SETS OF UNIT VECTORS WITH SMALL SUBSET SUMS

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ABSTRACT. We say that a family $\{\boldsymbol{x}_i \mid i \in [m]\}$ of vectors in a Banach space X satisfies the *k-collapsing condition* if $\|\sum_{i\in I} \boldsymbol{x}_i\| \leq 1$ for all k-element subsets $I \subseteq \{1,2,\dots,m\}$. Let $\overline{\mathscr{C}}(k,d)$ denote the maximum cardinality of a k-collapsing family of unit vectors in a d-dimensional Banach space, where the maximum is taken over all spaces of dimension d. Similarly, let $\overline{\mathscr{C}\mathscr{B}}(k,d)$ denote the maximum cardinality if we require in addition that $\sum_{i=1}^m \boldsymbol{x}_i = \boldsymbol{o}$. The case $k=2$ was considered by Füredi, Lagarias and Morgan (1991). These conditions originate in a theorem of Lawlor and Morgan (1994) on geometric shortest networks in smooth finite-dimensional Banach spaces. We show that $\overline{\mathscr{C}\mathscr{B}}(k,d) = \max\{k+1, 2d\}$ for all $k,d \geq 2$. The behaviour of $\overline{\mathscr{C}}(k,d)$ is not as simple, and we derive various upper and lower bounds for different ranges of k and d. These include the exact values $\overline{\mathscr{C}}(k,d) = \max\{k+1,2d\}$ in various cases.

We use a variety of tools from graph theory, convexity and linear algebra in the proofs: in particular the Hajnal-Szemerédi Theorem, the Brunn-Minkowski inequality, lower bounds for the rank of a perturbation of the identity matrix.

CONTENTS

1. INTRODUCTION

Let $[n]$ denote the set $\{1,2\dots,n\}$ and $\binom{S}{k}$ the set $\{A \subseteq S \mid |A| = k\}$. Let $d \geq 2$ and $m > k \geq 2$ be integers. Let $X = X^d$ denote a d-dimensional real Banach space

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with norm $\|\cdot\|$. Throughout the paper we use the term *Minkowski space* for finite-dimensional real Banach space.

Definition 1. *A family* $\{\boldsymbol{x}_i \mid i \in [m]\}$ *of* m (*not necessarily distinct*) *vectors in some Minkowski space* X *satisfies the* k-collapsing condition *if*

$$\Big\|\sum_{i\in I}\boldsymbol{x}_i\Big\| \leq 1 \quad \text{for all } I \in \binom{[m]}{k},$$

the full collapsing condition

$$\Big\|\sum_{i\in I}\boldsymbol{x}_i\Big\| \leq 1 \quad \text{for all } I \subseteq [m],$$

the strong balancing condition *if*

$$\sum_{i=1}^{m}\boldsymbol{x}_i = \boldsymbol{o},$$

and the weak balancing condition *if*

$$\boldsymbol{o} \text{ is in the relative interior of } \operatorname{conv}\{\boldsymbol{x}_i \mid i \in [m]\}.$$

In this paper we study the k-collapsing condition with or without the strong balancing condition. In previous work by Füredi, Lagarias, Morgan, Lawlor and the present author [13, 24, 32, 33] the full collapsing condition and the 2-collapsing condition with or without the strong or the weak balancing condition were considered. In Section 1.1 below we survey these previous results in order to sketch a context for the work presented in this paper. New results are summarised in Section 1.2. The remainder of this paper is then given an overview in Section 1.3.

Notation. Denote the closed ball with centre $\boldsymbol{c}$ and radius r by

$$B(\boldsymbol{c}, r) = \{\boldsymbol{x} \in X \mid \|\boldsymbol{x}-\boldsymbol{c}\| \leq r\}.$$

The *unit ball* of X is $B_X := B(\boldsymbol{o}, 1)$. Denote the dual of X by X^*. The elements of X^* are the (continuous) *linear functionals* over X, that is, linear functions

$$\boldsymbol{x}^*\colon X \to \mathbb{R}, \quad \boldsymbol{x} \mapsto \langle \boldsymbol{x}^*, \boldsymbol{x}\rangle,$$

with norm

$$\|\boldsymbol{x}^*\|^* := \sup\{\langle \boldsymbol{x}^*, \boldsymbol{x}\rangle \mid \boldsymbol{x} \in B_X\}.$$

Any $\boldsymbol{x} \in X \setminus \{\boldsymbol{o}\}$ has a *dual unit vector*: a functional $\boldsymbol{x}^* \in X^*$ such that $\|\boldsymbol{x}^*\|^* = 1$ and $\langle \boldsymbol{x}^*, \boldsymbol{x}\rangle = \|\boldsymbol{x}\|$. It is well-known that if the norm of a finite-dimensional X is *smooth*, that is, if $\|\cdot\|$ is differentiable on $X \setminus \{\boldsymbol{o}\}$, then X^* is strictly convex, that is, the boundary of B_{X^*} does not contain a line segment. Also, if X is strictly convex, then X^* is smooth. Recall that a space is smooth iff any $\boldsymbol{x} \in X \setminus \{\boldsymbol{o}\}$ has a unique dual unit vector.

Let $p \in (1, \infty)$. The space $\mathbb{R}^d$ with the norm

$$\|(x_1, x_2, \dots, x_d)\|_p := \Big(\sum_{i=1}^{d}|x_i|^p\Big)^{1/p}$$

is denoted by ℓ_p^d and the space $\mathbb{R}^d$ with the norm

$$\|(x_1,x_2,\dots,x_d)\|_\infty := \max\{|x_i| \mid i\in[d]\}$$

by ℓ_∞^d.

1.1. **Previous work.** The conditions in Definition 1 occur in a theorem of Lawlor and Morgan [24] on geometric shortest networks in smooth Minkowski spaces.

Given a family $N=\{\boldsymbol{p}_i \mid i\in[n]\}$ of points in a Minkowski space X, a *Steiner tree* is a (finite) tree $T=(V,E)$ such that $N\subseteq V\subset X$. The points in $V\setminus N$ (if any) are called the *Steiner points* of T. The *length* $\ell(T)$ of a tree is the sum $\sum_{\boldsymbol{xy}\in E}\|\boldsymbol{x}-\boldsymbol{y}\|$ of the edge lengths. A *Steiner minimal tree* of N is a Steiner tree of N that minimises $\ell(T)$. By a compactness argument [8] any finite family of points in a Minkowski space has at least one Steiner minimal tree. The following theorem characterises the edges that are incident to a Steiner point of a Steiner minimal tree when the underlying Minkowski space is smooth.

Theorem 2 (Lawlor and Morgan [24])**.** *Let $N=\{\boldsymbol{p}_i \mid i\in[n]\}$ be a family of points, all different from the origin $\boldsymbol{o}$, in a smooth Minkowski space X. Let $\boldsymbol{p}_i^*$ be the dual unit vector of $\boldsymbol{p}_i$, $i\in[n]$. Then the Steiner tree that joins $\boldsymbol{o}$ to each $\boldsymbol{p}_i$ by straight-line segments is a Steiner minimal tree of N if and only if the family $\{\boldsymbol{p}_i^* \mid i\in[n]\}$ satisfies the full collapsing condition and the strong balancing condition in the dual space X^*.*

Since the dual of a smooth Minkowski space is strictly convex, a natural problem suggested by Theorem 2 is to find an upper bound on the cardinality of a family of unit vectors satisfying the full collapsing and strong balancing conditions in a strictly convex Minkowski space.

Theorem 3 (Lawlor and Morgan [24])**.** *Let $N=\{\boldsymbol{x}_i \mid i\in[n]\}$ be a family of unit vectors satisfying the full collapsing condition and the strong balancing condition in a d-dimensional strictly convex Minkowski space. Then $n\le d+1$.*

Combined with Theorem 2 this implies that the degree of a Steiner point in any Steiner minimal tree in a d-dimensional smooth Minkowski space is bounded above by $d+1$.

The following theorem characterises the edges incident to an arbitrary point of a Steiner minimal tree in a smooth Minkowski space. Observe that if $\boldsymbol{p}$ is a Steiner point of a Steiner minimal tree $T=(V,E)$ of the point family N, then T is still a Steiner minimal tree of $N\cup\{\boldsymbol{p}\}$ (but with $\boldsymbol{p}$ not a Steiner point anymore). Therefore, the condition in this characterisation should be logically weaker than the characterisation appearing in Theorem 2, and it turns out that the full balancing condition has to be dropped.

Theorem 4 ([33])**.** *Let $N=\{\boldsymbol{p}_i \mid i\in[n]\}$ be a family of points, all different from the origin $\boldsymbol{o}$, in a smooth Minkowski space X. Let $\boldsymbol{p}_i^*$ be the dual unit vector of $\boldsymbol{p}_i$, $i\in[n]$. Then the Steiner tree that joins $\boldsymbol{o}$ to each $\boldsymbol{p}_i$ by straight-line segments is a Steiner minimal tree of $N\cup\{\boldsymbol{o}\}$ if and only if the family $\{\boldsymbol{p}_i^* \mid i\in[n]\}$ satisfies the full collapsing condition in the dual space X^*.*

The following is a strengthening of Theorem 3:

Theorem 5 ([33]). *Let $N = \{\boldsymbol{x}_i \,|\, i \in [n]\}$ be a family of unit vectors in a d-dimensional strictly convex Minkowski space satisfying the strong collapsing condition. Then $n \leq d+1$.*

Therefore, all points in Steiner minimal tree in a smooth d-dimensional Minkowski space have degree at most $d+1$. Generalising Theorems 2 and 4 to non-smooth Minkowski spaces is much more involved. There the degrees of Steiner points can be as large as 2^d; see [36] for a further discussion. We now leave the original motivation of Steiner minimal trees behind and continue to survey previous work on the various collapsing and balancing conditions.

After the paper of Lawlor and Morgan [24], Füredi, Lagarias and Morgan [13] used classical combinatorial convexity to study these conditions. They showed the following.

