+T_0) = X^n(t)$ for $t\geq 0$ and for every $T>0$, X^n belongs to the space $\mathbb{M}^p(Y_T,\mathbb{F}^{T_0})$ and $X^n = \mathcal{U}_{[0,T]}(u(T_0)) + \Phi^n_T(X^n) + \Psi^{T_0,n}_T(X^n)$. Moreover, given any $T>0$ the process X^n is the unique solution to the evolution equation (4.11) on the time interval $[0,T]$. Moreover, if a local process $v = \big(v(T_0+t) = \tilde{X}(t)\,, t\in[0,\tau)\big)$ is a local solution to (4.11), then the processes X^n and $\bar{X} := \tilde{X}_{|[0,\tau)\times\Omega}$ are equivalent.*

Proof. The second statement is obvious in view of the definitions of the maps $\mathcal{U}_{[0,T]}$, Φ^n_T and $\Psi^{T_0,n}_T$. To prove the first part let us fix $T>0$. It follows from Propositions 4.5 and 4.7 that for every $t>0$, the map

$$\Lambda^n_t : \mathbb{M}^p(Y_t,\mathbb{F}^{T_0}) \ni X \mapsto \mathcal{U}_{[0,t]}(u(T_0)) + \Phi^n_t(X) + \Psi^{T_0,n}_t(X) \in \mathbb{M}^p(Y_t,\mathbb{F}^{T_0}) \tag{4.30}$$

is well defined and Lipschitz. Moreover, if t is sufficiently small (depending on n), this map is a strict contraction. Thus, there exists $\delta>0$ such that if $t\leq\delta$ then Λ^n_t is a $\frac{1}{2}$-contraction in the space $\mathbb{M}^p(Y_t,\mathbb{F}^{T_0})$. Let us define a sequence $(T_k)_{k=0}^\infty$ by $T_k = T_0 + k\delta$, $k\in\mathbb{N}^*$. By the previous conclusion there exists a process $X^{(n,1)}\in\mathbb{M}^p(Y_\tau,\mathbb{F}^{T_0})$ such that $X^{(n,1)} = \mathcal{U}_{[0,\delta]}(u(T_0)) + \Phi^n_\delta(X^{(n,1)}) + \Psi^{T_0,n}_\delta(X^{(n,1)})$. By the definition of the space $\mathbb{M}^p(Y_\delta,\mathbb{F}^{T_0})$, $X^{(n,1)}(\delta)$ belongs to the space $L^p(\Omega,\mathcal{F}_{T_0+\delta},\mathrm{H})$. By Propositions 4.5 and 4.7 we can find a unique process $X^{(n,2)}\in\mathbb{M}^p(Y_\delta,\mathbb{F}^{T_1})$ such that $X^{(n,2)} = \mathcal{U}_{[0,\delta]}(X^{(n,1)}(T_1)) + \Phi^n_\delta(X^{(n,2)}) + \Psi^{T_1,n}_\delta(X^{(n,2)})$. By induction, we can find a sequence $(X^{(n,k)})_{k=1}^\infty$, such that $X^{(n,k)}\in\mathbb{M}^p(Y_\delta,\mathbb{F}^{T_{k-1}})$ and $X^{(n,k)} = \mathcal{U}_{[0,\delta)}(X^{(n,k-1)}(T_{k-1})) + \Phi^n_\delta(X^{(n,k)}) + \Psi^{T_{k-1},n}_\delta(X^{(n,k)})$. Next, we define a process u^n as follows: let $u^n(T_0+t) = X^{(n,1)}(t)$, $t\in[0,\delta)$, and for $k = [\frac{t-T_0}{\delta}]+1$ and $0\leq t<\delta$, let $u^n(T_k+t) = X^{(n,k)}(t)$. The proof is concluded by observing that for every $T>0$, $u^n(T_0+t) = X^n(t)$ for some $X^n\in\mathbb{M}^p(Y_T,\mathbb{F}^{T_0})$ and $X^n = \mathcal{U}_{[0,T]}(u_0) + \Phi^n_T(X^n) + \Psi^{T_0,n}_T(X^n)$.
Finally,to prove the uniqueness, let us choose $\delta>0$ as above and put $\sigma_1 = \tau\wedge\delta$. Then the fixed point Theorem used above implies that the processes $X_{|[0,\sigma_1)\times\Omega}$ and $\tilde{X}_{|[0,\sigma_1)\times\Omega}$ are equivalent. By an inductive argument, if $k\in\mathbb{N}^*$ and $\sigma_k = \tau\wedge(k\delta)$, then the processes $X^n_{|[0,\sigma_k)\times\Omega}$ and $\tilde{X}_{|[0,\sigma_k)\times\Omega}$ are equivalent. Since $\sigma_k\nearrow\tau$, the result follows.

□

5. Existence of a local maximal mild solution to problem (4.5)

Our first aim is to prove the following result about the existence and uniqueness of a local maximal solution to problem to (4.5). Let T_0 be a finite accessible $\mathbb{F}$-stopping time and suppose that the assumption 4.1 on F and G are satisfied. Let $u(T_0)$ be a p-integrable H-valued $\mathcal{F}_{T_0}$ random variable. In the previous section (see Theorem 4.9) we proved for every $n \in \mathbb{N}$ the existence of a unique solution $u^n(T_0+.) = X^n(.)$ on $[0,\infty)$ to the problem (4.11). Let τ_n and $\hat{\tau}_n$ denote the $\mathbb{F}^{T_0}$ stopping times defined by

$$\tau_n = \inf\{t>0 : |X^n|_{Y_t} \geq n\} \wedge n, \qquad \hat{\tau}_n = \inf\{t>0 : |X^n|_{Y_t} \geq 2n\} \wedge n, \tag{5.1}$$

The following result establishes the existence and uniqueness of a local solution to (4.11).

Proposition 5.1. *Let $(X^n(t), t \geq 0)$ be the process introduced in Theorem 4.9. Then the process $(X^n(t)\,,\, t < \tau_n)$ is a local mild solution to problem (4.5) with the filtration is $\mathbb{F}^{T_0}$ and the Brownian Motion W^{T_0}.*

Proof. Obviously τ_n is an accessible $\mathbb{F}^{T_0}$ stopping time and the process X^n satisfies for all $t \geq 0$, $\mathbb{P}$-a.s.,

$$X^n(t) - U_t[u(T_0)] - \int_0^t U_{t-r}\big[\theta_n(|X^n|_{Y_r})F(X^n(r))\big]\,dr = I(t), \tag{5.2}$$

where $I(t) = \int_0^t U_{t-r}\big[\theta_n(|X^n|_{Y_r})G(X^n(r))\big]\,dW^{T_0}(r)$. Since the processes on both sides of the above equality are continuous, the equality still holds when the fixed deterministic time t is replaced by the random one $t\wedge\tau_n$. The definitions of θ_n and τ_n imply $\theta_n(|X^n|_{Y_{r\wedge\tau_n}}) = 1$, and hence the deterministic convolution above stopped at $t\wedge\tau_n$ is equal to $\int_0^t U_{t-r}1_{[0,\tau_n)}(r)F(X^n(r))\,dr$. Moreover by [14, §4.3] and [7, Lemma A.1] we infer

$$I(t\wedge\tau_n) = \int_0^t U_{t-r}\big[1_{[0,\tau_n)}(r)\theta_n(|X^n|_{Y_{r\wedge\tau_n}})G(X^n(r\wedge\tau_n))\big]\,dW^{T_0}(r). \tag{5.3}$$

Finally, as above $1_{[0,\tau_n)}(s)G(X^n(s\wedge\tau_n)) = G(X^n(s))$, so that $I(t) = I_{\tau_n}(G(X^n))(t)$ where $I_{\tau_n}(G(X^n))(t)$ is defined in (4.9). This concludes the proof. □

The previous existence result will be supplemented by the following local uniqueness result in which we will use the notation introduced in [21] (see Theorem VI.5).

Lemma 5.2. *Suppose that the assumptions of Proposition 5.1 are satisfied. Assume that k,n are two natural numbers such that $n \leq k$. Then $\tau_n \leq \tau_k$ a.s. and the processes $u^n|_{[T_0,T_0+\tau_n)\times\Omega}$ and $u^k|_{[T_0,T_0+\tau_n)\times\Omega}$ are equivalent.*

Proof. Let us fix $k,n\in\mathbb{N}$ such that $n\leq k$ and put

$$\tau_{k,n} = \inf\{t>0 : |u^k|_{Y_{[0,t]}} \geq n\} \wedge k.$$

Then obviously $\tau_{k,n} \leq \tau_k$ and by repeating the argument from the proof of Proposition 5.1 the process $\big(u^k(t)\,,\, t < T_0+\tau_{k,n}\big)$ is a local solution of the problem (5.2). But by Proposition 5.1, the process $\big(u^n(t)\,,\, t \leq T_0+\tau_n\big)$ is also a local solution of the problem (4.11). The proof is thus completed by applying the uniqueness part of Theorem 4.9. □

In the following Theorem we will prove the existence of a unique local mild solution to (4.5). An important feature of this result is that we can estimate from below the length of the existence time interval with a lower bound, which depends on the p-th moment of the H-norm of the initial data, on a "large" subset of Ω whose probability does not depend on this moment. This property is used later on in proving that the H-norm of solution converges to ∞ as the time converges to the lifespan, provided it is finite.

