

Research report

MA12-03

University of Leicester
Department of Mathematics

2012

New Conjectures about Zeroes of Riemann's Zeta Function

YURI MATIYASEVICH

St.Petersburg Department
of Steklov Institute of Mathematics
of Russian Academy of Sciences

<http://logic.pdmi.ras.ru/~yumat>

Abstract

It is well known that zeroes of Riemann's zeta function encode a lot of number-theoretical information, in particular, about the distribution of prime numbers via Riemann's and von Mangoldt's formulas for $\pi(x)$ and $\psi(x)$. The goal of this paper is to present numerical evidence for a (presumably new and not yet proved) method for revealing all divisors of all natural numbers from the zeroes of the zeta function.

This text is essentially a written version of the talk [3] given by the author at the Department of Mathematics of University of Leicester, UK on June 18, 2012. This talk was based on more intensive computations made after previous author's talk [4] on the same subject given originally at the Mathematical Institute of the University of Oxford on January 26, 2012. The new numerical data indicate that some of conjectures stated in Oxford are, most likely, wrong.

“The physicist George Darwin used to say that every once in a while one should do a completely crazy experiment, like blowing the trumpet to the tulips every morning for a month. Probably nothing will happen, but if something did happen, that would be a stupendous discovery.”

Ian Hacking [1, p.154]

Riemann’s *zeta function* can be defined by the following Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \dots \quad (1)$$

The series converges for $s > 1$ and diverges at $s = 1$ where it turns into the harmonic series:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \quad (2)$$

The zeta function is named after Georg Friedrich Bernhard Riemann but it was previously studied by Leonhard Euler. His interest was initiated by the so-called *Basel Problem* posed by Pietro Mengoli in 1644, who asked the following question: *What is the value of the sum*

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \dots ? \quad (3)$$

In our notation (3) is nothing else but $\zeta(2)$.

Euler at first calculated more than a dozen of decimal digits of the sum and found that

$$\zeta(2) = 1.64493406684822644\dots \quad (4)$$

(this wasn’t an easy exercise because the series (3) converges very slowly but he invented what is nowadays called *Euler–Maclaurin summation*). Then somehow Euler knew that

$$\frac{\pi^2}{6} = 1.64493406684822644\dots \quad (5)$$

and made a natural conjecture that

$$\zeta(2) = \frac{\pi^2}{6}. \quad (6)$$

In 1735 Euler gave his first “proof” of this equality but it was not rigorous by today’s standards. Later he returned to this problem several times and gave a number of quite rigorous proofs.

Euler didn’t stop by merely answering the original question asked by Mengoli, but continued his investigations and found the following values of $\zeta(s)$ for other values of the argument:

$$\zeta(4) = \frac{1}{90}\pi^4, \quad (7)$$

$$\zeta(6) = \frac{1}{945}\pi^6, \quad (8)$$

$$\zeta(8) = \frac{1}{9450}\pi^8, \quad (9)$$

$$\zeta(10) = \frac{691}{638512875}\pi^{10}, \quad (10)$$

$$\zeta(12) = \frac{2}{18243225}\pi^{12}, \quad (11)$$

$$\zeta(14) = \frac{3617}{325641566250}\pi^{14}. \quad (12)$$

But what are the strange numerators in the values of $\zeta(10)$ and $\zeta(14)$?

No doubts that Euler immediately recognized these numbers as numerators of the so-called *Bernoulli numbers*. Named after Jacob Bernoulli, these numbers can be defined in many ways, in particular, from the coefficients in the Taylor expansion

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{1}{k!} B_k x^k. \quad (13)$$

The values of the initial Bernoulli numbers with even indices are:

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad (14)$$

$$B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \quad \dots$$

The values of the Bernoulli numbers with odd indices are “much simpler”:

$$B_1 = -\frac{1}{2}, \quad B_3 = B_5 = B_7 = B_9 = B_{11} = \dots = 0. \quad (15)$$

After substituting (14) into (6)–(12), it is easy to guess what the other factors are:

$$\begin{aligned}
\zeta(2) &= \frac{1}{6}\pi^2 = \frac{2^1 B_2}{2!}\pi^2, \\
\zeta(4) &= \frac{1}{90}\pi^4 = -\frac{2^3 B_4}{4!}\pi^4, \\
\zeta(6) &= \frac{1}{945}\pi^6 = \frac{2^5 B_6}{6!}\pi^6, \\
\zeta(8) &= \frac{1}{9450}\pi^8 = -\frac{2^7 B_8}{8!}\pi^8, \\
\zeta(10) &= \frac{691}{638512875}\pi^{10} = \frac{2^9 B_{10}}{10!}\pi^{10}, \\
\zeta(12) &= \frac{2}{18243225}\pi^{12} = -\frac{2^{11} B_{12}}{12!}\pi^{12}, \\
\zeta(14) &= \frac{3617}{325641566250}\pi^{14} = \frac{2^{13} B_{14}}{14!}\pi^{14}.
\end{aligned} \tag{16}$$

and Euler gave the general formula

$$\zeta(2k) = (-1)^{k+1} \frac{2^{2k-1} B_{2k}}{(2k)!} \pi^{2k}, \quad k = 1, 2, \dots \tag{17}$$

Euler also asked a question that might look stupid: *What should the value of $\zeta(0)$ be?* His answer was:

$$\zeta(0) = 1^0 + 2^0 + 3^0 + \dots = 1 + 1 + 1 + \dots = -\frac{1}{2}. \tag{18}$$

Euler's argumentation, important for our further considerations, was based on considering the following function:

$$\eta(s) = (1 - 2 \cdot 2^{-s})\zeta(s) \tag{19}$$

$$= (1 - 2 \cdot 2^{-s})(1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \dots) \tag{20}$$

$$= 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \dots - 2 \cdot 2^{-s} - 2 \cdot 4^{-s} - \dots \tag{21}$$

$$= 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \dots \tag{22}$$

The alternating Dirichlet series (22) has an advantage over the series (1): the former converges for $s > 0$. For $s = 0$ it becomes the series

$$1 - 1 + 1 - 1 + \dots \tag{23}$$

with partial sums alternating between 1 and 0; assuming that its “value” is $\frac{1}{2}$, one obtains (18) from (19).

Continuing in this style, Euler got the values

$$\zeta(-1) = 1^1 + 2^1 + 3^1 + \dots = 1 + 2 + 3 + \dots = -\frac{1}{12}, \quad (24)$$

$$\zeta(-2) = 1^2 + 2^2 + 3^2 + \dots = 1 + 4 + 9 + \dots = 0, \quad (25)$$

$$\zeta(-3) = 1^3 + 2^3 + 3^3 + \dots = 1 + 8 + 27 + \dots = \frac{1}{120}, \quad (26)$$

and in general

$$\zeta(-m) = -\frac{B_{m+1}}{m+1} \quad m = 0, 1, \dots \quad (27)$$

Both (17) and (27) contain Bernoulli numbers; putting $m = 2k - 1$ one can eliminate these numbers getting the identity

$$\zeta(1 - 2k) = (-1)^k 2^{1-2k} \pi^{-2k} (2k - 1)! \zeta(2k), \quad k = 1, 2, \dots \quad (28)$$

Riemann began to study $\zeta(s)$ for s being a complex number. The series (1) converges in the semiplane $\Re(s) > 1$ only, but Riemann [5] analytically extended it to the entire complex plane except for the point $s = 1$, the only (and simple) pole of the zeta function.

It turned out that values of $\zeta(s)$ for non-positive integers indicated by Euler as (27) coincide with the values obtained via analytical continuation. In particular, according to (15),

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = \dots = 0, \quad (29)$$

and the negative even integers are called *trivial zeroes* of the zeta function. Riemann [5] proved that this function has no other real zeroes.

Also, the equality (28) obtained by Euler for integer arguments of the zeta function can be extended to complex arguments after selecting the proper counterpart for the factor $(-1)^k$. Riemann [5] proved that

$$\zeta(1 - s) = \cos\left(\frac{\pi s}{2}\right) 2^{1-s} \pi^{-s} \Gamma(s) \zeta(s) \quad (30)$$

and this identity is known as the *functional equation* for the zeta function.

The cosine function can be expressed via the gamma function as

$$\cos\left(\frac{\pi s}{2}\right) = \frac{\pi}{\Gamma(-\frac{s}{2} + \frac{1}{2}) \Gamma(\frac{s}{2} + \frac{1}{2})}, \quad (31)$$

and for the gamma function we have the duplication formula

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2} + \frac{1}{2}\right). \quad (32)$$

Substituting (31) and (32) into (30), we see after simple transformations that

$$\pi^{-\frac{1-s}{2}}(-s)\Gamma\left(\frac{1-s}{2} + 1\right) \zeta(1-s) = \pi^{-\frac{s}{2}}(s-1)\Gamma\left(\frac{s}{2} + 1\right) \zeta(s). \quad (33)$$

The right hand side of (33) is nowadays usually denoted as $\xi(s)$. It is easy to check that the left hand side of (33) is just $\xi(1-s)$, that is, in terms of this function the functional equation has the following nice form:

$$\xi(1-s) = \xi(s). \quad (34)$$

The function $\xi(s)$ is entire; its zeroes are exactly *non-trivial* (i.e., non-real) zeroes of $\zeta(s)$.

Let us follow Riemann and make a change of the variable:

$$s = \frac{1}{2} + it, \quad t = \left(\frac{1}{2} - s\right) i \quad (35)$$

and define

$$\Xi(t) = \xi\left(\frac{1}{2} + it\right). \quad (36)$$

The functional equation (34) implies that $\Xi(t)$ is an even function:

$$\Xi(t) = \xi\left(\frac{1}{2} + it\right) = \xi\left(1 - \left(\frac{1}{2} + it\right)\right) = \Xi(-t). \quad (37)$$

In terms of this function Riemann stated his famous hypothesis.

The Riemann Hypothesis [5]. *All zeroes of $\Xi(t)$ are real numbers.*

Riemann established a relationship between prime numbers and zeroes of the zeta function by giving a (somewhat complicated) expression for $\pi(x)$ —the number of primes below x —via certain sums over these zeroes. A simpler form of such relationship was given by Hans

Carl Friedrich von Mangoldt in [6]. It gives an expression for the function $\psi(x)$ introduced by Pafnutij L'vovich Chebyshev as

$$\psi(x) = \sum_{\substack{q \leq x \\ q \text{ is a power} \\ \text{of prime } p}} \ln(p) \quad (38)$$

$$= \ln(\text{LCM}(1, 2, \dots, \lfloor x \rfloor)). \quad (39)$$

This is a step function jumping by $\ln(p)$ at every prime number p and every prime power:

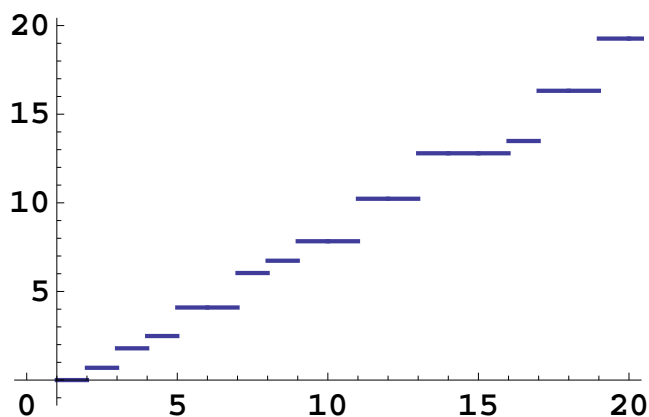


Figure 1: Chebyshev's function $\psi(x)$

Theorem (von Mangoldt [6]). For any non-integer $x > 1$

$$\psi(x) = x - \sum_{\xi(\rho)=0} \frac{x^\rho}{\rho} - \sum_{n=1}^{\infty} \frac{x^{-2n}}{-2n} - \ln(2\pi). \quad (40)$$

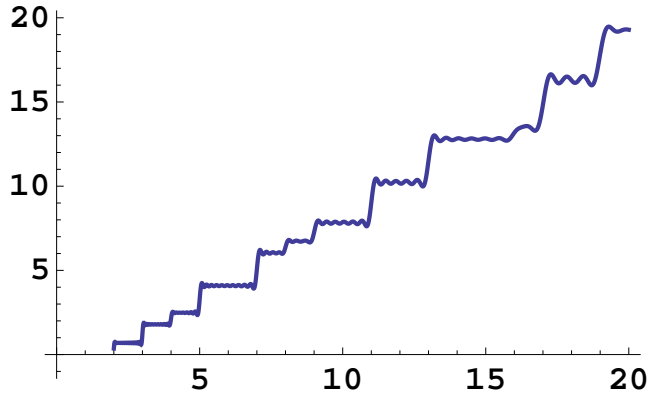


Figure 2: Plot of $x - \sum_{\substack{|\rho| < 200 \\ \xi(\rho) = 0}} \frac{x^\rho}{\rho} - \sum_{n=1}^{\infty} \frac{x^{-2n}}{-2n} - \ln(2\pi)$

Traditionally, (40) is viewed as an identity between well-defined left and right hand sides, but it can be interpreted in a different way. Imagine that we know nothing about prime numbers, even their definition, but have at our disposal sufficiently many initial zeroes of the zeta function. In such a case we could “discover” prime numbers just by looking at the plot of the truncated right hand side of (40) (see Figure 2). The prime numbers can be revealed from such a picture either by looking at integers in the vicinity of which the function has a big jump (they will be powers of primes) or by looking at the sizes of jumps (they will be close to natural logarithms of primes). To reveal more and more primes we would need to use more and more zeroes of the zeta function.