Theorem 6 (Füredi, Lagarias and Morgan [13]). *Let $N = \{\boldsymbol{x}_i \,|\, i \in [n]\}$ be a family of unit vectors in a d-dimensional Minkowski space X satisfying the 2-collapsing and weak balancing conditions. Then $n \leq 2d$, with equality only if N consists of a basis of X and its negative.*

They also mention without proof that if N is a family of $2d$ unit vectors in a d-dimensional Minkowski space satisfying the full collapsing and the strong balancing condition, then the space is isometric to ℓ_∞^d. We extend the above theorem to the k-collapsing condition, but with the strong balancing condition instead of the weak one (Theorem 17), and with a completely different proof.

For strictly convex norms Füredi, Lagarias and Morgan [13] obtained the following stronger conclusion (thus weakening the hypotheses of Theorem 3 in a different way from Theorem 5).

Theorem 7 (Füredi, Lagarias and Morgan [13]). *Let $N = \{\boldsymbol{x}_i \,|\, i \in [n]\}$ be a family of unit vectors in a d-dimensional strictly convex Minkowski space satisfying the 2-collapsing condition and the weak balancing condition. Then $n \leq d+1$.*

Without any balancing condition or condition on the norm, they showed the following:

Theorem 8 (Füredi, Lagarias and Morgan [13]). *Let $N = \{\boldsymbol{x}_i \,|\, i \in [n]\}$ be a family of unit vectors in a d-dimensional Minkowski space X satisfying the 2-collapsing condition. Then $n \leq 3^d - 1$.*

This exponential behaviour for the 2-collapsing condition without any balancing condition is necessary:

Theorem 9 (Füredi, Lagarias and Morgan [13]). *For each sufficiently large $d \in \mathbb{N}$ there exists a strictly convex and smooth d-dimensional Minkowski space with a family N of at least 1.02^d unit vectors that satisfies the following strengthened 2-collapsing condition: $\|\boldsymbol{x}+\boldsymbol{y}\| < 1$ for all $\{\boldsymbol{x},\boldsymbol{y}\} \in \binom{N}{2}$.*

We construct similar exponential lower bounds for the k-collapsing condition (Theorem 29).

Using the Brunn-Minkowski inequality we improved the upper bound of Theorem 8 as follows.

Theorem 10 ([32]). *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors in a d-dimensional Minkowski space X satisfying the* 2*-collapsing condition. Then $n \leq 2^{d+1} + 1$.*

In this paper we combine the Brunn-Minkowski inequality with the Hajnal-Szemerédi Theorem from graph theory to extend the above theorem to the k-collapsing condition (Theorem 27). In [13] it was asked whether there is an upper bound polynomial in d for the size of a collection of unit vectors in a d-dimensional Minkowski space satisfying the strong collapsing condition without any balancing condition. This was subsequently answered as follows:

Theorem 11 ([32]). *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors in a d-dimensional Minkowski space X satisfying the strong collapsing condition. Then $n \leq 2d$, with equality if and only if X is isometric to ℓ_∞^d, with N corresponding to $\{\pm \boldsymbol{e}_i \mid i \in [d]\}$ under any isometry.*

The analogous theorem for the strictly convex case is as follows:

Theorem 12 ([33]). *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors in a d-dimensional strictly convex Minkowski space X satisfying the full collapsing condition. Then $n \leq d+1$. If, in addition, the balancing condition is not satisfied then $n \leq d$.*

The full collapsing condition is closely connected to certain notions from the local theory of Banach spaces. The *absolutely summing constant* or the 1*-summing constant* $\pi_1(X)$ of a Minkowski space X is defined to be the infimum of all $c > 0$ satisfying

$$\sum_{i=1}^{m} \|\boldsymbol{x}_i\| \leq c \max_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^{m} \varepsilon_i \boldsymbol{x}_i \Big\|$$

where $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in X$. It is clear that $2\pi_1(X)$ is an upper bound to the number of unit vectors that satisfy the full collapsing condition. Deschaseaux [10] showed that $\pi_1(X) \leq d$ with equality iff X is isometric to ℓ_∞^d. This gives another proof of Theorem 11, apart from the characterisation of N. Franchetti and Votruba [12] showed that if X is 2-dimensional then $2\pi_1(X)$ equals the perimeter of the unit circle. By a result of Gołąb [26], the perimeter of the unit circle is less than 4 unless X is isometric to ℓ_∞^2. This implies the 2-dimensional case of Deschaseaux's theorem.

For $q \geq 2$, the cotype q constant $\kappa_q(X)$ of a Minkowski space X is defined to be the infimum of all $c > 0$ such that

$$\Big(\sum_{i=1}^{m} \|\boldsymbol{x}_i\|^q \Big)^{1/q} \leq c \operatorname*{avg}_{\varepsilon_i = \pm 1} \Big(\Big\| \sum_{i=1}^{m} \varepsilon_i \boldsymbol{x}_i \Big\|^2 \Big)^{1/2}$$

where $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in X$. It is again straightforward that $(2\kappa_q(X))^q$ is an upper bound for the number of vectors satisfying the full collapsing condition. For instance, bounds on the cotype 2 constants for ℓ_p^d (essentially consequences of the Khinchin inequalities) gives upper bounds independent on the dimension for fixed $p \in [1, \infty)$. Details may be found in [33].

A more general question was asked by Sidorenko and Stechkin [30, 31] and Katona and others [18, 19, 20, 21, 22], where the '≤ 1' in the collapsing conditions is replaced by '$\leq \delta$' or '$< \delta$'. In this direction work was done in the previously cited papers as well as [34]. We do not pursue this generalisation here, instead leaving it for a later investigation, as it will be seen that the arguments in this paper are already quite involved.

1.2. **Overview of new results.** In this paper we only consider the k-collapsing condition and strong balancing condition.

Definition 13. *For any $k \geq 2$, define $\mathscr{C}_k(X)$ to be the largest m such that a family of m vectors in X of norm at least 1 exist satisfying the k-collapsing condition. Also, define $\mathscr{C}\mathscr{B}_k(X)$ to be the largest m such that a family of m vectors in X of norm at least 1 exist satisfying the k-collapsing condition and the strong balancing condition.*

Next define the numbers

$$\overline{\mathscr{C}}(k,d) := \max\left\{\mathscr{C}_k(X^d) \,\middle|\, X^d \text{ is a } d\text{-dimensional Minkowski space}\right\},$$

$$\underline{\mathscr{C}}(k,d) := \min\left\{\mathscr{C}_k(X^d) \,\middle|\, X^d \text{ is a } d\text{-dimensional Minkowski space}\right\},$$

$$\overline{\mathscr{C}\mathscr{B}}(k,d) := \max\left\{\mathscr{C}\mathscr{B}_k(X^d) \,\middle|\, X^d \text{ is a } d\text{-dimensional Minkowski space}\right\},$$

$$\underline{\mathscr{C}\mathscr{B}}(k,d) := \min\left\{\mathscr{C}\mathscr{B}_k(X^d) \,\middle|\, X^d \text{ is a } d\text{-dimensional Minkowski space}\right\}.$$

It is not difficult to see that $\overline{\mathscr{C}}(k,d)$ and $\overline{\mathscr{C}\mathscr{B}}(k,d)$ are always finite. Although the vectors occurring in Theorems 2 to 12 are unit vectors, we weaken this to vectors of norm at least 1 in the above definition. Indeed it turns out that the quantities $\overline{\mathscr{C}}(k,d)$ and $\overline{\mathscr{C}\mathscr{B}}(k,d)$ stay exactly the same whether we require the vectors to be unit vectors or of norm ≥ 1. See Corollary 38 in Section 5 for this non-trivial fact.

Since we have assumed $d \geq 2$, as already mentioned, it follows that for any value of $k \geq 2$ there exist $k+1$ unit vectors that satisfy the strong balancing condition, hence also the k-collapsing condition. Therefore, $\mathscr{C}_k(X^d) \geq \mathscr{C}\mathscr{B}_k(X^d) \geq k+1$ for any d-dimensional X^d, as long as $k \geq 2$ and $d \geq 2$. In Section 2 we show that these inequalities cannot be improved in general:

Proposition 14. $\mathscr{C}_k(\ell_2^d) = \mathscr{C}\mathscr{B}_k(\ell_2^d) = k+1$ *for any $k \geq 2$ and $d \geq 2$.*

Consequently,

Corollary 15. $\underline{\mathscr{C}}(k,d) = \underline{\mathscr{C}\mathscr{B}}(k,d) = k+1$ *for all $k, d \geq 2$.*

The family of d unit vectors and their negatives $\{\pm \boldsymbol{e}_1, \dots, \pm \boldsymbol{e}_d\}$ shows that

$$\mathscr{C}_k(\ell_\infty^d) \geq \mathscr{C}\mathscr{B}_k(\ell_\infty^d) \geq 2d$$

for all $d \geq 2$ and $k \geq 2$. Therefore, $\overline{\mathscr{C}}(k,d) \geq \overline{\mathscr{C}}(k,d) \geq \max\{k+1, 2d\}$. In Section 2 we show the following:

Proposition 16. *For any $k \geq 2$ and $d \geq 2$,*

$$\mathscr{C}_k(\ell_\infty^d) = \mathscr{C}\mathscr{B}_k(\ell_\infty^d) = \max\{k+1, 2d\}.$$

It turns out that this is the extremal case for the quantity $\overline{\mathscr{CB}}(k,d)$.

Theorem 17. *For any $k \geq 2$ and $d \geq 2$,*

$$\overline{\mathscr{CB}}(k,d) = \max\{k+1, 2d\}.$$

For each $d \geq 3$ and $k \leq 2d-2$ there exist infinitely many d-dimensional X^d such that $\mathscr{CB}_k(X^d) = 2d$. (For $d = 2$ and $k = 2$, the only space satisfying $CB_2(X^2) = 4$ is $X^2 = \ell_\infty^2$ up to isometry.)

If $d \geq 2$, $2 \leq k \leq 2d-2$ and $\mathscr{CB}_k(X^d) = 2d$, then any set of $2d$ vectors of norm at least 1 satisfying the k-collapsing and strong balancing conditions are necessarily unit vectors consisting of a basis of X^d and its negative.

The proof uses a reduction to $m \times m$ matrices which are in a weak sense perturbations of the identity matrix, together with results on lower bounds of the ranks of such matrices [1, 4]. In order to apply these lower bounds we also have to solve a certain convex optimization problem.