Theorem 5.3. *Let us assume that Assumptions 3.1 and 4.1 be satisfied. Then for every $\mathcal{F}_{T_0}$-measurable* H*-valued p-integrable random variable $u(T_0)$ there exits a local process $X = \big(X(t), t \in [0,T_1)\big)$ which is the unique local mild solution to the problem* (4.5) *with the filtration $\mathbb{F}^{T_0}$ and the Brownian W^{T_0}. Moreover, given $R > 0$ and $\varepsilon > 0$ there exists $\tau(\varepsilon, R) > 0$, such that for every $\mathcal{F}_{T_0}$-measurable* H*-valued random variable $u(T_0)$ satisfying $\mathbb{E}|u(T_0)|^p_{\mathrm{H}} \le R^p$, one has $\mathbb{P}\big(T_1 \ge \tau(\varepsilon,R)\big) \ge 1-\varepsilon$.*

Proof. The first part follows from Proposition 5.1. The proof of the second part seems to be new with respect to the existing literature.
Let us fix $\varepsilon > 0$; choose N such that $N \ge 2\varepsilon^{-1/p}$.
Fix $\omega \in \Omega$ such that $|u(T_0)(\omega)|_{\mathrm{H}} < \infty$. Then for $C := \sup_{t\in\mathbb{R}} |U(t)|_{L(H)}$, we have $|U_t u(T_0)(\omega)|_{\mathrm{H}} \le C|u(T_0)(\omega)|_{\mathrm{H}}$ and by Assumption 3.1 we deduce that $\big(\int_0^T \|U_t u(T_0)(\omega)\|^p_E dt\big)^{1/p} \le C^p\, \tilde{C}_p(T)\ |u(T_0)(\omega)|_{\mathrm{H}}$. This implies that

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |U_t u(T_0)|^p_{\mathrm{H}} + \int_0^T \|U_t u(T_0)\|^p_{\mathrm{E}}\, dt\Big) \le C^p\,[1+\tilde{C}_p(T)^p]\ \mathbb{E}|u(T_0)|^p_{\mathrm{H}}.$$

Moreover, Lemmas 4.4 and 4.6 imply that given $X \in \mathbb{M}^p(Y_T, \mathbb{F}^{T_0})$, one has $|\Phi^n_T(X) + \Psi^{T_0,n}_T(X)|_{\mathbb{M}^p(Y_T,\mathbb{F}^{T_0})} \le K_n(T)$, with $\sup_{n\le n_0} K_n(T) \to 0$ as $T \to 0$ for every $n_0 \in \mathbb{N}^*$. Furthermore, the map $\tilde{C}_p(.)$ is non decreasing and $\tilde{C}_p(T) \to 0$ as $T \to 0$. Hence one may choose $\delta_1(\varepsilon)$ such that $[1+\tilde{C}_p(\delta_1(\varepsilon))^p]^{1/p} \le \frac{3}{2}$.
Let us put $n = NR$ for some "large" N to be chosen later and choose $\delta_2(\varepsilon) > 0$ such that $K_n(\delta_2(\varepsilon)) \le \frac{1}{2}R$. Let Λ^n_T be the map defined by (4.30). Since $\mathbb{E}(|u(T_0)|^p_{\mathrm{H}})^{1/p} \le R$, we deduce that for $T \le \delta_1(\varepsilon) \wedge \delta_2(\varepsilon)$, the range of Λ^n_T is included in the ball of $\mathbb{M}^p(Y_T, \mathbb{F}^{T_0})$ of radius $(3C+1)R/2$. Furthermore, Propositions 4.5 and 4.7 show that there exists $\delta_3(\varepsilon)$ such that Λ^n_T is a strict contraction of $\mathbb{M}^p(Y_T, \mathbb{F}^{T_0})$ if $T \le \delta_3(\varepsilon)$. Hence, if one lets $\tau(\varepsilon,R) = \delta_1(\varepsilon) \wedge \delta_2(\varepsilon) \wedge \delta_3(\varepsilon) \wedge \frac{n}{2}$, the unique fixed point X^n of the map $\Lambda^n_{\tau(\varepsilon,R)}$ is such that $\mathbb{E}\big(|X^n|^p_{Y_{\tau(\varepsilon,R)}}\big)^{1/p} = \|X^n\|_{M^p(Y_{\tau(\varepsilon,R)},\mathbb{F}^{T_0})} \le (3C+1)R/2$. Proposition 5.1 shows that the process $(X^n(t), t \le \tau_n)$ is a local mild solution to problem (4.5). Furthermore, by the definition of the stopping time τ_n, the set $\{\tau_n < \tau(\varepsilon,R)\}$ is contained in the set $\{\|X^n\|_{Y_{\tau(\varepsilon,R)}} \ge n\}$. Since by the Chebyshev's inequality $\mathbb{P}\big(\|X^n\|_{Y_{\tau(\varepsilon,R)}} \ge n\big) \le \big((3C+1)/2\big)^p N^{-p} \le \varepsilon$ provided that N is chosen large enough, we infer that $\mathbb{P}\big(\tau_n < \tau(\varepsilon,R)\big) \le \varepsilon$.
Therefore, $\mathbb{P}\big(\tau_n \ge \tau(\varepsilon,R)\big) \ge 1-\varepsilon$ and the stopping time $T_1 = \tau_n$ satisfies the requirements of the Theorem; this concludes the proof. ☐

Assume that $T_0 = 0$, $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathrm{H})$ and let τ_n be defined by (5.1). Set

$$\tau_\infty(\omega) := \lim_{n\nearrow\infty} \tau_n(\omega),\ \omega \in \Omega,$$

Then $U^n = X^n$ and Lemma 5.2 implies that the following identity uniquely defines a local process $(u(t)\,,\, t < \tau_\infty)$ as follows:

$$u(t,\omega) := u^n(t,\omega), \ \text{ if } t < \tau_n(\omega),\ \omega \in \Omega. \tag{5.4}$$

We can now prove the existence and uniqueness of a maximal solution to our abstract evolution equation (4.5); this is the main result of this section.

Theorem 5.4. *Assume that the Assumptions 3.5, 3.1 and 4.1 are satisfied with $2 \le \beta < p$ and $a \in [1, \frac{p}{2})$. Assume also that* E *is a martingale type* 2 *Banach space. Then for every finite and accessible $\mathbb{F}$-stopping time T_0 and every $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathrm{H})$, the process $u = (u(t)\,,\, t < \tau_\infty)$ defined by* (5.4) *is the unique local maximal solution to equation* (4.5). *Moreover,*

$\mathbb{P}\big(\{\tau_\infty < \infty\} \cap \{\sup_{t<\tau_\infty} |u(t)|_{\mathrm{H}} < \infty\}\big) = 0$ *and on* $\{\tau_\infty < \infty\}$, $\limsup_{t\to\tau_\infty} |u(t)|_{\mathrm{H}} = +\infty$ *a.s.*

Proof. We will follow some ideas from [4, Theorem 4.10]. The process $(u(t), t < \tau_\infty)$ is such that $\mathbb{P}$-a.s. we have $|u|_{Y_t} \to \infty$ as $t \nearrow \tau_\infty$ on the set $\{\tau_\infty < \infty\}$ and this solution is hence maximal. We now prove the last part which is not obvious since it requires to prove that the H norm of $u(t)$ does not remain bounded.

Let us argue by contradiction and assume that $\mathbb{P}\big(\{\tau_\infty < \infty\} \cap \{\sup_{t<\tau_\infty} |u(t)|_{\mathrm{H}} < \infty\}\big) = 4\varepsilon > 0$. Choosing R large enough, we may and do assume that

$$\mathbb{P}\big(\{\tau_\infty < \infty\} \cap \{\sup_{t<\tau_\infty} |u(t)|_{\mathrm{H}} \le R\}\big) = 3\varepsilon > 0.$$

Let $\sigma_R = \inf\{t \ge 0 : |u(t)|_{\mathrm{H}} \ge R\} \wedge \tau_\infty$; then σ_R is a predictable stopping time and the $\mathcal{F}_{\sigma_R}$-measurable set $\tilde{\Omega} = \{\sigma_R = \tau_\infty < \infty\}$ is such that $\mathbb{P}(\tilde{\Omega}) \ge 3\varepsilon$.

Let v_0 denote the $\mathcal{F}_{\sigma_R}$-measurable variable defined by $v_0 = u(\sigma_R)$ on $\tilde{\Omega}$ and $v_0 = 0$ otherwise. Then Theorem 5.3 implies the existence of a positive time $\tau_{\varepsilon,R}$ such that the $\mathcal{F}_t^{T_0}$ solution $X(t)$ to the evolution equation $idX(t) + \Delta X(t)dt = F(X(t))dt + G(X(t))dW^{T_0}(t)$ with initial condition v_0 has a solution on some time interval $[0, T_1)$ with $\mathbb{P}(T_1 \ge \tau_{\varepsilon,R}) \ge 1 - \varepsilon$. Let v be the predictable process defined by:

$$v(t,\cdot) = \begin{cases} u(t,\cdot) & \text{if} \quad t \le \sigma_R(\cdot) < \tau_\infty(\cdot), \\ u(t,\cdot) & \text{on} \quad \{t > \sigma_R(\cdot)\} \cap \tilde{\Omega}^c, \\ X(t - \sigma_R(\cdot)) & \text{on} \quad \{t > \sigma_R(\cdot)\} \cap \tilde{\Omega} \end{cases} .$$

Therefore, $0 < \mathbb{E}\big(|v|_{Y_{\tau_\infty + \frac{1}{2}\delta_{\varepsilon,R}}} 1_{\tilde{\Omega}}\big) < \infty$, which contradicts the definition of τ_∞; this concludes the proof. $\square$

6. Abstract Stochastic NLS in the Stratonovich form

Multiplying equation (4.5) by $-i$, we obtain the following form of it:

$$du(t) = i\Big[Au(t) - F(u)\Big]\,dt + (-i)\,G(u)\,dW(t),\ t \ge 0.$$

Now we suppose that the stochastic term is in the Stratonovich form, i.e. formally

$$du(t) = i\Big[Au(t) - F(u)\Big]\,dt + (-i)\,G(u)\,\circ\,dW(t),\ t \ge 0. \tag{6.1}$$

Below we will present a rigorous approach to equation (6.1). We assume that the assumptions of Theorem 5.4 are satisfied. In order for this problem to make sense, we need to make stronger assumption on the map G. To be precise, we require the following assumptions.

Assumption 6.1. *The Hilbert space* K *is such that*

$$\mathrm{K} \subset \mathcal{R} := \mathrm{H} \cap \mathrm{E},$$

and the natural embedding $\Lambda : \mathrm{K} \hookrightarrow \mathcal{R}$ *is* γ*-radonifying.*

Assumption 6.2. *The map* $G : \mathcal{R} \to \mathcal{L}(\mathcal{R}, \mathcal{R})$ *is of real* $\mathcal{C}^1$*-class.*

Note that the above Assumptions 6.1 and 6.2 imply that the naturally induced map

$$G : \mathcal{R} \ni u \mapsto G(u) \circ \Lambda \in \mathrm{R}(\mathrm{K}, \mathcal{R}),$$

which can be identified with the original map $G : \mathcal{R} \to \mathcal{L}(\mathcal{R}, \mathcal{R})$, is of real $\mathcal{C}^1$-class and satisfies

$$|G(u)|_{R(\mathrm{K},\mathcal{R})} \le |G(u)|_{\mathcal{L}(\mathcal{R},\mathcal{R})}\, |\Lambda|_{R(K,\mathcal{R})}.$$

Furthermore, for $u \in \mathcal{R}$, the Fréchet derivative $G'(u) = d_uG \in \mathcal{L}\big(\mathcal{R}, \mathcal{L}(\mathcal{R}, \mathcal{R})\big)$ is $\mathbb{R}$-linear.