The main novelty of the present research is the discovery of a new (and not yet proved) way to reveal prime numbers from the zeroes of the zeta function. It was an unexpected result of my “mathematical blowing the trumpet”, and I have to start explaining what it was.

Assuming that all zeroes of $\Xi(t)$ are real and simple, let them be denoted $\pm\gamma_1, \pm\gamma_2, \dots$, with $0 < \gamma_1 < \gamma_2 < \dots$. Thus the non-trivial zeroes of $\zeta(s)$ are $\frac{1}{2} \pm i\gamma_1, \frac{1}{2} \pm i\gamma_2, \dots$

Suppose that we have found $\gamma_1, \gamma_2, \dots, \gamma_{N-1}$; how could these numbers be used for calculating an (approximate) value of the next zero γ_N ?	(41)
---	------

It was a rather strange idea to seek an answer to this question

because known initial zeroes are distributed rather irregularly:

$$\gamma_1 = 14.1347\dots \quad \gamma_2 = 21.0220\dots \quad (42)$$

$$\gamma_3 = 25.0109\dots \quad \gamma_4 = 30.4249\dots \quad (43)$$

$$\gamma_5 = 32.9350\dots \quad \gamma_6 = 37.5861\dots \quad (44)$$



Figure 3: Initial zeroes of $\Xi(t)$

Nevertheless, let us try to give an answer to the question (41).

A natural idea is to approximate $\Xi(t)$ by some simpler even function also having zeroes at the points $\pm\gamma_1, \dots, \pm\gamma_{N-1}$. One way to construct such a function is to consider an *interpolating determinant* with some even functions f_1, f_2, \dots

$$\begin{vmatrix} f_1(\gamma_1) & \dots & f_1(\gamma_{N-1}) & f_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ f_N(\gamma_1) & \dots & f_N(\gamma_{N-1}) & f_N(t) \end{vmatrix}. \quad (45)$$

Clearly, it vanishes for $t = \pm\gamma_1, \dots, \pm\gamma_{N-1}$ because for such values of t the determinant contains two equal columns.

Selecting $f_n(t) = t^{2(n-1)}$ we would obtain just an interpolating polynomial

$$C \prod_{n=1}^{N-1} (t^2 - \gamma_n^2) \quad (46)$$

having no other zeroes and hence useless for our goal.

Let us consider a modelling situation where the interpolating polynomial does the job. If $\gamma_1^*, \gamma_2^*, \dots$ are zeroes of the function

$$\Xi^*(t) = \sum_{k=1}^N f_k(t) \quad (47)$$

then the determinant

$$\begin{vmatrix} f_1(\gamma_1^*) & \dots & f_1(\gamma_{N-1}^*) & f_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ f_N(\gamma_1^*) & \dots & f_N(\gamma_{N-1}^*) & f_N(t) \end{vmatrix} \quad (48)$$

vanishes as soon as t is equal to *any* zero of $\Xi^*(t)$ because for such a t the rows of the determinant sum up to the zero row.

But our case is more complicated. First, we are looking for zeroes of a function defined by an infinite number of summands:

$$\Xi(t) = \sum_{n=1}^{\infty} \alpha_n(t), \quad (49)$$

where

$$\alpha_n(t) = -\frac{\pi^{-\frac{1}{4}-\frac{it}{2}} (t^2 + \frac{1}{4}) \Gamma(\frac{1}{4} + \frac{it}{2})}{2n^{\frac{1}{2}+it}}. \quad (50)$$

Second, the summands $\alpha_n(t)$ aren't even.

We will overcome the second difficulty in a quite formal way. According to the functional equation (30),

$$\Xi(t) = \Xi(-t) = \sum_{n=1}^{\infty} \alpha_n(-t) \quad (51)$$

so we can write

$$\Xi(t) = \sum_{n=1}^{\infty} \beta_n(t), \quad (52)$$

where

$$\begin{aligned} \beta_n(t) &= \frac{\alpha_n(t) + \alpha_n(-t)}{2} & (53) \\ &= -\frac{\pi^{-\frac{1}{4}+\frac{it}{2}} (t^2 + \frac{1}{4}) \Gamma(\frac{1}{4} - \frac{it}{2})}{4n^{\frac{1}{2}-it}} - \\ &\quad \frac{\pi^{-\frac{1}{4}-\frac{it}{2}} (t^2 + \frac{1}{4}) \Gamma(\frac{1}{4} + \frac{it}{2})}{4n^{\frac{1}{2}+it}} & (54) \end{aligned}$$

are surely even functions. There is a “small problem”: the series (49) converges for $\Im(t) < -\frac{1}{2}$, the series (51) converges for $\Im(t) > \frac{1}{2}$, so the series (52) converges nowhere.

However, for each interpolating determinant we need only finitely many even functions, so we define the main object of our study as

$$\Delta_N(t) = \begin{vmatrix} \beta_1(\gamma_1) & \dots & \beta_1(\gamma_{N-1}) & \beta_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ \beta_N(\gamma_1) & \dots & \beta_N(\gamma_{N-1}) & \beta_N(t) \end{vmatrix}. \quad (55)$$

Here are some results of calculations:

$$\begin{aligned}\Delta_{47}(\mathbf{138.11604}) &= -2.18497 \dots \cdot 10^{-1216} < 0 \\ \gamma_{47} &= \mathbf{138.1160420545334} \dots \\ \Delta_{47}(\mathbf{138.11605}) &= +4.68242 \dots \cdot 10^{-1216} > 0\end{aligned}$$

We see that a zero of $\Delta_{47}(t)$ has 8 decimal digits coinciding with digits of γ_{47} . For $N = 220$ there are already 15 coinciding decimal digits:

$$\begin{aligned}\Delta_{220}(\mathbf{427.208825084074}) &= -1.92776 \dots \cdot 10^{-17793} < 0 \\ \gamma_{220} &= \mathbf{427.20882508407458052814} \dots \\ \Delta_{220}(\mathbf{427.208825084075}) &= +9.85564 \dots \cdot 10^{-17794} > 0\end{aligned}$$

For $N = 400$ the number of common digits increases to 38:

$$\begin{aligned}\Delta_{400}(\mathbf{679.74219788252821771952593891126999534}) &= \\ &= -2.95319 \dots \cdot 10^{-52001} < 0 \\ \gamma_{400} &= \mathbf{679.7421978825282177195259389112699953456135514} \dots \\ \Delta_{400}(\mathbf{679.74219788252821771952593891126999535}) &= \\ &= +1.78976 \dots \cdot 10^{-52001} > 0\end{aligned}$$

Moreover, $\Delta_N(t)$ allows us to calculate good approximations not only to the next not yet used zero γ_N but to γ_{N+k} as well for values of k that are not too large. Here are some examples¹:

$$\begin{aligned}\Delta_{47}(\mathbf{139.7362}) &= +1.27744 \dots \cdot 10^{-1216} > 0 \\ \gamma_{48} &= \mathbf{139.736208952121} \dots \\ \Delta_{47}(\mathbf{139.7363}) &= -9.88309 \dots \cdot 10^{-1216} < 0\end{aligned}$$

$$\begin{aligned}\Delta_{47}(\mathbf{141.12370}) &= -1.85988 \dots \cdot 10^{-1217} < 0 \\ \gamma_{49} &= \mathbf{141.1237074040211} \dots \\ \Delta_{47}(\mathbf{141.12371}) &= +2.40777 \dots \cdot 10^{-1217} > 0\end{aligned}$$

$$\begin{aligned}\Delta_{47}(\mathbf{143.11184}) &= +8.00594 \dots \cdot 10^{-1218} > 0 \\ \gamma_{50} &= \mathbf{143.1118458076206} \dots \\ \Delta_{47}(\mathbf{143.11185}) &= -8.98353 \dots \cdot 10^{-1218} < 0\end{aligned}$$

$$\begin{aligned}\Delta_{220}(\mathbf{428.127914076616}) &= +3.30722 \dots \cdot 10^{-17792} > 0 \\ \gamma_{221} &= \mathbf{428.12791407661668211030} \dots \\ \Delta_{220}(\mathbf{428.127914076617}) &= -1.28498 \dots \cdot 10^{-17792} < 0\end{aligned}$$

¹Tables showing the number of digits common to $(N+k)$ th zero of $\Xi(t)$ and a zero of $\Delta_N(t)$ for diverse N and k can be found in [2].

$$\begin{aligned} \Delta_{220}(430.3287454309386) &= -1.08026 \dots \cdot 10^{-17794} < 0 \\ \gamma_{222} &= 430.328745430938636669926\dots \\ \Delta_{220}(430.3287454309387) &= +1.56602 \dots \cdot 10^{-17793} > 0 \\ \dots\dots\dots \\ \Delta_{220}(441.683199201) &= -3.85957 \dots \cdot 10^{-17794} < 0 \\ \gamma_{230} &= 441.68319920118902387\dots \\ \Delta_{220}(441.683199202) &= +1.39118 \dots \cdot 10^{-17793} > 0 \\ \dots\dots\dots \\ \Delta_{220}(442.90454630) &= +6.07254 \dots \cdot 10^{-17795} > 0 \\ \gamma_{231} &= 442.9045463026094494\dots \\ \Delta_{220}(442.90454631) &= -4.92952 \dots \cdot 10^{-17794} < 0 \end{aligned}$$

$$\begin{aligned} \Delta_{400}(681.8949915331518891094524110813676572) &= \\ & \quad -3.24940 \dots \cdot 10^{-52001} < 0 \\ \gamma_{401} &= 681.894991533151889109452411081367657278562874\dots \\ \Delta_{400}(681.8949915331518891094524110813676573) &= \\ & \quad +3.00725 \dots \cdot 10^{-52001} > 0 \end{aligned}$$

$$\begin{aligned} \Delta_{400}(682.602735019750545487540836644871189) &= \\ & \quad +6.60550 \dots \cdot 10^{-52001} > 0 \\ \gamma_{402} &= 682.60273501975054548754083664487118909593475\dots \\ \Delta_{400}(682.602735019750545487540836644871190) &= \\ & \quad -4.20554 \dots \cdot 10^{-52000} < 0 \end{aligned}$$

$$\begin{aligned} \dots\dots\dots \\ \Delta_{400}(740.573807447295010515) &= +6.77206 \dots \cdot 10^{-52005} > 0 \\ \gamma_{446} &= 740.57380744729501051503597159\dots \\ \Delta_{400}(740.573807447295010516) &= -8.96545 \dots \cdot 10^{-52006} < 0 \\ \dots\dots\dots \\ \Delta_{400}(741.75733557294167327) &= -1.55647 \dots \cdot 10^{-52004} < 0 \\ \gamma_{447} &= 741.7573355729416732758611620\dots \\ \Delta_{400}(741.75733557294167328) &= +2.91156 \dots \cdot 10^{-52005} > 0 \end{aligned}$$

Determinant $\Delta_{12000}(t)$ has zeroes having more than 2000 common decimal digits with $\gamma_{12000}, \gamma_{12001}, \dots, \gamma_{12010}$.

The great accuracy of the approximative values of the zeroes of $\Xi(t)$ was a first surprising outcome of the calculations. So how is it possible that initial summands from a *divergent* series (52) produce so many correct digits? I don't have a full explanation of this phenomenon; two heuristic "reasons" will be presented below.