Conjecture 18. *If X^d is a strictly convex d-dimensional Minkowski space then*

$$\overline{\mathscr{CB}}_k(X^d) \leq \max\{k+1, d+1\}.$$

This conjecture is true for $k = 2$ [13]. Also, for each $d \geq 2$ there exists a strictly convex d-dimensional space with $d+1$ unit vectors satisfying the strong collapsing condition. Thus this conjecture would give the best possible estimate if true.

Question 19. *Can the strong balancing condition in Theorem 17 be replaced by the weak balancing condition?*

Again, we know that the answer is yes when $k = 2$ [13].

Estimating $\overline{\mathscr{C}}(k,d)$ is much harder. The same proof techniques work up to a certain extent and the details become much trickier.

Theorem 20. *For $k \geq 2$ let γ_k be the unique (positive) solution to*

$$(1+x)^{1/x}\left(1+\frac{1}{x}\right) = k^2.$$

Then $\mathrm{e}/k^2 < \gamma_k < \mathrm{e}/(k^2 - \mathrm{e})$ and

$$\overline{\mathscr{C}}(k,d) < 1.33 k^{2\gamma_k d + 2}.$$

If $k < \sqrt{d}$ then

$$\overline{\mathscr{C}}(k,d) < \frac{k}{\sqrt{d}} k^{2\gamma_k d + 2}.$$

In particular, if $k = c\sqrt{d}$ with $c < 1$, then $\overline{\mathscr{C}}(k,d) = O(d^{1+\mathrm{e}/c^2})$ as $d \to \infty$.

See Table 1 for the first few values of γ_k. The next theorem gives a slightly sharper result for k a small multiple of $\sqrt{d}$. See also the lower bound of Theorem 30 below.

k	γ_k	$k^{2\gamma_k d}$	$(1+\frac{2}{k})^d$	$\left(1+\frac{1}{2(2k+1)^2}\right)^d$
2	1	4^d	2^d	1.02^d
3	0.3541686	2.178^d	1.667^d	1.0102^d
4	0.1854203	1.672^d	1.5^d	1.0062^d
5	0.1149225	1.448^d	1.4^d	1.0041^d
6	0.0784510	1.325^d	1.334^d	1.0029^d
7	0.0570503	1.249^d	1.286^d	1.0022^d
8	0.0433914	1.198^d	1.25^d	1.0017^d
9	0.0341301	1.162^d	1.223^d	1.0013^d

TABLE 1. Values of γ_k with upper bounds of Theorems 20 and 27 and the lower bound of Theorem 29. The values of γ_k are rounded to the nearest decimal, of $k^{2\gamma_k}$ and $1+2/k$ are rounded up and of $1+1/(2(2k+1)^2)$ are rounded down.

Theorem 21. *For any $\varepsilon > 0$ and $p \in \mathbb{N}$, $p \geq 2$ there exists d_0 and $c > 0$ such that for all $d > d_0$, if*

$$\left((p!)^{-1/(2p)} + \varepsilon\right)\sqrt{d} < k \leq \sqrt{d}$$

then $\overline{\mathscr{C}}(k,d) < cd^p$.

For larger k we obtain almost optimal results. In particular, we obtain the exact result $\overline{\mathscr{C}}(k,d) = 2d$ for $(\sqrt{6}-2)d + O(1) < k < 2d - \sqrt{d/2}$.

Theorem 22. *Let $k \geq 3$ and $d \geq 2$.*

(1) *If $\sqrt{d} < k \leq \frac{d+1}{2}$ then $\overline{\mathscr{C}}(k,d) \leq \frac{2d(k-1)^2}{k^2-d} = 2d\left(1+\frac{d-2k+1}{k^2-d}\right)$.*
(2) *If $-2d+\sqrt{6d^2+3d+1} \leq k \leq 2d-\sqrt{d/2}$ then $\overline{\mathscr{C}}(k,d) = 2d$.*
(3) *If $d \geq 3$ and $k > 2d - \sqrt{d/2}$ then $\overline{\mathscr{C}}(k,d) \leq k + \frac{1+\sqrt{2d-3}}{2}$.*

For values of d up to 7 as $k \to \infty$ the same methods as used in proving Theorems 17, 20, 21 and 22 give the following exact values.

Theorem 23. *$\overline{\mathscr{C}}(k,d) = \max\{k+1, 2d\}$ in the following cases:*

(1) *$d = 2$ and $k \geq 2$,*
(2) *$d \in \{3,4,5\}$ and $k \geq 3$,*
(3) *$d = 6$ and $k \in \{3,\dots,10\} \cup \{17,\dots\}$,*
(4) *$d = 7$ and $k \in \{3,\dots,12\} \cup \{41,\dots\}$.*

The proof method used for the above theorem gives no information for $d \geq 8$ and k large. (The estimate $\overline{\mathscr{C}}(2,3) \leq 9$ is also obtained in the proof.) For arbitrary d, as long as k is large, we obtain the following using a completely different technique.

Theorem 24. *If $k \gg d^{d+2}$ then $\overline{\mathscr{C}}(k,d) = k+1$.*

The proof of this theorem uses geometric tools from convexity, in particular the Brunn-Minkowski inequality and the theorem of Carathéodory. The estimate

$\Omega(d^{d+2})$ is most likely not best possible, but we need at least $k \geq 2d-1$ for the conclusion of this theorem to hold, as shown by the example of $k \leq 2d-2$ and the family $\{\pm \boldsymbol{e}_i \mid i \in [d]\}$ in ℓ_∞^d.

Conjecture 25. *$\overline{\mathscr{C}}(k,d) = k+1$ whenever $k \geq 2d-1$.*

By Theorem 23 the above conjecture holds for $d \leq 5$. The next conjecture has non-empty content for $d \geq 8$.

Conjecture 26. *$\overline{\mathscr{C}}(k,d) = 2d$ if $2d - \sqrt{d/2} \leq k \leq 2d-2$.*

Since Theorem 22 gives $\overline{\mathscr{C}}(k,d) = 2d$ for $(\sqrt{6}-2)d + O(1) < k < 2d - \sqrt{d/2}$, it is likely that the bound in Conjecture 26 already holds for some range of $k < 0.45d$. On the other hand, as implied by Theorem 30 below, we need at least $k > (\frac{1}{2} + o(1))\sqrt{d}$.

We show the following upper bound using a method closely related to the proof of Theorem 24. We still use the Brunn-Minkowski inequality, but combine it the Hajnal-Szemerédi theorem from graph theory:

Theorem 27. *For any $k,d \geq 2$, $\overline{\mathscr{C}}(k,d) \leq k(1+\frac{2}{k})^d + k - 1$.*

Asymptotically for fixed k as $d \to \infty$, this bound is better when $k \leq 5$ while for $k \geq 6$ Theorem 20 is better. See Table 1 for a comparison between the upper bounds given by Theorem 20 and Theorem 27 for $k = 2, \dots, 8$.

Related to Proposition 14 is the following result on spaces that are close to Euclidean space. Denote the (multiplicative) *Banach-Mazur distance* between two Minkowski spaces X and Y of the same dimension by $d_{\mathrm{BM}}(X,Y)$.

Proposition 28. *Let $D = d_{\mathrm{BM}}(X^d, \ell_2^d)$. Then for any $k > D^2$,*

$$\mathscr{C}_k(X^d) \leq \frac{k^2 - D^2}{k - D^2} = k + D^2 + \frac{D^4 - D^2}{k - D^2}.$$

Its proof is at the end of Section 2. By John's theorem (see [15] for a modern account), $d_{\mathrm{BM}}(X^d, \ell_2^d) \leq \sqrt{d}$, from which follows $\mathscr{C}_k(X^d) \leq k + d + \frac{d^2-d}{k-d}$ if $k > d$. This estimate is worse, however, than the estimates of Theorems 22 and 23 for all $k > d$. On the other hand, if $D = d_{\mathrm{BM}}(X, \ell_2^d)$ is sufficiently small, then Proposition 28 may give bounds better than Theorems 22. In particular, Proposition 28 is better than Theorem 22 in the case $d < k \leq 2d - \sqrt{d/2}$ if $d_{\mathrm{BM}}(X, \ell_2^d) \leq \sqrt{\frac{(2d-k)k}{2d-1}}$, and in the case $k > 2d - \sqrt{d/2}$ if $d_{\mathrm{BM}}(X, \ell_2^d) \leq (d/2)^{1/4}$.

We now turn to lower bounds. The first, generalising Theorem 9, uses a simple greedy construction of sets of Euclidean unit vectors that are almost orthogonal.

Theorem 29. *For all $k \geq 2$ and sufficiently large d depending on k, there exists a strictly convex and smooth d-dimensional Minkowski space X^d such that*

$$\mathscr{C}_k(X^d) \geq \left(1 + \frac{1}{2(2k+1)^2}\right)^d.$$

The proof will in fact give a norm that is C^∞ away from the origin. This lower bound for $\overline{\mathscr{C}}(k,d)$ almost matches the upper bound from Theorem 20 asymptotically in the sense that as $k\to\infty$ and $d \ggg \log k$ it implies that $\overline{\mathscr{C}}(k,d)^{1/d}-1 \gg 1/k^2$, while Theorem 20 implies that $\overline{\mathscr{C}}(k,d)^{1/d}-1 \ll (\log k)/k^2$. See the last column in Table 1. (Note that since $\overline{\mathscr{C}}(k,d)\geq k+1$, we need d to grow with k in order to have $\lim_{k\to\infty}\overline{\mathscr{C}}(k,d)=1$, and in fact $\lim_{k\to\infty}(k+1)^{1/d}=1$ iff $d\ggg \log k$.)

The second lower bound uses an algebraic construction of almost orthogonal Euclidean vectors.

