GLOBAL SOLUTIONS OF THE RANDOM VORTEX FILAMENT EQUATION

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ABSTRACT. In this article we prove the existence of a global solution for the random vortex filament equation. Our work gives a positive answer to a question left open in recent publications: Berselli and Gubinelli [5] showed the existence of global solution for a smooth initial condition while Bessaih, Gubinelli, Russo [6] proved the existence of a local solution for a general initial condition.

In this article we prove the existence of a global solution for the following random vortex filament equation

$$\frac{d\gamma}{dt} = u^{\gamma(t)}(\gamma(t)), \quad t \in [0,\infty) \tag{0.1}$$

$$\gamma(0) = \gamma_0, \tag{0.2}$$

where the initial condition $\gamma_0 : [0,1] \to \mathbb{R}^3$ is a geometric ν-rough path (for some $\nu \in (\frac{1}{3}, 1)$), see Assumption 2.7. Here $\gamma : [0,\infty) \to \mathcal{D}_{\gamma_0} \subset \mathcal{C}$ is some trajectory in the subset $\mathcal{D}_{\gamma_0}$ of $\mathcal{C}$ of continuous closed curves in $\mathbb{R}^3$, u^Y, $Y \in \mathcal{D}_{\gamma_0} \subset \mathcal{C}$ is a vector field given by

$$u^Y(x) = \int_Y \nabla\phi(x-y) \times dy. \tag{0.3}$$

where $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function which satisfies certain assumptions (see Hypothesis 3.1). The exact meaning of the line integral above and set $\mathcal{D}_{\gamma_0}$ we consider will be explained below. Equation (0.1) appears in the fluid dynamics in the theory of three dimensional Euler equations. It is well known that for two dimensional Euler equations vorticity $\vec{\omega} = \operatorname{curl} \vec{u}$ is transported along the flow of the liquid. The situation changes drastically in three dimensional case. Additional "stretching" term in the equation defining vorticity leads to possibility of blow up of the vorticity. Furthermore, a result of Beale, Kato, and Majda [2] suggests that a possible singularity of Euler equations appears when the vorticity field of the fluid blows up. Consequently, understanding the behaviour of vorticity of ideal fluid is one of the most important problems in fluid dynamics.

The properties of the motion of the vorticity has been studied for the last 150 years starting from the works of Helmholtz [22] and Kelvin [23]. It has been suggested by Kelvin to use Biot-Savart law

$$\vec{u}(\vec{x}) = \int \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3} \times \vec{\omega}(\vec{y})d\vec{y}, \quad \vec{x},\vec{y} \in \mathbb{R}^3$$

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where $\times$ denotes the vector product, combined with assumption that vorticity is supported by some smooth curve γ

$$\omega(\vec{x},t) = \Gamma \int_0^1 \delta(\vec{x}-\gamma(s,t))\frac{\partial\gamma(s,t)}{\partial s}ds, x \in \mathbb{R}^3, t \geq 0$$

and definition of the flow

$$\begin{cases} \frac{d\vec{X}_t(\vec{x})}{dt} &= \vec{u}(\vec{X}_t(\vec{x}),t), \quad t \geq 0, \\ \vec{X}_0(\vec{x}) &= \vec{x}. \end{cases} \tag{0.4}$$

to formally deduce the following filament equation

$$\frac{\partial\gamma}{\partial t}(s,t) = -\frac{\Gamma}{4\pi}\int_0^1 \frac{\gamma(s,t)-\gamma(r,t)}{|\gamma(s,t)-\gamma(r,t)|^3} \times \frac{\partial\gamma(r,t)}{\partial r}dr. \tag{0.5}$$

The assumption that vorticity is supported by some curve is coherent with numerical simulations of 3D turbulent fluids which show that regions of large vorticity have a form of a "filament", see, for instance [3], [29].

Equation (0.5) has singularity when r is close to s and the initial curve γ is smooth. As a consequence, the energy of the solution of this equation given by the formula

$$E(t) = \frac{\Gamma^2}{8\pi}\int_0^1\int_0^1 \frac{1}{|\gamma(s,t)-\gamma(r,t)|}\frac{\partial\gamma(r,t)}{\partial r}\cdot\frac{\partial\gamma(s,t)}{\partial s}dsdr, t \geq 0$$

is infinite for any smooth curve $\gamma(\cdot,t)$. Hence different methods have been employed to avoid singularity. For instance, Gallavotti [18] motivated by finiteness of the energy integral for Brownian Motion[1], considers non smooth initial curves γ_0 while Rosenhead [28] has suggested to use the following model

$$\frac{\partial\gamma}{\partial t}(s,t) = -\frac{\Gamma}{4\pi}\int_0^1 \frac{\gamma(s,t)-\gamma(r,t)}{(|\gamma(s,t)-\gamma(r,t)|^2+\mu^2)^{3/2}} \times \frac{\partial\gamma(r,t)}{\partial r}dr. \tag{0.6}$$

Problem ((0.1)-(0.3)) has been studied by Berselli and Bessaih [4] and then by Berselli and Gubinelli [5]. It contains equation (0.6) considered by Rosenhead as a very particular case when

$$\phi(\vec{x}) = \frac{\Gamma}{(|\vec{x}|^2+\mu^2)^{\frac{1}{2}}}, \vec{x} \in \mathbb{R}^3, \mu > 0.$$

Equation (0.1) is in fact a nonlinear PDE for a function $\gamma : [0,\infty)\times[0,1] \to \mathbb{R}^3$. A natural setting for the well-posedness of the Cauchy problem is obtained by requiring the vector field u to be well defined and Lipshitz in the space variable. To this effect the approach followed in [4] is to set up the equation as an evolution problem in the Sobolev space H^1 of closed curves in $\mathbb{R}^3$ with square integrable first derivative (with respect to the parameter). This approach implies that the vector field (once lifted to H^1) does not allow good estimates to have global existence. This is ultimately due to the fact that in $3d$ incompressible flows vortices strech and undergo a complex dynamics and that a priori this could lead to a explosion of

[1] defined as double stochastic integral

the H^1 norm. Such a difficulty should be compared to the more stable behavior of 2d vortex points which, under incompressible flows, are simply transported along the flow lines. Exploiting the conservation of the kinetic energy of the flow and a control of the velocity field generated by the vortex line via the associated kinetic energy Berselli and Gubinelli [5] showed the existence of global solution to equation (0.1) with initial conditions in H^1. After that Bessaih, Gubinelli, Russo [6], partly motivated by the random filament models suggested by Gallavotti [18] and Chorin [12], considered the above evolution problem when the initial condition is a random closed curve which for definiteness, in the above paper, has been taken to be the sample of a Brownian loop (a Brownian motion starting at $0 \in \mathbb{R}^3$ and conditioned to return to $0 \in \mathbb{R}^3$ at time 1). In this case it is no more possible to set up the problem in the Sobolev space H^1 since Brownian trajectories almost surely do not belong to this space. A more serious problem is the meaning to give to the generalized Biot-Savart relation (0.3) since the line integral along a Brownian trajectory is a notoriously difficult object to define. Stochastic integration (à la Itô or Stratonovich) does not provide a good framework to study this problem and [6] identified the natural setting to have a well posed problem by considering the evolution as an equation on the space of rough paths.

Rough path theory has been introduced by T. J. Lyons in the seminal paper [25] (see also [17,26,27]) as a way to overcome certain difficulties of stochastic integration theories and have a robust analytical framework to solve stochastic differential equations and similar problems involving integration of non-regular vector-fields. It turns out that rough paths theory and in particular the notion of *controlled paths* introduced in [20] allows to give a natural interpretation to the Biot-Savart relation (0.3) and obtain a well-posed problem. Using this approach Bessaih, Gubinelli, Russo [6] could obtain existence of a local solution to equation (0.1) when initial condition is a closed curve of Hölder class with exponent $\nu \in (\frac{1}{3}, 1]$ (suitably lifted to rough path space).

The aim of the present paper is to extend the energy method of Berselli and Gubinelli to the rough path setting to obtain the global existence of solution of the equation (0.1) when the initial condition is a (geometric) rough path, thus completing the analysis of [6].

Recently there have been some progress in the study of evolution equations in the space of (controlled) rough paths. In particular Hairer [21] showed how to use rough path theory to have well-posedness of a multidimensional Burgers type equation driven by additive space-time white noise. The key technical tool to obtain these results has been the observation that the non-linear term in the Burgers equation has the same structure of the Biot-Savart relation (0.3) and thus can be similarly handled via rough paths techniques.

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1. NOTATION

In this section we present the controlled path framework introduced by Gubinelli [20]. Let V be a fixed Banach space. We introduce the following two objects, where S^1 denotes the unit circle:

$$\begin{aligned} C_n(V) &= \{f \in C((S^1)^n, V) : \ f(t,t,\cdots,t) = 0, t \in S^1\}, \\ C_*(V) &= \bigcup_{k\in\mathbb{N}} C_k(V). \end{aligned}$$

The operator δ is defined by

$$\begin{aligned} \delta &: C_n(V) \to C_{n+1}(V), \\ (\delta g)(t_1,\cdots,t_{n+1}) &= \sum_{i=1}^{n+1}(-1)^i g(t_1,\cdots,\hat{t}_i,\cdots,t_{n+1}), \ \ n \in \mathbb{N} \end{aligned}$$

satisfies the following fundamental property

$$\delta\delta = 0,$$

where $\delta\delta$ is understood as an operator from $C_n(V)$ to $C_{n+2}(V)$. Thus δ induces a complex and we can denote

$$\begin{aligned} \mathcal{Z}C_k(V) &= C_k(V) \cap \ker \delta, \\ \mathcal{B}C_k(V) &= C_k(V) \cap \operatorname{im} \delta. \end{aligned}$$

To avoid confusion we will use notation δ_n for operator $\delta : C_n(V) \to C_{n+1}(V)$. Furthermore, it can be noticed that

$$\operatorname{im} \delta_n = \ker \delta_{n+1},$$

i.e. $\mathcal{Z}C_{k+1}(V) = \mathcal{B}C_k(V)$.

