THE FLUID DYNAMICS OF JAMES CLERK MAXWELL

H.K.MOFFATT
Trinity College, Cambridge, CB2 1TQ, UK

1. Introduction

We tend to think of Maxwell's genius entirely in relation to his creation of electromagnetic theory (as summarised by Maxwell's equations) on the one hand, and to his seminal development of the kinetic theory of gases (leading to the Maxwell-Boltzmann distribution, the derivation of transport coefficients like viscosity and thermal diffusivity for gases and associated validation of the continuum model) on the other. To do so is perhaps to neglect his extraordinary insights in other directions, notably in the dynamics of fluids, a field in which his publications were few but nevertheless groundbreaking in their originality. Some of these insights were not published at the time, but are now available through the wonderful three-volume publication of *The Scientific Letters and Papers of James Clerk Maxwell* (Harman 1990, 1995, 2003) on which I draw freely in the following discussion. Maxwell's insights relate to centrifugal instability, to fluid-dynamical stability both in general and in particular application to the problem of Saturn's rings, to Kelvin's knotted vortices (in relation to which Maxwell played the role of amused commentator) and to the problem of Lagrangian particle displacement in an unsteady Eulerian flow, for which he provided perhaps the first explicit example. Fluid-dynamical thinking was in fact intrinsic to Maxwell's development of electromagnetic theory in the 1850s, and provided the initial framework upon which he constructed this theory, as evidenced by his first major paper in the subject *On Faraday's lines of force.* His endeavours to express observed phenomena in mechanical terms lay at the heart of his thinking, and may help us to understand the development of a mind of extraordinary boldness and originality.

2. Centrifugal instability

In Volume II of the *Scientific Papers and Letters of James Clerk Maxwell* (Harman 1995), one may find the following *Draft question on the stability of vortex motion*[1] as set by Maxwell, who had been appointed a Moderator (i.e. External Examiner in modern parlance) for the Cambridge Mathematical Tripos 1866:

> *A mass M of fluid is running round a circular groove or channel of radius a with velocity u. An equal mass is running round another channel of radius b with velocity v. The one channel is made to expand and the other*

[1]This question is number 101 in a collection of problems in Maxwell's notebook 2, which may be found in the King's College London Archives; for a set of solutions to these problems, see Fuller (1986); see also <http://www.clerkmaxwellfoundation.org/SmithsPrizeSolutions2008 2 14.pdf> for solutions to a remarkable range of problems set by Maxwell in the Smith's Prize competition 1879, for which he was Examiner.

to contract till their radii are exchanged. Show that the work expended in effecting the change is

$$-\frac{1}{2}\left(\frac{u^2}{b^2}-\frac{v^2}{a^2}\right)(a^2-b^2)M.$$

Hence show that the motion of a fluid in a circular whirlpool will be stable or unstable according as the areas described by particles in equal times increase or diminish from centre to circumference.

The final sentence of this question (the sting in the tail!) is particularly noteworthy: readers familiar with the theory of centrifugal instability will recognise that it embodies what is generally known as *Rayleigh's criterion* (Rayleigh 1917), namely that a flow of an ideal fluid with velocity field $\mathbf{u} = (0, v(r), 0)$ in cylindrical polar coordinates (r, θ, z) is stable or unstable to axisymmetric disturbances according as the circulation $k(r) = 2\pi r v(r)$ increases or decreases with distance r from the axis. Rayleigh discusses the problem of stability first through analogy with the simpler problem of the stability of hydrostatic equilibrium of an incompressible fluid of variable density in which the density is a function of height. Then, appealing implicitly to the circulation theorem of Kelvin (1869), he goes on to say:

We may also found our argument upon a direct consideration of the kinetic energy T of the motion. For T is proportional to $\int v^2 r\,dr$, or $\int k^2\,dr^2/r^2$. Suppose now that two rings of fluid, one with $k = k_1$ and $r = r_1$ and the other with $k = k_2$ and $r = r_2$ where $r_2 > r_1$ and of equal areas dr_1^2 or dr_2^2 are interchanged. The corresponding increment in T is represented by

$$(dr_1^2 = dr_2^2)(k_2^2/r_1^2 + k_1^2/r_2^2 - k_1^2/r_1^2 - k_2^2/r_2^2) = dr^2(k_2^2 - k_1^2)(r_1^{-2} - r_2^{-2}),$$

and is positive if $k_2^2 > k_1^2$; so that a circulation always increasing outwards makes T a minimum and thus ensures stability.