The determinant $\Delta_N(t)$ is a linear combination of the first N summands from (52) with numerical coefficients equal to corresponding signed minors of the matrix from (55):

$$\Delta_N(t) = \begin{vmatrix} \beta_1(\gamma_1) & \dots & \beta_1(\gamma_{N-1}) & \beta_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ \beta_N(\gamma_1) & \dots & \beta_N(\gamma_{N-1}) & \beta_N(t) \end{vmatrix} \quad (56)$$

$$= \sum_{n=1}^N \tilde{\delta}_{N,n} \beta_n(t), \quad (57)$$

where

$$\tilde{\delta}_{N,n} = (-1)^{N+n} \begin{vmatrix} \beta_1(\gamma_1) & \dots & \beta_1(\gamma_{N-1}) \\ \vdots & \ddots & \vdots \\ \beta_{n-1}(\gamma_1) & \dots & \beta_{n-1}(\gamma_{N-1}) \\ \beta_{n+1}(\gamma_1) & \dots & \beta_{n+1}(\gamma_{N-1}) \\ \vdots & \ddots & \vdots \\ \beta_N(\gamma_1) & \dots & \beta_N(\gamma_{N-1}) \end{vmatrix}. \quad (58)$$

Since we are only interested in the zeroes of $\Delta_N(t)$, we can consider the *normalized coefficients*

$$\delta_{N,n} = \frac{\tilde{\delta}_{N,n}}{\tilde{\delta}_{N,1}}, \quad (59)$$

in particular, $\delta_{N,1} = 1$. The function

$$\sum_{n=1}^N \delta_{N,n} \beta_n(t) \quad (60)$$

has the same zeroes as $\Delta_N(t)$.

As a first example, let us look at the normalized coefficients $\delta_{47,n}$:

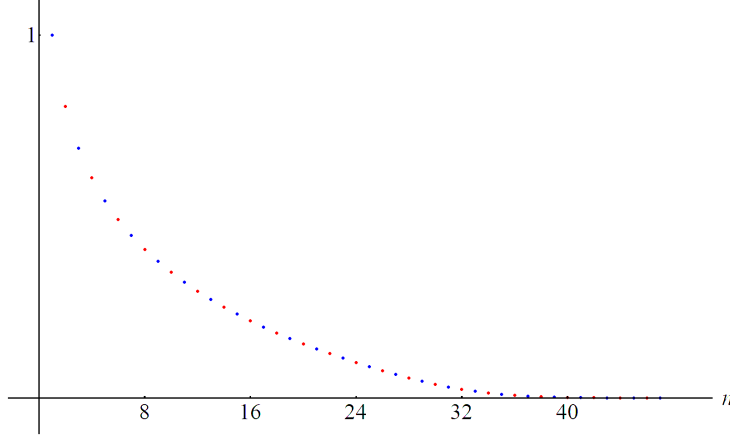


Figure 4: Normalized coefficients $\delta_{47,n}$

They lie on a smoothly decaying curve. Thus

$$\sum_{n=1}^{47} \delta_{47,n} \beta_n(t) = \frac{\Delta_{47}(t)}{\tilde{\delta}_{47,1}} \quad (61)$$

is not a *sharp* but a *smooth* truncation of the divergent series (52). It is known that smooth truncation can accelerate convergence of a series – compare

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{k+1}}{k} \quad (62)$$

oscillating with amplitude of order k^{-1} , with

$$\begin{aligned} \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^k}{k-1} + \frac{1}{2} \frac{(-1)^{k+1}}{k} = \\ \frac{1}{2} + \frac{1}{2} \left(\frac{1}{1} - \frac{1}{2} \right) - \frac{1}{2} \left(\frac{1}{2} - \frac{1}{3} \right) + \dots - \frac{(-1)^k}{2} \left(\frac{1}{k-1} - \frac{1}{k} \right) \end{aligned} \quad (63)$$

oscillating with amplitude of order k^{-2} . Moreover, smooth truncation can even transform a divergent series into a convergent one – compare

$$1 - 1 + 1 - \dots + (-1)^{k+1} = \frac{1}{2} + \frac{(-1)^{k+1}}{2} \quad (64)$$

with

$$1 - 1 + 1 - \dots + (-1)^k + \frac{(-1)^{k+1}}{2} = \frac{1}{2}. \quad (65)$$

So the smoothness of the truncation in (61) might be the first “reason” why the summands of the divergence series (52) are useful for calculation of the zeroes.

The curve on which the coefficients $\delta_{47,n}$ lie looks like a logarithmic curve

$$\delta_{47,n} \approx 1 + \lambda_{47} \log(n) \quad (66)$$

with some parameter λ_{47} , this being better seen on the plot of the same coefficients but with logarithmic scale:

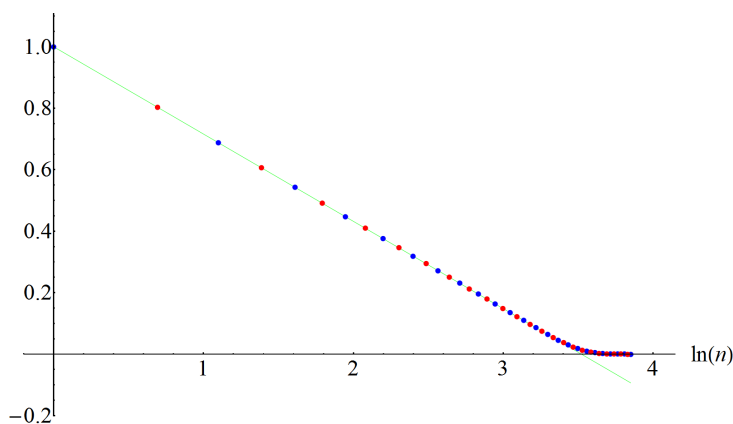


Figure 5: Normalized coefficients $\delta_{47,n}$ with logarithmic scale

We see that the initial coefficients lie approximately on a straight line.

The pictures for $N = 48, 49, 50, 51$ look very similar to the picture for $N = 47$ from Figure 5. Normalized coefficients $\delta_{52,n}$ and $\delta_{53,n}$ show slightly different behaviour of trailing coefficients:

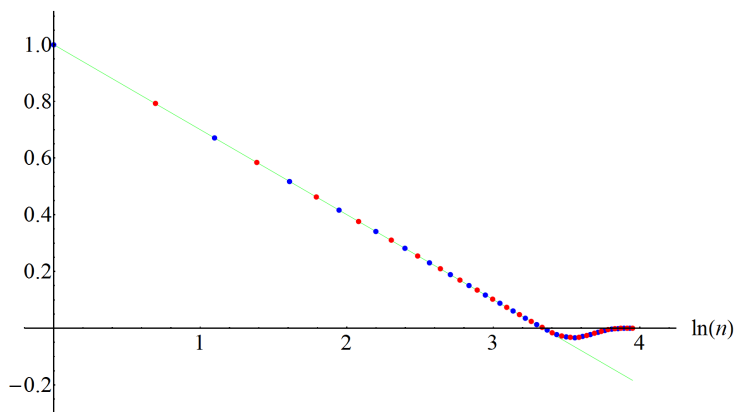


Figure 6: Normalized coefficients $\delta_{52,n}$ with logarithmic scale

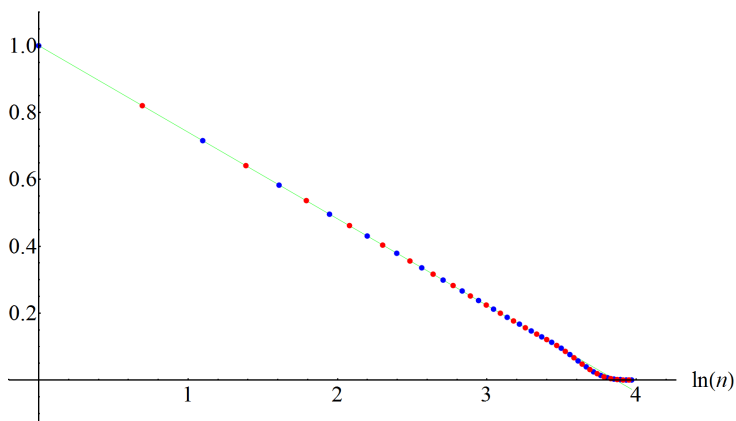


Figure 7: Normalized coefficients $\delta_{53,n}$ with logarithmic scale

At first it seems that we capture the behaviour of the normalized coefficients, but what a surprise we see for $N = 54$:

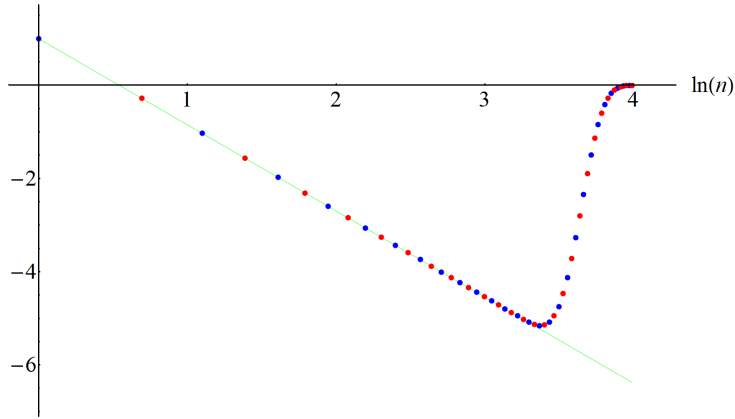


Figure 8: Normalized coefficients $\delta_{54,n}$ with logarithmic scale

Most of the coefficients are negative and greater than 1 in absolute value. Nevertheless, the initial coefficients do lie on a straight line and $\Delta_{54}(t)$ gives good predictions for a few of the next zeroes $\gamma_{54}, \gamma_{55}, \dots$. I have no explanation why 54 is such a special number – cases $N = 55, 56, 57$ look again similar to cases $N = 47, \dots, 53$.

We skip² now many similar pictures and jump to $N = 130$ where we observe something radically new:

²A catalog of pictures of $\delta_{N,n}$ for many N can be found in [2]

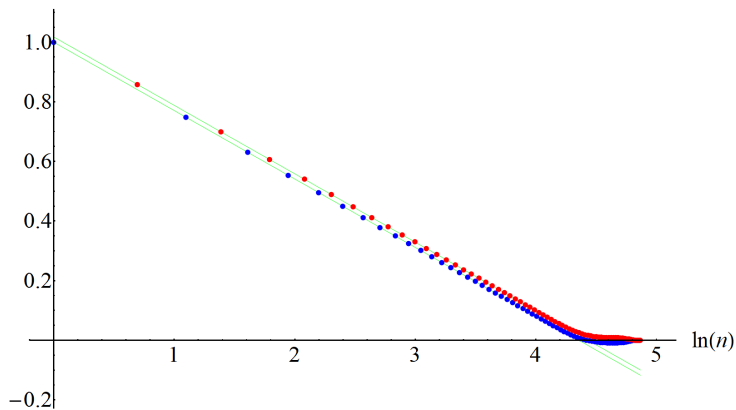


Figure 9: Normalized coefficients $\delta_{130,n}$ with logarithmic scale

Now the initial coefficients lie on two parallel lines rather than on a single line, so instead of an analog of (66) for $N = 130$ we should use approximation

$$\delta_{130,n} \approx 1 + \mu_{130,2} \text{dom}_2(n) + \lambda_{130} \log(n) \quad (67)$$

where $\text{dom}_m(k)$ is the characteristic function of divisibility:

$$\text{dom}_m(k) = \begin{cases} 1, & \text{if } m \mid k, \\ 0, & \text{otherwise.} \end{cases} \quad (68)$$

Such a splitting is more transparent for $N = 169$ and $N = 180$:

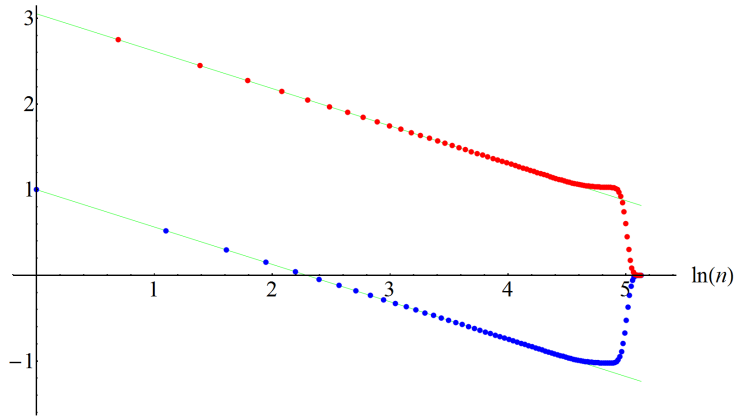


Figure 10: Normalized coefficients $\delta_{169,n}$ with logarithmic scale

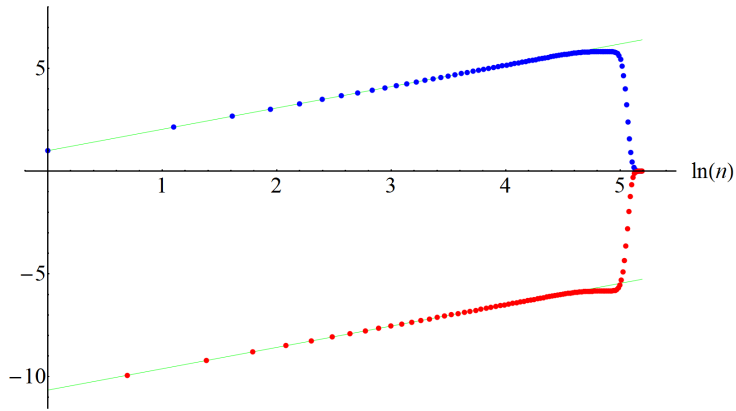


Figure 11: Normalized coefficients $\delta_{180,n}$ with logarithmic scale

Now we again skip many values of N and jump to $N = 220$:

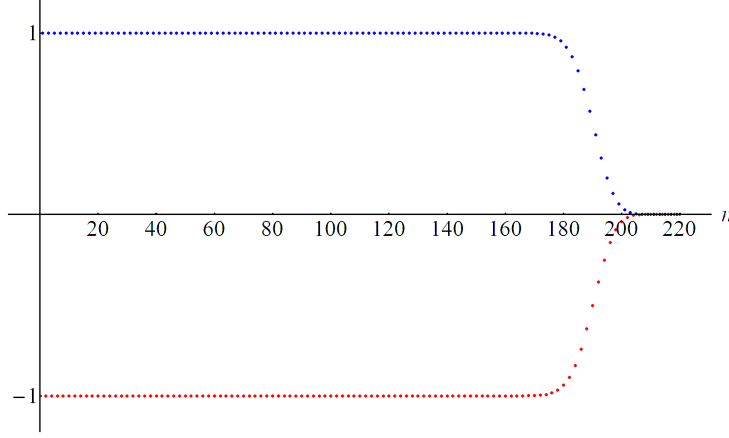


Figure 12: Normalized coefficients $\delta_{220,n}$

We see that in the approximation

$$\delta_{220,n} \approx 1 + \mu_{220,2} \text{dom}_2(n) + \lambda_{220} \log(n) \quad (69)$$

we should take $\mu_{220,2}$ close to -2 and λ_{220} close to 0 . In other words,

$$\sum_{n=1}^{220} \delta_{220,n} \beta_n(t) = \frac{\Delta_{220}(t)}{\tilde{\delta}_{220,1}} \quad (70)$$

is a smooth truncation not of the divergent series (52) but of the convergent (for real t) alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \beta_n(t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\alpha_n(t) + \alpha_n(-t)}{2} \quad (71)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \left(t^2 + \frac{1}{4} \right) \left(\frac{\pi^{-\frac{1}{4} + \frac{it}{2}} \Gamma\left(\frac{1}{4} - \frac{it}{2}\right)}{4n^{\frac{1}{2} - it}} + \frac{\pi^{-\frac{1}{4} - \frac{it}{2}} \Gamma\left(\frac{1}{4} + \frac{it}{2}\right)}{4n^{\frac{1}{2} + it}} \right) \quad (72)$$

$$= \frac{1}{4} \left(t^2 + \frac{1}{4} \right) \pi^{-\frac{1}{4} + \frac{it}{2}} \Gamma\left(\frac{1}{4} - \frac{it}{2}\right) \sum_{n=1}^{\infty} (-1)^{n-1} n^{-\frac{1}{2} + it} + \frac{1}{4} \left(t^2 + \frac{1}{4} \right) \pi^{-\frac{1}{4} - \frac{it}{2}} \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \sum_{n=1}^{\infty} (-1)^{n-1} n^{-\frac{1}{2} - it} \quad (73)$$

$$\begin{aligned}
&= \frac{1}{4} \left(t^2 + \frac{1}{4}\right) \pi^{-\frac{1}{4} + \frac{it}{2}} \Gamma\left(\frac{1}{4} - \frac{it}{2}\right) \eta\left(\frac{1}{2} + it\right) + \\
&\quad \frac{1}{4} \left(t^2 + \frac{1}{4}\right) \pi^{-\frac{1}{4} - \frac{it}{2}} \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \eta\left(\frac{1}{2} - it\right)
\end{aligned} \tag{74}$$

where $\eta(s)$ is defined by (22). Euler introduced this function in order to assign a value to the zeta function outside the area of convergence of the Dirichlet series, and that convergence gives a second “reason” (besides the smoothness of the truncation) why Δ_{220} is so good for prediction the values $\gamma_{220}, \gamma_{221}, \dots$. It is very remarkable that here the function η emerges by itself, just from our calculation of the determinants (58) as if they were as clever as Euler was.

Still some natural questions remain open. The switching from divergent (52) to convergent (71) partly “explains” why $\Delta_{220}(t)$ is able to “predict” the values of γ_{220} and further zeroes of $\Xi(t)$ but *why does this also happen for, say, $N = 47$?* The series (71) converges on the critical line rather slowly, so *why are the zeroes of $\Delta_{220}(t)$ so close to those of $\Xi(t)$?*

For many values of N after 220 the pictures look very similar to the case $N = 220$. To avoid the “non-informative” part of the plot we can start making cuts in the abscissae:

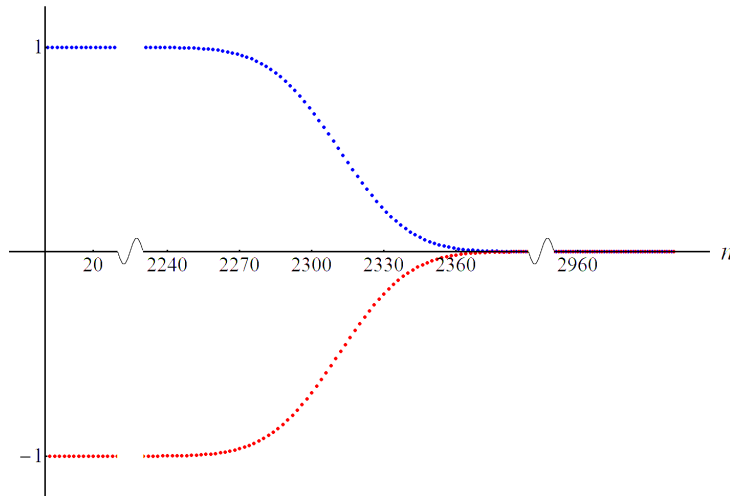


Figure 13: Normalized coefficients $\delta_{3000,n}$

However, for some sporadic values of N the pictures are different:

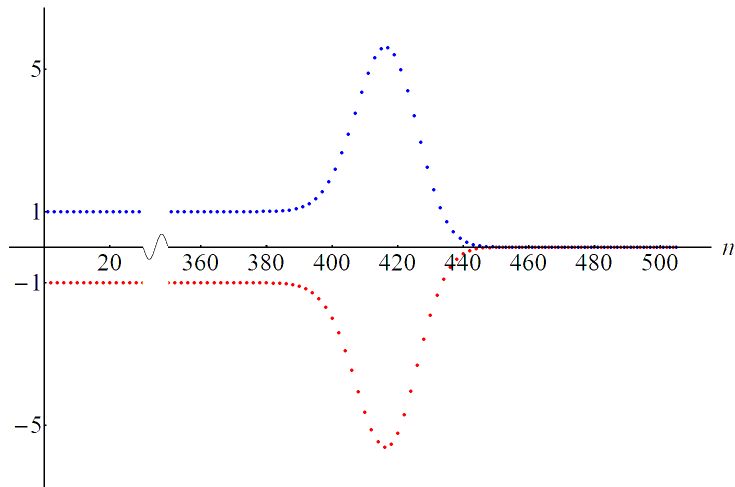


Figure 14: Normalized coefficients $\delta_{505,n}$

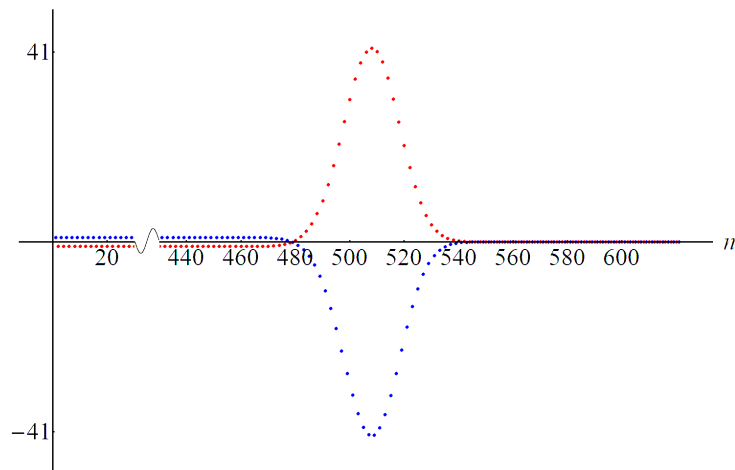


Figure 15: Normalized coefficients $\delta_{621,n}$

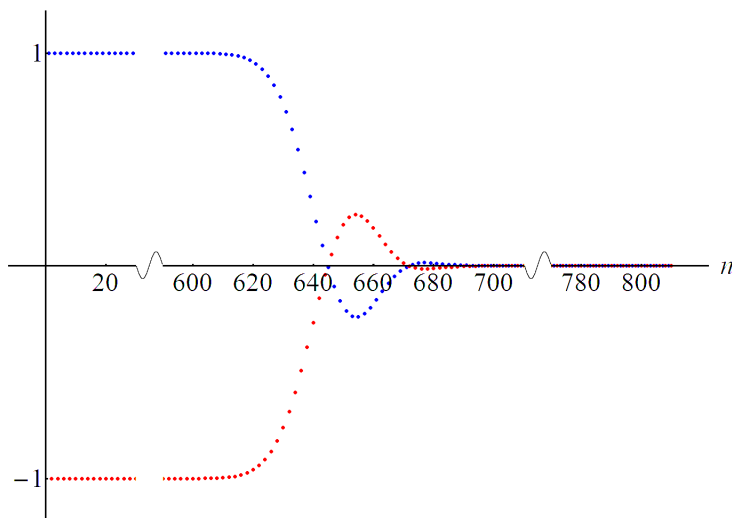


Figure 16: Normalized coefficients $\delta_{810,n}$

In spite of such sporadic “misbehaviour” of some pictures, the author stated in his previous talk [4] the following conjectures.

Conjecture 1.³ *For every fixed odd (even) value of n the normalized coefficients $\delta_{N,n}$ have, as $N \rightarrow \infty$, a limit equal to 1 (respectively, equal to -1).*

Conjecture 2.³ *For every ν such that $1 \leq \nu$ the sequence*

$$\delta_{[\nu],1}, \dots, (-1)^{n-1} \delta_{[\nu n],n}, \dots \quad (75)$$

has certain limiting value $\delta(\nu)$.

Informally, Conjecture 2 tells us that if pictures are properly scaled, then with the growth of N the dots will approach two (smooth?) curves—the plots of $\delta(x^{-1})$ and $-\delta(x^{-1})$ for $x \in (0, 1)$.

Conjecture 1 and 2 were stated on the base of pictures for $N \leq 1200$, however subsequent calculation for $N \leq 12000$ suggests that these conjectures most likely are wrong—just look at Figure 17:

³Now these conjectures are expected to be wrong—see below.

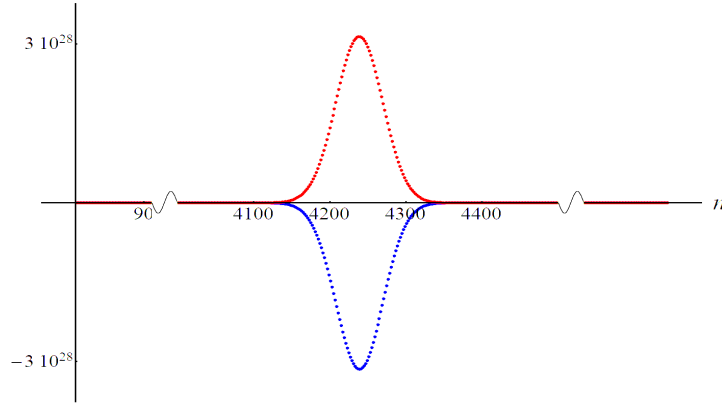


Figure 17: Normalized coefficients $\delta_{5600,n}$

Pictures for many other big values of N look similar to the picture on Figure 17. In other words, while the case $N = 621$ on Figure 15 was sporadic amid pictures looking like cases $N = 220$ and $N = 3000$ on Figures 12 and 13, for bigger N the picture on Figure 17 becomes typical.