We will mainly consider the cases $n = 1$ and $n = 2$. Then the operator δ takes the following form

$$\delta_1 g(t,s) = g(t) - g(s), \delta_2 h(t,u,s) = h(t,s) - h(t,u) - h(u,s), t,u,s \in S^1.$$

We will use special topology in spaces $C_2(V)$ and $C_3(V)$. Let, for $\mu, \rho > 0$,

$$\begin{aligned} |f|_\mu &= \sup_{a,b\in S^1} \frac{|f(a,b)|_V}{|a-b|^\mu}, \ \ f \in C_2(V), \\ C_2^\mu(V) &= \{f \in C_2(V) : \ |f|_\mu < \infty\} \\ |g|_{\rho,\mu} &= \sup_{a,b,c\in S^1} \frac{|g(a,b,c)|_V}{|a-b|^\rho |b-c|^\mu}, \ \ g \in C_3(V), \\ C_3^{\rho,\mu}(V) &= \{f \in C_3(V) : \ |f|_{\rho,\mu} < \infty\}, \ \ |\sum_i h_i|_\mu := \sum_i |h_i|_{\rho_i,\mu-\rho_i} \\ C_3^\mu(V) &= \{f \in C_3(V) : \exists (h_i), \ \exists (\rho_i) \subset (0,\mu) : \ f = \sum_i h_i, |f|_\mu < \infty\}, \\ \mathcal{Z}C_3^\mu(V) &= C_3^\mu(V) \cap \mathcal{Z}C_3(V), \\ \mathcal{Z}C_3^{1+}(V) &= \bigcup_{\mu>1} \mathcal{Z}C_3^\mu(V), \\ C_2^{1+}(V) &= \bigcup_{\mu>1} C_2^\mu(V). \end{aligned}$$

Then following fundamental proposition has been proved in [20]:

Proposition 1.1. *There exists an unique linear map $\Lambda : \mathcal{Z}C_3^{1+}(V) \to C_2^{1+}(V)$ such that*

$$\delta\Lambda = \mathrm{id}_{\mathcal{Z}C_3^{1+}(V)} .$$

Furthermore, for any $\mu > 1$, this map is continuous from $\mathcal{Z}C_3^{\mu}(V)$ to $C_2^{\mu}(V)$ and we have

$$\|\Lambda h\|_{\mu} \leq \frac{1}{2^{\mu}-2}\|h\|_{\mu}, h \in \mathcal{Z}C_3^{1+}(V).$$

Now, we define class of paths for which rough path integral will be defined.

Definition 1.2. *Assume that $\nu \in (0,1)$. Let us fix $X \in C^{\nu}(S^1, V)$. We say that path $Y \in C(S^1, V)$ is weakly controlled by X if there exist functions $Z \in C^{\nu}(S^1, L(V,V))$ and $R \in C_2^{2\nu}(V)$ such that*

$$Y(\xi) - Y(\eta) = Z(\eta)(X(\xi) - X(\eta)) + R(\xi, \eta), \xi, \eta \in S^1, \tag{1.1}$$

Let $\mathcal{D}_X$ be the set of pairs (Y, Z), where $Y \in C(S^1, V)$ is a path weakly controlled by X, and $Z \in C^{\nu}(S^1, L(V,V))$ is such that $R \in C_2^{2\nu}(V)$, where R is defined by representation (1.1), i.e.

$$R(\xi, \eta) = Y(\xi) - Y(\eta) - Z(\eta)(X(\xi) - X(\eta)), \xi, \eta \in S^1. \tag{1.2}$$

Let us notice that $\mathcal{D}_X$ is a vector space. Let us define semi-norm $\|\cdot\|_{\mathcal{D}_X}$ in $\mathcal{D}_X$ by

$$\|(Y,Z)\|_{\mathcal{D}_X} = |Z|_{C^{\nu}} + \|R\|_{C_2^{2\nu}}, \tag{1.3}$$

where Furthermore, let us define a norm $\|\cdot\|^*_{\mathcal{D}_X}$ in $\mathcal{D}_X$ by

$$\|(Y,Z)\|^*_{\mathcal{D}_X} = \|Y\|_{\mathcal{D}_X} + |Y|_{C(S^1,V)}. \tag{1.4}$$

one can prove that $(\mathcal{D}_X, \|\cdot\|^*_{\mathcal{D}_X})$ is a Banach space. From now on we will denote elements of $\mathcal{D}_X$ by (Y, Y') and the corresponding function R will be denoted by R^Y. We will often omit to specify Y' when it is clear from the context and write $\|Y\|_{\mathcal{D}_X}$ instead of $\|(Y, Y')\|_{\mathcal{D}_X}$.

2. Definition and Properties of Rough Path integrals

In this section we define rough path integral and state some of its properties. We mainly follow [20] and [6]. We assume that $V = \mathbb{R}^3$.

Definition 2.1. *Let $\Pi : \mathcal{D}_X \ni (Y, Z) \mapsto Y \in C(S^1, \mathbb{R}^3)$ be the natural projection.*

We will need following properties of the $\mathcal{D}_X$, see [20].

Lemma 2.2. $\Pi(\mathcal{D}_X) \subset C^{\nu}(S^1, \mathbb{R}^3)$.

Proof of Lemma 2.2. It follows from equality (1.1)

$$\|Y\|_{C^{\nu}} \leq \|Y\|^*_{\mathcal{D}_X}(1 + \|X\|_{C^{\nu}}). \tag{2.1}$$

The proof is complete. □

Lemma 2.3. *Let $\phi \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ and $(Y, Z) \in \mathcal{D}_X$. Then*

$$(W, W') := (\phi(Y), \phi'(Y)Z) \in \mathcal{D}_X \tag{2.2}$$

and the remainder has the following representation

$$\begin{aligned} R^W(\xi,\eta) &= \phi'(Y(\xi))R(\xi,\eta) + (Y(\eta)-Y(\xi)) \\ &+ \int_0^1 [\nabla\phi(Y(\xi)+r(Y(\eta)-Y(\xi))) - \nabla\phi(Y(\xi))]dr, \quad \xi,\eta\in S^1. \end{aligned} \tag{2.3}$$

where R is the remainder for Y w.r.t. X given by (1.2)*. Furthermore, there exists a constant $K \geq 1$ such that*

$$\|\phi(Y)\|_{\mathcal{D}_X} \leq K\|\nabla\phi\|_{C^1}\|Y\|_{\mathcal{D}_X}(1+\|Y\|_{\mathcal{D}_X})(1+\|X\|_{C^\nu})^2. \tag{2.4}$$

Moreover, if $(\tilde{Y},\tilde{Z}) \in \mathcal{D}_{\tilde{X}}$ and

$$(\tilde{W},\tilde{W}') := (\phi(\tilde{Y}), \phi'(\tilde{Y})\tilde{Z})$$

then

$$\begin{aligned} |W'-\tilde{W}'|_{C^\nu} + |R^W - R^{\tilde{W}}|_{C_2^{2\nu}} + |W-\tilde{W}|_{C^\nu} \leq \\ C(|X-\tilde{X}|_{C^\nu} + |Y'-\tilde{Y}'|_{C^\nu} + |R^Y - R^{\tilde{Y}}|_{C_2^{2\nu}} + |Y-\tilde{Y}|_{C^\nu}) \end{aligned} \tag{2.5}$$

with

$$C = K\|\phi\|_{C^3}(1+\|X\|_{C^\nu}+\|\tilde{X}\|_{C^\nu})^3|(1+\|Y\|_{\mathcal{D}_X}+\|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})^2. \tag{2.6}$$

In the case $X=\tilde{X}$ we have

$$\begin{aligned} \|\phi(Y)-\phi(\tilde{Y})\|_{\mathcal{D}_X} \leq & K\|\nabla\phi\|_{C^2}\|Y\|_{\mathcal{D}_X} \\ & (1+\|Y\|_{\mathcal{D}_X}+\|\tilde{Y}\|_{\mathcal{D}_X})^2(1+\|X\|_{C^\nu})^4\|Y-\tilde{Y}\|_{\mathcal{D}_X}. \end{aligned} \tag{2.7}$$

Proof of Lemma 2.3. See [20], Proposition 4 for all statements of the Lemma, except (2.3) (which is actually also proven, though not stated explicitly). Let us show (2.3). Denote $y(r) = Y(\xi)+r(Y(\eta)-Y(\xi)), r\in[0,1]$. Then

$$\begin{aligned} \phi(y(1)) - \phi(y(0)) &= \int_0^1 \phi'(y(r))y'(r)dr \\ &= \sum_k (Y^k(\eta)-Y^k(\xi)) \int_0^1 \frac{\partial\phi}{\partial x_k}(y(r))dr \\ &= \sum_k \frac{\partial\phi}{\partial x_k}(Y(\xi))(Y^k(\eta)-Y^k(\xi)) \\ &+ \sum_k (Y^k(\eta)-Y^k(\xi)) \int_0^1 [\frac{\partial\phi}{\partial x_k}(y(r)) - \frac{\partial\phi}{\partial x_k}(Y(\xi))]dr \\ &= \sum_{k,l} \frac{\partial\phi}{\partial x_k}(Y(\xi))(Y')^{kl}(X^l(\eta)-X^l(\xi)) + \sum_k \frac{\partial\phi}{\partial x_k}(Y(\xi))(R^Y)^k(\xi,\eta) \\ &+ \sum_k (Y^k(\eta)-Y^k(\xi)) \int_0^1 [\frac{\partial\phi}{\partial x_k}(y(r)) - \frac{\partial\phi}{\partial x_k}(Y(\xi))]dr, \end{aligned} \tag{2.8}$$

and the result follows. □

Now we will define an integral of a weakly controlled by X path Y w.r.t. to another weakly controlled by X path Z. For this aim we will need one more definition.