Theorem 30. *For any $d\in\mathbb{N}$ let $q=q_d$ be the largest prime power such that $d\geq q^2-q+1$. (By the Prime Number Theorem, $q_d\sim\sqrt{d}$ as $d\to\infty$.) Then for each $c\in\mathbb{N}$ and $k\geq 2$ satisfying $c\leq q-2$ and*

$$k\leq \frac{q-1}{2c}-\frac{1}{2}\quad\left(\sim\frac{\sqrt{d}}{2c}\right)$$

there exists a d-dimensional Minkowski space X^d such that

$$\mathscr{C}_k(X^d)\geq q^{c+2}\quad(\sim d^{1+c/2}\text{ as } d\to\infty).$$

In particular, when $k\leq(\frac{1}{2}+o(1))\sqrt{d}$ as $d\to\infty$ we have $\overline{\mathscr{C}}_k(d)\gg d^{3/2}$. The lower bound of Theorem 30 is better than that of Theorem 29 when $k\gg\sqrt{d}/\log d$. For k a small multiple of $\sqrt{d}$, Theorems 20 and 21 give an upper bound polynomial in d while Theorem 30 gives a lower bound polynomial in d, but with a gap between the degrees of the polynomials. Nevertheless, Theorem 30 matches the bound in Theorem 20 in a similar sense as in the discussion after Theorem 29, in that it implies that $\overline{\mathscr{C}}(k,d)^{1/d}-1\gg(\log k)/(ck^2)$ as $k\to\infty$ and $k\sim\sqrt{d}/(2c)$, $c\in\mathbb{N}$.

1.3. **Organisation of the paper.** In Section 2 we use elementary combinatorial arguments involving coordinates and inner products to prove Proposition 16 on ℓ_∞^d, Proposition 14 on ℓ_2^d and Proposition 28 on spaces close to ℓ_2^d. In Section 3 we use the Brunn-Minkowski inequality and the Hajnal-Szemerédi Theorem to prove Theorem 27. This is followed in Section 4 by a proof of Theorem 24 which is along similar lines. In addition to the Brunn-Minkowski inequality it uses a metric consequence of Carathéodory's Theorem that may be of independent interest (Lemma 34). Then in Section 5 we reformulate the notion of a k-collapsing collection of vectors in terms of matrices. There we also prove a sharp version of a well-known result that bounds the rank of a matrix from below (Lemma 39). These results are applied in Section 6, where Theorem 17 is proved, and Section 7 where Theorems 22 and 23 are proved. These proofs are all very technical and involve an application of Lemma 39 combined with convex optimisation. In Section 8 Theorems 20 and 21 are proved. The arguments are similar as in Sections 6 and 7 and use in addition a well-known bound on the rank of an integer Hadamard power of a matrix (Lemma 42). In Section 9 we derive the lower bounds of Theorems 29 and 30.

2. THE SUP-NORM AND EUCLIDEAN NORM

Proposition 31. *If $S = \{\boldsymbol{x}_i \mid i \in [m]\} \subset \ell_\infty^d$ is a k-collapsing collection of $m > k+1$ vectors of norm at least 1, then $m \leq 2d$. If furthermore $m = 2d$, then $S = \{\pm \boldsymbol{e}_1, \dots, \pm \boldsymbol{e}_d\}$.*

Proof. Suppose that there exist a coordinate $j \in [d]$ and two distinct indices $i \in [m]$ such that $\boldsymbol{x}_i(j) \geq 1$. Without loss of generality, $\boldsymbol{x}_{m-1}(1), \boldsymbol{x}_m(1) \geq 1$. By the k-collapsing condition, for any $I \in \binom{[m-2]}{k-2}$,

$$\sum_{i \in I} \boldsymbol{x}_i(1) \leq -2 + \sum_{i \in I \cup \{m-1,m\}} \boldsymbol{x}_i(1) \leq -2 + \left\| \sum_{i \in I \cup \{m-1,m\}} \boldsymbol{x}_i \right\|_\infty \leq -1.$$

Fix a $J \in \binom{[m-2]}{k}$ (note that $k \leq m-2$). It follows that

$$\binom{k-1}{k-3} \sum_{i \in J} \boldsymbol{x}_i(1) = \sum_{I \in \binom{J}{k-2}} \sum_{i \in I} \boldsymbol{x}_i(1) \leq -\binom{k}{k-2},$$

which gives

$$\sum_{i \in J} \boldsymbol{x}_i(1) \leq -\binom{k}{k-2} / \binom{k-1}{k-2} = -k/(k-2) < -1$$

and $\|\sum_{i \in J} \boldsymbol{x}_i\|_\infty > 1$, contradicting the k-collapsing condition for the set J. Therefore, for each coordinate $j \in [d]$ there is at most one index $i \in [m]$ such that $\boldsymbol{x}_i(j) \geq 1$. Similarly, there is at most one $i \in [m]$ such that $\boldsymbol{x}_i(j) \leq -1$. Therefore, there are at most $2d$ pairs $(i, j) \in [m] \times [d]$ such that $|\boldsymbol{x}_i(j)| \geq 1$. On the other hand, since $\|\boldsymbol{x}_i\|_\infty \geq 1$ for each $i \in [m]$, there are at least m such pairs, which gives $m \leq 2d$.

If we now assume that $m = 2d$, then for each $j \in [d]$ there is exactly one $i \in [m]$ such that $\boldsymbol{x}_i(j) \geq 1$, and exactly one $i \in [m]$ such that $\boldsymbol{x}_i(j) \leq -1$. It follows that we may renumber the $\boldsymbol{x}_i$ such that $\boldsymbol{x}_{2i-1}(i) \geq 1$ and $\boldsymbol{x}_{2i}(i) \leq -1$ for each $i \in [d]$. By the k-collapsing condition we have for any $J \in \binom{[m-2]}{k-1}$ that

$$\sum_{i \in J} \boldsymbol{x}_i(d) + 1 \leq \sum_{i \in J \cup \{2d-1\}} \boldsymbol{x}_i(d) \leq \left\| \sum_{i \in J \cup \{2d-1\}} \boldsymbol{x}_i \right\|_\infty \leq 1,$$

hence $\sum_{i \in J} \boldsymbol{x}_i(d) \leq 0$. Similarly, $\sum_{i \in J} \boldsymbol{x}_i(d) \geq 0$. Therefore, $\sum_{i \in J} \boldsymbol{x}_i(d) = 0$ for each $J \in \binom{[m-2]}{k-1}$. Since $k-1 < m-2$, it follows that $\boldsymbol{x}_i(d) = 0$ for all $i \in [m-2]$. Similarly, $\boldsymbol{x}_i(j) = 0$ for all i, j such that $i \notin \{2j-1, 2j\}$. We conclude that $\boldsymbol{x}_{2i-1} = \boldsymbol{e}_i$ and $\boldsymbol{x}_{2i} = -\boldsymbol{e}_i$ for all $i \in [d]$. ☐

Proof of Proposition 16. We have already observed that

$$\mathscr{C}_k(\ell_\infty^d) \geq \mathscr{C}\mathscr{B}_k(\ell_\infty^d) \geq \max\{k+1, 2d\}.$$

Proposition 31 implies that $\mathscr{C}_k(\ell_\infty^d) \leq \max\{k+1, 2d\}$. ☐

The next lemma occurs in an equivalent form in [18, Lemma 5].

Lemma 32. *Let $k \geq 2$ be an integer and $\lambda \in (0, \sqrt{k})$. Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ be vectors in any inner product space such that $\|x_i\| \geq 1$ for all $i \in [m]$ and*

$$\Big\|\sum_{i \in I} \boldsymbol{x}_i\Big\| \leq \lambda \quad \textit{for all} \quad I \in \binom{[m]}{k}. \tag{1}$$

Then

$$m \leq \frac{k^2 - \lambda^2}{k - \lambda^2}.$$

Proof. Square (1) and sum over all $I \in \binom{[m]}{k}$ to obtain

$$\begin{aligned}
\binom{m}{k}\lambda^2 &\geq \binom{m-1}{k-1}\sum_{i=1}^{m}\|\boldsymbol{x}_i\|^2 + \binom{m-2}{k-2}\sum_{\{i,j\}\in\binom{[m]}{2}} 2\langle \boldsymbol{x}_i, \boldsymbol{x}_j\rangle \\
&= \left(\binom{m-1}{k-1} - \binom{m-2}{k-2}\right)\sum_{i=1}^{m}\|\boldsymbol{x}_i\|^2 + \Big\|\sum_{i=1}^{m}\boldsymbol{x}_i\Big\|^2 \\
&\geq \left(\binom{m-1}{k-1} - \binom{m-2}{k-2}\right)m + 0,
\end{aligned}$$

which simplifies to the conclusion of the theorem. □

Proof of Proposition 14. For the upper bound, set $\lambda = 1$ in Lemma 32. For the lower bound, note that since $d \geq 2$ there exist $k+1$ unit vectors that sum to $\boldsymbol{o}$. □

Proof of Proposition 28. By the definition of Banach-Mazur distance there exist coordinates such that

$$\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\| \leq D\|\boldsymbol{x}\|_2 \quad \text{for all } \boldsymbol{x} \in X^d.$$

Then the conclusion follows immediately from Lemma 32 with $\lambda = D$. □

3. THE BRUNN-MINKOWSKI INEQUALITY AND GRAPH COLOURINGS

The proofs of Theorems 24 and 27 are very similar, but that of Theorem 27 is somewhat more straightforward and we consider it first. We first discuss the three main tools used in its proof. The first is the dimension-independent version of the Brunn-Minkowski inequality (see Ball [5].) Denote the volume (or d-dimensional Lebesgue measure) of a measurable set $A \subseteq \mathbb{R}^d$ by $\operatorname{vol}(A)$.

Brunn-Minkowski inequality. *If $A, B \subset \mathbb{R}^d$ are compact sets and $0 < \lambda < 1$, then*

$$\operatorname{vol}(\lambda A + (1-\lambda)B) \geq \operatorname{vol}(A)^{\lambda}\operatorname{vol}(B)^{1-\lambda}.$$

Induction immediately gives the following version for k sets:

k-fold Brunn-Minkowski inequality. *Let $A_1, A_2, \dots, A_k \subset \mathbb{R}^d$ be compact and $\lambda_1, \lambda_2, \dots, \lambda_k > 0$ such that $\sum_{i=1}^{k}\lambda_i = 1$. Then*

$$\operatorname{vol}(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_k A_k) \geq \prod_{i=1}^{k}\operatorname{vol}(A_i)^{\lambda_i}.$$

The second tool is the Hajnal-Szemerédi Theorem. A *k-colouring* of a graph $G=(V,E)$ is a function $f\colon V\to[k]$ such that $f(x)\neq f(y)$ whenever $xy\in E$. The k-colouring partitions the vertex set V into *colour classes* $f^{-1}(i)$, $i\in[k]$. A k-colouring of a graph on m vertices is called *equitable* if each colour class has size $\lfloor m/k\rfloor$ or $\lceil m/k\rceil$. The following result was originally a conjecture of Erdős [11]. Although the original proof [16] was quite complicated and long, there is now a relatively simple, compact proof, due to Kierstead and Kostochka [23].