Here Lord Rayleigh (Nobel Laureate 1904) in effect provides a correct solution to the problem that had been set by Maxwell for undergraduates taking the Tripos Examination half a century earlier; he would presumably have been given an α-mark for his answer, had he provided it then. Actually, Lord Rayleigh (then John William Strutt) had been a candidate for Mathematical Tripos just one year earlier (1865) and was Senior Wrangler (i.e. top of the list) in that examination.

When Maxwell set his examination question in 1865, he of course did not have access to Kelvin's circulation theorem, proved four years later. He presumably thought in terms of conservation of the angular momentum of the fluid in his "circular groove", and this is equivalent to conservation of circulation for the axisymmetric displacements that he envisaged. His phrase "the area described by particles . . . " clearly means the area swept out by a radial line (z = constant, θ = constant) from the axis of symmetry to the particle in question, i.e. $rv(r)\delta t$ in time δt (reminiscent of Kepler's phrase for planetary motion "equal areas in equal times"), so again proportional to circulation. Furthermore, Maxwell posed the question in terms of the "work done" in effecting the displacement, but since this work done must convert to a corresponding change in the kinetic energy of the fluid (assumed incompressible), Maxwell's criterion for stability is in effect identical with that of Rayleigh.

Viscosity has a stabilising influence for this problem, as recognised by Rayleigh, so that for a viscous fluid, the condition that circulation decrease outwards is necessary but not sufficient for instability to axisymmetric disturbances. The effect of viscosity was brilliantly analysed and demonstrated experimentally by G.I.Taylor (1923), in a work that has provided the impetus for a huge body of research on the problem of centrifugal instability (see for example Stuart 1958, Lin 1966, Drazin & Reid 1981, Koschmieder 1993, Iooss 1994). Maxwell's 1866 question for the Mathematical Tripos reveals his seminal insight into this important branch of hydrodynamic stability theory.

3. Stability of steady Euler flows

But now let us go back a bit earlier to 1854 when Maxwell, then aged 23, was himself subjected to the rigours of the Mathematical Tripos; in this, he came Second Wrangler, second to E.J.Routh who subsequently distinguished himself as the greatest Cambridge mathematics tutor of his day. Following the Tripos, Maxwell continued to work as a graduate student, aiming at a Fellowship of Trinity College, to which he was elected just one year later.

One of his preoccupations during this formative year of graduate study was the *Stability of Fluid Motion*, on which subject he revealed his ideas in a remarkable letter to William Thomson (later Lord Kelvin) dated 15 May 1855 (article 66 in Harman 1990). Here Maxwell first considers the steady two-dimensional flow of an ideal incompressible fluid for which the velocity field $\mathbf{u}$ is given in terms of a streamfunction $\psi(x, y)$ by $\mathbf{u} = (-\partial\psi/\partial y, \partial\psi/\partial x)$, a notation introduced by Stokes, Lucasian Professor of Mathematics, at whose feet Maxwell must have sat during his undergraduate years. Maxwell first re-derives the condition for steady flow $\nabla^2\psi = f(\psi)$ for some function $f(\cdot)$ (Stokes 1842). He then writes:

> *To determine the stability or instability of this steady motion we must give it a perfectly general derangement & determine whether it will or will not tend to return to its original state.*