Definition 2.4. *Assume that $\nu > \frac{1}{3}$. We say that couple $\mathbb{X} = (X, \mathbb{X}^2)$, $X \in C^\nu(S^1, \mathbb{R}^3)$, $\mathbb{X}^2 \in C_2^{2\nu}(L(\mathbb{R}^3, \mathbb{R}^3))$ is a ν-rough path if the following condition is satisfied:*

(2.9)

$$\mathbb{X}^2(\xi,\rho) - \mathbb{X}^2(\xi,\eta) - \mathbb{X}^2(\eta,\rho) = (X(\xi) - X(\eta)) \otimes (X(\eta) - X(\rho)), \xi, \eta, \rho \in S^1$$

Remark 2.5. If $\nu > 1$ and $\mathbb{X}$ is a ν-rough path, then $\mathbb{X}$ is identically pair of constants $(X(0), 0)$. Indeed, in this case X is Hölder function with exponent more than 1 i.e. constant $X(0)$. Hence, $\mathbb{X}^2 = 0$.

Remark 2.6. If $\nu \in (\frac{1}{2}, 1]$ then $\mathbb{X}^2$, the second component of a ν-rough path $\mathbb{X} = (X, \mathbb{X}^2)$, is uniquely determined by its first component. Indeed, for $i, j = 1, 2, 3$,

$$\mathbb{X}^{2,ij}(\xi,\eta) = \int_\xi^\eta (X^i_\rho - X^i_\eta) dX^j_\rho, \xi, \eta \in S^1, \tag{2.10}$$

where the integral is understood in the sense of Young, see [30]. One can show that $\mathbb{X}^2$ defined by formula (2.10) satisfies conditions of Definition 2.4. Let us show the uniqueness of $\mathbb{X}^2$. Assume that there exists another $\mathbb{X}^2_1$ which satisfies definition 2.4. Put $G(\xi) = \mathbb{X}^2(\xi, 0) - \mathbb{X}^2_1(\xi, 0)$. Then by condition 2.9

$$\mathbb{X}^2(\xi,\rho) - \mathbb{X}^2_1(\xi,\rho) = G(\xi) - G(\rho),$$

and, since $\mathbb{X}^2 \in C_2^{2\nu}$, G is a Hölder function of order bigger than 1. Hence, $G = 0$. Therefore, $\mathbb{X}^2_1 = \mathbb{X}^2$.

Note that by identity (2.9) it follows that $\mathbb{X}^2(\xi, \xi) = 0, \xi \in S^1$.

Assumption 2.7. *We say that our ν-rough path $(X, \mathbb{X}^2)$ is an geometric ν-rough path if there exist a sequence $(X_n, \mathbb{X}^2_n)$ such that*

$$X_n \in C^\infty(S^1, \mathbb{R}^3),$$

$$\mathbb{X}^{2,ij}_n(\xi,\eta) = \int_\xi^\eta (X^i_n(\rho) - X^i_n(\eta)) dX^j_n(\rho), \xi, \eta \in S^1, i, j = 1, 2, 3,$$

and

$$\lim_{n\to\infty} \left[|X_n - X|_{C^\nu} + |\mathbb{X}^2_n - \mathbb{X}^2|_{C_2^{2\nu}} \right] = 0. \tag{2.11}$$

Example 2.8. Let $\{B_t\}_{t\in[0,1]}$ be the standard 3-dimensional Brownian bridge such that $B_0 = B_1 = x_0$ and let $\mathbb{B}^{2,ij}$, $i, j = 1, 2, 3$, be the area processes defined by

$$\mathbb{B}^{2,ij}(\xi,\eta) = \int_\xi^\eta (B^i_\rho - B^i_\eta) dB^j_\rho,$$

where the integral can be understood either in the Stratonovich or in the Itô sense. Then, the couple $(B, \mathbb{B}^2)$ is a ν-rough path see [6, p.1849]. Moreover, if the integral is understood in Stratonovich sense it is geometric ν-rough path. Indeed, it follows from Theorem 3.1 in [16] that one can approximate X with piecewise linear dyadic X'_n in the sense of assumption 2.7a.s..

From now on we suppose that the geometric ν-rough path $\mathbb{X} = (X, \mathbb{X}^2)$ and the corresponding Banach space $\mathcal{D}_X$ are fixed. For a finite partition $\pi = \{\xi_0 = \xi < \xi_1 < \cdots < \xi_n = \eta\}$ be of the interval $[\xi, \eta]$, let $d(\pi) = \sup_i |\xi_{i+1} - \xi_i|$ denote the mesh of the partition π.

Lemma 2.9. *Let If $Y, Z \in \mathcal{D}_X$ then the limit*

$$\lim_{d(\pi)\to 0} \sum_{i=0}^{n-1} [Y(\xi_i)(Z(\xi_{i+1}) - Z(\xi_i)) + Y'(\xi_i)Z'(\xi_i)\mathbb{X}^2(\xi_{i+1}, \xi_i)] \tag{2.12}$$

exists and is denoted by definition by

$$\int_\xi^\eta Y dZ.$$

Proof of Lemma 2.9. See [20], Theorem 1. □

Remark 2.10. In the case of $\nu > \frac{1}{2}$ the line integral defined in the Lemma 2.9 is reduced to the Young definition of the line integral $\int Y dZ$. Indeed, it is enough to notice that second term in formula (2.12) is of the order $O(|\xi_{i+1} - \xi_i|^{2\nu})$, $2\nu > 1$. Obviously, line integral does not depend upon Y', Z' in this case.

Lemma 2.11. *Assume $Y, W \in \mathcal{D}_X$, $\tilde{Y}, \tilde{W} \in \mathcal{D}_{\tilde{X}}$. Define maps $Q, \tilde{Q} : (S^1)^2 \to \mathbb{R}$ by the following identities*

$$Q(\eta, \xi) := \int_\xi^\eta Y dW - Y(\xi)(W(\eta) - W(\xi)) - Y'(\xi)W'(\xi)\mathbb{X}^2(\eta, \xi), \eta, \xi \in S^1, \tag{2.13}$$

$$\tilde{Q}(\eta, \xi) := \int_\xi^\eta \tilde{Y} d\tilde{W} - \tilde{Y}(\xi)(\tilde{W}(\eta) - \tilde{W}(\xi)) - \tilde{Y}'(\xi)\tilde{W}'(\xi)\tilde{\mathbb{X}}^2(\eta, \xi), \eta, \xi \in S^1. \tag{2.14}$$

Then $Q, \tilde{Q} \in C_2^{3\nu}$.
Moreover, there exists constant $C = C(\nu) > 0$ such that for all $Y, W \in \mathcal{D}_X$

$$\|Q\|_{C_2^{3\nu}} \le C(1 + \|X\|_{C^\nu} + \|\mathbb{X}^2\|_{C_2^{2\nu}})\|Y\|_{\mathcal{D}_X}\|W\|_{\mathcal{D}_X}. \tag{2.15}$$

Furthermore,

$$\|Q - \tilde{Q}\|_{C_2^{3\nu}} \le C(1 + \|X\|_{C^\nu} + \|\mathbb{X}^2\|_{C_2^{2\nu}}) \\ ((\|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})\varepsilon_W + (\|W\|_{\mathcal{D}_X} + \|\tilde{W}\|_{\mathcal{D}_{\tilde{X}}})\varepsilon_Y + \varepsilon_X). \tag{2.16}$$

where

$$\varepsilon_Y = |Y' - \tilde{Y}'|_{C^\nu} + |R^Y - R^{\tilde{Y}}|_{C_2^{2\nu}} + |Y - \tilde{Y}|_{C^\nu},$$

$$\varepsilon_W = |W' - \tilde{W}'|_{C^\nu} + |R^W - R^{\tilde{W}}|_{C_2^{2\nu}} + |W - \tilde{W}|_{C^\nu},$$

$$\varepsilon_X = (\|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})(\|W\|_{\mathcal{D}_X} + \|\tilde{W}\|_{\mathcal{D}_{\tilde{X}}})(|X - \tilde{X}|_{C^\nu} + |\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{C_2^{2\nu}}).$$

Proof of Lemma 2.11. See [20], Theorem 1. For formula (2.16) see [20], p.104, formula (27). □