By the Kwapień-Szymański Theorem [24] we can assume that an ONB $\{e_j\}_{j\geq 1}$ of K can be chosen (and fixed for the remainder of the article) in such a way that

$$\sum_{j\geq 1} |\Lambda e_j|^2_{\mathcal{R}} < \infty. \tag{6.2}$$

Let us recall that for a bilinear $\phi \in \mathbb{L}_2(\mathcal{R};\mathcal{R})$, we put

$$\mathrm{tr}_{\mathrm{K}}(\phi) := \sum_{j\geq 1} \phi(\Lambda e_j, \Lambda e_j) \in \mathcal{R}. \tag{6.3}$$

Following [6] we can define the Stratonovich differential $-iG(u)\circ dW(t)$ as follows:

$$\begin{aligned} -iG(u)\circ dW(t) &= -iG(u)\, dW(t) + \frac{1}{2}\mathrm{tr}_{\mathrm{K}}\big(-iG'(u)\big)\big(-iG(u)\big)\, dt \\ &= -iG(u)\, dW(t) + \frac{1}{2}\mathrm{tr}_{\mathrm{K}}\big(iG'(u)\big)\big(iG(u)\big)\, dt. \end{aligned} \tag{6.4}$$

If for $u\in\mathcal{R}$, we denote by $\mathcal{M}(u) = (iG'(u))(iG(u))$ the element of $\mathcal{L}\big(\mathcal{R},\mathcal{L}(\mathcal{R},\mathcal{R})\big) \equiv \mathbb{L}_2(\mathcal{R};\mathcal{R})$ defined by

$$\mathcal{M}(u)(h_1,h_2) = (iG'(u))(iG(u))(h_1,h_2) = (iG'(u)(iG(u)h_1)h_2, \quad h_1,h_2\in\mathcal{R}, \tag{6.5}$$

then equation (6.1) can be reformulated in the following way:

$$du = \Big[iAu - iF(u) + \frac{1}{2}\mathrm{tr}_{\mathrm{K}}\big(\mathcal{M}(u)\big)\Big]\, dt + iG(u)dW(t). \tag{6.6}$$

Given real-valued maps ϕ,ψ we write $\phi\lesssim\psi$ if there exists a constant c such that $\phi\leq c\psi$. We write $\phi\eqsim\psi$ to express that $\phi\lesssim\psi$ and $\psi\lesssim\phi$.
Let us recall that although $\mathcal{R}$ is a complex Banach space, below we will treat it as a real Banach space. The following result states the equivalence of the $\mathbb{L}_2(\mathcal{R};\mathcal{R})$ norm of $\mathcal{M}(u)$ and of $G'(u)G(u)$. Its proof, which is is straightforward, is omitted.

Lemma 6.3. *Assume that the multiplication by i is a bounded real linear map in the real Banach space $\mathcal{R}$. Then we have, for all $u,v\in\mathcal{R}$,*

$$\begin{aligned} |\mathcal{M}(u)|_{\mathbb{L}_2(\mathcal{R};\mathcal{R})} &\eqsim |G'(u)(G(u))|_{\mathbb{L}_2(\mathcal{R};\mathcal{R})}, \\ |\mathcal{M}(u)-\mathcal{M}(v)|_{\mathbb{L}_2(\mathcal{R};\mathcal{R})} &\eqsim |G'(u)(G(u)) - G'(v)(G(v))|_{\mathbb{L}_2(\mathcal{R};\mathcal{R})}, \end{aligned}$$

where $G'(u)(G(u))(h_1,h_2) = G'(u)\big(G(u)h_1\big)h_2$ for $h_1,h_2\in\mathcal{R}$.

Since $\mathcal{R}\hookrightarrow \mathrm{H}$ continuously and the trace

$$\mathrm{tr}_{\mathrm{K}} : \phi\in\mathbb{L}_2(\mathcal{R};\mathcal{R}) \to \mathrm{tr}_{\mathrm{K}}\big(\phi\big)\in\mathcal{R}$$

is a linear and bounded map, we can find $C>0$ such that

$$\big|\mathrm{tr}_{\mathrm{K}}\mathcal{M}(u) - \mathrm{tr}_{\mathrm{K}}\mathcal{M}(v)\big|_H \leq C|\mathcal{M}(u)-\mathcal{M}(v)|_{\mathbb{L}_2(\mathcal{R};\mathcal{R})}, \quad u,v\in\mathcal{R}.$$

Definition 6.4. *We say that a process u is a local (resp. local maximal, global) solution to equation* (6.1) *if and only if it is a local (resp. local maximal, global) solution to the Itô equation* (6.6)*, that is equation* (4.5)*, with the map F being replaced by $F_1 = F + \frac{i}{2}\mathrm{tr}_{\mathrm{K}}\Big[\mathcal{M}\Big]$.*

We now state the stronger version of Assumption 4.1(iii).

Assumption 6.5. *Let $p\in(2,\infty)$, $a\in[1,\frac{p}{2})$ and $\gamma\in[1,p)$. The map $G:\mathcal{R}\to\mathcal{L}(\mathcal{R},\mathcal{R})$ is of class $\mathcal{C}^1$ and such that for some positive constant C, and all $u,v\in\mathcal{R}$, with M having been defined in* (6.5)

$$|G(u)|_{\mathcal{L}(\mathcal{R},\mathrm{H})} \leq C\Big[1 + \|u\|^a_{\mathrm{E}} + \big(1+\|u\|^{a-1}_{\mathrm{E}}\big)|u|_{\mathrm{H}}\Big] \tag{6.7}$$

$$|G(u)-G(v)|_{\mathcal{L}(\mathcal{R},\mathrm{H})} \le C\big(1+\|u\|_{\mathrm{E}}^{a-1}+\|v\|_{\mathrm{E}}^{a-1}\big)|u-v|_{\mathrm{H}}$$
$$+\,C\big(1+\|u\|_{\mathrm{E}}^{(a-2)^+}+\|v\|_{\mathrm{E}}^{(a-2)^+}\big)\big(1+|u|_{\mathrm{H}}+|v|_{\mathrm{H}}\big)\|u-v\|_{\mathrm{E}}, \tag{6.8}$$
$$\Big|\mathcal{M}(u)\Big|_{\mathbb{L}_2(\mathcal{R};\mathrm{H})} \le C\Big[1+\|u\|_{\mathrm{E}}^{\gamma}+\big(1+\|u\|_{\mathrm{E}}^{\gamma-1}\big)|u|_{\mathrm{H}}\Big], \tag{6.9}$$
$$\Big|\mathcal{M}(u)-\mathcal{M}(v)\Big|_{\mathbb{L}_2(\mathcal{R};\mathrm{H})} \le C\big(1+\|u\|_{\mathrm{E}}^{\gamma-1}+\|v\|_{\mathrm{E}}^{\gamma-1}\big)|u-v|_{\mathrm{H}}$$
$$+\;\; C\big(1+\|u\|_{\mathrm{E}}^{(\gamma-2)^+}+\|v\|_{\mathrm{E}}^{(\gamma-2)^+}\big)\big(1+|u|_{\mathrm{H}}+|v|_{\mathrm{H}}\big)\|u-v\|_{\mathrm{E}}. \tag{6.10}$$

Theorem 5.4 immediately yields the following result.

Theorem 6.6. *Assume that the Assumptions 3.5, 3.1 and 4.1 are satisfied with* $2\le\beta<p$ *and* $a\in[1,\frac{p}{2})$ *and also Assumptions 6.1, 6.2 and 6.5 are satisfied. Assume also that* E *is a martingale type* 2 *Banach space and* $W=(W_t)_{t\ge 0}$ *is an* $\mathcal{R}=\mathrm{H}\cap\mathrm{E}$*-valued Wiener process defined on some filtered probability space* $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ *satisfying the usual assumptions. Then for every* $u_0\in L^p(\Omega,\mathcal{F}_0,\mathrm{H})$ *there exists a unique local process* $u=\big(u(t)\,,\,t<\tau_\infty\big)$ *which is the local maximal solution to equation* (6.1)*. Moreover,* $\mathbb{P}\big(\{\tau_\infty<\infty\}\cap\{\sup_{t<\tau_\infty}|u(t)|_{\mathrm{H}}<\infty\}\big)=0$ *and on* $\{\tau_\infty<\infty\}$, $\limsup_{t\to\tau_\infty}|u(t)|_{\mathrm{H}}=+\infty$ *a.s.*

7. Stochastic NSEs: the local existence

In this section we will formulate results about the existence and the uniqueness of solutions to the stochastic NLS equation, about an equation in the Itô (respectively Stratonovich) formulation. The former one will be based on Theorem 5.4 and the latter on Theorem 6.6. For simplicity we formulate it for $d=2$. One can also prove a similar result for $d>2$ but since in the latter case we do not know whether the solution is global or blows up in finite time, we have decided to leave it out. Thus we assume that M is a 2-dimensional, compact riemannian manifold and Δ is the Laplace Beltrami operator on M. We assume that some numbers p,q satisfy the scaling admissible condition (with $d=2$)

$$\frac{2}{p}+\frac{2}{q}=1. \tag{7.1}$$

We choose $s\in(1-\frac{1}{p},1]$ and put $\hat{s}:=s-\frac{1}{p}$. Since $s-\frac{1}{p}>\frac{d}{q}$, we infer that the Sobolev space $W^{\hat{s},q}(M)$ is embedded into the space $\mathcal{C}(M)$ of continuous (and hence bounded) functions on M; the latter is a Banach space equipped with the L^∞-norm.
In this section we let $\mathrm{H}=H^{s,2}(M):=H^{s,2}(M,\mathbb{C})$ and $\mathrm{E}=W^{\hat{s},q}(M):=W^{\hat{s},q}(M,\mathbb{C})$, where $\mathbb{C}$ is identified with $\mathbb{R}^2$. Finally, as in Assumption 6.1, we denote by $\mathcal{R}$ the following real Banach space

$$\mathcal{R}=H^{s,2}(M)\cap W^{\hat{s},q}(M)=\mathrm{H}\cap\mathrm{E}.$$

In order to study the diffusion operator G we need the following result which follows from Corollary 2.2 and Theorem 2.3.

