Normality and differentiability

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A recent theorem of Brattka, Miller and Nies [1] shows that a real number r in the unit interval is computably random if and only if every nondecreasing computable function from the unit interval to the real numbers is differentiable at r. Here we establish a counterpart result that characterizes *normality to a given base* in terms of differentiability of *functions computable with finite transducers* (injective finite state automata).

For a real number r we consider the unique expansion in base b of the form $r = \lfloor x \rfloor + \sum_{n=1}^{\infty} a_n b^{-n}$ where the integers $0 \leq a_n < b$, and $a_n < b-1$ infinitely many times. This last condition over a_n ensures a unique representation of every rational number. Let us recall that Borel's original definition of normality in [2] is equivalent to the following simpler one [3].

Definition. A real number r is simply normal to a given base b if each digit in $\{0, 1, .., (b-1)\}$ occurs with the same limiting frequency $1/b$ in the expansion of r in base b. A number is *normal to base* b if it is simply normal to the each base b^i, for very positive integer i.

For a finite set of symbols $\mathcal{A}$ we write $\mathcal{A}^*$ and $\mathcal{A}^\omega$ to denote, respectively, the set of finite and infinite sequences of symbols in $\mathcal{A}$,

Definition. (1) A *finite state transducer* is a 4-uple $C = \langle \mathcal{Q}, q_0, \delta, o \rangle$, where $\mathcal{Q}$ is a finite set of states, $q_0 \in \mathcal{Q}$ is the initial state, $\delta : \mathcal{Q} \times \mathcal{A} \to \mathcal{Q}$ is the transition function and $o : \mathcal{Q} \times \mathcal{A} \to \mathcal{A}^*$ is the output function. A finite state transducer processes the input symbols according to the current state q. When a symbol $a \in \mathcal{A}$ is read, the automaton moves to state $\delta(q, a)$ and outputs $o(q, a)$. The extension of δ and o to process strings are $\delta^* : \mathcal{Q} \times \mathcal{A}^* \to \mathcal{Q}$ and $o^* : \mathcal{Q} \times \mathcal{A}^* \to \mathcal{A}^*$ such that, for $a \in \mathcal{A}$, $s \in \mathcal{A}^*$ and λ the empty string, $\delta^*(q, \lambda) = q$, $\delta^*(q, as) = \delta^*(\delta(q, a), s)$, and $o^*(q, \lambda) = \lambda$, $o^*(q, as) = o(q, a)o^*(\delta(q, a), s)$.. The extension of o to infinite sequences $o^* : \mathcal{Q} \times A^\omega \to A^\omega$ is $o^*(q, x) = \lim_{k \to \infty} o(q, x[1..k])$.

(2) The function $f_C : A^\omega \to A^\omega$ computed by $C = \langle \mathcal{Q}, q_0, \delta, o \rangle$ is $f_C(x) = o^*(q_0, x)$.

(3) A function $f : A^\omega \to A^\omega$ is computable by a finite state transducer when $f = f_C$ for some finite state transducer C. A function $f : A^\omega \to \mathbb{R}$ is computable by a finite state transducer when $f = \text{conv}(f_C)$ for some finite state transducer C, where $\text{conv} : A^\omega \to \mathbb{R}$ is the usual map $\text{conv}(x) = \sum_{i \geq 1} t^{-i} x[i]$, with t the cardinality of $\mathcal{A}$.

The following example shows that the obvious definition of differentiability is not appropriate for our purposes.

Example. Let $I = \langle q, q, \pi_1, \pi_2 \rangle$ where π_1 and π_2 are respectively the projections functions of the first and second argument. So, the function $f_I : \{0,1\}^\omega \to \mathbb{R}$ is the identify function mapped to the unit interval. The obvious definition of differentiability would yield $\lim_{k\to\infty} 2^{-k}(\mathrm{conv}(\pi_2^*(q, x[1..k-1]1)) - \mathrm{conv}(\pi_2^*(q, x[1..k-1]0))) = 1$. Now, let $C = \langle \{q, r_0, r_1\}, q, \delta, o \rangle$ such that for $a, b \in \mathbf{2}$, $\delta(q, b) = r_b, \delta(r_b, a) = q, o(q, b) = \lambda, o(r_b, a) = ba$. It is easy to check that $f_C : \{0,1\}^\omega \to \mathbb{R}$ is also the identify function mapped to the unit interval. However, $\lim_{k\to\infty} 2^{-k}(\mathrm{conv}(o^*(q, x[1..k-1]0)) - \mathrm{conv}(o^*(q, x[1..k-1]0)))$ does not exist for any x.

Definition. The differential of a non-decreasing function $f : A^\omega \to \mathbb{R}$ at x is

$$Df(x) = \lim_{k\to\infty} t^{-k}(f(x[1..k]1^\omega) - f(x[1..k]0^\omega)) = \lim_{k\to\infty} \mu(f(T_{x[1..k]}))/\mu(T_{x[1..k]}),$$

where t is the cardinality of $\mathcal{A}$, $T_s = \{sx : x \in A^\omega\}$ is the cone defined by the string s, and $f(T_s) = \{f(sx) : x \in A^\omega\}$. We say that f is differentiable at x if $Df(x)$ exists.

Now we can formulate the announced result.

Theorem. A real number r is normal to a given base b if, and only if, every real valued non-decreasing function computable by a finite state transducer is differentiable at the expansion of r in base b.

The proof relies in the characterization of normal sequences as those incompressible by information lossless finite state compressors, result that follows from [6, 4, 5]. An adaptation is needed to deal with the non-decreasing condition.

References

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