By Lemmata 2.9 and 2.3 for any $A \in C^2(\mathbb{R}^3, L(\mathbb{R}^3,\mathbb{R}^3)), Y \in \mathcal{D}_X$ we can a define a map $V^Y : \mathbb{R}^3 \to \mathbb{R}$ by invoking rough path integral as follows

$$V^Y(x) := \int_{S^1} A(x-Y)dY, x \in \mathbb{R}^3. \tag{2.17}$$

We have following bounds on its regularity:

Lemma 2.12. *Let $Y \in \mathcal{D}_X$, $\tilde{Y} \in \mathcal{D}_{\tilde{X}}$, then there exists $C_1 = C_1(\nu), C_2 = C_2(\mathbb{X})$ such that for any integer $n \geq 0$,*

$$\|\nabla^n V^Y\|_{L^\infty} \leq 4C_1C_2^3\|\nabla^{n+1}A\|_{C^1}\|Y\|^2_{\mathcal{D}_X}(1+\|Y\|_{\mathcal{D}_X}) \tag{2.18}$$

and

$$\|\nabla^n V^Y - \nabla^n V^{\tilde{Y}}\|_{L^\infty} \leq C(\nu)|A|_{C^{n+3}}C_X^4(1+\|Y\|_{\mathcal{D}_X}+\|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})^3 \tag{2.19}$$
$$(|X-\tilde{X}|_{C^\nu}+|\mathbb{X}^2-\tilde{\mathbb{X}}^2|_{C_2^{2\nu}}+|Y'-\tilde{Y}'|_{C^\nu}+|R^Y-R^{\tilde{Y}}|_{C_2^{2\nu}}+|Y-\tilde{Y}|_{C^\nu}),$$

where

$$C_X = 1+|X|_{C^\nu}+|\tilde{X}|_{C^\nu}+|\mathbb{X}^2|_{C_2^{2\nu}}+|\tilde{\mathbb{X}}^2|_{C_2^{2\nu}}.$$

In the case of $X = \tilde{X}$, inequality (2.19) *can be rewritten as*

$$\|\nabla^n V^Y - \nabla^n V^{\tilde{Y}}\|_{L^\infty} \leq 16C_1C_2^3\|\nabla^{n+1}A\|_{C^2}\|Y\|_{\mathcal{D}_X}(1+\|Y\|_{\mathcal{D}_X})^2\|Y-\tilde{Y}\|^*_{\mathcal{D}_X}. \tag{2.20}$$

Proof of Lemma 2.12. Inequalities (2.18) and (2.20) were proved in [6], Lemma 7. Now we will show (2.19). It is enough to consider the case of $n = 0$. By formulae (2.15) and (2.14) we have

$$\begin{aligned}
V^Y - V^{\tilde{Y}}(x) &= A(x-Y(0))(Y(1)-Y(0)) \\
&- A(x-\tilde{Y}(0))(\tilde{Y}(1)-\tilde{Y}(0)) \\
&+ (A(x-Y))'(0)Y'(0)\mathbb{X}^2(0,1) \\
&- (A(x-\tilde{Y}))'(0)\tilde{Y}'(0)\tilde{\mathbb{X}}^2(0,1) \\
&+ Q^x(0,1)-\tilde{Q}^x(0,1)
\end{aligned}$$

where Q^x and $\tilde{Q}^x$ (given by formulae (2.15) and (2.14)) satisfy inequality (2.16) and we have identified S^1 with $[0,1]$. Therefore, $Y(1) = Y(0)$, $\tilde{Y}(1) = \tilde{Y}(0)$. Hence, we have

$$\begin{aligned}
|V^Y - V^{\tilde{Y}}|_{L^\infty} \leq \sup_x |(A(x-Y))'(0)Y'(0)\mathbb{X}^2(0,1)-(A(x-\tilde{Y}))'(0)\tilde{Y}'(0)\tilde{\mathbb{X}}^2(0,1)| \\
+ \sup_x |Q^x(0,1)-\tilde{Q}^x(0,1)|.
\end{aligned} \tag{2.21}$$

For the first term on the r.h.s. we have

(2.22)

$$
\begin{aligned}
&|(A(x-Y))'(0)Y'(0)\mathbb{X}^2(0,1) - (A(x-\tilde{Y}))'(0)\tilde{Y}'(0)\tilde{\mathbb{X}}^2(0,1)| \\
&\quad \le |(dA(x-Y(0))Y'(0)Y'(0) - dA(x-\tilde{Y}(0))\tilde{Y}'(0)\tilde{Y}'(0))\mathbb{X}^2(0,1)| \\
&\quad + |dA(x-\tilde{Y}(0))\tilde{Y}'(0)\tilde{Y}'(0)(\mathbb{X}^2(0,1) - \tilde{\mathbb{X}}^2(0,1))| \\
&\quad \le |\mathbb{X}^2|_{C_2^{2\nu}} |dA(x-Y(0))Y'(0)Y'(0) - dA(x-\tilde{Y}(0))\tilde{Y}'(0)\tilde{Y}'(0)| \\
&\quad + |A|_{C^2}|Y'|^2_{L^\infty}|\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{C_2^{2\nu}} \\
&\quad \le |\mathbb{X}^2|_{C_2^{2\nu}}|Y'|^2_{L^\infty}|A|_{C^2}|Y'-\tilde{Y}'|_{L^\infty} + |\mathbb{X}^2|_{C_2^{2\nu}}|A|_{C^1}(|Y'|_{L^\infty} + |\tilde{Y}'|_{L^\infty})|Y'-\tilde{Y}'|_{L^\infty} \\
&\quad + |A|_{C^2}|Y'|^2_{L^\infty}|\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{C_2^{2\nu}}.
\end{aligned}
$$

By (2.16) we can estimate second term as follows

$$
\begin{aligned}
|Q^x - \tilde{Q}^x|_{C_2^{3\nu}} \le C\Big[&(\|A(x-Y)\|_{\mathcal{D}_X} + \|A(x-\tilde{Y})\|_{\mathcal{D}_{\tilde{X}}})\varepsilon_Y \\
&+ (\|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})\varepsilon_A + \varepsilon_X\Big].
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_Y &= |Y-\tilde{Y}|_{C^\nu} + |Y'-\tilde{Y}'|_{C^\nu} + |R^Y - R^{\tilde{Y}}|_{C_2^{2\nu}}, \\
\varepsilon_A &= |A(x-Y) - A(x-\tilde{Y})|_{C^\nu} + |A(x-Y)' - A(x-\tilde{Y})'|_{C^\nu} \\
&\quad + |R^{A(x-Y)} - R^{A(x-\tilde{Y})}|_{C_2^{2\nu}}, \\
\varepsilon_X &= (\|A(x-Y)\|_{\mathcal{D}_X} + \|A(x-\tilde{Y})\|_{\mathcal{D}_{\tilde{X}}}) \\
&\quad \times (\|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})(|X-\tilde{X}|_{C^\nu} + |\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{C_2^{2\nu}}).
\end{aligned}
$$

By formula (2.5) we can estimate ε_A as follows

$$
\begin{aligned}
&|\varepsilon_A| \le K|A|_{C^3}(1+|X|_{C^\nu} + |\tilde{X}|_{C^\nu})^3(1+|Y|_{D_X} + |\tilde{Y}|_{D_{\tilde{X}}})^2 \\
&\quad \times (|X-\tilde{X}|_{C^\nu} + |Y-\tilde{Y}|_{C^\nu} + |Y'-\tilde{Y}'|_{C^\nu} + |R^Y - R^{\tilde{Y}}|_{C_2^{2\nu}}). \qquad (2.23)
\end{aligned}
$$

By inequality (2.4) we infer that

$$
\|A(x-Y)\|_{\mathcal{D}_X} \le K|A|_{C^2}|Y|_{D_X}(1+|Y|_{D_X})(1+|X|_{C^\nu})^2, \qquad (2.24)
$$

and similarly,

$$
\|A(x-\tilde{Y})\|_{\mathcal{D}_{\tilde{X}}} \le K|A|_{C^2}|\tilde{Y}|_{D_{\tilde{X}}}(1+|\tilde{Y}|_{D_{\tilde{X}}})(1+|\tilde{X}|_{C^\nu})^2. \qquad (2.25)
$$

Therefore, combining (2.23) with (2.23), (2.24) and (2.25) we get

(2.26)

$$
\begin{aligned}
&|Q^x - \tilde{Q}^x|_{C_2^{3\nu}} \le C(\nu)|A|_{C^{n+3}}(1+|X|_{C^\nu} + |\tilde{X}|_{C^\nu})^4(1+\|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})^3 \\
&(|X-\tilde{X}|_{C^\nu} + |\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{C_2^{2\nu}} + |Y'-\tilde{Y}'|_{C^\nu} + |R^Y - R^{\tilde{Y}}|_{C_2^{2\nu}} + |Y-\tilde{Y}|_{C^\nu}).
\end{aligned}
$$

Hence, the result follows from (2.22) and (2.26). □

We will denote for any $Y \in \mathcal{D}_X$, $\tilde{Y} \in \mathcal{D}_{\tilde{X}}$

$$
|Y-\tilde{Y}|_D = |X-\tilde{X}|_{C^\nu} + |\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{C_2^{2\nu}} + |Y'-\tilde{Y}'|_{C^\nu} + |R^Y - R^{\tilde{Y}}|_{C_2^{2\nu}} + |Y-\tilde{Y}|_{C^\nu}.
$$

3. Random Filaments evolution problem

Let $\mathcal{D}_{\mathbb{X},T} = C([0,T],\mathcal{D}_X)$, where has been defined after (1.1), be a vector space with the usual supremum norm

$$|F|_{\mathcal{D}_{\mathbb{X},T}} = \sup_{t\in[0,T]} |F(t)|^*_{\mathcal{D}_X}. \tag{3.1}$$

Obviously $\mathcal{D}_{\mathbb{X},T}$ is a Banach space. Assume also that the function ϕ appeared in the formula (0.3) satisfies following hypothesis.