Lemma 7.1. *Under the above assumptions the pointwise multiplication map*

$$\Pi:\mathcal{R}\times\mathcal{R}\ni(u,h)\mapsto uh\in\mathcal{R} \tag{7.2}$$

is bilinear and continuous. The same assertion holds for $\mathcal{R}=H^{s,2}(M)\cap L^\infty(M)$.

The following assumption will play an essential rôle in the next section as well as in the last part of Theorem 7.6, which is the main result of the current section.

Assumption 7.2. *There exists a function* $\tilde{g}:[0,+\infty)\to\mathbb{R}$ *of class* $\mathcal{C}^1$ *such that*

$$g(z)=\tilde{g}(|z|^2)z,\;\; z\in\mathbb{C}. \tag{7.3}$$

We will consider the "generalized" Nemytski map $\tilde{G}$ associated with g (see [6]), that is with $\mathcal{R} = \mathrm{H} \cap \mathrm{E}$,

$$\tilde{G} : \mathcal{R} \ni u \mapsto \big\{h \mapsto \Pi(g(u), h)\big\} \in \mathcal{L}(\mathcal{R}, \mathcal{R}). \tag{7.4}$$

The aim of this section is to prove the existence and uniqueness of a maximal solution to

$$idu(t) + \Delta u(t)\, dt = f(u)\, dt + g(u)\, dW(t), \quad u(0) = u_0, \tag{7.5}$$

or to its Stratonovich formulation

$$idu(t) + \Delta u(t)\, dt = f(u)\, dt + g(u) \circ dW(t), \quad u(0) = u_0. \tag{7.6}$$

By a solution of equation (7.5) (resp. (7.6)), we mean a solution to its abstract version (4.5) (resp. (6.6)) defined in terms of the Nemytski map $\tilde{G}$. As proved in the previous section, the Stratonovich formulation requires to identify $\mathcal{M}$ as the Nemytski map corresponding to function $z \mapsto (ig'(z))(ig(z))$. This will be a consequence of the following general result.

Lemma 7.3. *Assume that $(H, |\cdot|)$ is a real separable Hilbert space and that $\mathcal{I} : H \to H$ is a bounded linear operator such that $\mathcal{I}^2 = -Id$, $\langle z, \mathcal{I}z\rangle = 0, z \in H$. Let $\varphi : H \to H$ be function of the form $\varphi(z) = \tilde{\varphi}(|z|^2)\mathcal{I}z$, $z \in H$, where $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ is differentiable. Then, φ is $\mathbb{R}$-differentiable and*

$$\big(\varphi'(z)\big)\big(\varphi(z)\big) = \big(d_z\varphi\big)\big(\varphi(z)\big) = -|\tilde{\varphi}(|z|^2)|^2\, z, \ \forall z \in H. \tag{7.7}$$

Proof. Let $y, z \in H$; then we have $[\varphi'(z)](y) = 2\tilde{\varphi}'(|z|^2)\langle z, y\rangle \mathcal{I}z + \tilde{\varphi}(|z|^2)\mathcal{I}y$. Therefore, given $z \in H$, we deduce

$$\begin{aligned}
[\varphi'(z)](\varphi(z)) =& 2\tilde{\varphi}'(|z|^2)\langle z, \tilde{\varphi}(|z|^2)\mathcal{I}z\rangle \mathcal{I}z + \tilde{\varphi}(|z|^2)\mathcal{I}\varphi(z)\\
=& 2\tilde{\varphi}'(|z|^2)\tilde{\varphi}(|z|^2)\langle z, \mathcal{I}z\rangle \mathcal{I}z |\tilde{\varphi}(|z|^2)|^2\mathcal{I}^2 z = -|\tilde{\varphi}(|z|^2)|^2 z.
\end{aligned}$$

This completes the proof of (7.7). □

Since we identify $\mathbb{C}$ with $\mathbb{R}^2$ and the operator of multiplication by i with the operator $\mathcal{I} : \mathbb{R}^2 \ni (x, y) \mapsto (-y, x) \in \mathbb{R}^2$, we deduce the following result.

Corollary 7.4. *Assume that a function $g : \mathbb{C} \to \mathbb{C}$ satisfies Assumption 7.2 for a differentiable function $\tilde{g} : \mathbb{R} \to \mathbb{R}$. Then g is $\mathbb{R}$-differentiable and, with $\langle\cdot,\cdot\rangle$ (resp. $|\cdot|$) denoting the scalar product (resp. the euclidian norm), in $\mathbb{C} \cong \mathbb{R}^2$, we have for all $z \in \mathbb{C}$,*

$$m(z) := \big((ig)'(z)\big)\big(ig(z)\big) = -\big(\tilde{g}(|z|^2)\big)^2 z, \quad \langle \big((ig)'(z)\big)\big(ig(z)\big), z\rangle = -|g(z)|^2. \tag{7.8}$$

In particular we get the formulation of $\mathrm{tr}_{\mathrm{K}}\big(\mathcal{M}(u)\big)$, where $\mathcal{M}$ is defined by (6.5). Let Π be the bilinear map defined in (7.2) and $\Lambda : K \to \mathcal{R}$ denotes the natural embedding (which is a gamma-radonifying operator) and $\big(e_j\big)_{j\geq 1}$ is a complete orthonormal system of K satisfying (6.2) and consisting of real valued functions. Then it follows from the definition (6.3) of the trace and the Kwapień-Szymański result (6.2) that

$$\mathfrak{p} := \mathrm{tr}_{\mathrm{K}}(\Pi) = \sum_{j\geq 1} (\Lambda e_j)^2 \in H^{s,2}(M, \mathbb{R}) \cap W^{\hat{s},q}(M, \mathbb{R}) \subset \mathcal{R}. \tag{7.9}$$

Let us make another useful observation. Let m be defined in (7.8) and $\mathbb{M}$ be the Nemytski map corresponding to the function $m : \mathbb{C} \to \mathbb{C}$, that is $\mathbb{M}(u) = m \circ u$, $u \in \mathcal{R}$. Then

$$\mathcal{M}(u)(h_1, h_2) = \mathbb{M}(u)h_1h_2, \quad \text{for } u, h_1, h_2 \in \mathcal{R}. \tag{7.10}$$

Furthermore, we have the following:

Lemma 7.5. *Assume that a function $g:\mathbb{C}\to\mathbb{C}$ satisfies Assumption 7.2 for a differentiable function $\tilde g:\mathbb{R}\to\mathbb{R}$. Then the maps G and $\mathbb{M}$ transform $\mathcal{R}$ to $\mathcal{L}(\mathcal{R},\mathcal{R})$ and $\mathcal{R}$ respectively, and, for every $u\in\mathcal{R}$,*

$$\mathrm{tr}_{\mathrm{K}}(\mathcal{M}(u))=\Pi\big(\mathfrak{p},\mathbb{M}(u)\big),\tag{7.11}$$

$$\mathrm{Re}\,\langle \mathrm{tr}_{\mathrm{K}}\mathcal{M}(u),u\rangle_{L^2(M)}=-\int_M |g(u(x))|^2\mathfrak{p}(x)\,dx=-\|G(u)\Lambda\|^2_{R(\mathrm{K},L^2(M))},\tag{7.12}$$

$$\mathrm{Re}\,\langle \nabla \mathrm{tr}_{\mathrm{K}}\mathcal{M}(u),\nabla u\rangle=-\int_M |\tilde g(|u(x)|^2)|^2\mathrm{Re}\,\langle u(x)\nabla\mathfrak{p}(x),\nabla u(x)\rangle\,dx\tag{7.13}$$

$$-4\int_M(\tilde g'\tilde g)(|u(x)|^2)\mathrm{Re}\,\langle u(x)\nabla u(x),\nabla u\rangle\mathfrak{p}(x)\,dx-2\int_M\tilde g(|u(x)|^2)^2\mathfrak{p}(x)|\nabla u(x)|^2\,dx.$$

Proof of Lemma 7.5. Since both g and m are functions of $\mathcal{C}^1$-class and the point-wise multiplication in $\mathcal{R}$ is a bounded bilinear map, Proposition 2.1 implies that G and $\mathbb{M}$ are well defined maps from $\mathcal{R}$ to $\mathcal{L}(\mathcal{R},\mathcal{R})$ and $\mathcal{R}$. Using (6.5) we deduce that for $u\in\mathcal{R}$,

$$\mathrm{tr}_{\mathrm{K}}\Big[\mathcal{M}(u)\Big]=\sum_{j\ge 1}\big[\big((ig)'(ig)\big)\circ u\big](\Lambda e_j)^2\in\mathcal{R}.$$

Then the definition (7.8) of the function m concludes the proof of identity (7.11). Moreover the second identity in (7.8) yields

$$\mathrm{Re}\,\langle \mathrm{tr}_{\mathrm{K}}\mathcal{M}(u),u\rangle_{L^2(M)}=\sum_j\int_M \mathrm{Re}\,\langle (ig)'(u(x))\big(ig(u(x))\big)\big((\Lambda e_j)(x)\big)^2,u(x)\rangle dx$$

$$=-\sum_j\int_M\big((\Lambda e_j)(x)\big)^2|g(u(x))|^2\,dx=-\sum_j|\tilde G(u)\Lambda e_j|^2_{L^2(M)}.$$

This concludes the proof of (7.12); that of (7.13) is similar. $\square$

Recall that m is defined by (7.8). The above results show that the Stratonovich equation (7.6) can be written in the following Itô form:

$$du(t)=\Big[iAu(t)-if(u(t))+\frac12\mathfrak{p}\,m(u(t))\Big]\,dt-ig(u(t))\,dW(t).\tag{7.14}$$

We now prove the existence and uniqueness of a maximal solution to equations (7.5) and (7.6) - or (7.14). This is the main result of this section.