Hajnal-Szemerédi theorem. *Let G be a graph with m vertices and maximum degree Δ. Then for any $k>\Delta$, G has an equitable k-colouring.*

The third tool is the following simple consequence of the triangle inequality.

Lemma 33. *Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_k$ be vectors of norm at least 1 in a normed space such that*

$$\Big\|\sum_{i=1}^{k}\boldsymbol{x}_i\Big\|\leq 1.$$

Then for each $i\in[k]$ there exists $j\in[k]\setminus\{i\}$ such that $\left\|\boldsymbol{x}_i-\boldsymbol{x}_j\right\|\geq 1$.

Proof. Note that

$$k\boldsymbol{x}_i=\sum_{j=1}^{k}\boldsymbol{x}_j+\sum_{\substack{j=1\\j\neq i}}^{k}(\boldsymbol{x}_i-\boldsymbol{x}_j).$$

Take norms and apply the triangle inequality and the hypotheses:

$$\begin{aligned}k\leq\|k\boldsymbol{x}_i\|&\leq\Big\|\sum_{j=1}^{k}\boldsymbol{x}_j\Big\|+\sum_{\substack{j=1\\j\neq i}}^{k}\left\|\boldsymbol{x}_i-\boldsymbol{x}_j\right\|\\&\leq 1+\sum_{\substack{j=1\\j\neq i}}^{k}\left\|\boldsymbol{x}_i-\boldsymbol{x}_j\right\|.\end{aligned}$$

Thus we have a lower bound for the average distance between $\boldsymbol{x}_i$ and the other points:

$$1\leq\frac{1}{k-1}\sum_{\substack{j=1\\j\neq i}}^{k}\left\|\boldsymbol{x}_i-\boldsymbol{x}_j\right\|,$$

which gives $1\leq\left\|\boldsymbol{x}_i-\boldsymbol{x}_j\right\|$ for some $j\neq i$. □

Proof of Theorem 27. Let $V=\{\boldsymbol{x}_i\,|\,i\in[m]\}\subset X^d$ be a k-collapsing family with each $\|\boldsymbol{x}_i\|\geq 1$. Define a graph G on V by joining $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ if $\left\|\boldsymbol{x}_i-\boldsymbol{x}_j\right\|<1$. By Lemma 33, the maximum degree Δ of G is at most $k-2$. By the Hajnal-Szemerédi Theorem, G has an equitable k-colouring. This gives a partition $I_1,\dots,I_k$ of $[m]$ such that each $|I_t|\in\{q,q+1\}$, where $q:=\lfloor m/k\rfloor$, and such that $\left\|\boldsymbol{x}_i-\boldsymbol{x}_j\right\|\geq 1$ whenever i,j are distinct elements from the same I_t. For each $t\in[k]$ let

$$S_t=\bigcup_{j\in I_t}B\Big(\boldsymbol{x}_j,1/2\Big).$$

Then

$$\operatorname{vol}(S_t) = (1/2)^d \, |I_t| \operatorname{vol}(B). \tag{2}$$

By the k-collapsing property,

$$\frac{1}{k}(S_1 + \cdots + S_k) \subseteq B\left(\boldsymbol{o}, \frac{1}{k} + \frac{1}{2}\right). \tag{3}$$

Substitute (2) and (3) into the k-fold Brunn-Minkowski inequality

$$\prod_{t=1}^{k} \operatorname{vol}(S_t)^{1/k} \leq \operatorname{vol}\Big(\frac{1}{k}(S_1 + \cdots + S_k)\Big),$$

to obtain

$$\left(\prod_{t=1}^{k} |I_t|\right)^{1/k} \leq \left(1 + \frac{2}{k}\right)^d.$$

Set $r = m - kq$. Then there are r sets I_t of cardinality $q+1$ and $k-r$ of cardinality q. Therefore,

$$\left(\left(\frac{m-r}{k} + 1\right)^r \left(\frac{m-r}{k}\right)^{k-r}\right)^{1/k} \leq \left(1 + \frac{2}{k}\right)^d. \tag{4}$$

Instead of minimising the left-hand side over all $r \in \{0, 1, \ldots, k-1\}$, we just use the weakening

$$\frac{m-r}{k} \leq \left(1 + \frac{2}{k}\right)^d$$

to obtain

$$m \leq k\left(1 + \frac{2}{k}\right)^d + r \leq k\left(1 + \frac{2}{k}\right)^d + k - 1. \qquad \square$$

By taking more care in minimising the left-hand side of (4) it is possible to find a slightly better upper bound. However, the inequality $\overline{\mathscr{C}}(k,d) \leq k\left(1 + \frac{2}{k}\right)^d$ cannot be obtained from (4). For example, the values $d = 4$, $m = 19$, $k = 6$ satisfy (4), but not $m \leq k\left(1 + \frac{2}{k}\right)^d$. (Of course $\overline{\mathscr{C}}(6,4) = 8$ by Theorem 23.)

4. The Brunn-Minkowski inequality and Carathéodory's theorem

In this section we consider k-collapsing sets when $k \ggg d$ as $d \to \infty$. We use the Brunn-Minkowski inequality in much the same way as before, but now coupled with Carathéodory's theorem from combinatorial convexity.

Carathéodory's Theorem. *Let $\boldsymbol{p}$ be in the convex hull of a family $\{\boldsymbol{x}_i \mid i \in I\}$ of points in $\mathbb{R}^d$. Then $\boldsymbol{p} \in \operatorname{conv}\{\boldsymbol{x}_i \mid i \in J\}$ for some $J \subseteq I$ with $|J| \leq d+1$.*

Carathéodory's theorem is used to prove the following auxiliary result. The technique is very similar to an argument in [37] that bounds the number of vertices of an edge-antipodal polytope.

Lemma 34. *Let $d \geq 2$, $n \geq 1$ and $\{\boldsymbol{x}_i \,|\, i \in [n]\} \subset X^d$ such that $\|\boldsymbol{x}_i\| \geq 1$ for each $i \in [n]$ and*

$$\operatorname{diam}\{\boldsymbol{x}_i \,|\, i \in [n]\} < 1 + 1/d. \tag{5}$$

Then

$$\Big\| \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i \Big\| > 1/d^2. \tag{6}$$

Proof. Let $P := \operatorname{conv}\{\boldsymbol{x}_i \,|\, i \in [n]\}$. By convexity the centroid $\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i \in P$. Choose $\boldsymbol{p} \in P$ of minimum norm. It is sufficient to prove that $\|\boldsymbol{p}\| > 1/d^2$. Suppose that $\boldsymbol{p} = \boldsymbol{o}$. Then by Carathéodory's Theorem, $\boldsymbol{o}$ is a convex combination of a subfamily of at most $d+1$ of the $\boldsymbol{x}_i$, that is, $\boldsymbol{o} = \sum_{i \in J} \lambda_i \boldsymbol{x}_i$ where $J \subseteq [n]$, $|J| \leq d+1$, $\lambda_i \geq 0$ for each $i \in J$, and $\sum_{i \in J} \lambda_i = 1$. Note that $|J| \geq 2$. For any $j \in J$,

$$-\boldsymbol{x}_j = \sum_{i \in J \setminus \{j\}} \lambda_i (\boldsymbol{x}_i - \boldsymbol{x}_j)$$

and by the triangle inequality,

$$1 \leq \sum_{i \in J \setminus \{j\}} \lambda_i \big\|\boldsymbol{x}_i - \boldsymbol{x}_j\big\| \leq \sum_{i \in J \setminus \{j\}} \lambda_i \operatorname{diam} P = (1 - \lambda_j) \operatorname{diam} P.$$

Summing over $j \in J$, we obtain $|J| \leq (|J|-1) \operatorname{diam} P$ and $\operatorname{diam} P \geq \frac{|J|}{|J|-1} \geq \frac{d+1}{d}$. However,

$$\operatorname{diam} P = \operatorname{diam}\{\boldsymbol{x}_i \,|\, i \in [n]\} < 1 + 1/d$$

by assumption, a contradiction. It follows that $\boldsymbol{p} \neq \boldsymbol{o}$. Thus $\boldsymbol{p}$ is on some facet of P. We now apply Carathéodory's Theorem to the affine span of this facet, of dimension $< d$:

$$\boldsymbol{p} = \sum_{i \in J} \lambda_i \boldsymbol{x}_i \quad \text{where } J \subseteq [n],\ |J| \leq d,\ \lambda_i \geq 0 \text{ for each } i \in J, \text{ and } \sum_{i \in J} \lambda_i = 1.$$

If $|J| = 1$ then $\boldsymbol{p} = \boldsymbol{x}_i$ for some $i \in [n]$ and $\|\boldsymbol{p}\| \geq 1 > 1/d^2$. Without loss of generality we may then assume that $|J| \geq 2$. It follows that for each $j \in J$,

$$\boldsymbol{p} - \boldsymbol{x}_j = \sum_{i \in J \setminus \{j\}} \lambda_i (\boldsymbol{x}_i - \boldsymbol{x}_j)$$

and as before,

$$\begin{aligned} 1 - \|\boldsymbol{p}\| \leq \big\|\boldsymbol{x}_j\big\| - \|\boldsymbol{p}\| \leq \big\|\boldsymbol{p} - \boldsymbol{x}_j\big\| &\leq \sum_{i \in J \setminus \{j\}} \lambda_i \big\|\boldsymbol{x}_i - \boldsymbol{x}_j\big\| \\ &\leq \sum_{i \in J \setminus \{j\}} \lambda_i \operatorname{diam} P = (1 - \lambda_j) \operatorname{diam} P. \end{aligned}$$

Sum over $j \in J$ to obtain (since $|J| \geq 2$) that

$$(1 - \|\boldsymbol{p}\|)\,|J| \leq (|J|-1) \operatorname{diam} P < (|J|-1)(1 + 1/d)$$

and

$$1 - \|\boldsymbol{p}\| < \frac{|J|-1}{|J|}(1 + 1/d) \leq \frac{d-1}{d}(1 + 1/d) = 1 - 1/d^2.$$

It follows that $\|\boldsymbol{p}\| > 1/d^2$. □

The right-hand side of (6), while perhaps not the best possible function of d, is at least of the right order as shown by the following example. Let $\varepsilon > 0$ be small and fixed and let $d \geq 3$. Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_d$ be the standard unit basis of ℓ_1^d and set

$$\boldsymbol{x}_i = \left(\frac{d+1}{2d} - \frac{(d-2)\varepsilon}{d-1}\right)\boldsymbol{e}_i - \left(\frac{d-1}{2d(d-2)} + \frac{\varepsilon}{d-1}\right)\sum_{j=1}^{d} \boldsymbol{e}_j \quad \text{for } i \in [d].$$

Then it is easily checked that $\{\boldsymbol{x}_i \mid i \in [d]\}$ satisfies the hypotheses of Lemma 34: $\|\boldsymbol{x}_i\|_1 = 1$ for all $i \in [d]$ and

$$\operatorname{diam}\{\boldsymbol{x}_i\} = 1 + \frac{1}{d} - \frac{2(d-2)}{d-1}\varepsilon < 1 + \frac{1}{d}.$$

On the other hand,

$$\left\|\frac{1}{d}\sum_{i=1}^{d} \boldsymbol{x}_i\right\|_1 = \frac{1}{d(d-2)} + 2\varepsilon,$$

which is within $O(1/d^2)$ of $1/d^2$.