Hypothesis 3.1.(i) *$\phi : \mathbb{R}^3 \to \mathbb{R}$ is even function.*

(ii) *the Fourier transform of ϕ is real and non-negative function:*

$$\hat{\phi}(k) \geq 0, \quad k \in \mathbb{R}^3$$

(iii)

$$\int_{\mathbb{R}^3} (1+|k|^2)^2 \hat{\phi}(k) dk < \infty$$

Example 3.2. The function $\phi = \phi_\mu$, $\mu > 0$ defined by

$$\phi(\cdot) = \frac{\Gamma}{(|\cdot|^2+\mu^2)^{\frac{1}{2}}}$$

is smooth and satisfies Hypothesis 3.1, see p.6 of [5].

Then the following local existence and uniqueness Theorem for problem (0.1)-(0.3) has been proved in [6], see Theorem 3,p.1842.

Theorem 3.3. *Assume $\phi \in C^6(\mathbb{R}^3,\mathbb{R})$, $\nu \in (\frac{1}{3},1)$, $\mathbb{X} = (X,\mathbb{X}^2)$ is a ν-rough path, $\gamma_0 \in \mathcal{D}_X$. Then there exists a time $T_0 = T_0(\nu, |\phi|_{C^5}, \mathbb{X}) > 0$ such that the problem* (0.1)-(0.3) *has unique solution in the space $\mathcal{D}_{\gamma_0,T_0} \subset \mathcal{D}_{\mathbb{X},T_0}$.*

Our aim is to prove global existence of solution of the problem (0.1)-(0.3) under assumptions of Theorem 3.3 and additional hypothesis 3.1 i.e. we shall prove

Theorem 3.4. *Assume that γ_0 is a geometric ν-rough path, $\phi \in C^6(\mathbb{R}^3,\mathbb{R})$ satisfies satisfies Hypothesis 3.1 and $\nu \in (\frac{1}{3},1)$. Then for every $T > 0$, the problem* (0.1)-(0.3) *has unique solution in $\mathcal{D}_{\gamma_0,T}$.*

We will need the following definition.

Definition 3.5. *Let $\phi \in C^4(\mathbb{R}^3,\mathbb{R})$, $\gamma \in \mathcal{D}_X$. Put*

$$\mathcal{H}^\phi_X(\gamma) = \frac{1}{2}\int_{S^1} \vec{\phi}^\gamma(\gamma(\xi))\cdot d\vec{\gamma}(\xi), \tag{3.2}$$

where

$$\vec{\phi}^\gamma(x) = \int_{S^1} \phi(x-\gamma(\eta))d\vec{\gamma}(\eta).$$

$\mathcal{H}^\phi_X(\gamma)$ is called the energy of path γ. We will omit ϕ below.

Remark 3.6. Definition (3.2)-(3.5) is well posed. Indeed, by Lemma 2.12 $\vec{\psi}^\gamma \in C^2(\mathbb{R}^3,\mathbb{R}^3)$ and, therefore, it follows by Lemma 2.3 that $\vec{\psi}^\gamma \circ \gamma \in \mathcal{D}_X$.

Remark 3.7. Assume that $\nu > \frac{1}{2}$ and $\gamma \in C^1(S^1, \mathbb{R}^3)$. Then by Remark 2.10 the line integrals in the definition of energy are understood in the sense of Young and

$$\mathcal{H}^{\phi}_X(\gamma) = \frac{1}{2}\int\limits_{S^1}\int\limits_{S^1} \phi(\vec{\gamma}(\xi) - \vec{\gamma}(\eta))(\frac{d\vec{\gamma}}{d\xi}(\xi), \frac{d\vec{\gamma}}{d\eta}(\eta))d\xi d\eta. \tag{3.3}$$

Lemma 3.8. *Assume $\phi \in C^4(\mathbb{R}^3, \mathbb{R})$. Then there exists constant $C = C(\nu, \mathbb{X})$ such that for all $\gamma \in \mathcal{D}_X$*

$$|\mathcal{H}_X(\gamma)| \leq C|\phi|_{C^4}|\gamma|^4_{\mathcal{D}_X}(1 + |\gamma|_{\mathcal{D}_X})^2. \tag{3.4}$$

Moreover, the map $\mathcal{H}_X : \mathcal{D}_X \to \mathbb{R}$ is locally Lipshitz i.e. for any $R > 0$ there exists $C = C(R)$ such that for any $\gamma, \tilde{\gamma} \in \mathcal{D}_X$, $|\gamma|_{\mathcal{D}_X} \leq R$, $|\tilde{\gamma}|_{\mathcal{D}_X} \leq R$ we have

$$|\mathcal{H}_X(\gamma) - \mathcal{H}_X(\tilde{\gamma})| \leq C(R)|\gamma - \tilde{\gamma}|^*_{\mathcal{D}_X}. \tag{3.5}$$

Furthermore, for any $R > 0$ there exists $C = C(R)$ such that for any $\gamma \in \mathcal{D}_X$, $\tilde{\gamma} \in \mathcal{D}_{\tilde{X}}$,

$$|\gamma|_{\mathcal{D}_X} \leq R, |\tilde{\gamma}|_{\mathcal{D}_{\tilde{X}}} \leq R,$$
$$C_{X,\tilde{X}} = |X|_{C^\nu} + |\tilde{X}|_{C^\nu} + |\mathbb{X}^2|_{C_2^{2\nu}} + |\tilde{\mathbb{X}}^2|_{C_2^{2\nu}} < R$$

we have

$$|\mathcal{H}_X(\gamma) - \mathcal{H}_{\tilde{X}}(\tilde{\gamma})| \leq C(R)(|X - \tilde{X}|_{C^\nu} + |\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{C_2^{2\nu}} + |\gamma - \tilde{\gamma}|_{C^\nu} + |\gamma' - \tilde{\gamma}'|_{C^\nu} + |R^\gamma - R^{\tilde{\gamma}}|_{C_2^{2\nu}}). \tag{3.6}$$

Proof of Lemma 3.8. First we will show inequality (3.4). By representation (2.13) we have

$$\begin{aligned}\mathcal{H}_X(\gamma) &= \frac{1}{2}(\vec{\psi}^\gamma(\gamma(0))(\vec{\gamma}(1) - \vec{\gamma}(0))) \\ &+ \left[d\vec{\psi}^\gamma(\gamma(0))\gamma'(0)\right]\gamma'(0)\mathbb{X}^2(1,0) + Q(0,1) \\ &= I + II + III, \gamma \in \mathcal{D}_X.\end{aligned}$$

Since $\vec{\gamma}(1) = \vec{\gamma}(0)$ we infer that $I = 0$. Concerning the second term by Lemma 2.12 we have the following estimate

$$\begin{aligned}|II| &\leq \|\mathbb{X}^2\|_{C_2^{2\nu}}\|\nabla\vec{\psi}\|_{L^\infty}|\gamma'|^2_{L^\infty} \\ &\leq C(\nu, \mathbb{X})|\phi|_{C^3}|\gamma|^4_{\mathcal{D}_X}(1 + |\gamma|_{\mathcal{D}_X}).\end{aligned} \tag{3.7}$$

For third term we infer from inequality (2.15)

$$|III| \leq \|Q\|_{C_2^{3\nu}} \leq C(\nu, \mathbb{X})|\vec{\psi}(\vec{\gamma})|_{\mathcal{D}_X}|\vec{\gamma}|_{\mathcal{D}_X}. \tag{3.8}$$

Then by Lemmas 2.3 and 2.12 we have

$$\begin{aligned}|\vec{\psi}^\gamma(\vec{\gamma})|_{\mathcal{D}_X} &\leq C(\nu, \mathbb{X})|\psi^\gamma|_{C^2}|\vec{\gamma}|_{\mathcal{D}_X}(1 + |\vec{\gamma}|_{\mathcal{D}_X}) \\ &\leq C(\nu, \mathbb{X})|\phi|_{C^4}|\vec{\gamma}|^3_{\mathcal{D}_X}(1 + |\vec{\gamma}|_{\mathcal{D}_X})^2\end{aligned} \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) we get inequality (3.4).