It is difficult to follow Maxwell's subsequent reasoning, but by physical arguments that do not bear too close examination he in fact arrives at the correct conclusion, namely that

$$f'(\psi) > 0 \text{ is the condition of stability.}$$

This is altogether astonishing, because the result as stated remained unpublished (and therefore unknown) for more than a century, and was not rediscovered and rigorously proved until the work of Arnold (1966; see also Moffatt 1986) through consideration of the second variation of kinetic energy K under 'isovortical perturbations', i.e. perturbations that transport the vortex lines of the basic flow as though 'frozen in the fluid', in conformity with the constraint of Euler's equations (Helmholtz 1858). The first variation of kinetic energy $\delta^1 K$ vanishes for any steady Euler flow $\mathbf{u}^E$ with vorticity $\boldsymbol{\omega}^E = \nabla \times \mathbf{u}^E$ (this much was known to Kelvin), and the second variation under an arbitrary volume-preserving isovortical displacement $\boldsymbol{\eta}(\mathbf{x})$ is given by

$$\delta^2 K = \frac{1}{2}\int [(\boldsymbol{\eta}\times\boldsymbol{\omega}^E)_s^2 - (\boldsymbol{\eta}\times\boldsymbol{\omega}^E)_s \cdot \nabla\times(\boldsymbol{\eta}\times\mathbf{u}^E)]\, dV\,,$$

where the suffix s denotes the 'solenoidal projection' of the vector field concerned, and the integration is over the fluid domain. In the case of two-dimensional perturbations of a two-dimensional flow with streamfunction ψ satisfying $\nabla^2\psi = f(\psi)$, this expression reduces to

$$\delta^2 K = \frac{1}{2}\int [(\boldsymbol{\eta}\times\boldsymbol{\omega}^E)_s^2 + (\boldsymbol{\eta}\times\mathbf{u}^E)^2 f'(\psi)] dxdy\,,$$

so that a sufficient condition for $\delta^2 K$ to be positive for all admissible $\boldsymbol{\eta}$ (and so for the flow to be stable) is indeed that $f'(\psi) > 0$.

So here again, Maxwell demonstrates profound physical insight; although his argument is obscure to the point of incomprehensibility (perhaps inevitably so because this was well before Helmholtz's (1858) recognition of the key role played by vorticity), he nevertheless arrives at the correct conclusion, as ultimately confirmed more than 100 years later by Arnold, who was quite unaware of Maxwell's prior deliberations on this problem.

During this same pivotal year 1855, Maxwell wrote his first major paper on electromagnetism *On Faraday's lines of force* (Maxwell 1856) in which he develops the analogy between the lines of force of a magnetic field and the streamlines of the flow of an incompressible fluid. He introduces this fluid with due caution:

> *The substance here treated of must not be assumed to possess any of the properties of ordinary fluids except those of freedom of movement and resistance to compression. It is not even a hypothetical fluid which is introduced to explain actual phenomena. It is merely a collection of imaginary properties which may be employed for establishing certain theorems in pure mathematics in a way more intelligible to many minds and more applicable to physical problems than that in which algebraic symbols alone are used. The use of the word "Fluid" will not lead us into error, if we remember that it denotes a purely imaginary substance with the following property:* The portion of fluid which at any instant occupies a given volume, will at any succeeding instant occupy an equal volume. *This law expresses the incompressibility of the fluid and furnishes us with a convenient measure of its quantity, namely its volume.*

In this way Maxwell leads the reader to conceive of flow in the abstract, a concept that mathematicians a century later would come to describe in terms of 'volume-preserving diffeomorphisms' (Arnold 1966). Many might still prefer Maxwell's more primitive terminology! The evolution of Maxwell's theory of electromagnetism over the years 1855 to 1862 was largely guided by his efforts to express the theory in fluid mechanical terms, although ultimately the great triumph of the theory was that he was able to discard the artificial and, as it turned out, irrelevant fluid-mechanical framework.