Again, episodically there exceptional cases:

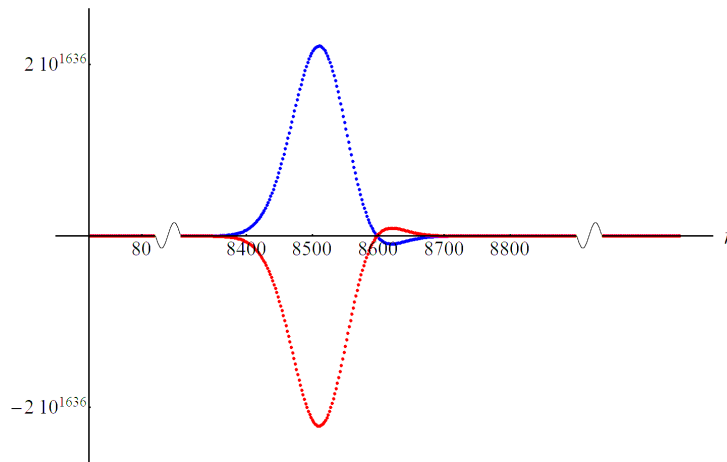


Figure 18: Normalized coefficients $\delta_{11428,n}$

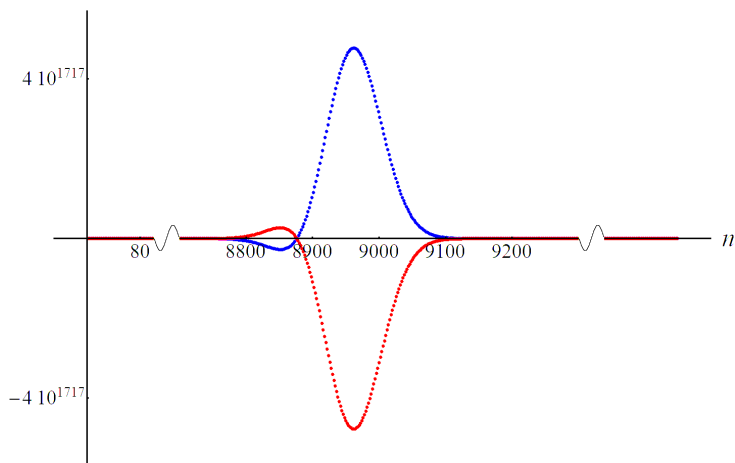


Figure 19: Normalized coefficients $\delta_{11981,n}$

The large maximal values $\delta_{N,n}$ for big N are, of course, due to our normalization (59). In order to be able to see what happens on the whole range of n we can now apply logarithmic scaling for the ordinate:

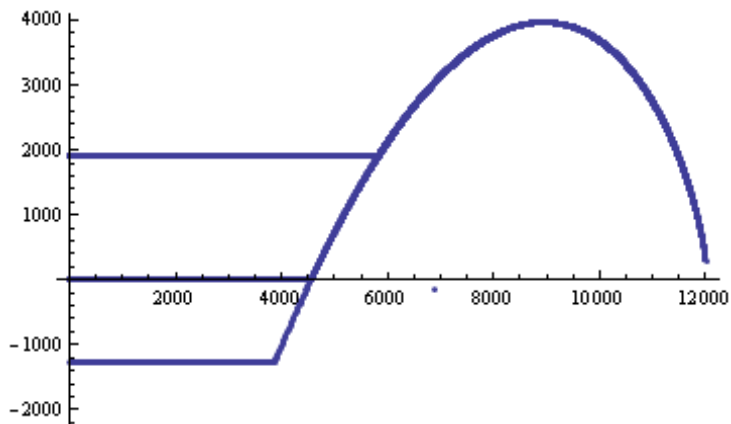


Figure 20: Values of $\ln(|\delta_{12000,n}|)$

The dot at $n = 6892$ corresponds to the place where values of $N_{12000,n}$ with odd and with even indices change their signs (like it

happens for $N = 621$ on Figure 15). It is remarkable that three horizontal lines end at a mysterious parabolic looking curve. Now we proceed to analyze the nature of these horizontal lines.

In fact, this phenomenon occurs already for much smaller values of N but there it is less noticeable. The next two Figures show normalized coefficients $\delta_{321,n}$ at first with full range on the ordinate and then with cuts in this axe enabling us to use different scale and show the areas marked on the first figure in more details:

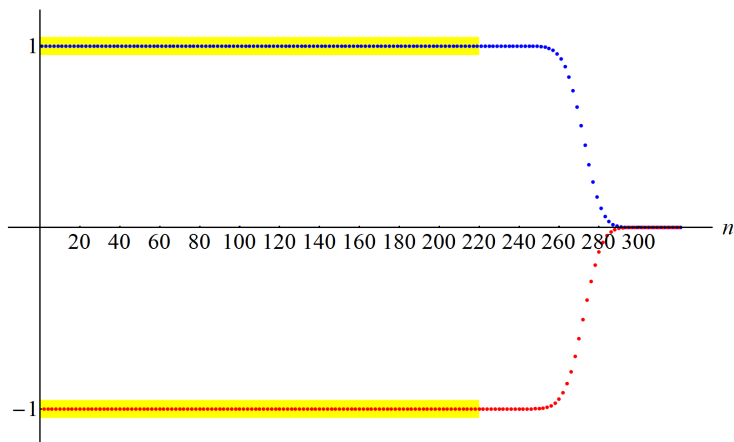


Figure 21: Normalized coefficients $\delta_{321,n}$

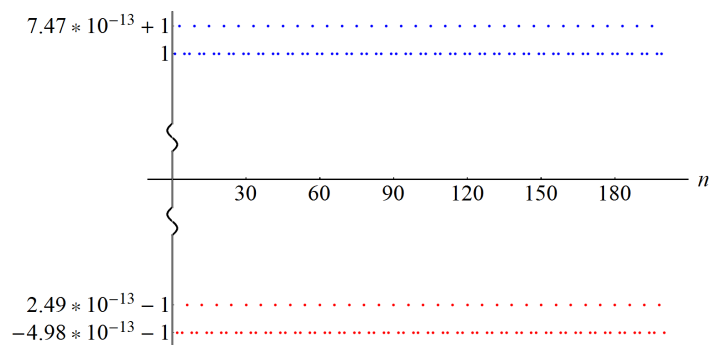


Figure 22: Normalized coefficients $\delta_{321,n}$

We see that actually the plot splits into 4 parallel lines so the initial normalized coefficients are better approximated as

$$\delta_{321,n} \approx 1 + \mu_{321,2} \text{dom}_2(n) + \mu_{321,3} \text{dom}_3(n) + \lambda_{321} \log(n) \quad (76)$$

with

$$\mu_{321,2} = -2 - 4.98 \dots \cdot 10^{-13}, \quad \mu_{321,3} = 7.47 \dots \cdot 10^{-13}, \quad (77)$$

$$\lambda_{321} = -3.33 \dots \cdot 10^{-18}. \quad (78)$$

In other words, after the first splitting depending on $n \bmod 2$ there is a second splitting depending on $n \bmod 3$ but with much smaller amplitude.

If fact, in general case there is a fine structure of further splittings depending on $n \bmod 4$, $n \bmod 5$, \dots and initial normalized coefficients are better approximated as

$$\delta_{N,n} \approx \sum_m \mu_{N,m} \text{dom}_m(n) + \lambda_N \log(n). \quad (79)$$

The weights $\mu_{N,m}$ become smaller and smaller and in order to visualize further splitting we should make more cuttings in the ordinate and increase its scale. However, there is another way to see splittings for bigger moduli. Namely, we can consider *averaged normalized coefficients*

$$\delta_{N,n,a} = \frac{\delta_{N,n} + \dots + \delta_{N,n+a-1}}{a}. \quad (80)$$

For example, for $a = 2$ we have:

$$\delta_{N,n,2} = \frac{\delta_{N,n} + \delta_{N,n+1}}{2} \quad (81)$$

$$\approx 1 + \mu_{N,2} \frac{\text{dom}_2(n) + \text{dom}_2(n+1)}{2} + \mu_{N,3} \frac{\text{dom}_3(n) + \text{dom}_3(n+1)}{2} + \lambda_N \frac{\log(n(n+1))}{2} \quad (82)$$

$$= 1 + \frac{\mu_{N,2}}{2} + \mu_{N,3} \frac{\text{dom}_3(n) + \text{dom}_3(n+1)}{2} + \lambda_N \frac{\log(n(n+1))}{2}. \quad (83)$$

In other words, the dependence on $n \bmod 2$ disappears and we can observe the dependence on $n \bmod 3$:

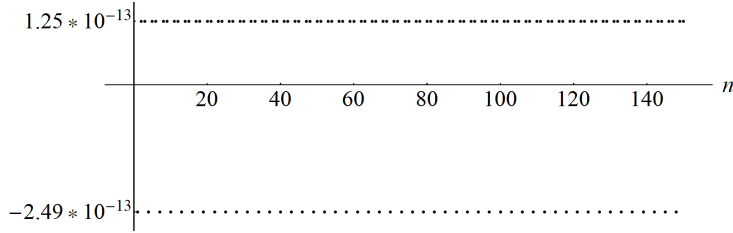


Figure 23: Averaged normalized coefficients $\delta_{321,n,2}$

For $N = 999$ the second splitting is even smaller, namely, $\mu_{999,3} = 2.26 \dots \cdot 10^{-41}$ but otherwise the picture is exactly the same as in Figure 23:

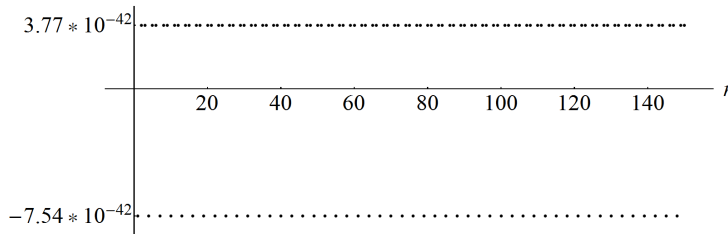


Figure 24: Averaged normalized coefficients $\delta_{999,n,2}$

Taking the average over 6 consecutive coefficients, we eliminate dependence both on $n \bmod 2$ and on $n \bmod 3$ and can see the splitting depending on $n \bmod 4$ with $\mu_{999,4} = 1.75 \dots \cdot 10^{-90}$:

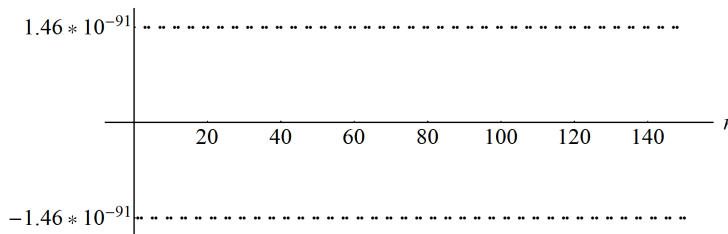


Figure 25: Averaged normalized coefficients $\delta_{999,n,6}$

With $a = 12$ we can see even smaller splitting with $\mu_{999,5} = -3.09 \dots \cdot 10^{-127}$:

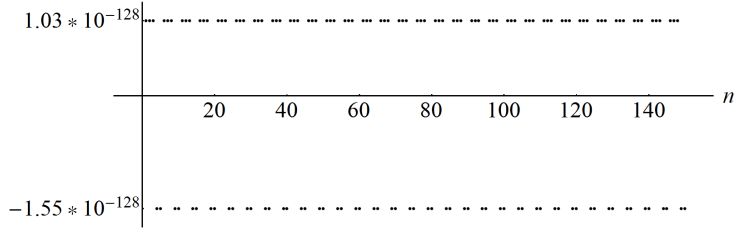


Figure 26: Averaged normalized coefficients $\delta_{999,n,12}$

However, with $a = 60$ we cannot see the splitting for $n \bmod 7$:

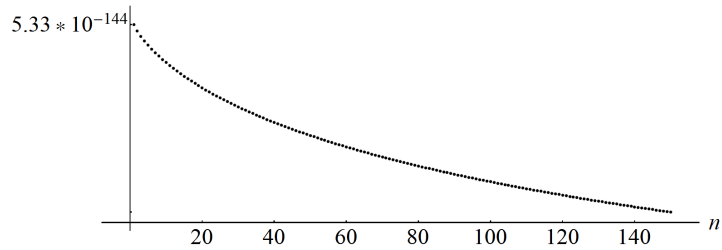


Figure 27: Averaged normalized coefficients $\delta_{999,n,60}$

This can be explained as follows: the value $\lambda_{999} = -1.73 \dots \cdot 10^{-144}$ is bigger than the value $\mu_{999,7} = 9.13 \dots \cdot 10^{-170}$ and for this reason we see a logarithmic-like curve on Figure 27.