Now we will prove inequality (3.6). By formula (2.14) we have

$$\begin{aligned}\mathcal{H}_X(\gamma) - \mathcal{H}_{\tilde{X}}(\tilde{\gamma}) &= \frac{1}{2}\Big[(\nabla\psi^\gamma(\gamma(0))\gamma'(0)\gamma'(0) - \nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))\tilde{\gamma}'(0)\tilde{\gamma}'(0))\mathbb{X}^2(1,0) \\ &+ \nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))\tilde{\gamma}'(0)\gamma'(0)(\mathbb{X}^2(1,0) - \tilde{\mathbb{X}}^2(1,0)) + Q(0,1) - \tilde{Q}(0,1)\Big] \\ &= I + II + III\end{aligned} \tag{3.10}$$

The first term in (3.10) can be represented as follows

$$\begin{aligned} I &= (\nabla\psi^{\gamma}(\gamma(0))\gamma'(0)\gamma'(0) - \nabla\psi^{\tilde\gamma}(\tilde\gamma(0))\tilde\gamma'(0)\tilde\gamma'(0))\mathbb{X}^2(1,0) \\ &= \big[(\nabla\psi^{\gamma}(\gamma(0)) - \nabla\psi^{\tilde\gamma}(\tilde\gamma(0)))\gamma'(0)\gamma'(0) \\ &+ \nabla\psi^{\tilde\gamma}(\tilde\gamma(0))(\gamma'(0)-\tilde\gamma(0))\gamma'(0) \\ &+ \nabla\psi^{\tilde\gamma}(\tilde\gamma(0))\tilde\gamma(0)(\gamma'(0)-\tilde\gamma(0))\big]\,\mathbb{X}^2(1,0) = A+B+C \end{aligned} \tag{3.11}$$

The first term in (3.11) can be estimated as follows

$$\begin{aligned} |A| &= |(\nabla\psi^{\gamma}(\gamma(0)) - \nabla\psi^{\tilde\gamma}(\tilde\gamma(0)))\gamma'(0)\gamma'(0)\mathbb{X}^2(1,0)| \\ &\le \|\mathbb{X}^2\|_{C_2^{2\nu}}|\gamma|^2_{\mathcal{D}_X}(|\nabla\psi^{\gamma}(\gamma(0)) \\ &- \nabla\psi^{\gamma}(\tilde\gamma(0))| + |\nabla\psi^{\gamma}(\tilde\gamma(0)) - \nabla\psi^{\tilde\gamma}(\tilde\gamma(0))|) \\ &\le \|\mathbb{X}^2\|_{C_2^{2\nu}}|\gamma|^2_{\mathcal{D}_X}(|\psi^{\gamma}|_{C^2}|\gamma(0) \\ &- \tilde\gamma(0)| + C_X^4|\phi|_{C^4}(1+|\gamma|_{\mathcal{D}_X}+|\tilde\gamma|_{\mathcal{D}_{\tilde X}})^3|\gamma-\tilde\gamma|_D \\ &\le KC_X^4|\phi|_{C^4}(1+|\gamma|_{\mathcal{D}_X}+|\tilde\gamma|_{\mathcal{D}_{\tilde X}})^3|\gamma-\tilde\gamma|_D, \end{aligned} \tag{3.12}$$

where second inequality follows from inequality (2.19) and third one from inequality (2.18). For second term we have by inequality (2.18)

$$\begin{aligned} |B| &\le C\|\mathbb{X}^2\|_{C_2^{2\nu}}|\gamma|_{\mathcal{D}_X}|\phi|_{C^3}|\tilde\gamma|^2_{\mathcal{D}_X}(1+|\tilde\gamma|_{\mathcal{D}_X})|\gamma-\tilde\gamma|_D \\ &\le CC_X(1+|\gamma|_{\mathcal{D}_X}+|\tilde\gamma|_{\mathcal{D}_{\tilde X}})^3|\gamma-\tilde\gamma|_D. \end{aligned} \tag{3.13}$$

Similarly, we have for third term

$$|C| \le C(\nu,\mathbb{X},|\gamma|_{\mathcal{D}_X},|\tilde\gamma|_{\mathcal{D}_X})|\gamma-\tilde\gamma|_D. \tag{3.14}$$

The term II in (3.10) can be estimated as follows

$$\begin{aligned} |II| &\le |\nabla\psi^{\tilde\gamma}|_{L^\infty}|\tilde\gamma|^2_{\mathcal{D}_X}|\gamma-\tilde\gamma|_D \\ &\le C_X^3|\phi|_{C^3}(1+|\gamma|_{\mathcal{D}_X}+|\tilde\gamma|_{\mathcal{D}_{\tilde X}})^3|\gamma-\tilde\gamma|_D. \end{aligned}$$

Thus it remains to estimate third term of equality (3.10). We have by inequality (2.16)

$$\begin{aligned} |Q(0,1)-\tilde Q(0,1)| &\le \|Q-\tilde Q\|_{C_2^{3\nu}} \\ &\le C_X\Big[(|\psi^{\tilde\gamma}(\tilde\gamma)|_{\mathcal{D}_{\tilde X}} + |\psi^{\gamma}(\gamma)|_{\mathcal{D}_X})|\gamma-\tilde\gamma|_D \\ &+ (|\tilde\gamma|_{\mathcal{D}_{\tilde X}} + |\tilde\gamma|_{\mathcal{D}_{\tilde X}})|\psi^{\tilde\gamma}(\tilde\gamma)-\psi^{\gamma}(\gamma)|_D \\ &+ (|\psi^{\tilde\gamma}(\tilde\gamma)|_{\mathcal{D}_{\tilde X}} + |\psi^{\gamma}(\gamma)|_{\mathcal{D}_X})(|\tilde\gamma|_{\mathcal{D}_{\tilde X}} + |\tilde\gamma|_{\mathcal{D}_{\tilde X}}) \\ &\times(|X-\tilde X|_{C^\nu} + |\mathbb{X}^2-\tilde{\mathbb{X}}^2|_{C_2^{2\nu}})\Big] \end{aligned}$$

By inequality (2.18), the term $|\psi^{\tilde\gamma}(\tilde\gamma)|_{\mathcal{D}_X}$ is bounded by the constant $C = C(\nu,\mathbb{X},|\tilde\gamma|_{\mathcal{D}_X})$. Therefore, to prove estimate (3.5) it is enough to show that there exists constant $C = C(\nu,\mathbb{X},R)$ such that for $\gamma,\tilde\gamma\in\mathcal{D}_X$ with $|\gamma|_{\mathcal{D}_X},|\tilde\gamma|_{\mathcal{D}_X}\le R$

$$|\psi^{\tilde\gamma}(\tilde\gamma)-\psi^{\gamma}(\gamma)|_D \le C|\tilde\gamma-\gamma|_D. \tag{3.15}$$

By the triangle inequality we have

$$\begin{aligned} |\psi^{\tilde\gamma}(\tilde\gamma)-\psi^{\gamma}(\gamma)|_D &\le |\psi^{\tilde\gamma}(\tilde\gamma)-\psi^{\tilde\gamma}(\gamma)|_D + |\psi^{\tilde\gamma}(\gamma)-\psi^{\gamma}(\gamma)|_D \\ &= I+II. \end{aligned} \tag{3.16}$$

The first term can be estimated by using inequality (2.5) as follows

$$|I| \le KC_X^3 |\psi^{\tilde\gamma}|_{C^3}(1+|\tilde\gamma|_{\mathcal{D}_{\tilde X}} + |\gamma|_{\mathcal{D}_X})^2 |\tilde\gamma-\gamma|_D. \tag{3.17}$$

By inequality (2.18) we have

$$|\psi^{\tilde\gamma}|_{C^3} \le C|\phi|_{C^5}|\tilde\gamma|^2_{\mathcal{D}_X}(1+|\tilde\gamma|_{\mathcal{D}_X}) \tag{3.18}$$

Combining (3.17) and (3.18) we get necessary estimate for I. It remains to find an estimate for term II. By inequalities (2.4) and (2.20) we have

$$\begin{aligned} II &= |\psi^{\tilde\gamma}(\gamma) - \psi^{\gamma}(\gamma)|_D \\ &\le (1+|X|_\nu)|\psi^{\tilde\gamma}(\gamma) - \psi^{\gamma}(\gamma)|_{\mathcal{D}_X} \\ &\le K|\nabla\psi^{\tilde\gamma} - \nabla\psi^{\gamma}|_{C^1}|\gamma|_{\mathcal{D}_X}(1+|\gamma|_{\mathcal{D}_X})(1+|X|_\nu)^3 \\ &\le K|\phi|_{C^5}C_X^7(1+|\gamma|_{\mathcal{D}_X}+|\tilde\gamma|_{\mathcal{D}_{\tilde X}})^5|\tilde\gamma-\gamma|_D. \end{aligned} \tag{3.19}$$

Hence the inequality (3.6) follows. Inequality (3.5) is a consequence of inequality (3.6). □

Corollary 3.9. *Under assumptions of Lemma 3.8 and assumption 2.7 the energy function $\mathcal{H}_X : \mathcal{D}_X \to \mathbb{R}$ is continuous. Furthermore, for any $\gamma \in \mathcal{D}_X$*

$$0 \le \mathcal{H}_X(\gamma) < \infty.$$

Proof of Corollary 3.9. We only need to show that $\mathcal{H}_X(\gamma) \ge 0$, for any $\gamma \in \mathcal{D}_X$. Other statements of the Corollary easily follow from Lemma 3.8. Fix $n \in \mathbb{N}$. Let $C(0) = \cup_{i=1}^{n^6} C(k_i)^n$ be a partition of the cube $C(0)$ with center 0 and length n^2 on the cubes $C(k_i)^n$ of the length of $\frac{1}{n}$ with centers k_i and nonintersecting interiors. Define $\hat\phi^n(k) = \hat\phi(k_i)$, $k \in C(k_i)^n$ and 0 otherwise. Consequently, define

$$\phi^n(x) = \sum_{i=1}^{n^6} \hat\phi(k_i) \int\limits_{C(k_i)^n} e^{i<k,x>} dk, x \in \mathbb{R}^3$$

Then $\phi = \lim\limits_{n\to\infty} \phi^n$ in C_b^4 because of the Assumption 3.1. Consequently,

$$\mathcal{H}_X^\phi(\gamma) = \lim_{n\to\infty} \mathcal{H}_X^{\phi_n}(\gamma). \tag{3.20}$$

Now formula

$$\mathcal{H}_X^z(\gamma) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} \hat z(k) \Big| \int\limits_{S^1} e^{i(k,\gamma_s)} d\gamma_s \Big|^2 dk, \gamma \in \mathcal{D}_X. \tag{3.21}$$

is correct for $z = \phi^n$, because ϕ^n is a sum of a finite number of Fourier modes. Therefore,

$$\mathcal{H}_X^{\phi_n}(\gamma) \ge 0. \tag{3.22}$$

Thus, the result follows from identity (3.20) and inequality (3.22). □

Now we are going to show that energy is a local integral of motion for problem (0.1)-(0.3).