4. The stability of Saturn's rings

In October 1856 Maxwell took up his position as Professor of Natural Philosophy at Marischal College, Aberdeen, and immediately set to work on the problem of the motion and stability of Saturn's rings, the subject that had been posed for the 1857 Adams Prize at Cambridge University. He was the only candidate to make significant progress on the problem, and duly won the prize. He continued to work on the vexing problem of the stability of the rings, and the revised version of this work was published two years later (Maxwell 1859).

I wish to comment here on only a small portion of this work contained in §23 of the paper[2], in which Maxwell focuses on a model problem: the instability of a uniform layer of liquid of great extent compared to its thickness, subject only to the effects of self-gravitation. The best-known example of this type of instability is the 'Jeans instability' of a large extent of compressible fluid subject to self-gravitation (Jeans 1902); so here again, we have a situation in which Maxwell anticipates by several decades an instability mechanism of fundamental significance in cosmology, the difference being that for Maxwell it is the mobility of the free surfaces of the liquid, rather than its compressibility, that permits effective horizontal compression or dilatation of the liquid layer. Self-gravitation promotes two effects: the propagation of (stable) gravity waves on both surfaces of the layer; and the horizontal 'bunching up' of the layer which can decrease gravitational potential energy and is therefore destabilising. Maxwell recognised the competing influence of these effects and obtained the correct criterion for instability, a veritable *tour de force* at that time. If sinusoidal perturbations of the layer of wavelength $2\pi/m$ are considered, then it is found by linearised analysis that these perturbations (assumed symmetric about the central plane of the layer) grow like $\exp \sigma t$, where

$$\sigma^2 = 2\pi G\rho \tanh mb \,[1 + \exp(-2mb) - 2mb] \,,$$

in which $2b$ is the thickness of the layer, ρ the density of the liquid, and G the universal gravitational constant. This function is positive for $2mb < 1.278$, indicating instability for this range of wave-numbers m. Maxwell gave the value 1.147 here, not very accurate, but his conclusion that the layer is gravitationally unstable to perturbations of wavelength $\lambda = 2\pi/m$ sufficiently large compared with the layer thickness $2b$ was certainly correct. The growth rate of the instability is in fact maximal when $mb = 0.294$, or $\lambda/2b = 10.69$.

When $mb > 1.278$, σ^2 is negative and the perturbation modes are oscillatory with frequency $\omega = \mathrm{i}\sigma$; indeed for large mb, the above formula gives $\omega^2 \sim gm$ where $g(= 4\pi G\rho b)$ is the inward gravitational acceleration at the free surface of the liquid layer. This is just the familiar dispersion relation for gravity waves on deep water, a result already well-known to Stokes (1847).

5. Knotted vortices

In 1867, Kelvin published in the Philosophical Magazine his paper *On Vortex Atoms*, which must have caused quite a stir in the scientific community. It was in this paper that he mentions his visit to Peter Guthrie Tait's laboratory in Edinburgh, where he witnessed Tait's demonstration of the production of vortex rings by ejection of air from an orifice, the rings being visualised by smoke. Kelvin's subsequent realisation that any knot in a vortex tube in an ideal fluid would be of permanent form led him to hypothesise

[2] I am grateful to David Forfar who drew my attention to Maxwell's ingenious treatment of this problem.

that the existence and character of the known elements of the periodic table might be explained at the atomic level in terms of 'vortex knots' of increasingly complex structure in the ether, an ideal fluid imagined to permeate all space; and that the frequency spectra associated with these elements might then be explained in terms of the frequencies of vibration of such vortex knots.