Theorem 7.6. *Assume that M is a compact riemannian manifold of dimension $d=2$. Assume that $f:\mathbb{C}\to\mathbb{R}$ is of real $\mathcal{C}^1$-class satisfying, for some $\beta\ge 2$ and all $y,z\in\mathbb{C}$,*

$$|f(y)|\le C(1+|y|^\beta),\ |f'(y)|\le C(1+|y|^{\beta-1}),\ |f'(y)-f'(z)|\le C(1+|y|^{\beta-2}+|z|^{\beta-2})|y-z|.\tag{7.15}$$

Assume that $g:\mathbb{C}\to\mathbb{R}$ is of real $\mathcal{C}^1$-class satisfying, for some $a\ge 1$ and all $y,z\in\mathbb{C}$,

$$|g(y)|\le C(1+|y|^a),\ |g'(y)|\le C(1+|y|^{a-1}),\ |g'(y)-g'(z)|\le C(1+|y|^{(a-2)^+}+|z|^{(a-2)^+})|y-z|.\tag{7.16}$$

Assume that $p>\beta\vee(2a)$ and $q>2$ satisfy the scaling admissible condition (7.1). Assume that $s\in(1-\frac1p,1]$ and let $W=(W(t)\,,\,t\ge 0)$ be an $H^{s,2}(M)\cap W^{s-\frac1p,q}(M)$-valued Wiener process defined on some filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ satisfying the usual assumptions.

Then for every $u_0\in L^p(\Omega,\mathcal{F}_0,H^{s,2}(M))$ there exist a local process $u=(u(t)\,,\,t<\tau_\infty)$ whose trajectories are $H^{s,2}(M)$-valued continuous and locally p-integrable with values in

$W^{s-\frac{1}{p},q}(M)$, *that is the unique local maximal solution to* (7.5) *Moreover,* $\mathbb{P}\big(\{\tau_\infty<\infty\}\cap\{\sup_{t<\tau_\infty}|u(t)|_{H^{s,2}(M)}<\infty\}\big)=0$ *and*

$$\limsup_{t\to\tau_\infty}|u(t)|_{H^{s,2}(M)}=+\infty \text{ a.s. on } \{\tau_\infty<\infty\}.$$

Suppose furthermore that $p>\beta\vee(2a)\vee\gamma$, *that* g *satisfies Assumption 7.2 and that* $m(z)=-\tilde g(|z|^2)^2z$ *satisfies the following condition for some* $\gamma\geq 2$ *and all* $y,z\in\mathbb{C}$*:*

$$|m(y)|\leq C(1+|y|^\gamma),\ |m'(y)|\leq C(1+|y|^{\gamma-1}),\ |m'(y)-m'(z)|\leq C(1+|y|^{(\gamma-2)}+|z|^{(\gamma-2)})|y-z|. \tag{7.17}$$

Then the same conclusion as above holds for the equation (7.6) *in the Stratonovich form.*

Remark 7.7. Let $\tilde g$ be of class $\mathcal{C}^2$ such that for some constants $C>0$, $\alpha_0\geq 0$, and some constants α_i, $i=1,2$ one has for all $r\geq 0$:

$$|\tilde g(r)|\leq C(1+r^{\alpha_0}),\quad |\tilde g'(r)|\leq C(1+r^{\alpha_1}),\quad |\tilde g''(r)|\leq C(1+r^{\alpha_2}).$$

Then if $g(z)=\tilde g(|z|^2)z$ and $m(z)=\tilde g(|z|^2)^2z$, the function g satisfies condition (7.16) with $a\geq(2\alpha_0+1)\vee(2\alpha_1+3)\geq 1$ and $a\geq 2\alpha_2+5$ if $a>2$. Furthermore, the function m defined by $m(z)=-\tilde g(|z|^2)^2z$ satisfies (7.17) with $\gamma=(2a-1)\vee 2$.

Proof of Theorem 7.6. We only consider the case $s\in(1-\frac{1}{p},1)$. The case $s=1$ can be dealt with analogously.
We put $\mathcal{H}_0=L^2(M)$, $\mathrm{E}_0=L^q(M)$, $\mathrm{H}_0=H^{\frac{1}{p},2}(M)$, $A=\Delta$ with $D(A)=H^{2,2}(M)$ and $\mathbf{U}=\big(U_t\big)_{t\in\mathbb{R}}$, the unitary group on $L^2(M)$ generated by iA. Then Assumption 3.1 is satisfied. Next we put $\hat s=s-\frac{1}{p}$. Since by the assumptions $s-\frac{1}{p}>1-\frac{2}{p}=\frac{2}{q}>0$, we infer that $\hat s>0$. Finally we put $\mathcal{H}=A^{-\hat s}(\mathcal{H}_0)=H^{s,2}(M)$, $\mathrm{H}=A^{-\hat s}(\mathrm{H}_0)=H^{s+\frac{1}{p},2}(M)$ and $\mathrm{E}=W^{\hat s,q}(M)$. Then Assumption 4.1(i) is satisfied and by Lemma 3.3, Assumption 3.1 is satisfied as well. Note that since $q>2$, the space $W^{\hat s,q}(M)$ is bigger that $H^{\hat s,2}(M)$. Moreover, since again $\hat s>\frac{2}{q}$, in view of Theorem 2.3, the Nemytski maps F satisfies Assumption 4.1(ii).
As above we infer that G satisfies Assumption 4.1(iii); see also Proposition 6.4 in [6], where a weaker version of our results from section 2 was used.
We conclude that the problem (7.5) is a special case of the problem (4.5) for the above choice of spaces H and E. Therefore, the first result follows by applying Theorem 5.4.

We now turn to the Stratonovich evolution equation (7.6) written in terms of an Itô integral in (7.14). We only need to show that the function $\mathcal{M}$ defined in (6.5) satisfies Assumption 6.5, and in particular inequalities (6.9) and (6.10). First notice that if g satisfies (7.16) then G satisfies the first part of Assumption 6.5 with the same spaces E and H. If Assumption 7.2 is satisfied, the map $\mathbb{M}:\mathcal{R}\to\mathcal{R}$ is Lipschitz on balls and since m satisfies the assumption (7.15) with some parameter $\gamma=2a-1$, we deduce that $\mathbb{M}$ satisfies the second part of Assumption 6.5 with the same choice of spaces E and H. This completes the proof. ☐

Remark 7.8. Although $q>2$ and $s>\hat s$, since $s-1<\hat s-\frac{2}{q}$, we cannot deduce that $H^{1,2}(M)\subset W^{\hat s,q}(M)$; see e.g. Theorem [35, Theorem 4.6.1]. In fact, in view of the Sobolev embedding Theorem, $H^{s,2}(M)$ is not a subset of $H^{\hat s,q}(M)$. On the other hand, we believe that although $\hat s-\frac{d}{q}>s-\frac{d}{2}$, but $q>2$ and $s>\hat s$, it is not true that $H^{\hat s,q}(M)\subset H^{s,2}(M)$. Hence, the two Banach spaces $H^{s,2}(M)$ and $H^{\hat s,q}(M)$ are not included in one another.

8. Existence of a global solution $H^{1,2}$-valued solution to the Stochastic NLS in the Stratonovich form

8.1. **Preliminaries.** As in section 7, we assume below that M is a 2-dimensional, compact riemannian manifold and Δ is the Laplace Beltrami operator on M. In the previous sections we considered the stochastic nonlinear Schrödinger equation (7.5) with the initial data u_0 belonging to the Sobolev space $H^{s,2}(M)$ for some $s \leq 1$. In this section we will consider the problem of global existence for $s = 1$. We at first rewrite the non linear Schrödinger equation (7.5) with a Stratonovich integral and then prove that the $L^2(M)$ norm of the solution is preserved. We finally conclude by means of the Khashmiski Theorem with the energy function playing the rôle of the Lyapunov function.
The following notations already, used in section 7 for any $s \in (1-\frac{1}{p}, 1]$, will be used in the entire section for $s = 1$. Given $\theta \in (0,1]$ and $r \in [1,+\infty)$, we put $W^{\theta,r}(M) := W^{\theta,r}(M,\mathbb{C})$ where $\mathbb{C}$ is identified to $\mathbb{R}^2$. Let (p,q) be a pair of positive numbers which satisfies the scaling admissible condition (7.1), that is $\frac{2}{p}+\frac{2}{q} = 1$. We set $s = 1$, $\hat{s} = 1-\frac{1}{p}$, $\mathrm{H} = H^{1,2}(M)$, $\mathrm{E} = W^{\hat{s},q}(M)$, $\|u\|_{\hat{s},q} = \|u\|_{W^{\hat{s},q}(M)}$ and $\mathcal{R} = \mathrm{H} \cap \mathrm{E}$. Since $q > 2$, $\hat{s}q > 2$ and hence we deduce that $W^{\hat{s},q}(M) \subset L^\infty(M)$. We use the notation for the scalar product in $L^2(M) := L^2(M;\mathbb{C})$:

$$\langle u, v\rangle = \int_M \mathrm{Re}\,\big(u(x)\overline{v(x)}\big)dx, \quad u, v \in L^2(M).$$

We will consider the stochastic NLS equation in Stratonovich form, that is for $u_0 \in H^{1,2}(M)$,

$$idu(t) + \Delta u(t)dt = f(u)dt + g(u) \circ dW(t), \quad u(0) = u_0. \tag{8.1}$$

To prove the global existence of the solution to the the NLS equation (8.1), we need to impose conditions on the noise W, on the diffusion coefficient g and on the non-linearity f stonger that those made in the previous section.

Assumption 8.1. *Thus we suppose that $(W(t), t \geq 0)$ is a real $W^{1,2s_0}(M,\mathbb{R}) \cap W^{\hat{s},q}(M,\mathbb{R})$-valued Wiener process, for some $s_0 > 1$.*

Let K be the reproducing kernel Hilbert space of the law of the $H^{1,2s_0}(M,\mathbb{R}) \cap W^{\hat{s},q}(M,\mathbb{R})$-valued random variable $W(1)$. Then the embedding $\Lambda : \mathrm{K} \to H^{1,2s_0}(M,\mathbb{R}) \cap W^{\hat{s},q}(M,\mathbb{R})$ is γ radonifying.

Lemma 8.2. *For $x \in M$ let $\mathfrak{p}(x) = \mathrm{tr}_K(\Pi)(x) = \sum_j |\Lambda e_j(x)|^2$ and $\mathfrak{q}(x) = \sum_{j\geq 1} |\nabla \Lambda e_j(x)|^2$. Then $\mathfrak{p} \in L^\infty(M)$ and $\mathfrak{q} \in L^1(M)$. Furthermore, $\sum_{j\geq 1} \|\nabla \Lambda e_j\|^2_{L^{2s_0}} < \infty$.*

Proof. By the Kwapień-Szymański Theorem [24], we can assume that the ONB $\{e_j\}_{j=1}^\infty$ is chosen in such a way that $\sum_{j\geq 1} \|\Lambda e_j\|^2_{H^{1,2s_0}(M)\cap W^{\hat{s},q}(M)} < \infty$. Since $W^{\hat{s},q}(M) \subset L^\infty(M)$ we deduce that $\mathfrak{p}$ is bounded. Furthermore, $\sum_{j\geq 1} \|\nabla \Lambda e_j\|^2_{L^1(M)} \leq \sum_{j\geq 1} \|\nabla \Lambda e_j\|^2_{L^{s_0}(M)} < \infty$ and therefore, the series $\sum_{j\geq 1} |\nabla \Lambda e_j|^2$ is absolutely convergent in $L^1(M)$ as claimed; this concludes the proof. □

In this section, we suppose that g satsfies the following stronger version of Assumption 7.2.