Lemma 3.10. *Let $\gamma \in \mathcal{D}_{\gamma_0,T_0}$ be a local solution of problem* (0.1)-(0.3) *(such a solution exists by Theorem 3.3). Then*

$$\frac{d\mathcal{H}_{\gamma_0}(\gamma(s))}{ds} = 0, s \in [0, T_0).$$

Proof of Lemma 3.10. Since $\gamma(0) = \gamma_0 \in \mathcal{D}_{\gamma_0}$ is a geometric rough path there exist sequence $\{\gamma_0^n\}_{n=1}^{\infty} \in C^{\infty}(S^1, \mathbb{R}^3)$ such that

$$|\gamma_0^n - \gamma_0|_{C^\nu} + |\gamma^2 - \gamma^2|_{C_2^{2\nu}} \to 0, n \to \infty.$$

Now $\gamma_0^n \in \mathcal{D}_{\gamma_0^n}$, $\gamma_0 \in \mathcal{D}_{\gamma_0}$. Thus we can put $(\gamma_0^n)' = (\gamma_0)' = 1$ and $R^{\gamma_0^n} = R^{\gamma_0} = 0$. Hence we deduce that

$$|\gamma_0^n - \gamma_0|_D \to 0, n \to \infty.$$

Denote by $\gamma^n \in C([0,\infty), \mathbf{H}^1(S^1, \mathbb{R}^3))$ the global solution of problem (0.1)-(0.3) with initial condition γ_0^n. Existence of such solution has been proved in Theorem 2 of [5]. Then according to [5] (Theorem 4, p.1846) we have

$$\lim_{n\to\infty} \sup_{t\in[0,T_0]} |\gamma^n(t) - \gamma(t)|_D = 0.$$

Therefore, by the continuity of the energy functional $\mathcal{H}_{\gamma_0}$ we have

$$\mathcal{H}_{\gamma_0}(\gamma(s)) = \lim_{n\to\infty} \mathcal{H}_{\gamma_0^n}(\gamma^n(s)), s \in [0, T_0]. \tag{3.23}$$

Furthermore, by Lemma 2 of [5], we have

$$\mathcal{H}_{\gamma_0^n}(\gamma^n(s)) = \mathcal{H}_{\gamma_0^n}(\gamma_0^n), s \in [0, T_0]. \tag{3.24}$$

As a result, combining (3.23) and (3.24) we get statement of the Lemma. □

Let us recall the definition (0.3) of the vector field u^Y generated by a ν-rough path Y

$$u^Y(x) = \int_Y \nabla\phi(x-y) \times dy, Y \in \mathcal{D}_X. \tag{3.25}$$

Now we will show that if energy functional of Y is bounded then associated velocity field is a smooth function. We have

Lemma 3.11 (See Lemma 3 in [5]). *For any $n \in \mathbb{Z}, n \geq 0$, we have following bound*

$$\|\nabla^n u^\gamma\|_{L^\infty}^2 \leq \frac{1}{(2\pi)^3} \left[\int_{\mathbb{R}^3} |\vec{k}|^{2(1+n)} \hat{\phi}(\vec{k}) d\vec{k} \right] \mathcal{H}_X(\gamma), \gamma \in \mathcal{D}_X, \tag{3.26}$$

provided that the integral $\int_{\mathbb{R}^3} |\vec{k}|^{2(1+n)} \hat{\phi}(\vec{k}) d\vec{k}$ is finite and $\phi \in C^{n+4}(\mathbb{R}^3, \mathbb{R}^3)$.

Proof of Lemma 3.11. For a smooth curve γ Lemma 3.11 has been proved in [5], see Lemma 3. In the general case, when $\gamma \in \mathcal{D}_X$, it is enough to notice that both sides of inequality (3.26) are locally Lipshitz and therefore, continuous w.r.t. distance $d(Y, \tilde{Y}) := |Y - \tilde{Y}|_{\mathcal{D}_X}$, $Y \in \mathcal{D}_X$, $\tilde{Y} \in \mathcal{D}_{\tilde{X}}$. Indeed, continuity of $\mathcal{H}_X$ has been proven in Lemma 3.8 and continuity of $\|\nabla^n u^\gamma\|_{L^\infty}$ follows from Lemma 2.12. □

Now we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. According to Theorem 3.3 there exists unique local solution of problem (0.1)-(0.3). Then, we can find $T^* > 0$ and a unique maximal local solution $\gamma : [0, T^*) \to \mathcal{D}_{\gamma_0}$ and

$$\lim_{t \nearrow T^*} \|\gamma(t)\|_{\mathcal{D}_{\gamma_0}} = \infty, \tag{3.27}$$

see e.g. [11]. Notice that we will have

$$\frac{d\mathcal{H}_{\gamma_0}(\gamma(s))}{ds} = 0, s \in [0, T^*). \tag{3.28}$$

Indeed, by Theorem 3.3 for any $t_0 \in [0, T^*)$ there exists unique local solution $\tilde{\gamma}$ of problem (0.1), (0.3) with initial condition $\gamma(t_0)$ on segment $[t_0, t_0 + \varepsilon_{t_0}]$ for some $\varepsilon_{t_0} > 0$. Therefore, $\gamma = \tilde{\gamma}$ on the segment $[t_0, t_0 + \varepsilon_{t_0}]$. Hence,

$$\frac{d\mathcal{H}_{\gamma_0}(\gamma(s))}{ds} = 0, s \in [t_0, t_0 + \varepsilon_{t_0}], t_0 \in [0, T^*),$$

and identity (3.28) follows. We need to show that $T^* = \infty$. Therefore, it is enough to prove

$$\sup_{t \in [0, T^*)} \|\gamma(t)\|_{\mathcal{D}_{\gamma_0}} < \infty.$$

Indeed, by contradiction with (3.27), the result will follow. In the rest of the proof we show such estimate. We recall that

$$\gamma(t) = \gamma_0 + \int_0^t u^{\gamma(s)}(\gamma(s)) ds. \tag{3.29}$$

Firstly we have

$$\begin{aligned} |\gamma(t)|_{L^\infty} &\leq |\gamma_0|_{L^\infty} + \int_0^t |u^{\gamma(s)}|_{L^\infty} ds \\ &\leq |\gamma_0|_{L^\infty} + C \int_0^t \mathcal{H}_{\gamma_0}(\gamma(s)) ds \\ &\leq |\gamma_0|_{L^\infty} + C\mathcal{H}_{\gamma_0}(\gamma_0) t, t \in [0, T^*). \end{aligned} \tag{3.30}$$

It follows from (3.29) that

$$\gamma'(t) = \gamma_0' + \int_0^t \nabla u^{\gamma(s)}(\gamma(s)) \gamma'(s) ds, t \in [0, T^*). \tag{3.31}$$

Therefore, by Lemmas 3.10 and 3.11

$$\begin{aligned} |\gamma'(t)|_{L^\infty} &\leq |\gamma_0'|_{L^\infty} + \int_0^t |\nabla u^{\gamma(s)}|_{L^\infty} |\gamma'(s)|_{L^\infty} ds \\ &\leq |\gamma_0'|_{L^\infty} + \int_0^t C\mathcal{H}_{\gamma_0}^{\frac{1}{2}}(\gamma(s)) |\gamma'(s)|_{L^\infty} ds \\ &= |\gamma_0'|_{L^\infty} + \int_0^t C\mathcal{H}_{\gamma_0}^{\frac{1}{2}}(\gamma_0) |\gamma'(s)|_{L^\infty} ds, t \in [0, T^*). \end{aligned} \tag{3.32}$$

Then by the Gronwall inequality we infer our second estimate

$$|\gamma'(t)|_{L^\infty} \leq |\gamma_0'|_{L^\infty} e^{C\mathcal{H}_{\gamma_0}^{\frac{1}{2}}(\gamma_0) t}, t \in [0, T^*). \tag{3.33}$$

We will need one auxiliary estimate. We have

$$
\begin{aligned}
|\gamma(t)|_{C^\nu} &\leq |\gamma_0|_{C^\nu} + \int_0^t |u^{\gamma(s)}(\gamma(s))|_{C^\nu}|ds \\
&\leq |\gamma_0|_{C^\nu} + \int_0^t |\nabla u^{\gamma(s)}|_{L^\infty}|\gamma(s)|_{C^\nu} ds \\
&\leq |\gamma_0|_{C^\nu} + \int_0^t C\mathcal{H}^{\frac{1}{2}}_{\gamma_0}(\gamma(s))|\gamma(s)|_{C^\nu} ds \\
&= |\gamma_0|_{C^\nu} + \int_0^t C\mathcal{H}^{\frac{1}{2}}_{\gamma_0}(\gamma_0)|\gamma(s)|_{C^\nu} ds, t \in [0, T^*).
\end{aligned}
\tag{3.34}
$$