Maxwell refers to this paper with his usual pawky humour in his letter to Tait of 13 November 1867 (Harman 1995, 275):

> *Thomson has set himself to spin the chains of destiny out of a fluid plenum ... and I saw you had put your calculus in it too. May you both prosper and disentangle your formulae in proportion as you entangle your worbles. But I fear that the simplest indivisible* whorl *is either two embracing worbles or a worble embracing itself ....*

And three weeks later (4 December 1867, Harman 1995, 276), a further letter from Maxwell to Tait, mainly about the properties of certain plane curves; but he adds, as if by afterthought, *I have amused myself with knotted curves for a day or two.* He goes on to state the formula (due to Gauss, but which Maxwell had evidently here derived independently – Ricca & Nipoti 2011)[3] for the linking number n of two closed curves C and C':

$$\int_C \int_{C'} \frac{(\mathbf{x}-\mathbf{x}')\cdot d\mathbf{x}\times d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = 4\pi n.$$

He points out that although $n \neq 0$ implies linkage, the condition $n = 0$ does not necessarily imply that C and C' are unlinked, and he gives two examples of nontrivial links with $n = 0$ — examples that later became known as the Whitehead link and the Borromean rings. In the final paragraph of the letter, he gives a number of examples of knotted curves described in parametric form, and the manner in which the knot type can change as a parameter is varied. What an astonishing burst of creativity in that *day or two* of desultory amusement!

There is no doubt that Maxwell retained an interest in Tait's development of knot theory during the 1870s (Ricca & Weber, 2012). It was doubtless the monumental study of Tait in 1877 (Tait 1898) as much as Kelvin's knotted vortices that stimulated Maxwell to compose his "Paradoxical Ode" *To* [the imaginary philosopher] *Hermann Stoffcraft PhD*, the first verse of which reads:

> *My soul's an amphicheiral knot*
> *Upon a liquid vortex wrought*
> *By intellect in the unseen residing.*
> *While thou dost like a convict sit*
> *With marlinespike untwisting it*
> *Only to find my knottiness abiding:*
> *Since all the tools for my untying*
> *In four-dimensioned space are lying,*
> *Where playful fancy intersperses*
> *Whole avenues of universes;*

[3] Maxwell used the full Cartesian notation current at the time; here, I use the more compact modern vector notation.

Where Klein and Clifford fill the void
With one unbounded finite homaloid,
Whereby the Infinite is hopelessly destroyed.

This poem, Maxwell's last, was written shortly before his death from cancer in 1879. It touches on current scientific and philosophical ideas in equal measure; an illuminating interpretation is provided by Silver (2008). The phrase "whole avenues of universes" has a certain resonance with recent speculations (see, for example, Carr 2007) concerning the concept of the 'multiverse'; playful fancy indeed!

Although Kelvin's vortex knots held great promise, he was unable to find any steady stable solutions of the Euler equations having a knotted character, and he was gradually forced to abandon the theory. If, instead of vortex tubes, he had chosen to focus on magnetic flux tubes in a perfectly conducting (i.e. ideal) fluid medium, the result might have been very different, for we know now that stable magnetostatic equilibria of arbitrarily complex topology do exist, albeit with imbedded tangential discontinuities (Moffatt 1985). All the ingredients for such a 'complementary' theory were already available in the 1870s as a result of Maxwell's development of electromagnetism; but the realisation that the magnetic field is frozen in a perfectly conducting fluid (and so its topology conserved) had to await the development of magnetohydrodynamics in the 20th century, spearheaded by the work of Alfvén (1942), and the discovery of the conservation of magnetic helicity and its interpretation as a measure of knottedness of magnetic flux tubes (Woltjer 1958, Moffatt 1969).

6. Lagrangian particle displacement

In an unsteady fluid flow, the streamline pattern is continuously changing in time, and the (Lagrangian) trajectory of a fluid particle is quite distinct from the instantaneous streamline on which it finds itself at any instant. This was recognised by Maxwell, who provided perhaps the first explicit calculation of the Lagrangian displacement of a fluid particle in an unsteady flow field. This was the two-dimensional potential flow $\mathbf{u} = \nabla\phi$ generated by the steady motion of a circular cylinder (with boundary $(x-Ut)^2+y^2 = a^2$, say) in a fluid at rest 'at infinity'. The streamline pattern is that of a virtual dipole at the centre of the cylinder. The cylinder moves along the x-axis with velocity U, and each fluid particle follows a trajectory that depends upon its initial distance y from this axis. Maxwell succeeded in calculating the particle paths in terms of elliptic functions, and in plotting these out by hand, with the comment *The curves thus drawn appear to be as near the truth as I could get without a much greater amount of labour.* The labour already invested was certainly sufficient to demonstrate quite beautifully the essential distinction between an unsteady streamline pattern and the resulting field of particle displacements.