One way to see the splitting depending on $n \bmod 7$ is to increase the value of N . Another possibility is to eliminate the contribution of logarithmic summands by subtracting them:

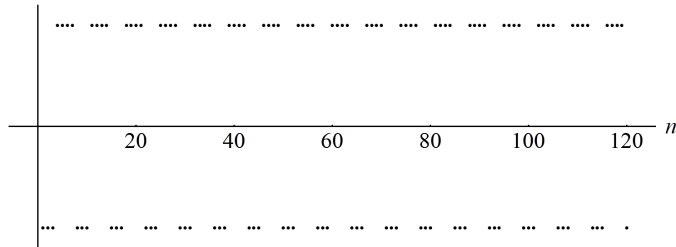


Figure 28: Differences $\delta_{999,n,60} - \lambda_{999} \log(\Gamma(n+60) - \Gamma(n))/60$

Table 1: Values of $\mu_{N,2}$, $\mu_{N,3}$, and $\mu_{N,5}$

N	$\mu_{N,2}$	$\mu_{N,3}$	$\mu_{N,5}$
400	$-2 + 3.77 \dots 10^{-16}$	$-5.66 \dots 10^{-16}$	$-1.58 \dots 10^{-28}$
800	$-2 - 6.04 \dots 10^{-33}$	$+9.06 \dots 10^{-33}$	$+7.17 \dots 10^{-100}$
1200	$-2 + 4.77 \dots 10^{-50}$	$-7.16 \dots 10^{-50}$	$-4.35 \dots 10^{-155}$
1600	$-2 - 2.21 \dots 10^{-66}$	$+3.32 \dots 10^{-66}$	$-2.90 \dots 10^{-212}$
2000	$-2 - 5.98 \dots 10^{-84}$	$+8.98 \dots 10^{-84}$	$+2.42 \dots 10^{-268}$
2400	$-2 - 4.64 \dots 10^{-102}$	$+6.97 \dots 10^{-102}$	$-7.12 \dots 10^{-326}$
2800	$-2 + 1.15 \dots 10^{-117}$	$-1.73 \dots 10^{-117}$	$+4.92 \dots 10^{-383}$
3200	$-2 - 3.00 \dots 10^{-134}$	$+4.50 \dots 10^{-134}$	$+2.36 \dots 10^{-442}$
3600	$-2 + 1.55 \dots 10^{-151}$	$-2.32 \dots 10^{-151}$	$-1.31 \dots 10^{-498}$
4000	$-2 - 8.05 \dots 10^{-168}$	$+1.20 \dots 10^{-167}$	$-7.14 \dots 10^{-557}$
4400	$-2 + 9.65 \dots 10^{-185}$	$-1.44 \dots 10^{-184}$	$+1.96 \dots 10^{-614}$
4800	$-2 - 9.40 \dots 10^{-201}$	$+1.41 \dots 10^{-200}$	$+7.29 \dots 10^{-671}$
5200	$-2 - 5.63 \dots 10^{-215}$	$+8.45 \dots 10^{-215}$	$+1.00 \dots 10^{-726}$
5600	$-2 - 1.39 \dots 10^{-204}$	$+2.09 \dots 10^{-204}$	$-2.54 \dots 10^{-757}$
6000	$-2 + 2.19 \dots 10^{-165}$	$-3.29 \dots 10^{-165}$	$-3.55 \dots 10^{-759}$
6400	$-2 - 3.36 \dots 10^{-105}$	$+5.04 \dots 10^{-105}$	$-4.80 \dots 10^{-740}$
6800	$-2 + 4.84 \dots 10^{-26}$	$-7.26 \dots 10^{-26}$	$-8.73 \dots 10^{-703}$
7200	$-9.60 \dots 10^{66}$	$+1.44 \dots 10^{67}$	$-9.27 \dots 10^{-652}$
7600	$-6.31 \dots 10^{170}$	$+9.47 \dots 10^{170}$	$-2.25 \dots 10^{-588}$
8000	$-3.97 \dots 10^{285}$	$+5.96 \dots 10^{285}$	$+4.81 \dots 10^{-516}$
8400	$+2.02 \dots 10^{408}$	$-3.03 \dots 10^{408}$	$+8.15 \dots 10^{-435}$
8800	$+5.57 \dots 10^{538}$	$-8.36 \dots 10^{538}$	$-6.24 \dots 10^{-346}$
9200	$+2.98 \dots 10^{673}$	$-4.47 \dots 10^{673}$	$+1.31 \dots 10^{-249}$
9600	$-3.52 \dots 10^{818}$	$+5.29 \dots 10^{818}$	$-1.81 \dots 10^{-149}$
10000	$+1.34 \dots 10^{966}$	$-2.01 \dots 10^{966}$	$-6.61 \dots 10^{-43}$
10400	$+6.40 \dots 10^{1049}$	$-9.60 \dots 10^{1049}$	$-1 - 2.36 \dots 10^{-69}$
10800	$-4.75 \dots 10^{1090}$	$+7.13 \dots 10^{1090}$	$-1 - 1.18 \dots 10^{-183}$
11200	$+2.29 \dots 10^{1030}$	$-3.44 \dots 10^{1030}$	$-1 + 3.39 \dots 10^{-302}$
11600	$-4.82 \dots 10^{927}$	$+7.24 \dots 10^{927}$	$-1 - 7.95 \dots 10^{-424}$
12000	$+6.21 \dots 10^{828}$	$-9.32 \dots 10^{828}$	$-1 - 2.38 \dots 10^{-548}$

Now we can return to the case $N = 12000$ exhibited on Figure 20. For a long range of N coefficients $\mu_{N,2}$ were approaching -2 and coefficients $\mu_{N,3}, \mu_{N,4}, \dots$ were approaching 0 supporting Conjecture 1, but then their behavior changed; see Table 1. So in Figure 20 we

observe the same splitting but now on a different scale: the top horizontal line corresponds to n divisible by 2 or by 3, the horizontal line at the bottom corresponds to $n \equiv \pm 5 \pmod{30}$ and the middle line corresponds to the remaining values of n .

Coefficients $\mu_{N,n}$ were introduced above for approximating initial normalized coefficients $\delta_{N,n}$. Surprisingly, these numbers turn out to have number-theoretical significance: they “know” what divisors every natural number has, in particular, what numbers are primes.

This “knowledge” can be revealed from the δ 's by seeking “almost linear” relations between them. By this we mean equalities of the form

$$r_1\delta_{N,1} + r_2\delta_{N,2} + \cdots + r_m\delta_{N,m} = \epsilon \quad (84)$$

where r_1, r_2, \dots, r_m are rational numbers with small denominators and ϵ is small in comparison with $|\delta_{N,1}| + |\delta_{N,2}| + \cdots + |\delta_{N,m}|$.

One series of such “almost linear” relations has the form

$$\nu_{N,n} = \epsilon \quad (85)$$

where

$$\nu_{N,n} = \sum_{k=1}^n \frac{\mu_{N,k}}{k}. \quad (86)$$

Here

$$\mu_{N,1} = \delta_{N,1} = 1, \quad (87)$$

while for $n > 1$

$$\mu_{N,n} = \delta_{N,n} - \mu_{N,k(n,1)} - \cdots - \mu_{N,k(n,D_n)} \quad (88)$$

for some integers $k(n,1), \dots, k(n,D_n)$ such that

$$1 = k(n,1) < \cdots < k(n,D_n), \quad 1 \leq D_n < n. \quad (89)$$

For $n \geq 3$ partial sums

$$\mu_{N,n,m} = \delta_{N,n} - \mu_{N,k(n,1)} - \cdots - \mu_{N,k(n,m)}, \quad (90)$$

being very small too, will supply another series of “almost linear” relations between $\delta_{N,n}$.

The numbers $D_n, k(n,1), \dots, k(n,D_n)$ have a clear number-theoretical meaning so we could give a direct definition of them now (and shall do so later), but it is more instructive to see how their values can be defined inductively on the basis of the values of $\delta_{N,n}$ and the

values of $\mu_{N,n}$ and $\nu_{N,n}$ defined previously—we shall be trying to get small values for $\nu_{N,n}$ and $\mu_{N,n,m}$.

Case n = 1 is done by (87).

Case n = 2. According to (89), $D_2 = 1$ and respectively, according to (88) and (86),

$$\mu_{N,2} = \delta_{N,2} - \mu_{N,1}, \quad (91)$$

$$= -\delta_{N,1} + \delta_{N,2}, \quad (92)$$

$$\nu_{N,2} = \frac{\delta_{N,1}}{2} + \frac{\delta_{N,2}}{2}. \quad (93)$$

For example, for $N = 3200$ this gives

$$\mu_{3200,2} = -2.0000000 \dots \cdot 10^0 \quad (94)$$

and the following “almost linear” relation:

$$\nu_{3200,2} = -1.5020306 \dots \cdot 10^{-134}. \quad (95)$$

Case n = 3. If we wish the value of $\nu_{3200,3} = \nu_{3200,2} + \frac{1}{3}\mu_{3200,3}$ to be even smaller than (95), then the value of $\mu_{3200,3}$ should be close to

$$-3\nu_{3200,2} = 4.5060920 \dots \cdot 10^{-134}. \quad (96)$$

We observe that already

$$\mu_{3200,3,1} = \delta_{3200,1} - \mu_{3200,1} \quad (97)$$

$$= 4.5060920 \dots \cdot 10^{-134} \quad (98)$$

looks to be very close to (96), indeed, they have more than 170 common decimal digits:

$$\frac{-3\nu_{3200,2}}{\mu_{3200,3,1}} = 1 - 1.8048583 \dots \cdot 10^{-171}. \quad (99)$$

So we put $D_3 = 1$ and get respectively

$$\mu_{N,3} = \delta_{N,3} - \mu_{N,1} \quad (100)$$

$$= -\delta_{N,1} + \delta_{N,3}, \quad (101)$$

$$\nu_{N,3} = \frac{\delta_{N,1}}{6} + \frac{\delta_{N,2}}{2} + \frac{\delta_{N,3}}{3}. \quad (102)$$

For $N = 3200$ this gives the following “almost linear” relations:

$$\mu_{3200,3} = 4.5060920 \dots \cdot 10^{-134}, \quad (103)$$

$$\nu_{3200,3} = -2.7109526 \dots \cdot 10^{-305}. \quad (104)$$

Case $\mathbf{n} = 4$. If we wish the value of $\nu_{3200,4} = \nu_{3200,3} + \frac{1}{4}\mu_{3200,4}$ to be even smaller than (104), then the value of $\mu_{3200,4}$ should be close to

$$-4\nu_{3200,3} = 1.0843810 \dots \cdot 10^{-304}. \quad (105)$$

We observe that now

$$\mu_{3200,4,1} = \delta_{3200,1} - \mu_{3200,1} \quad (106)$$

$$= -2.0000000 \dots \cdot 10^0 \quad (107)$$

is much bigger than (104). On the other hand, the value of $\mu_{3200,4,1}$ is very close to the value of $\mu_{3200,2}$ given in (94), indeed, they have more than 300 common decimal digits:

$$\frac{\mu_{3200,2}}{\mu_{3200,4,1}} = 1 - 5.4219052 \dots \cdot 10^{-305}. \quad (108)$$

So we put $k(4, 2) = 2$ and observe that now

$$\mu_{3200,4,2} = \mu_{3200,4,1} - \mu_{3200,2} \quad (109)$$

$$= 1.0843810 \dots \cdot 10^{-304} \quad (110)$$

looks to be very close to (105), indeed, they have more than 170 common decimal digits:

$$\frac{-4\nu_{3200,3}}{\mu_{3200,4,1}} = 1 - 1.8048583 \dots \cdot 10^{-171}. \quad (111)$$