Thus, by the Gronwall inequality we get

$$
|\gamma(t)|_{C^\nu} \leq |\gamma_0|_{C^\nu} e^{C\mathcal{H}^{\frac{1}{2}}_{\gamma_0}(\gamma_0)t}, t \in [0, T^*). \tag{3.35}
$$

Now we can estimate C^ν norm of γ'. We have

$$
\begin{aligned}
|\gamma'(t)|_{C^\nu} &\leq |\gamma_0'|_{C^\nu} + \int_0^t |\nabla u^{\gamma(s)}(\gamma(s))\gamma'(s)|_{C^\nu} ds \\
&\leq |\gamma_0'|_{C^\nu} + \int_0^t (|\nabla u^{\gamma(s)}|_{L^\infty}|\gamma'(s)|_{C^\nu} + |\gamma'(s)|_{L^\infty}|\nabla u^{\gamma(s)}(\gamma(s))|_{C^\nu}) ds \\
&\leq |\gamma_0'|_{C^\nu} + \int_0^t (|\nabla u^{\gamma(s)}|_{L^\infty}|\gamma'(s)|_{C^\nu} + |\gamma'(s)|_{L^\infty}|\nabla^2 u^{\gamma(s)}|_{L^\infty}|\gamma(s)|_{C^\nu}) ds \\
&\leq |\gamma_0'|_{C^\nu} \\
&\quad + \int_0^t (C\mathcal{H}^{\frac{1}{2}}_{\gamma_0}(\gamma_0)(|\gamma'(s)|_{C^\nu} + |\gamma_0'|_{L^\infty}|\gamma_0|_{C^\nu} e^{C\mathcal{H}_{\gamma_0}(\gamma_0)s})) ds, t \in [0, T^*),
\end{aligned}
\tag{3.36}
$$

where last inequality follows from Lemmas 3.10 and 3.11. Then by the Gronwall inequality we get the third estimate

$$
|\gamma'(t)|_{C^\nu} \leq (|\gamma_0'|_{C^\nu} + |\gamma_0'|_{L^\infty}|\gamma_0|_{C^\nu}) e^{C\mathcal{H}_{\gamma_0}(\gamma_0)t}, t \in [0, T^*). \tag{3.37}
$$

It remains to find an estimate for $|R^{\gamma(t)}|_{2\nu}$. We have

$$
R^{\gamma(t)} = R^{\gamma_0} + \int_0^t R^{u^{\gamma(s)}(\gamma(s))} ds, t \in [0, T^*). \tag{3.38}
$$

By identity (2.3) we have for $s \in [0, T^*)$

$$R^{u^{\gamma(s)}(\gamma(s))}(\xi,\eta) = \nabla u^{\gamma(s)}(\gamma(s,\xi)) R^{\gamma(s)}(\xi,\eta) + \sum_k (\gamma^k(s,\eta) - \gamma^k(s,\xi)) \times$$

$$\int_0^1 \left[\frac{\partial u^{\gamma(s)}}{\partial x_k}(\gamma(s,\xi) + r(\gamma(s,\eta) - \gamma(s,\xi))) - \frac{\partial u^{\gamma(s)}}{\partial x_k}(\gamma(s,\xi)) \right] dr. \tag{3.39}$$

Therefore,

$$|R^{u^{\gamma(s)}(\gamma(s))}|_{\tilde{C}^{2\nu}} \le |\nabla u^{\gamma(s)}|_{L^\infty} |R^{\gamma(s)}|_{\tilde{C}^{2\nu}} + \frac{1}{2} |\gamma(s)|^2_{C^\nu} |\nabla^2 u^{\gamma(s)}|_{L^\infty}, s \in [0, T^*). \tag{3.40}$$

Thus, by inequalities (3.40) and (3.35) we have for $t \in [0, T^*)$

$$\begin{aligned} |R^{\gamma(t)}|_{\tilde{C}^{2\nu}} &\le |R^{\gamma_0}|_{\tilde{C}^{2\nu}} + \int_0^t (|\nabla u^{\gamma(s)}|_{L^\infty} |R^{\gamma(s)}|_{\tilde{C}^{2\nu}} + \frac{1}{2} |\gamma(s)|^2_{C^\nu} |\nabla^2 u^{\gamma(s)}|_{L^\infty}) ds \\ &\le |R^{\gamma_0}|_{\tilde{C}^{2\nu}} + \int_0^t (|\nabla u^{\gamma(s)}|_{L^\infty} |R^{\gamma(s)}|_{\tilde{C}^{2\nu}} + |\gamma_0|_{C^\nu} e^{C \mathcal{H}^{\frac{1}{2}}_{\gamma_0}(\gamma_0) t} |\nabla^2 u^{\gamma(s)}|_{L^\infty}) ds \\ &\le |R^{\gamma_0}|_{\tilde{C}^{2\nu}} + C(|\gamma_0|_{C^\nu}, \mathcal{H}_{\gamma_0}(\gamma_0)) e^{C \mathcal{H}^{\frac{1}{2}}_{\gamma_0}(\gamma_0) t} \\ &\quad + \int_0^t C \mathcal{H}^{\frac{1}{2}}_{\gamma_0}(\gamma_0) |R^{\gamma(s)}|_{\tilde{C}^{2\nu}} ds, \end{aligned} \tag{3.41}$$

where in the last inequality we used Lemmas 3.10 and 3.11. Hence, by the Gronwall Lemma we get

$$|R^{\gamma(t)}|_{\tilde{C}^{2\nu}} \le (|R^{\gamma_0}|_{\tilde{C}^{2\nu}} + C(|\gamma_0|_{C^\nu}, \mathcal{H}_{\gamma_0}(\gamma_0)) e^{C \mathcal{H}^{\frac{1}{2}}_{\gamma_0}(\gamma_0) t})) e^{C \mathcal{H}^{\frac{1}{2}}_{\gamma_0}(\gamma_0) t}, t \in [0, T^*), \tag{3.42}$$

and combining estimates (3.30), (3.33), (3.37), and (3.42) we prove following a'priori estimate

$$|\gamma(t)|_{\mathcal{D}_{\gamma_0}} \le K(1 + \mathcal{H}_{\gamma_0}(\gamma_0))(1 + |\gamma_0|_{\mathcal{D}_{\gamma_0}}) |\gamma_0|_{\mathcal{D}_{\gamma_0}} e^{C \mathcal{H}_{\gamma_0}(\gamma_0) t}, t \in [0, T^*), \tag{3.43}$$

and the result follows. □

4. Future directions of research

It would be interesting to consider problem (0.1)-(0.3) with added white noise i.e. to consider problem

$$d\gamma(t) = u^{\gamma(t)}(\gamma(t)) dt + \sqrt{2\nu} dw_t, \nu > 0, t \in [0, T] \tag{4.1}$$

$$\gamma(0) = \gamma_0, \tag{4.2}$$

where γ_0 is a geometric ν-rough path, vector field of velocity u^Y is given by (0.2) and w_t is $\mathcal{D}_{\gamma_0}$–valued Wiener process. This model would correspond to Navier-Stokes equations rather than Euler equations. There are two possible mathematical frameworks for the model.

First one is to make change of variables

$$\alpha(t) = \gamma(t) - \sqrt{2\nu} w_t, t \in [0, T].$$

Then we can fix $\{w_t\}_{t\geq 0}$ and system (4.1)-(4.2) is reformulated as follows

$$\frac{d\alpha}{dt} = u^{\alpha(t)+\sqrt{2\nu}w_t}(\alpha(t) + \sqrt{2\nu} w_t), \nu > 0, t \in [0, T] \tag{4.3}$$

$$\alpha(0) = \gamma_0. \tag{4.4}$$

Now the problem (4.3)-(4.4) is ordinary differential equation (ODE) with random coefficients in $\mathcal{D}_{\gamma_0}$ and it can be studied by methods of the theory of random dynamical systems, see [1] and [15]. This approach works only in the case of additive noise.

Second approach is to consider problem (4.1)-(4.2) as SDE in Banach space $\mathcal{D}_{\gamma_0}$. Then, we can consider more general system with multiplicative noise:

$$d\gamma(t) = u^{\gamma(t)}(\gamma(t))dt + \sqrt{2\nu} G(\gamma) dw_t, \nu > 0, t \in [0, T] \tag{4.5}$$

$$\gamma(0) = \gamma_0. \tag{4.6}$$

The problem which appear here is to define Stochastic integral in the Banach space $\mathcal{D}_{\gamma_0}$. Stochastic calculus in M-type 2 Banach spaces developed in works [13]-[14], [8], [10] does not work in this situation. It seems that it is necessary to try to alter definition of $\mathcal{D}_{\gamma_0}$ to be able to apply the theory.

Other possible direction of research is the theory of connections on infinite dimensional manifolds, see [19], [7], [24]. In [9] the authours claimed, see p. 251 therein, that it is possible to define the topological space of Gawędzki's [19] line bundle over the set of rough loops in the sense of Lyons [25]. Since the trajectories of the Brownian loop are almost surely rough paths, this allows us to define the topological space of Gawędzki's line bundle over the Brownian bridge, because it is possible to define the integral of a one-form over a rough path. It would be interesting to write down a complete proof of this claim. The theory presented in this article could be seen as a first step in realizing such a programme.

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