Assumption 8.3. *There exists a bounded function $\tilde{g} : [0,+\infty) \to \mathbb{R}$ of class $\mathcal{C}^1$ such that*

$$g(z) = \tilde{g}(|z|^2)z, \; z \in \mathbb{C}, \tag{8.2}$$

Furthermore, we assume that the function g satisfies the conditions (7.16) with $a = 1$ and the function $m : \mathbb{C} \to \mathbb{C}$ defined by $m(z) = -\tilde{g}(|z|^2)^2\, z$ satisfies condition (7.17) with $\gamma \geq 2$.

An example of function $\tilde{g}$ such that the function g defined by (8.2) satisfies Assumption 8.3 is a bounded function of class $\mathcal{C}^2$ such that $\sup_{r>0}(1+r)|\tilde{g}'(r)| < \infty$ and $\sup_{r>0} r^{\frac{3}{2}}|\tilde{g}''(r)| < \infty$, for instance, $\tilde{g}(r)=\frac{\ln(1+r)}{C+\ln(1+r)}$ for $r>0$ and $C>0$. Indeed, the conditions in Remark 7.7 are satisfied with $\alpha_0=0$, $\alpha_1=-1$ and $\alpha_2=-\frac{3}{2}$, which yields (7.16) for g with $a=1$ while m satisfies (7.17) with $\gamma=2$.

We study the global existence for two types of equation (8.1) depending on the non-linear term f, which is defocusing or focusing. Precise assumptions will be described below, but let us mention that a typical example of the former is when $f(u)=|u|^2u$ while a typical example of the latter is when $f(u)=-|u|u$.

Assumption 8.4. *We assume that $f:\mathbb{C}\to\mathbb{R}$ is of the form*

$$f(z)=\tilde{f}(|z|^2)z,\ z\in\mathbb{C}, \tag{8.3}$$

where the function $\tilde{f}:[0,+\infty)\to\mathbb{R}$ satisfies one of the following two cases.

Case 1: defocusing nonlinearity. *The function $\tilde{f}$ satisfies either (a) or (b):*

(a) *There exist a natural number N and real number a_k, $k=0,\cdots,N$, with $a_N>0$, such that $\tilde{f}(r)=\sum_{k=0}^N a_k r^k$ for every $r\in\mathbb{R}$.*

(b) *There exist $C>0$ and $\sigma\in[\frac{1}{2},\infty)$ such that $\tilde{f}(r)=Cr^\sigma$ for every $r\in\mathbb{R}$.*

Case 2: focusing nonlinearity. *There exist $C>0$ and $\sigma\in[\frac{1}{2},1)$ such that for every $r\in\mathbb{R}$, $\tilde{f}(r)=-Cr^\sigma$.*

This assumption yields the following result, whose straightforward proof is omitted.

Lemma 8.5. *If Assumption 8.4 is satisfied, then function f satisfies inequalities* (7.15) *with $\beta=2N+1\geq 2$ in the defocusing case 1(a) and with $\beta=2\sigma+1\geq 2$ in the focusing case or the defocusing case 1(b).*

Thus, Lemma 7.5 and equation (6.6) imply that in this framework we can reformulate equation (8.1) as

$$du(t)=i\Big[\Delta u(t)-F(u)\Big]\,dt+\frac{1}{2}\mathfrak{p}\mathbb{M}(u)\,dt+(-i)\tilde{G}(u)\,dW(t), \tag{8.4}$$

where $\tilde{G}$ is the generamized Nemytski map defined by (7.4), $\mathbb{M}(u)=m\circ u$ is the Nemytski map associatied with $m(z)=-\tilde{g}(|z|^2)^2z$ and $\mathfrak{p}=\sum_j(\Lambda e_j)^2$ is defined in Lemma 8.2.
Let us fix $u_0\in L^p(\Omega,\mathcal{F}_0,H^{1,2}(M))$ and observe that the assumptions of Theorem 8.12 imply that Theorem 7.6 can be applied. Hence equation (8.4) has a unique local maximal solution $u=\big(u(t),t\in[0,\tau_\infty)\big)$. We will show that our assumptions on f, g and on the noise W are sufficient to ensure that the explosion time τ_∞ is a.s. infinite. This will be achieved by proving some conservation laws in the next two subsections.
Let us recall that according to Theorem 7.6 $\lim_{t\nearrow\tau_\infty}|u(t)|_{H^{1,2}}=\infty$ $\mathbb{P}$-a.s. on $\{\tau_\infty<\infty\}$. Hence, the following stopping times are well defined (and finite on $\{\tau_\infty<\infty\}$):

$$\tilde{\tau}_k:=\inf\big\{t\in[0,\tau_\infty):|u(t)|_{H^{1,2}}\geq k\big\}. \tag{8.5}$$

The aim of this section is to prove that $\tau_\infty=\infty$ a.s. The proof of this result will be given in several steps. Recall that $\mathrm{H}=H^{1,2}(M)$ and $\mathrm{E}=W^{1-\frac{1}{p},q}(M)$. The main two steps are described in the following two sections. The first one is the a.s. conservation of the $L^2(M)$ norm due to the Stratonovich integral and the deterministic conservation law. The second one is the use of a Lyapounov function. Note that unlike the deterministic case, the Itô-Stratonovich correction term implies that the expected value of this Lyapounov function is not preserved. However, it remains bounded and this implies that the expected value of the H-norm of the maximal solution remains bounded, which forbids the explosion time to be finite.

8.2. **Preservation of the L^2-norm.** We at first prove that the $L^2(M)$-norm of this solution is almost surely constant in time This extends classical results for the deterministic NLS equation (see [12] for the case of compact manifolds) as well as [17] deBouard Debussche for the flat stochastic NLS equation.

Lemma 8.6. *Assume that f and g satisfy the Assumptions 8.4 and 8.3 respectively, p and q satisfy the scaling admissibility condition $\frac{2}{p}+\frac{2}{q}=1$. Let $(W(t), t\geq 0)$ be an $\mathrm{H}\cap\mathrm{E}$-valued Wiener process and $u_0\in H$. Then $|u(t)|_{L^2(M)}=|u_0|_{L^2(M)}$, for all $t\in[0,\tau_\infty)$, $\mathbb{P}$-almost surely.*

Proof. Let $\left(\tilde\tau_k\right)_k$ denote the approximating sequence of the stopping time τ_∞ defined by (8.5). Suppose that we have proved that for each $t\geq 0$ and $k\in\mathbb{N}$, $|u(t\wedge\tilde\tau_k)|_{L^2(M)}=|u_0|_{L^2(M)}$ $\mathbb{P}$-almost surely. Then it follows that there exists a set $\hat\Omega\subset\Omega$ of full $\mathbb{P}$-measure such that for each $\omega\in\hat\Omega$, $|u(t,\omega)|_{L^2(M)}=|u_0|_{L^2(M)}$ for all $t\in\mathbb{Q}\cap[0,\tau(\omega))$. Thus, since for all $\omega\in\hat\Omega$ the map $[0,\tau(\omega))\ni t\mapsto u(t,\omega)\in L^2(M)$ is continuous, the result will follow. To prove the conservation of the $L^2(M)$-norm, let us consider the functional

$$\Phi: L^2(M)\ni u\mapsto \frac{1}{2}|u|^2_{L^2(M)}=\frac{1}{2}\int_M u(x)\overline{u(x)}\,dx\in\mathbb{R}.$$

where dx denotes the integration with respect to the riemannian volume measure on M. The function Ψ is of real-$\mathcal{C}^\infty$ class and for all $u,v,v_1,v_2\in L^2(M)$, we have

$$\begin{aligned}\Phi'(u)(v) &= d_u\Phi(v)=\mathrm{Re}\,\langle u,v\rangle_{L^2}=\int_M \mathrm{Re}\left(u(x)\overline{v(x)}\right)dx,\\ \Phi''(u)(v_1,v_2) &= d_u^2\Phi(v_1,v_2)=\mathrm{Re}\,\langle v_1,v_2\rangle_{L^2}=\int_M\mathrm{Re}\left(v_1(x)\overline{v_2(x)}\right)dx.\end{aligned}$$

Let us now assume, for purely pedagogical reasons, that u is a strong solution. Applying the Itô formula we obtain for each $t\in\mathbb{R}_+$ and every $k\in\mathbb{N}$, $\mathbb{P}$-almost surely,

$$\begin{aligned}&\Phi(u(t\wedge\tilde\tau_k))-\Phi(u_0)=-\int_0^t 1_{[0,\tilde\tau_k)}(s)\langle\Phi'(u(s)),iG(u(s))\rangle_{L^2}\,dW(s)\\ &\quad+\int_0^t 1_{[0,\tilde\tau_k)}(s)\Big\langle\Phi'(u(s)),i\big[\Delta u(s)-F(u(s))\big]+\frac{1}{2}\mathrm{tr}_K(\Pi)\mathbb{M}(u(s))\Big\rangle_{L^2}ds\\ &\quad+\frac{1}{2}\int_0^t 1_{[0,\tilde\tau_k)}(s)\mathrm{tr}_K\Big[\Phi''(u(s))\big(iG(u(s)),iG(u(s))\big)\Big]\,ds\\ &=\int_0^t 1_{[0,\tilde\tau_k)}(s)\mathrm{Re}\,\langle u(s),i\Delta u(s)\rangle_{L^2}\,ds-\int_0^t 1_{[0,\tilde\tau_k)}(s)\mathrm{Re}\,\langle u(s),iF(u(s))\rangle_{L^2}\,ds\\ &\quad+\frac{1}{2}\mathfrak{p}\int_0^t 1_{[0,\tilde\tau_k)}(s)\mathrm{Re}\,\langle u(s),\mathbb{M}(u(s))\rangle_{L^2}\,ds\\ &\quad-\int_0^t 1_{[0,\tilde\tau_k)}(s)\mathrm{Re}\,\langle u(s),iG(u(s))\rangle_{L^2}\,dW(s)+\frac{1}{2}\int_0^t 1_{[0,\tilde\tau_k)}(s)\sum_{j\geq 1}|G(u(s))\Lambda e_j|^2_{L^2}\,ds.\end{aligned}$$

Next we make the following three observations.