A slight modification of this example also shows that the right-hand side of (5) cannot be increased: There exist d-dimensional Minkowski spaces with $d+1$ unit vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{d+1}$ such that $\operatorname{diam}\{\boldsymbol{x}_i\} = 1 + 1/d$ although $\sum_{i=1}^{d+1} \boldsymbol{x}_i = \boldsymbol{o}$. Indeed, let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{d+1}$ be the standard unit basis of ℓ_1^{d+1} and consider the subspace

$$X = \left\{ (\alpha_1, \dots, \alpha_{d+1}) \,\middle|\, \sum_{i=1}^{d+1} \alpha_i = 0 \right\}.$$

Let

$$\boldsymbol{x}_i = \frac{d+1}{2d}\boldsymbol{e}_i - \frac{1}{2d}\sum_{j=1}^{d+1} \boldsymbol{e}_j \quad (i \in [d+1]).$$

It is again easy to see that the $\boldsymbol{x}_i$ are unit vectors in X, $\left\|\boldsymbol{x}_i - \boldsymbol{x}_j\right\|_1 = 1 + 1/d$ for distinct $i, j \in [d+1]$, and $\sum_{i=1}^{d+1} \boldsymbol{x}_i = \boldsymbol{o}$. It may seem strange that the centroid of the vectors can jump from the origin to a point bounded away from the origin by $1/d^2$ when the diameter decreases from $1 + 1/d$. However, a similar phenomenon can be demonstrated even in Euclidean space. Consider a regular simplex inscribed in the unit sphere of ℓ_2^d. Then it is not possible to continuously move the $d+1$ vertices an arbitrarily small distance while remaining on the sphere so as to reduce the diameter of the simplex. The diameter will increase at first and when it has eventually decreased below the diameter of the original equilateral simplex, the centroid will be bounded away from the origin.

Proof of Theorem 24. Suppose that $\mathscr{C}_k(X^d) \geq k+2$. Let $\{\boldsymbol{x}_i \mid i \in [k+2]\} \subset X^d$ be a k-collapsing collection of vectors of norm at least 1. Our aim is to show that $k = O(d^{d+2})$.

Let $\boldsymbol{s} := \sum_{i=1}^{k+2} \boldsymbol{x}_i$. The k-collapsing condition gives an upper bound to the norm of $\boldsymbol{s}$ as follows: Since

$$\sum_{S \in \binom{[k+2]}{k}} \sum_{i \in S} \boldsymbol{x}_i = \binom{k+1}{k-1} \sum_{i=1}^{k+2} \boldsymbol{x}_i = \binom{k+1}{k-1} \boldsymbol{s},$$

the triangle inequality gives

$$\binom{k+1}{k-1}\|\boldsymbol{s}\| \le \sum_{S\in\binom{[k+2]}{k}} \Big\|\sum_{i\in S}\boldsymbol{x}_i\Big\| \le \binom{k+2}{k},$$

and

$$\|s\| \le \binom{k+2}{k}\Big/\binom{k+1}{k-1} = 1+2/k. \tag{7}$$

Without loss of generality some $\|\boldsymbol{x}_i\| = 1$. For each $j\in[k+2]\setminus\{i\}$ the k-collapsing condition implies that $\big\|(\boldsymbol{s}-\boldsymbol{x}_i)-\boldsymbol{x}_j\big\| \le 1$, and again by the triangle inequality,

$$\big\|\boldsymbol{x}_j\big\| \le 1+\|\boldsymbol{s}\|+\|\boldsymbol{x}_i\| \le 3+2/k. \tag{8}$$

Let $\varepsilon>0$ (to be fixed later). Define a graph G on $[k+2]$ by joining i and j whenever $\big\|\boldsymbol{x}_i-\boldsymbol{x}_j\big\| < \varepsilon$. Let $C\subseteq[k+2]$ be the set of all isolated vertices of G. Suppose for the moment that $|C|\ge 2$. Partition C into two parts as equally as possible: $C=C_1\cup C_2$ with $C_1\cap C_2=\varnothing$, $||C_1|-|C_2||\le 1$. Let

$$S_t = \bigcup_{j\in C_t} B(\boldsymbol{x}_j,\varepsilon/2) \quad \text{for } t=1,2.$$

Then

$$\operatorname{vol}(S_t) = |C_t|\,(\varepsilon/2)^d \operatorname{vol}(B).$$

Also, $S_1+S_2\subseteq B(\boldsymbol{s},1+\varepsilon)$, which gives

$$\operatorname{vol}\left(\frac{1}{2}S_1+\frac{1}{2}S_2\right) \le \left(\frac{1+\varepsilon}{2}\right)^d \operatorname{vol}(B).$$

By the Brunn-Minkowski inequality,

$$\operatorname{vol}\left(\frac{1}{2}S_1+\frac{1}{2}S_2\right) \ge \operatorname{vol}(S_1)^{1/2}\operatorname{vol}(S_2)^{1/2} = \sqrt{|C_1|\cdot|C_2|}\left(\frac{\varepsilon}{2}\right)^d \operatorname{vol}(B).$$

It follows that

$$\frac{|C|-1}{2} < \sqrt{|C_1|\cdot|C_2|} \le \left(1+\frac{1}{\varepsilon}\right)^d$$

and

$$|C| < 2\left(1+\frac{1}{\varepsilon}\right)^d + 1. \tag{9}$$

This bound clearly also holds if $|C|<2$.

Next consider the complement $C' := [k+2]\setminus C$, consisting of the vertices of G of degree at least 1. For each $i\in C'$ there exists $i'\in[k+2]\setminus\{i\}$ such that $\|\boldsymbol{x}_i-\boldsymbol{x}_{i'}\|<\varepsilon$. We claim that

$$\operatorname{diam}\left\{\boldsymbol{x}_i \,\middle|\, i\in C'\right\} < 1+\varepsilon. \tag{10}$$

Consider distinct $i,j\in C'$. There exist $i',j'\in C'$ such that $i'\neq i$, $j'\neq j$, $\|\boldsymbol{x}_i-\boldsymbol{x}_{i'}\|<\varepsilon$ and $\|\boldsymbol{x}_j-\boldsymbol{x}_{j'}\|<\varepsilon$. Then by the triangle inequality and the k-collapsing condition,

$$\begin{aligned}\|2\boldsymbol{x}_i-2\boldsymbol{x}_j\| &= \|\boldsymbol{x}_i-\boldsymbol{x}_{i'}+\boldsymbol{x}_i+\boldsymbol{x}_{i'}-\boldsymbol{s}+\boldsymbol{s}-\boldsymbol{x}_j-\boldsymbol{x}_{j'}+\boldsymbol{x}_{j'}-\boldsymbol{x}_j\|\\ &\leq \|\boldsymbol{x}_i-\boldsymbol{x}_{i'}\|+\|\boldsymbol{x}_i+\boldsymbol{x}_{i'}-\boldsymbol{s}\|+\|\boldsymbol{s}-\boldsymbol{x}_j-\boldsymbol{x}_{j'}\|+\|\boldsymbol{x}_{j'}-\boldsymbol{x}_j\|\\ &< \varepsilon+1+1+\varepsilon,\end{aligned}$$

which shows (10). In order to apply Lemma 34 to $\{\boldsymbol{x}_i \mid i\in C'\}$ we set $\varepsilon=1/d$ and obtain that

$$\Big\|\sum_{i\in C'}\boldsymbol{x}_i\Big\| > \frac{|C'|}{d^2}=\frac{k+2-|C|}{d^2}. \tag{11}$$

On the other hand, by (7) and (8),

$$\begin{aligned}\Big\|\sum_{i\in C'}\boldsymbol{x}_i\Big\| &= \Big\|\boldsymbol{s}-\sum_{i\in C}\boldsymbol{x}_i\Big\| \leq \|\boldsymbol{s}\|+\sum_{i\in C}\|\boldsymbol{x}_i\|\\ &\leq 1+\frac{2}{k}+|C|\left(3+\frac{2}{k}\right).\end{aligned}$$

By (9) and the choice of ε, $|C|\leq 2(d+1)^d$. It follows that

$$\frac{k+2}{d^2}<1+\frac{2}{k}+|C|\left(3+\frac{2}{k}+\frac{1}{d^2}\right)=O(d^d). \qquad \square$$

5. Reformulation in terms of matrices

We now reduce the existence of a d-dimensional Minkowski space admitting vectors satisfying the k-collapsing condition and strong balancing condition to the existence of a matrix of rank at least d satisfying certain properties.