A further example of Maxwell's originality is to be found in his 1870 paper 'On hills and dales', in which he discusses in effect the topology of the contours of a scalar field in two dimensions. He does this in the context of the height of land above sea level; but the arguments are equally valid for any scalar field, for example the streamfunction $\psi(x, y)$ of any two-dimensional incompressible steady flow. Maxwell's thinking was topological in character: flexible, widely adaptable, and of great generality; in this as in other areas he was decades ahead of his time.

7. Postscript

While Maxwell's scientific achievements were truly phenomenal, it must also be recognised that he was not afraid to admit incomprehension in relation to some of the natural phenomena that he addressed. One of these concerned the origin and evolution of the Earth's magnetic field, which he discussed in Part IV, Chapter VIII of his great *Treatise on Electricity and Magnetism* (Maxwell 1873). Having commented on the secular variation of the field (by which he meant the slow wandering of the magnetic poles), he writes with an appropriate sense of wonder:

> *What cause, whether exterior to the earth or in its inner depths, produces such enormous changes in the earth's magnetism, that its magnetic poles move slowly from one part of the globe to another? When we consider that the intensity of the magnetisation of the great globe of the earth is quite comparable with that which we produce with much difficulty in our steel magnets, these immense changes in so large a body force us to conclude that we are not yet acquainted with one of the most powerful agents in nature, the scene of whose activity lies in those inner depths of the earth, to the knowledge of which we have so few means of access.*

The means of access were in due course provided by the penetrating power of seismology (see for example Bullen & Bolt 1985) which established the liquid-metal character of the outer core of the Earth, capable of sustaining the electric currents $\mathbf{j}(\mathbf{x}, t)$ that are the source of the magnetic field $\mathbf{B}(\mathbf{x}, t)$ whose variations we observe at the surface. Gravitational convection and the rotation of the Earth conspire to provide a turbulent (or at least random) flow $\mathbf{u}(\mathbf{x}, t)$ in the core with the property of nonzero helicity oppositely signed in north and south hemispheres, and it is now known that this property alone (implying a degree of knottedness of the vortex lines of the flow) in a body of fluid as large as the earth's outer core is sufficient to guarantee the growth of a large-scale magnetic field by a dynamo-instability mechanism (Moffatt 1979).

This dynamo instability depends on what are sometimes know as the pre-Maxwell equations (with displacement current filtered out):

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0, \quad \mathbf{j} = \nabla \times \mathbf{B},$$

coupled with Ohm's law in a moving medium of conductivity σ:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

Maxwell did have these essential ingredients at his disposal, so that this *most powerful agent in nature* was already imbedded within the theoretical structure that he provided in his 1873 treatise. The fact that a full century had to pass before the dynamo mechanism was finally teased out of this structure may perhaps be seen as yet further indication of just how far James Clerk Maxwell was ahead of his time.

Acknowledgements

I thank David Forfar, Alex Craik, Renzo Ricca, Julian Hunt and Raymond Flood for their most helpful comments on an earlier draft of this chapter.