So we put $D_4 = 2$ and get respectively

$$\mu_{N,4} = \delta_{N,4} - \mu_{N,1} - \mu_{N,2} \quad (112)$$

$$= -\delta_{N,2} + \delta_{N,4}, \quad (113)$$

$$\nu_{N,4} = \frac{\delta_{N,1}}{6} + \frac{\delta_{N,2}}{4} + \frac{\delta_{N,3}}{3} + \frac{\delta_{N,4}}{4}. \quad (114)$$

For $N = 3200$ this gives the following “almost linear” relation:

$$\mu_{3200,4} = 1.0843810 \dots \cdot 10^{-304}, \quad (115)$$

$$\nu_{3200,4} = -4.7221542 \dots \cdot 10^{-443}. \quad (116)$$

Case $\mathbf{n} = 5$. This case is similar to the case $n = 3$. We have

$$\frac{-5\nu_{3200,4}}{\mu_{3200,5,1}} = 1 - 6.2926086 \dots \cdot 10^{-106}, \quad (117)$$

so we put $D_5 = 1$ and define respectively

$$\mu_{N,5} = \delta_{N,5} - \mu_{N,1} \quad (118)$$

$$= -\delta_{N,1} + \delta_{N,5}, \quad (119)$$

$$\nu_{N,5} = -\frac{\delta_{N,1}}{30} + \frac{\delta_{N,2}}{4} + \frac{\delta_{N,3}}{3} + \frac{\delta_{N,4}}{4} + \frac{\delta_{N,5}}{5}. \quad (120)$$

For $N = 3200$ this gives the following “almost linear” relations:

$$\mu_{3200,5} = 2.3610771 \dots \cdot 10^{-442}, \quad (121)$$

$$\nu_{3200,5} = -2.9714668 \dots \cdot 10^{-548}. \quad (122)$$

Case n = 6. We wish the value of $\mu_{3200,6}$ to be close to

$$-6\nu_{3200,5} = 1.7828801 \dots \cdot 10^{-547}. \quad (123)$$

We observe that

$$\mu_{3200,6,1} = \delta_{3200,1} - \mu_{3200,1} \quad (124)$$

$$= -1.9999999 \dots \cdot 10^0 \quad (125)$$

is much bigger than (123) but is very close to the value of $\mu_{3200,2}$ given in (94), indeed,

$$\frac{\mu_{3200,2}}{\mu_{3200,6,1}} = 1 - 2.2530460 \dots \cdot 10^{-134}. \quad (126)$$

So we put $k(6, 2) = 2$ and now observe that

$$\mu_{3200,6,2} = \mu_{3200,6,1} - \mu_{3200,2} \quad (127)$$

$$= 4.5060920 \dots \cdot 10^{-134} \quad (128)$$

is very close to the value of $\mu_{3200,3}$ given in (103), indeed,

$$\frac{\mu_{3200,3}}{\mu_{3200,6,2}} = 1 - 3.9565994 \dots \cdot 10^{-414}. \quad (129)$$

So we put $k(6, 3) = 3$ and now observe that

$$\mu_{3200,6,3} = \mu_{3200,6,2} - \mu_{3200,3} \quad (130)$$

$$= 1.7828801 \dots \cdot 10^{-547} \quad (131)$$

is very close to the desired (123), indeed,

$$\frac{-6\nu_{3200,5}}{\mu_{3200,6,3}} = 1 - 2.0064209 \dots \cdot 10^{-82}. \quad (132)$$

So we put $D_6 = 3$ and define respectively

$$\mu_{N,6} = \delta_{N,6} - \mu_{N,1} - \mu_{N,2} - \mu_{N,3} \quad (133)$$

$$= \delta_{N,1} - \delta_{N,2} - \delta_{N,3} + \delta_{N,6}, \quad (134)$$

$$\begin{aligned} \nu_{N,6} &= \frac{2\delta_{N,1}}{15} + \frac{\delta_{N,2}}{12} + \frac{\delta_{N,3}}{6} + \frac{\delta_{N,4}}{4} + \frac{\delta_{N,5}}{5} \\ &\quad + \frac{\delta_{N,6}}{6}. \end{aligned} \quad (135)$$

For $N = 3200$ this gives the following ‘‘almost linear’’ relation:

$$\mu_{3200,6} = 1.7828801 \dots \cdot 10^{-547}, \quad (136)$$

$$\nu_{3200,6} = 5.9620134 \dots \cdot 10^{-630}. \quad (137)$$

Case $\mathbf{n} = 7$. This case is similar to the cases $n = 3$ and $n = 5$. We have

$$\frac{-7\nu_{3200,6}}{\mu_{3200,7,1}} = 1 - 1.5520609 \dots \cdot 10^{-62}, \quad (138)$$

so we put $D_7 = 1$ and define respectively

$$\mu_{N,7} = \delta_{N,7} - \mu_{N,1} \quad (139)$$

$$= -\delta_{N,1} + \delta_{N,7}, \quad (140)$$

$$\begin{aligned} \nu_{N,7} &= -\frac{\delta_{N,1}}{105} + \frac{\delta_{N,2}}{12} + \frac{\delta_{N,3}}{6} + \frac{\delta_{N,4}}{4} + \frac{\delta_{N,5}}{5} \\ &\quad + \frac{\delta_{N,6}}{6} + \frac{\delta_{N,7}}{7}. \end{aligned} \quad (141)$$

For $N = 3200$ this gives the following ‘‘almost linear’’ relations:

$$\mu_{3200,7} = -4.1734094 \dots \cdot 10^{-629}, \quad (142)$$

$$\nu_{3200,7} = -9.2534079 \dots \cdot 10^{-692}. \quad (143)$$

Case $\mathbf{n} = 8$. This case is similar to the case $n = 6$. We wish the value of $\mu_{3200,8}$ to be close to

$$-8\nu_{3200,7} = 7.4027263 \dots \cdot 10^{-691}. \quad (144)$$

We observe that

$$\mu_{3200,8,1} = \delta_{3200,1} - \mu_{3200,1} \quad (145)$$

$$= -2.0000000 \dots \cdot 10^0 \quad (146)$$

is much bigger than (144) but is very close to the value of $\mu_{3200,2}$ given in (94), indeed,

$$\frac{\mu_{3200,2}}{\mu_{3200,8,1}} = 1 - 5.4219052 \dots \cdot 10^{-305}. \quad (147)$$

So we put $k(8,2) = 2$ and now observe that

$$\mu_{3200,8,2} = \mu_{3200,8,1} - \mu_{3200,2} \quad (148)$$

$$= 1.0843810 \dots \cdot 10^{-304} \quad (149)$$

is very close to the value of $\mu_{3200,4}$ given in (115), indeed,

$$\frac{\mu_{3200,4}}{\mu_{3200,8,2}} = 1 - 6.8266836 \dots \cdot 10^{-387}. \quad (150)$$

So we put $k(8,3) = 3$ and now observe that

$$\mu_{3200,8,3} = \mu_{3200,8,2} - \mu_{3200,4} \quad (151)$$

$$= 7.4027263 \dots \cdot 10^{-691} \quad (152)$$

is very close to the the desired (144), indeed,

$$\frac{-8\nu_{3200,7}}{\mu_{3200,8,3}} = 1 - 3.1514980 \dots \cdot 10^{-47}. \quad (153)$$

So we put $D_8 = 3$ and define respectively

$$\mu_{N,8} = \delta_{N,8} - \mu_{N,1} - \mu_{N,2} - \mu_{N,4} \quad (154)$$

$$= -\delta_{N,4} + \delta_{N,8}, \quad (155)$$

$$\begin{aligned} \nu_{N,8} &= -\frac{\delta_{N,1}}{105} + \frac{\delta_{N,2}}{12} + \frac{\delta_{N,3}}{6} + \frac{\delta_{N,4}}{8} + \frac{\delta_{N,5}}{5} \\ &\quad + \frac{\delta_{N,6}}{6} + \frac{\delta_{N,7}}{7} + \frac{\delta_{N,8}}{8}. \end{aligned} \quad (156)$$

For $N = 3200$ this gives the following ‘‘almost linear’’ relation:

$$\mu_{3200,8} = 7.4027263 \dots \cdot 10^{-691}, \quad (157)$$

$$\nu_{3200,8} = -2.9162097 \dots \cdot 10^{-738}. \quad (158)$$

Case $n = 9$. If we wish the value of $\nu_{3200,9} = \nu_{3200,8} + \frac{1}{9}\mu_{3200,9}$ to be even smaller than (158), the value of $\mu_{3200,9}$ should be close to

$$-9\nu_{3200,8} = 2.6245887 \dots \cdot 10^{-737}. \quad (159)$$

We observe that

$$\mu_{3200,9,1} = \delta_{3200,1} - \mu_{3200,1} \quad (160)$$

$$= 4.5060920 \dots \cdot 10^{-134} \quad (161)$$

is much bigger than (158). On the other hand, the value of $\mu_{3200,9,1}$ is very close to the value of $\mu_{3200,3}$ given in (103), indeed:

$$\frac{\mu_{3200,3}}{\mu_{3200,9,1}} = 1 - 5.6391692 \dots \cdot 10^{-604}. \quad (162)$$

So we put $k(9, 2) = 3$ and examine now

$$\mu_{3200,9,2} = \mu_{3200,9,1} - \mu_{3200,3} \quad (163)$$

$$= 2.5410615 \dots \cdot 10^{-737}. \quad (164)$$

On the one hand, this value isn't sufficiently close to (159), on the other hand, it isn't close to any of $\mu_{3200,2}, \dots, \mu_{3200,8}$ either so we cannot proceed as before any longer.

In order to be able to continue, we need to increase the value of N . Here are the values of $\mu_{4800,2}, \dots, \mu_{4800,8}$ and $\nu_{4800,2}, \dots, \nu_{4800,8}$:

Table 2: $N = 4800$

n	$\mu_{4800,n}$	$\nu_{4800,n}$
2	$-2.0000000 \dots \cdot 10^0$	$-4.7021571 \dots \cdot 10^{-201}$
3	$1.4106471 \dots \cdot 10^{-200}$	$2.5665830 \dots \cdot 10^{-460}$
4	$-1.0266332 \dots \cdot 10^{-459}$	$-1.4584552 \dots \cdot 10^{-671}$
5	$7.2922764 \dots \cdot 10^{-671}$	$-1.3366569 \dots \cdot 10^{-836}$
6	$8.0199419 \dots \cdot 10^{-836}$	$9.4934119 \dots \cdot 10^{-966}$
7	$-6.6453883 \dots \cdot 10^{-965}$	$-1.0614379 \dots \cdot 10^{-1067}$
8	$8.4915037 \dots \cdot 10^{-1067}$	$6.3881904 \dots \cdot 10^{-1149}$

We see that these numbers are much smaller than the corresponding numbers for $N = 3200$. What is more important, now

$$\frac{-9\nu_{4800,8}}{\mu_{4800,9,2}} = 1 - 2.9558657 \dots \cdot 10^{-65}. \quad (165)$$

So we put $D_9 = 2$ and get respectively

$$\mu_{N,9} = \delta_{N,9} - \mu_{N,1} - \mu_{N,3} \quad (166)$$

$$= -\delta_{N,3} + \delta_{N,9}, \quad (167)$$

$$\begin{aligned} \nu_{N,9} &= -\frac{\delta_{N,1}}{105} + \frac{\delta_{N,2}}{12} + \frac{\delta_{N,3}}{18} + \frac{\delta_{N,4}}{8} + \frac{\delta_{N,5}}{5} \\ &\quad + \frac{\delta_{N,6}}{6} + \frac{\delta_{N,7}}{7} + \frac{\delta_{N,8}}{8} + \frac{\delta_{N,9}}{9}. \end{aligned} \quad (168)$$

For $N = 4800$ this gives the following “almost linear” relation:

$$\mu_{4800,9} = -5.7493714 \dots \cdot 10^{-1148}, \quad (169)$$

$$\nu_{4800,9} = 1.8882633 \dots \cdot 10^{-1213}. \quad (170)$$

Case $n = 10$. This case is similar to the cases $n = 6$ and 8 . We wish the value of $\mu_{3200,10}$ to be close to

$$-10\nu_{3200,9} = 9.2807969 \dots \cdot 10^{-739}. \quad (171)$$

We observe that

$$\mu_{3200,10,1} = \delta_{3200,1} - \mu_{3200,1} \quad (172)$$

$$= -2.0000000 \dots \cdot 10^0 \quad (173)$$

is much bigger than (171) but is very close to the value of $\mu_{3200,2}$ given in (94), indeed,

$$\frac{\mu_{3200,2}}{\mu_{3200,10,1}} = 1 - 1.1805385 \dots \cdot 10^{-442}. \quad (174)$$