Definition 35. *An $m\times m$ matrix $A=[a_{ij}]$ is called* k-collapsing *if the following conditions all hold:*

$$a_{i,i}\geq 1 \quad \text{for all } i\in[m], \tag{12}$$

$$|a_{i,j}|\leq a_{j,j} \text{ for all } i,j\in[m], \tag{13}$$

$$\text{and} \quad \Big|\sum_{j\in I}a_{i,j}\Big|\leq 1 \quad \text{for all } I\in\binom{[m]}{k} \text{ and } i\in[m]. \tag{14}$$

The matrix A is called balancing *if*

$$\sum_{j=1}^{m}a_{i,j}=0 \text{ for all } i\in[m]. \tag{15}$$

We say that A is normalised *if $a_{i,i}=1$ for all $i\in[m]$.*

Suppose that A does not have a 0 on its main diagonal. Then we define the normalisation *of A as the normalised matrix $\tilde{A}=[a_{i,j}/a_{i,i}]$ obtained by dividing row i of $A=[a_{i,j}]$ by $a_{i,i}$ for each $i\in[m]$.*

Note that the normalisation $\tilde{A}$ is normalised and has the same rank as A. If A is balancing, then its normalisation is also balancing. However, it is not clear whether the normalisation is k-collapsing if A is k-collapsing. The next lemma shows that this is the case if $k \leq m-2$.

Lemma 36. *Let $2 \leq k \leq m-2$. If $A = [a_{i,j}]$ is an $m \times m$ k-collapsing matrix, then so is its normalisation $\tilde{A} = [a_{i,j}/a_{i,i}]$.*

Proof. It is clear that conditions (12), (14) and (15) stay valid when row i is divided by $a_{i,i}$. We show that (13) also remains valid by proving that any k-collapsing matrix A already satisfies the stronger inequality

$$\left|a_{i,j}\right| \leq 1 \quad \text{for all distinct } i,j \in [m]. \tag{16}$$

Fix distinct $i,j \in [m]$, and choose any $I,J \in \binom{[m]}{k}$ such that $I \setminus J = \{i\}$ and $J \setminus I = \{j\}$. By (14), $\sum_{s\in I} a_{i,s} \leq 1$ and $\sum_{s\in J} a_{i,s} \geq -1$. Subtract these two inequalities to obtain $a_{i,i} - a_{i,j} \leq 2$, hence

$$a_{i,j} \geq a_{i,i} - 2 \geq 1 - 2 = -1.$$

Before proving that $a_{i,j} \leq 1$, we first show that

$$S_i := \{s \in [m] \,|\, a_{i,s} > 0\}$$

contains at most $k-1$ elements. Suppose this is false. Then choose any $I \in \binom{S_i}{k}$ such that $i \in I$. By (14), $\sum_{s\in I} a_{i,s} \leq 1$, from which follows $\sum_{s\in I\setminus\{i\}} a_{i,s} \leq 1 - a_{i,i} \leq 0$, a contradiction. Thus

$$|[m] \setminus (S_i \cup \{j\})| \geq m-k \geq 2,$$

and we may choose two distinct indices $i',j' \in [m] \setminus \{i,j\}$ such that $a_{i,i'} \leq 0$ and $a_{i,j'} \leq 0$. Choose any $I,I' \in \binom{[m]}{k}$ such that $I \setminus I' = \{i,j\}$ and $I' \setminus I = \{i',j'\}$. By (14), $\sum_{s\in I} a_{i,s} \leq 1$ and $\sum_{s\in I'} a_{i,s} \geq -1$. Subtract these inequalities to obtain $a_{i,i} + a_{i,j} - a_{i,i'} - a_{i,j'} \leq 2$. Therefore,

$$a_{i,j} \leq 2 - a_{i,i} + a_{i,i'} + a_{i,j'} \leq 2 - 1 + 0 + 0. \qquad \square$$

Lemma 36 does not hold if $k = m-1$ and $m \geq 5$. For example, the matrix

$$\begin{bmatrix} 1 & 2 & -\varepsilon & -\varepsilon & \cdots & -\varepsilon \\ 1 & 2 & -\varepsilon & -\varepsilon & \cdots & -\varepsilon \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

with $\varepsilon = 2/(m-3)$ is an $(m-1)$-collapsing $m \times m$ matrix, but its normalisation is not $(m-1)$-collapsing. However, for $k = 2 = m-1$ and $k = 3 = m-1$ the normalisation of an $m \times m$ k-collapsing matrix is still k-collapsing. This can be shown easily as in the proof of Lemma 36.

The next lemma is the promised reduction from Minkowski spaces to matrices.

Lemma 37. *Let $2 \leq k < m$ and $d \geq 2$. The following two statements are equivalent:*

(1) *There exist a d-dimensional Minkowski space X and $\boldsymbol{x}_1,\dots,\boldsymbol{x}_m \in X$ with $\|\boldsymbol{x}_i\| \geq 1$ for all $i \in [m]$ satisfying the k-collapsing condition* [*and strong balancing condition*].
(2) *There exists an $m \times m$ k-collapsing* [*and balancing*] *matrix of rank at most d.*

The following two statements are also equivalent:

(1) *There exist a d-dimensional Minkowski space X and $\boldsymbol{x}_1,\dots,\boldsymbol{x}_m \in X$ with $\|\boldsymbol{x}_i\| = 1$ for all $i \in [m]$ satisfying the k-collapsing condition* [*and strong balancing condition*].
(2) *There exists a normalised $m \times m$ k-collapsing* [*and balancing*] *matrix of rank at most d.*

Proof. We only prove the first statement as the proof of the second statement only requires minor modifications.

($\Longrightarrow$) Let a d-dimensional X be given with vectors $\boldsymbol{x}_1,\dots,\boldsymbol{x}_m$ satisfying $\|\boldsymbol{x}_i\| \geq 1$ for all $i \in [m]$ and the k-collapsing condition. Choose a dual unit vector $\boldsymbol{x}_i^*$ for each $\boldsymbol{x}_i$. Then for each $i \in [m]$, $\boldsymbol{y}_i = [\langle \boldsymbol{x}_1^*, \boldsymbol{x}_i\rangle, \dots, \langle \boldsymbol{x}_m^*, \boldsymbol{x}_i\rangle]^{\mathsf{T}}$ is a vector in ℓ_∞^m of norm $\|\boldsymbol{y}_i\|_\infty = \langle \boldsymbol{x}_i^*, \boldsymbol{x}_i\rangle = \|\boldsymbol{x}_i\| \geq 1$, and $\{\boldsymbol{y}_1,\dots,\boldsymbol{y}_m\}$ also satisfies the k-collapsing condition. In other words, in the $m \times m$ matrix $[\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j\rangle]$ the value of each diagonal entry is at least 1 (giving (12)); the value of each entry is dominated by the diagonal entry in its column (giving (13)); and the sum of any k entries in the same row lies in the interval $[-1,1]$ (giving (14)). In addition, if $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_k\}$ satisfies the strong balancing condition, the matrix $[\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j\rangle]$ will clearly also satisfy the balancing condition.

Since we have the factorisation

$$[\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j\rangle]_{i,j\in[m]} = [\boldsymbol{x}_1^*,\dots,\boldsymbol{x}_m^*]^{\mathsf{T}}[\boldsymbol{x}_1,\dots,\boldsymbol{x}_m]$$

into matrices of rank at most d, $[\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j\rangle]$ has rank at most d.

($\Longleftarrow$) Conversely, if we start off with an $m \times m$ k-collapsing [and balancing] matrix A with rank at most d, then its columns are m vectors of norm at least 1 in ℓ_∞^m satisfying the k-collapsing condition [and balancing condition]. Since they span a subspace of ℓ_∞^m of dimension at most d, we may take any d-dimensional subspace of ℓ_∞^m that contains the column space of A. $\square$

Corollary 38. *Let $2 \leq k < m$ and $d \geq 2$. There exists a d-dimensional Minkowski space that contains a k-collapsing* [*and balancing*] *set of m vectors of norm ≥ 1 iff there exists a d-dimensional Minkowski space that contains a k-collapsing* [*and balancing*] *set of m unit vectors.*

Proof. The case $k = m-1$ is trivial, as there exist $k+1$ unit vectors that sum to $\boldsymbol{o}$ if $d \geq 2$. The case $k \leq m-2$ follows from Lemma 36 and 37. $\square$

The following result gives a lower bound for the rank of a square matrix in terms of its Frobenius norm and trace. This bound is usually productive if the matrix has relatively large positive entries on the diagonal.

Lemma 39. *Let $A = [a_{ij}]$ be any $n \times n$ matrix with complex entries. Then*

$$\Big|\sum_{i=1}^{n} a_{ii}\Big|^2 \le \operatorname{rank}(A)\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} \left|a_{ij}\right|^2\Big). \tag{17}$$

Equality holds in (17) *if and only if A is a normal matrix and all its non-zero eigenvalues are equal. If A is a real matrix then equality holds in* (17) *if and only if A is symmetric and all its non-zero eigenvalues are equal.*

The case when A is real and symmetric is an exercise in Bellman [7, p. 137]. Various combinatorial and geometric applications may be found in [1, 2, 3, 4, 6, 28]. If A is not symmetric, it can be replaced by $A + A^{\mathsf{T}}$, of rank at most $2\operatorname{rank}(A)$, without decreasing $|\sum_i a_{ii}|^2 / \sum_{i,j} \left|a_{ij}\right|^2$. Thus assuming the validity of (17) only for real symmetric matrices gives a lower bound for the rank of A which is weaker by a factor of 2. This weakening is usually of no concern in applications. However, we do need the sharp estimate (17) for general (real) matrices, in order to obtain the sharp and almost sharp estimates in Theorems 17, 22 and 23.

Proof of Lemma 39. Let the non-zero eigenvalues of A be $\lambda_1, \dots, \lambda_r$. Since the result is trivial if $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{i,i} = 0$, we may assume without loss of generality that $r \ge 1$. By the Schur decomposition of a square matrix with complex entries [29] (see also [17, Theorem 2.3.1]) there exists an $n \times n$ unitary matrix U such that $C = [c_{ij}] := U^*AU$ is upper triangular. In particular, the eigenvalues of A are the diagonal entries of C, and

$$r \le \operatorname{rank}(C) = \operatorname{rank}(A). \tag{18}$$

Also,

$$\Big|\sum_{i=1}^{n} a_{i,i}\Big| = |\operatorname{tr}(A)| = \Big|\sum_{i=1}^{r} \lambda_i\Big| \le \sum_{i=1}^{r} |\lambda_i|, \tag{19}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \left|a_{i,j}\right|^2 = \operatorname{tr}(A^*A) = \operatorname{tr}(C^*C) = \sum_{i=1}^{n}\sum_{j=1}^{n} \left|c_{ij}\right|^2 \ge \sum_{i=1}^{r} |\lambda_i|^2. \tag{20}$$

(This inequality $\sum_i |\lambda_i|^2 \le \sum_{i,j} \left|a_{i,j}\right|^2$ was deduced by Schur in [29] using his decomposition.) Finally, by the Cauchy-Schwarz inequality,

$$\Big(\sum_{i=1}^{r} |\lambda_i|\Big)^2 \le r \sum_{i=1}^{r} |\lambda_i|^2, \tag{21}$$

and (17) follows from (18), (19), (20) and (21).