References

Alfvén, H. (1942) Existence of electromagnetic-hydrodynamic waves. *Nature* **150**, 405-406.
Arnold, V.I.(1966) Sur un principe variationelle pour les écoulements stationnaires des liquides parfaits et ces applications aux problèmes de stabilité non-linéaires. *J.Méc.* **5**, 29-43.
Bullen, K.E. & Bolt, B.A.(1985) *An Introduction to the Theory of Seismology* (4th Edn.) Cambridge University Press.
Carr, B. (Ed.) (2007) *Universe or Multiverse?* Cambridge University Press.
Drazin, P.G. & Reid, W.H. (1981) *Hydrodynamic Stability.* Cambridge University Press.
Fuller, A.T. (1986) James Clerk Maxwell's Cambridge manuscripts: extracts relating to control and stability — VI, *Int. J. Control*, **43**,1135-1168.
Harman, P.M. (1990, 1995, 2003) *The Scientific Letters and Papers of James Clerk Maxwell*, Cambridge University Press. Vol. 1(1990), Vol. 2(1995), Vol. 3(2003).
Helmholtz, H. (1858) On integrals of the hydrodynamical equations, which express vortex motion. *Crelle's Journal*, **55** (translated from German by P.G.Tait 1867 *Phil. Mag.* Suppl. **33**).
Jeans, J.H. (1902) The stability of a spherical nebula. *Phil. Trans. A* **312**, 1-53.
Koschmieder, E.L. (1993) *Bénard Cells and Taylor Vortices.* Cambridge University Press.
Iooss, G. (1994) *The Couette-Taylor Problem.* Springer-Verlag.
Kelvin, Lord (W. Thomson) (1867) On vortex atoms. *Phil. Mag.* **34**, 15-24.
Kelvin, Lord (W. Thomson) (1869) On vortex motion. *Trans. Roy. Soc. Edin.* **25**, 217-260.
Lin, C.C. (1966) *The Theory of Hydrodynamic Stability.* Cambridge University Press.
Maxwell, J.C. (1856) On Faraday's lines of force. *Scientific Papers of James Clerk Maxwell*, Cambridge University Press, **1**, 155-229.
Maxwell, J.C. (1859) On the stability of the motion of Saturn's rings. *Scientific Papers of James Clerk Maxwell*, Cambridge University Press, **1**, 286-377.
Maxwell, J.C. (1870) On the displacement in a case of fluid motion. *Scientific Papers of James Clerk Maxwell*, Cambridge University Press, **2**, 208-214.
Maxwell, J.C. (1870) On hills and dales. *Scientific Papers of James Clerk Maxwell*, Cambridge University Press, **2**, 233-240.
Moffatt, H.K. (1969) The degree of knottedness of tangled vortex lines. *J.Fluid Mech.* **35**, 117-129.
Moffatt, H.K. (1979) *Magnetic Field Generation in Electrically Conducting Fluids.* Cambridge University Press.
Moffatt, H.K. (1985) Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part I. Fundamentals. *J.Fluid Mech.* **159**, 359-378.
Moffatt, H.K. (1986) Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part II. Stability considerations. *J.Fluid Mech.* **166**, 359-378.
Ricca, R.L. & Nipoti, B. (2011) Gauss? linking number revisited. *J. Knot Theory & its Ram.* **20**, 1325-1343.
Ricca, R.L. & Weber, C. (2012) On the origin of knot theory and topological fluid mechanics, in preparation.
Rayleigh, Lord (1917) On the dynamics of revolving fluids. *Proc. Roy. Soc. Lond. A* **93**, 148-154.
Silver, D.S. (2008) The last poem of James Clerk Maxwell. *Notices of the AMS*, **55**, 1266-1270.
Stokes, G.G. (1842) On the steady motion of incompressible fluids. *Trans. Camb. Phil. Soc.* **7**, 439-454.
Stokes, G.G. (1847) On the theory of oscillatory waves. *Trans. Camb. Phil. Soc.* **8**, 441-455.
Stuart, J.T. (1958) On the non-linear mechanics of hydrodynamic stability. *J.Fluid Mech.* **4**, 1-21.
Tait, P.G. (1898) On knots I, II, III, *Scientific Papers* Vol.I, 273-347 .
Taylor, G.I. (1923) Stability of a viscous fluid contained between two rotating cylinders. *Phil. Trans. Roy. Soc.A* **223**, 289-343.
Woltjer, L. (1958) A theorem on force-free magnetic fields. *Proc. Nat. Acad. Sci.* **43**, 489-491.