So we put $k(10, 2) = 2$ and now observe that

$$\mu_{3200,10,2} = \mu_{3200,10,1} - \mu_{3200,2} \quad (175)$$

$$= 2.3610771 \dots \cdot 10^{-442} \quad (176)$$

is very close to the value of $\mu_{3200,5}$ given in (121), indeed,

$$\frac{\mu_{3200,5}}{\mu_{3200,10,2}} = 1 - 1.1642551 \dots \cdot 10^{-330}. \quad (177)$$

So we put $k(10, 3) = 3$ and now see that

$$\mu_{3200,10,3} = \mu_{3200,10,2} - \mu_{3200,5} \quad (178)$$

$$= -2.7488961 \dots \cdot 10^{-772} \quad (179)$$

is much smaller than the desired (171). In order to be able to continue, we again need to increase the value of N . We have

$$\frac{-10\nu_{4800,9}}{\mu_{4800,10,3}} = 1 - 2.8194018 \dots \cdot 10^{-36}, \quad (180)$$

so we put $D_{10} = 3$ and define respectively

$$\mu_{N,10} = \delta_{N,10} - \mu_{N,1} - \mu_{N,2} - \mu_{N,5} \quad (181)$$

$$= \delta_{N,1} - \delta_{N,2} - \delta_{N,5} + \delta_{N,10}, \quad (182)$$

$$\begin{aligned} \nu_{N,10} &= \frac{19\delta_{N,1}}{210} - \frac{\delta_{N,2}}{60} + \frac{\delta_{N,3}}{18} + \frac{\delta_{N,4}}{8} + \frac{\delta_{N,5}}{10} \\ &\quad + \frac{\delta_{N,6}}{6} + \frac{\delta_{N,7}}{7} + \frac{\delta_{N,8}}{8} + \frac{\delta_{N,9}}{9} + \frac{\delta_{N,10}}{10}. \end{aligned} \quad (183)$$

For $N = 4800$ this gives the following ‘‘almost linear’’ relation:

$$\mu_{4800,10} = -1.8882633 \dots \cdot 10^{-1212}, \quad (184)$$

$$\nu_{4800,10} = -5.3237731 \dots \cdot 10^{-1249}. \quad (185)$$

Case $n = 11$. This case is similar to the cases $n = 3, 5$, and 7 but requires even larger N than it was in the cases $n = 9$ and 10 . We have:

$$\frac{-11\nu_{8000,10}}{\mu_{8000,11,1}} = 1 - 9.1921885 \dots \cdot 10^{-82}. \quad (186)$$

So we put $D_{11} = 1$ and define respectively

$$\mu_{N,11} = \delta_{N,11} - \mu_{N,1} \quad (187)$$

$$= -\delta_{N,1} + \delta_{N,11}, \quad (188)$$

$$\begin{aligned} \nu_{N,11} &= -\frac{\delta_{N,1}}{2310} - \frac{\delta_{N,2}}{60} + \frac{\delta_{N,3}}{18} + \frac{\delta_{N,4}}{8} + \frac{\delta_{N,5}}{10} \\ &\quad + \frac{\delta_{N,6}}{6} + \frac{\delta_{N,7}}{7} + \frac{\delta_{N,8}}{8} + \frac{\delta_{N,9}}{9} + \frac{\delta_{N,10}}{10} \\ &\quad + \frac{\delta_{N,11}}{11}. \end{aligned} \quad (189)$$

For $N = 8000$ this gives the following ‘‘almost linear’’ relations:

$$\mu_{8000,11} = 3.7096353 \dots \cdot 10^{-1588}, \quad (190)$$

$$\nu_{8000,11} = -3.0999697 \dots \cdot 10^{-1670}. \quad (191)$$

Case $n = 12$. This case is similar to the cases $n = 6, 8$ and 10 but is twice as long. We wish the value of $\mu_{3200,12}$ to be close to

$$-12\nu_{3200,11} = 1.0704335 \dots \cdot 10^{-738}. \quad (192)$$

We observe that

$$\mu_{3200,12,1} = \delta_{3200,1} - \mu_{3200,1} \quad (193)$$

$$= -1.9999999 \dots \cdot 10^0 \quad (194)$$

is very close to the value of $\mu_{3200,2}$, namely,

$$\frac{\mu_{3200,2}}{\mu_{3200,12,1}} = 1 - 2.2530460 \dots \cdot 10^{-134}. \quad (195)$$

So we put $k(12, 2) = 2$ and now observe that

$$\mu_{3200,12,2} = \mu_{3200,12,1} - \mu_{3200,2} \quad (196)$$

$$= 4.5060920 \dots \cdot 10^{-134} \quad (197)$$

is very close to the value of $\mu_{3200,3}$, namely,

$$\frac{\mu_{3200,3}}{\mu_{3200,12,2}} = 1 - 2.6163259 \dots \cdot 10^{-439}. \quad (198)$$

So we put $k(12, 3) = 3$ and now observe that

$$\mu_{3200,12,3} = \mu_{3200,12,2} - \mu_{3200,3} \quad (199)$$

$$= 1.0843810 \dots \cdot 10^{-304} \quad (200)$$

is very close to the value of $\mu_{3200,4}$, namely,

$$\frac{\mu_{3200,4}}{\mu_{3200,12,3}} = 1 - 7.9483409 \dots \cdot 10^{-651}. \quad (201)$$

So we put $k(12, 4) = 4$ and now observe that

$$\mu_{3200,12,4} = \mu_{3200,12,3} - \mu_{3200,4} \quad (202)$$

$$= 1.7828801 \dots \cdot 10^{-547} \quad (203)$$

is very close to the value of $\mu_{3200,6}$, namely,

$$\frac{\mu_{3200,6}}{\mu_{3200,12,4}} = 1 - 1.3949963 \dots \cdot 10^{-267}. \quad (204)$$

So we put $k(12, 5) = 6$ and now see that

$$\mu_{3200,12,5} = \mu_{3200,12,4} - \mu_{3200,6} \quad (205)$$

$$= -2.4871113 \dots \cdot 10^{-814} \quad (206)$$

is much smaller than (192). We need once again increase N . We have

$$\frac{-12\nu_{8000,11}}{\mu_{8000,12,5}} = 1 - 1.7308919\dots \cdot 10^{-68}, \quad (207)$$

so we put $D_{12} = 5$ and define respectively

$$\mu_{N,12} = \delta_{N,12} - \mu_{N,1} - \mu_{N,2} - \mu_{N,3} - \mu_{N,4} - \mu_{N,6} \quad (208)$$

$$= \delta_{N,2} - \delta_{N,4} - \delta_{N,6} + \delta_{N,12}, \quad (209)$$

$$\begin{aligned} \nu_{N,12} = & -\frac{\delta_{N,1}}{2310} + \frac{\delta_{N,2}}{15} + \frac{\delta_{N,3}}{18} + \frac{\delta_{N,4}}{24} + \frac{\delta_{N,5}}{10} \\ & + \frac{\delta_{N,6}}{12} + \frac{\delta_{N,7}}{7} + \frac{\delta_{N,8}}{8} + \frac{\delta_{N,9}}{9} + \frac{\delta_{N,10}}{10} \\ & + \frac{\delta_{N,11}}{11} + \frac{\delta_{N,12}}{12}. \end{aligned} \quad (210)$$

For $N = 8000$ this gives the following “almost linear” relations:

$$\mu_{8000,12} = 3.7199637\dots \cdot 10^{-1669}, \quad (211)$$

$$\nu_{8000,12} = 5.3657127\dots \cdot 10^{-1738}. \quad (212)$$

From (87), (91), (100), (112), (118), (133), (139), (154), (166), (181), (187), and (208) we can guess the general pattern: D_n is 1 less than the number of divisors of n , and $k(n, 1), \dots, k(n, D_n)$ are all the divisors except n itself. Isn't striking that we came to these values just by comparing certain linear combinations of the numbers $\delta_{N,n}$?

Respectively, by the Möbius inversion formula, (92), (101), (113), (119), (134), (140), (155), (167), (182), (188), and (209) are just special cases of

$$\mu_{N,n} = \sum_{k|n} \mu\left(\frac{n}{k}\right) \delta_{N,k} \quad (213)$$

where $\mu(m)$ is the Möbius function.

In its turn, (87), (93), (102), (114), (120), (135), (141), (156), (168), (183), (189), and (210) are particular cases of

$$\nu_{N,n} = \sum_{k=1}^n \left(\frac{1}{k} \sum_{m=1}^{n/k} \frac{\mu(m)}{m} \right) \delta_{N,k}. \quad (214)$$

We saw three “typical” behaviors of the normalized coefficients $\delta_{N,n}$:

- “logarithmic” for “small” N like on Figure 5,

- “parallel” for “medium” N like on Figures 12,
- “normally distributed” for larger N like on Figure 17.

But is $N = 12000$ big enough to allow us make predictions about asymptotic behavior of the δ 's? Cannot the pictures change once (or even many times) again? Table 1 suggests that this might happen. The values of $\mu_{N,2} = \delta_{N,2} - 1$ were at first approaching -2 , then began to grow up rapidly in absolute value, then began to decrease, and it remains unclear what will be the limiting value of $\mu_{N,2}$ (and respectively of $\delta_{N,2}$) if such a limit exists at all. The values of $\mu_{N,5} = \delta_{N,5} - 1$ were at first approaching 0 , then began to approach -1 , so will -1 be the limiting value of $\mu_{N,5}$ or for bigger N the values of $\mu_{N,5}$ might began to grow up in absolute value as $\mu_{N,2}$ did? It would be very interesting to continue calculations for $N > 12000$ but this exceeds the computational resources I have at my disposal at present, international cooperation seems to be necessary.

Acknowledgements

This paper is based on intensive computer calculations.

The multiprecision values of the zeroes of the zeta function were computed on Mathematica and Sage. The author is grateful to Oleksandr Pavlyk, special functions developer at Wolfram Research, for performing part of the calculations, and to Dmitrii Pasechnik (NTU Singapore) for hand-on help with setting up and monitoring usage of Sage.

The author expresses his gratitude for the computational resources used to calculate these zeroes to:

- ArmNGI (Armenian National Grid Initiative Foundation),
- Isaac Newton Institute for Mathematical Sciences, UK,
- LACL (Laboratoire d'Algorithmique, Complexité et Logique de Université Paris-Est Créteil),
- LIAFA (Laboratoire d'Informatique Algorithmique: Fondements et Applications, supported jointly by the French National Center for Scientific Research (CNRS) and by the University Paris Diderot - Paris 7),
- SPIIRAS (St.Petersburg Institute for Informatics and Automation of RAS),
- Wolfram Research.

Most time-consuming part, calculation of the determinants (58), was performed on “Chebyshev” supercomputer of Moscow State University Supercomputing Center.

The author thanks Isaac Newton Institute for Mathematical Sciences also for the possibility to participate in the Programm “Semantics and Syntax: A Legacy of Alan Turing” and for financial support of author’s visits to Oxford [4] and Leicester [3].

The author thanks Jeremy Levesley for the careful reading of the manuscript, correcting English and a number of small errors.

References

- [1] I. Hacking. *Representing and Intervening*. Cambridge University Press, 1983. ISBN 0521282462.
- [2] Yu. Matiyasevich. *An artless method for calculating approximate values of zeros of Riemann’s zeta function*. WWW site of ongoing research. <http://logic.pdmi.ras.ru/~yumat/personaljournal/artlessmethod/artlessmethod.php>.
- [3] Yu. Matiyasevich. *New Conjectures about Zeroes Riemann’s zeta function*. Talk given at the Department of Mathematics of University of Leicester, UK, on June 18, 2012. http://localhost/public_html/personaljournal/artlessmethod/artlessmethod.php#leicester2012.
- [4] Yu. Matiyasevich. *New Conjectures about Zeroes Riemann’s zeta function*. Talk given at the Mathematical Institute of the University of Oxford on January 26, 2012. http://localhost/public_html/personaljournal/artlessmethod/artlessmethod.php#oxford2012.
- [5] B. Riemann. Über die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatsberichte der Berliner Akademie*, 1859. Included into: Riemann, B. *Gesammelte Werke*. Teubner, Leipzig, 1892; reprinted by Dover Books, New York, 1953. http://www.claymath.org/millennium/Riemann_Hypothesis/1859_manuscript/zeta.pdf, English translation <http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta/EZeta.pdf>.

- [6] H. von Mangoldt. Zu Riemanns Abhandlung 'Über die Anzahl der Primzahlen unter einer gegebenen Grösse'. *J. Reine Angew. Math.*, 114:255–305, 1895.