(1) Since Δ is self-adjoint in $L^2(M)$, we have $\mathrm{Re}\,\langle u(s),i\Delta u(s)\rangle_{L^2(M)}=0$.

(2) If H be the Nemytski map associated with h of the form $h(z)=\tilde h(|z|^2)z$, where $\tilde h:\mathbb{R}\to\mathbb{R}$, then

$$\mathrm{Re}\,\langle u(s),iH(u(s))\rangle_{L^2}=\int_M\tilde h(|u(s,x)|^2)\mathrm{Re}\left[u(s,x)\overline{iu(s,x)}\right]dx=0.$$

(3) Lemma 7.5 implies that $\mathfrak{p}\mathrm{Re}\,\langle u(s), \mathbb{M}(u(s))\rangle_{L^2} = -\sum_{j\geq 1} |G(u(s))\Lambda e_j|^2_{L^2}$.

Therefore, we infer that that for each $t \geq 0$ and every $k \in \mathbb{N}$, $\mathbb{P}$-almost surely, $\Phi(u(t\wedge\tau_k)) - \Phi(u_0) = 0$, that is $|u(t\wedge\tilde\tau_k)|_{L^2(M)} = |u_0|_{L^2(M)}$ $\mathbb{P}$-almost surely and the result follows.
A full proof can be made by replacing u by its Yosida approximation as it has been done for instance in [7]; see also [16] and [17] for a similar approach. $\square$

8.3. **The Lyapounov function.** As in the deterministic case, we will use some Lyapunov function. Let $\tilde F$ denote the antiderivative of $\tilde f$ such that $\tilde F(0) = 0$. In this section, we will consider two cases as in Assumption 8.4.

Case 1(a). We assume that $\tilde f$ is a polynomial of degree N with a positive leading coefficient. Hence $\tilde F(r) = a_{N+1} r^{N+1} + Q(r)$, where Q is a polynomial function of degree at most N such that $Q(0) = 0$ and $a_{N+1} > 0$. We have the following result.

Lemma 8.7. *Let $\tilde F$ and Q be polynomial functions as above. Then there exists a constant $C > 0$, and for every $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that for all $u \in L^{2N+2}(M) \supset \mathcal{R}$,*

$$\int_M |u(x)|^{2N+2} dx \leq C \int_M \tilde F(|u(x)|^2 dx + C \int_M |u(x)|^2 dx, \tag{8.6}$$

$$\Big| \int_M Q(|u(x)|^2)\, dx \Big| \leq \varepsilon \int_M |u(x)|^{2N+2} dx + C(\varepsilon) \int_M |u(x)|^2 dx. \tag{8.7}$$

Finally for $u \in H^{1,2}(M)$ we have

$$\int_M |\tilde f(|u(x)|^2)|u(x)|^2\, dx \leq C \int_M |\tilde F(|u(x)|^2)\, dx + C \int_M |u(x)|^2\, dx. \tag{8.8}$$

Proof. Since $\dim(M) = 2$, the Gagliardo-Nirenberg inequality implies $H^{1,2}(M) \subset \bigcap_{r\geq 2} L^r(M)$ and hence $\mathcal{R} \subset L^{2N+2}(M)$. Let us fix $\alpha > 0$ and $u \in L^{2N+2}(M)$. Then for $k = 2, \cdots, N$ by the Hölder and Young inequalities yield

$$\begin{aligned}\int_M |u(x)|^{2k} dx &\leq \Big(\int_M |u(x)|^{2(N+1)} dx \Big)^{\frac{k-1}{N}} \Big(\int_M |u(x)|^2 dx \Big)^{\frac{N+1-k}{N}} \\ &\leq \alpha\, \frac{k-1}{N} \int_M |u(x)|^{2(N+1)} dx + \frac{N+1-k}{N}\, \alpha^{-\frac{N+1-k}{N}} \int_M |u(x)|^2 dx.\end{aligned}$$

This concludes the proof of (8.7). Since a_N is positive, we have

$$\int_M |u(x)|^{2N+2} dx \leq \frac{1}{a_{N+1}} \int_M \tilde F(|u(x)|^2) dx - \frac{1}{a_{N+1}} \int_M Q(|u(x)|^2) dx.$$

Thus applying (8.7) with $\varepsilon = \frac{1}{2a_{N+1}}$ concludes the proof of (8.6). Finally, $\tilde f(r) r = a_N r^{N+1} + \tilde Q(r)$ where $\tilde Q$ is a polynomial of degree N. Hence (8.7) and (8.6) yield (8.8). $\square$

I have put the corollary inside the lemma
Case 1(b). We assume that $\tilde f(r) = C r^\sigma$ for $C > 0$ and $\sigma \geq \frac12$. Then $\tilde F \geq 0$ and thus

$$\int_M \tilde F(|u(x)|^2) dx \geq 0, \text{ for any } u \in \mathcal{R}.$$

Furthermore, since $r\tilde f(r) = C\tilde F(r)$, there exists some positive constant C such that

$$\int_M \tilde f(|u(x)|^2)|u(x)|^2\, dx \leq C \int_M |\tilde F(|u(x)|^2)|\, dx, \quad \forall u \in H^{1,2}(M). \tag{8.9}$$

Case 2. We assume that $\tilde{f}(r) = -Cr^\sigma$ for $C>0$ and $\sigma \in [\frac{1}{2},1)$. Then $\tilde{F}(r) = -\frac{C}{\sigma+1} r^{\sigma+1}$. The following lemma will be used to deal with this case.

Lemma 8.8. *Assume that $\alpha \in (1,3)$. Then for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for any $u \in H^{1,2}(M)$ (and for any $u \in \mathcal{R}$),*

$$\int_M |u(x)|^{\alpha+1} dx \le \varepsilon |\nabla u|_{L^2}^2 + C(\varepsilon) |u|_{L^2}^{\frac{4}{3-\alpha}}. \tag{8.10}$$

Furthermore, for $\sigma \in [\frac{1}{2},1)$ there exists $\tilde{C} > 0$ such that for every $u \in H^{1,2}(M)$,

$$\int_M |\tilde{f}(|u(x)|^2)| |u(x)|^2 \, dx \le \frac{1}{2} |\nabla u|_{L^2}^2 + \frac{1}{2} \int_M \tilde{F}(|u(x)|^2) dx + \tilde{C} |u|_{L^2}^{\frac{2}{1-\sigma}}. \tag{8.11}$$

Proof. The Gagliardo-Nirenberg and Young inequalities imply that for $u \in H^{1,2}(M)$ and $\varepsilon > 0$,

$$\int_M |u(x)|^{\alpha+1} dx \le C \, |\nabla u|_{L^2}^{\alpha-1} \, |u|_{L^2}^2 \le \varepsilon |\nabla u|_{L^2}^2 + C \varepsilon^{-\frac{2(\alpha-1)}{3-\alpha}} \, |u|_{L^2}^{\frac{4}{3-a}}.$$

This concludes the proof of (8.10). Finally, $|\tilde{f}(r)| r = c r^{\sigma+1}$, $r \ge 0$ and since $\sigma < 1$ we have

$$\int_M |\tilde{f}(|u(x)|^2)| |u(x)|^2 \, dx - \frac{1}{2} \int_M \tilde{F}(|u(x)|^2) dx = C\Big(1 + \frac{1}{2(\sigma+1)}\Big) \int_M |u(x)|^{2+2\sigma} \, dx.$$

Hence using (8.10) with $\varepsilon = \frac{1+\sigma}{2C(2\sigma+3)}$ concludes the proof. ☐

Let us assume that $\tilde{f}$ satisfy Assumption 8.4, hence either the conditions of Case **1(a)**, **1(b)** or **2** above. Let us define the map

$$\Psi : \mathcal{R} \ni u \mapsto \frac{1}{2} |\nabla u|_{L^2}^2 + \frac{1}{2} \int_M \tilde{F}(|u(x)|^2) dx \in \mathbb{R}. \tag{8.12}$$

Using Lemmas 8.7 or 8.8, it is easy to see that there exists a constant $C > 0$ such that $\Psi(u) + C|u|_{L^2}^2 \ge 0$ for all $u \in \mathcal{R}$. This proves the following

Corollary 8.9. *There exists a constant $c \ge 0$ such that*

$$|u|_{H^{1,2}}^2 \le 2\Psi(u) + c|u|_{L^2}^2, \quad u \in \mathcal{R}.$$

We will need the following result about the regularity of Ψ and some of its properties.

Lemma 8.10. *The function Ψ defined by (8.12) is of real $\mathcal{C}^2$-class with the second derivative bounded on balls; for all $u, v_1, v_2 \in \mathcal{R}$, we have*

$$\begin{aligned}
\Psi'(u)(v) &= \mathrm{Re} \int_M \nabla u(x) \overline{\nabla v(x)} \, dx + \int_M \tilde{f}(|u(x)|^2) \, \mathrm{Re}\, [u(x)\overline{v(x)}] \, dx, \\
\Psi''(u)(v,v) &= \int_M |\nabla v(x)|^2 \, dx + \int_M \tilde{f}(|u(x)|^2) |v(x)|^2 dx \\
&\quad + 2 \int_M \tilde{f}'(|u(x)|^2) \Big(\mathrm{Re}\, [u(x)\overline{v(x)}] \Big)^2 \, dx.
\end{aligned}$$

Moreover,

$$\langle \Psi'(u), i[\Delta u - F(u)] \rangle = 0, \quad u \in H^{2,2}(M), \tag{8.13}$$

$$\langle \Psi'(u), iG(u) \rangle = \int_M \mathrm{Re}\, \big(\nabla u(x) \overline{\nabla i g(u(x))} \big) dx, \quad u \in \mathcal{R}, \tag{8.14}$$

$$\mathrm{tr}_K \Psi''(u)\big(iG(u), iG(u)\big) \le 2 \int_M |g'(u(x)) \nabla u(x)|^2 \mathfrak{p}(x) \, dx + 2 \int_M |g(u(x))|^2 \mathfrak{q}(x) \, dx$$

$$+\int_M \tilde{f}(|u(x)|^2)|g(u(x))|^2\mathfrak{p}(x)\,dx \quad u\in\mathcal{R}, \tag{8.15}$$

where

Proof. The regularity of Ψ and the explicit expressions of its first and second derivatives follow from section 2, in particular from Remark 2.5. Note that if $\phi:\mathbb{R}\to\mathbb{R}$, we have

$$\mathrm{Re}\big(z\,\overline{i\phi(|z|^2)z}\,\big)=0, \quad \forall z\in\mathbb{C}. \tag{8.16}$$

Integration by parts implies that for every $u\in H^{2,2}(M)$, we have

$$\mathrm{Re}\int_M \nabla u(x)\overline{\nabla[i\Delta u(x)]}\,dx = -\mathrm{Re}\int_M \Delta u(x)\overline{i\Delta u(x)}\,dx = 0.$$

Thus (8.16) applied to $\phi=\tilde{f}$ yields (8.13). Equality (8.14) is a consequence of (8.16) applied with $\phi=\tilde{g}$.
We now prove the last assertion. The inclusion $\mathcal{R}\subset L^\infty(M)$ implies that for every $u\in\mathcal{R}$, $g\circ u\in L^\infty(M)$, and the integral $\int_M |g(u(x))|^2\mathfrak{q}(x)\,dx$ exists since $\mathfrak{q}\in L^1(M)$ as proved in Lemma 8.2.

By the definition of the trace, since $\Lambda e_j(x)\in\mathbb{R}$, using the Cauchy-Schwarz inequality and (8.16) with $\phi=\tilde{g}$, we deduce

$$\begin{aligned}
&\mathrm{tr}_K\Psi''(u)\big(iG(u),iG(u)\big)=\sum_{j\geq 1}\Psi''(u)\big(i(g\circ u)e_j, i(g\circ u)\Lambda e_j\big)\\
&= \sum_{j\geq 1}\int_M\Big|\nabla\Big(g(u(x))\Lambda e_j(x)\Big)\Big|^2dx+\sum_{j\geq 1}\int_M\tilde{f}(|u(x)|^2)\,|g(u(x))\Lambda e_j(x)|^2\,dx\\
&\quad +2\sum_{j\geq 1}\int_M\tilde{f}'(|u(x)|^2)\Big(\mathrm{Re}\,[u(x)\overline{iu(x)\tilde{g}(|u(x)|^2)}\Lambda e_j(x)]\Big)^2dx.\\
&\leq 2\int_M|g'(u(x))\nabla u(x)|^2\Big(\sum_{j\geq 1}|\Lambda e_j(x)|^2\Big)dx+2\int_M|g(u(x))|^2\Big(\sum_{j\geq 1}|\nabla\Lambda e_j(x)|^2\Big)dx\\
&\quad +\int_M\tilde{f}(|u(x)|^2)|g(u(x))|^2\Big(\sum_{j\geq 1}|\Lambda e_j(x)|^2\Big)dx.
\end{aligned}$$

This proves (8.15) and concludes the proof of the Lemma. $\square$

The following lemma gives an explicit expression of $\Psi(u(t))$, where $\big(u(t),t\in[0,\tau_\infty)\big)$ denotes the local maximal solution to (8.4). Note that, unlike in the deterministic case, the Itô-Stratonovich correction term yields that $\mathbb{E}(\Psi(u(t)))$ is not time invariant.

Lemma 8.11. *Assume that $(W(t),t\geq 0)$ is an $\mathcal{R}$-valued Wiener process. Then in the framework above, for every $t\geq 0$ and every $k\in\mathbb{N}^*$, we have*

$$\Psi(u(t\wedge\tilde{\tau}_k))=\Psi(u_0)-\int_0^{t\wedge\tilde{\tau}_k}\int_M\mathrm{Re}\,\big(\nabla u(s,x)\overline{\nabla ig(u(s,x))}\big)\,dx\,dW(s)+T(t\wedge\tilde{\tau}_k), \tag{8.17}$$

where

$$\begin{aligned}
T(t\wedge\tilde{\tau}_k)\leq &\int_0^{t\wedge\tilde{\tau}_k}\int_M|g'(u(s,x))|^2\,|\nabla u(s,x)|^2\,\mathfrak{p}(x)\,dx\,ds+\int_0^{t\wedge\tilde{\tau}_k}\int_M|g(u(s,x))|^2\,\mathfrak{q}(x)\,dx\,ds\\
&+\frac{1}{2}\int_0^{t\wedge\tilde{\tau}_k}\int_M\tilde{f}(|u(s,x)|^2)|g(u(s,x))|^2\mathfrak{p}(x)\,dx\,ds
\end{aligned} \tag{8.18}$$

Proof. According to Lemma 8.10 Ψ is of class $\mathcal{C}^2$. Thus the proof of (8.17) is done using the Itô Lemma for Yosida approximations of the solution and passing to the limit (see e.g. [7] for a more detailed justification). The upper estimate (8.18) of the "quadratic variation" is deduced from using (8.15). $\square$

8.4. **Existence of a global solution.** We can now state the main result of this section, proving that the non linear stochastic Schrödinger equation (8.1), or (8.4), has a unique global solution.

Theorem 8.12. *Assume that the function $\tilde{f}$ satisfy Assumption 8.4, that g satisties Assumption 8.3 and that the Wiener process $(W(t), t \geq 0)$ satisfies Assumption 8.1. Let β be defined in Lemma 8.5, $p > \beta \vee \gamma$ where γ is defined in Assumption 8.3, and let q be such that (p, q) satisfy the scaling admissible condition $\frac{2}{p} + \frac{2}{q} = 1$. Suppose furthermore that $u_0 \in H^{1,2}(M)$. Then the stochastic NLS equation* (8.1) *has a unique global solution whose trajectories belong a.s. to $\mathcal{C}([0, \infty), H^{1,2}(M))$.*

Proof. Let $u = \big(u(t)\,,\, t < \tau_\infty\big)$ belonging to $\mathbb{M}^p_{\text{loc}}(Y_{[0,\tau_\infty)})$, be the unique local maximal solution to the problem (8.1). Note that $\limsup_{t\to\tau_\infty} |u(t)|_{H^{1,2}} = +\infty$ a.s. on $\{\tau_\infty < \infty\}$; for an integer $k \geq 1$ recall that $\tilde{\tau}_k = \inf\{t \geq 0 : |u(t)|_{H^{1,2}} \geq k\}$. Using the Khashminskii test for non-explosions (see [23, Theorem III.4.1] for the finite-dimensional case) and arguing as in [7, page 7] it is sufficient to show that each $t > 0$, there exists a constant $C_t > 0$ such that

$$\mathbb{E}\Big(|u(t \wedge \tilde{\tau}_k)|^2_{H^{1,2}}\Big) \leq C_t, \quad \text{for every } k \in \mathbb{N}^*. \tag{8.19}$$

In view of Corollary 8.9 and Lemma 8.6 it is sufficient to find, for each $t > 0$, a constant $C_t > 0$ such that

$$\mathbb{E}\Big(\Psi(u(t \wedge \tilde{\tau}_k))\Big) \leq C_t, \;\; \text{for every } k \in \mathbb{N}^*. \tag{8.20}$$

Since $W^{1,2s_0}(M, \mathbb{R}) \cap W^{\hat{s},q}(M, \mathbb{R}) \subset \mathcal{R}$, the assumptions of Lemma 8.11 are satisfied. Hence, for each $t \in \mathbb{R}_+$ and every $k \in \mathbb{N}$, $\mathbb{P}$-almost surely,

$$\begin{aligned} \mathbb{E}\Psi(t \wedge \tilde{\tau}_k) \leq & \mathbb{E}\Psi(u_0) + C|\mathfrak{p}|_\infty \mathbb{E} \int_0^{t\wedge\tilde{\tau}_k} \int_M |\nabla u(s,x)|^2 \, dx \, ds \\ & + C \sum_{j\geq 1} \mathbb{E} \int_0^{t\wedge\tilde{\tau}_k} \int_M |u(s,x)|^2 |\nabla \Lambda e_j(x)|^2 \, dx \, ds \\ & + C|\mathfrak{p}|_\infty \, \mathbb{E} \int_0^{t\wedge\tilde{\tau}_k} \int_M \tilde{f}(|u(s,x)|^2)|u(s,x)|^2 \, dx \, ds. \end{aligned} \tag{8.21}$$

Let s_0^* denote the conjugate exponent to s_0. The Gagliardo-Nirenberg inequality proves that $H^{1,2}(M) \subset L^{2s_0^*}(M)$; Hölder's inequality, Lemma 8.2 and Corollary 8.9 imply that for $u \in H^{1,2}(M)$,

$$\sum_{j\geq 1} \int_M |u(x)|^2 |\nabla \Lambda e_j(x)|^2 \, dx \leq \|u\|^2_{L^{2s_0^*}} \sum_j \|\nabla \Lambda e_j\|^2_{L^{2s_0}} \leq C\big[\Psi(u) + |u|_{L^2(M)}\big].$$

Next, the inequalities (8.8), (8.9) and (8.11), imply the existence of positive constants C and δ such that for all $u \in H^{1,2}(M)$,

$$\int_M \tilde{f}(|u(x)|^2)|u(x)|^2 \, dx \leq C\big[\Psi(u) + |u|^\delta_{L^2(M)}\big].$$

Therefore, the conservation of energy proved in Lemma 8.6 and the above estimates imply the existence of an increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $C_t > 0$ such that

$$\begin{aligned} \mathbb{E}\Psi(t\wedge\tilde\tau_k) \le& \mathbb{E}\Psi(u_0) + C\mathbb{E}\int_0^{t\wedge\tilde\tau_k} \Psi(u(s))\,ds + C\mathbb{E}\int_0^{t\wedge\tilde\tau_k} \phi(|u(s\wedge\tilde\tau_k)|_{L^2(M)})\,ds \\ \le& \mathbb{E}\Psi(u_0) + C\mathbb{E}\int_0^t \Psi(u(s\wedge\tilde\tau_k))\,ds + C\phi(|u_0|_{L^2(M)}). \end{aligned} \tag{8.22}$$

The Gronwall Lemma yields that for some constant $C > 0$ the upper estimate

$$\mathbb{E}\Psi(t\wedge\tilde\tau_k) \le \big[\mathbb{E}\Psi(u_0) + Ct\phi(|u_0|_{L^2(M)})\big]e^{Ct}, \quad t \ge 0, \tag{8.23}$$

holds for every integer $k \ge 1$. This concludes the proof of (8.20) and hence that of the Theorem. $\square$

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