Suppose equality holds in (17). This gives equality in (18)–(21). Equality in (21) gives that all $|\lambda_i|$ are equal. Equality in (19) gives that all λ_i are positive multiples of each other. Therefore, all λ_i are equal. Equality in (20) gives that C is a diagonal matrix, hence A is normal. If A is real we furthermore obtain that the λ_i are real, since they are equal and their sum is the real number $\operatorname{tr}(A)$. Then $C = C^*$, hence $A^{\mathsf{T}} = A^* = A$ and A is symmetric.

Conversely, if A is normal, then C is diagonal, and equality holds in (18) and (20). If all the non-zero eigenvalues of A are equal, equality holds in (19) and (21), and we obtain equality in (17). $\square$

6. A TIGHT UPPER BOUND FOR $\mathscr{C}\mathscr{B}_k(X)$

In this section we prove Theorem 17. To show that $\mathscr{C}\mathscr{B}_k(X) \leq \max\{k+1, 2d\}$ for all d-dimensional X^d, it is sufficient by Lemmas 36 and 37 to prove that for any $m \times m$ normalised k-collapsing and balancing matrix $A = [a_{i,j}]$ of rank at most d we have that $m \leq 2d$ as long as $k \leq m-2$. By Lemma 39 it is sufficient to show that $|\sum_i a_{i,i}| / \sum_{i,j} |a_{i,j}|^2 \geq m/2$. Since $\sum_i a_{i,i} = m$, this is equivalent to $\sum_{i,j} a_{i,j}^2 \leq 2m$. Since $a_{i,i} = 1$, it will be sufficient to show that for any i,

$$\sum_{\substack{j=1\\ j\neq i}}^{m} a_{i,j}^2 \leq 1.$$

This is implied by the next lemma, which solves a convex maximisation problem with linear constraints.

Lemma 40. *Let $k, m \in \mathbb{N}$ such that $k \geq 2$ and $2k \leq m$. Then*

$$\max\left\{\sum_{i=1}^{m-1} x_i^2 \,\middle|\, \sum_{i=1}^{m-1} x_i = -1, \sum_{i\in I} x_i \leq 0 \text{ for all } I \in \binom{[m-1]}{k-1}\right\} = 1.$$

The maximum value $\sum_{i=1}^{m-1} x_i^2 = 1$ is attained under these constraints only if for some $j \in [m-1]$, $x_j = -1$ and $x_i = 0$ for all $i \in [m-1] \setminus \{j\}$.

Proof. The objective function $f(\boldsymbol{x}) := \sum_{i=1}^{m-1} x_i^2$, as well as the constraints

$$\sum_{i\in I} x_i \leq 0 \text{ for all } I \in \binom{[m-1]}{k-1} \tag{22}$$

and

$$\sum_{i=1}^{m-1} x_i = -1 \tag{23}$$

are symmetric in the variables $x_1, \dots, x_{m-1}$. Thus we may assume without loss of generality that

$$x_1 \geq x_2 \geq \cdots \geq x_{m-1}. \tag{24}$$

Then (22) becomes equivalent to the single inequality

$$\sum_{i=1}^{k-1} x_i \leq 0. \tag{25}$$

The $m-1$ linear inequalities in (24) and (25) define a polytope P in the hyperplane H of $\mathbb{R}^{m-1}$ defined by (23). Since the objective function f is convex, it attains its maximum on P at a vertex of P. Since the point in $\mathbb{R}^{m-1}$ with coordinates

$$x_i = \frac{-2i}{m(m-1)}, \quad i \in [m-1]$$

satisfies (24) and (25) with strict inequalities (as well as (23)), P has non-empty interior in H. It follows that P is an $(m-2)$-dimensional simplex, and it is easy to calculate its $m-1$ vertices, as follows.

Case I. If $x_1 = \cdots = x_{m-1}$ then

$$\boldsymbol{x} = \Big(\underbrace{\frac{-1}{m-1}, \dots, \frac{-1}{m-1}}_{m-1 \text{ times}}\Big)$$

and $f(\boldsymbol{x}) = 1/(m-1) < 1$.

Case II. If $x_1 = \cdots = x_t$ and $x_{t+1} = \cdots = x_{m-1}$ for some $t \in [m-2]$, and $\sum_{i=1}^{k-1} x_i = 0$, we distinguish between two subcases:

Subcase II.i. $t \leq k-1$. Then

$$\boldsymbol{x} = \Big(\underbrace{\frac{k-1-t}{t(m-k)}, \dots, \frac{k-1-t}{t(m-k)}}_{t \text{ times}}, \underbrace{\frac{-1}{m-k}, \dots, \frac{-1}{m-k}}_{m-1-t \text{ times}}\Big)$$

and

$$f(\boldsymbol{x}) = \frac{(k-1-t)^2}{t(m-k)^2} + \frac{m-1-t}{(m-k)^2}.$$

It is easy to check using the assumption $m \geq 2k$ that this expression is strictly less than 1 for all $t \in [k-1]$.

Subcase II.ii. $t \geq k$. Then

$$\boldsymbol{x} = \Big(\underbrace{0, \dots, 0}_{t \text{ times}}, \underbrace{\frac{-1}{m-1-t}, \dots, \frac{-1}{m-1-t}}_{m-1-t \text{ times}}\Big)$$

and

$$f(\boldsymbol{x}) = \frac{1}{m-1-t} \leq 1$$

with equality if and only if $t = m-2$, and then

$$\boldsymbol{x} = (0, \dots, 0, -1).$$

This shows that the maximum of f on P is 1, attained at only one point if the coordinates are in decreasing order. $\square$

Proof of Theorem 17. By Corollary 16, $\mathscr{CB}_k(\ell_\infty^d) = \max\{k+1, 2d\}$. In fact, if $k \leq 2d$, for any norm with unit vector basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_d\}$, the family $\{\pm\boldsymbol{e}_i \mid i \in [d]\}$ is k-collapsing if $\sum_{i\in I} \boldsymbol{e}_i$ is contained in the unit ball for all $I \subseteq [d]$ with $|I| \leq k$. When $d \geq 3$, any $\boldsymbol{o}$-symmetric convex body C that satisfies

$$P_k := \operatorname{conv}\left\{\pm\sum_{i\in I} \boldsymbol{e}_i \,\middle|\, I \subseteq [d], |I| \leq k\right\} \subseteq C \subseteq [-1,1]^d$$

is the unit ball of a norm $\|\cdot\|_C$ such that $\{\pm\boldsymbol{e}_i \mid i \in [d]\}$ is k'-collapsing in the norm $\|\cdot\|_C$ for all $k' = 2, \dots, k$, with $\|\boldsymbol{e}_i\|_C = 1$.

It remains to show that if X^d contains a family of $m \geq k+2$ vectors of norm at least 1 satisfying the k-collapsing condition and strong balancing condition, then $m \leq 2d$, and equality implies that the $m = 2d$ vectors are made up of a basis and its negative. (Note that this also takes care of the statement that $CB_2(X^2) = 4$ only if $X^2 = \ell_\infty^2$ up to isometry.) By the strong balancing condition we may assume without loss of generality that $k \leq m/2$.

Thus let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in X$ be a k-collapsing, strongly balancing family of vectors of norm at least 1. For each $\boldsymbol{x}_i$, let $\boldsymbol{x}_i^* \in X^*$ be a dual unit vector. By Proposition 37, $A = [a_{ij}] := [\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j \rangle]$ is an $m \times m$ k-collapsing and balancing matrix. We will show that the rank of this matrix is at most $m/2$, with equality implying that, after a permutation of the $\boldsymbol{x}_i$,

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}. \tag{26}$$

By Lemma 36 the normalisation $\tilde{A} = [\tilde{a}_{i,j}] := [a_{i,j}/a_{i,i}]$ is also k-collapsing and balancing. We want to show that $\sum_{j=1}^m \tilde{a}_{i,j}^2 \leq 2$ for all $i \in [m]$. Consider without loss of generality the last row

$$\tilde{a}_{m,1} \quad \tilde{a}_{m,2} \quad \dots \quad \tilde{a}_{m,m-1} \quad 1.$$

Let $x_i = \tilde{a}_{m,i}$ for $i \in [m-1]$. By the balancing condition, $\sum_{i=1}^{m-1} x_i = -1$. By the k-collapsing condition, for all $I \in \binom{[m-1]}{k-1}$,

$$\left| 1 + \sum_{i \in I} x_i \right| \leq 1,$$

hence $\sum_{i \in I} x_i \leq 0$. Lemma 40 now gives $\sum_{i=1}^{m-1} x_i^2 \leq 1$. It follows that $\sum_{j=1}^m \tilde{a}_{i,j}^2 \leq 2$ for each $i \in [m]$, and by Lemma 39,

$$d \geq \operatorname{rank}(A) = \operatorname{rank}(\tilde{A}) \geq \frac{m^2}{2m} = \frac{m}{2}.$$

This shows that $m \leq 2d$. Suppose now that $m = 2d$. Again by Lemmas 39 and 40, $\operatorname{rank}(A) = \operatorname{rank}(\tilde{A}) = d$, $\tilde{A}$ is a symmetric $2d \times 2d$ matrix, and each row of $\tilde{A}$ has a 1 on the diagonal, a -1 at some non-diagonal entry, and 0s everywhere else. Thus $\tilde{A} = I - P$, where P is a symmetric permutation matrix. The associated permutation must be an involution. It follows that after some permutation of the coordinates, $\tilde{A}$ is as in (26). Since $\tilde{A}$ has an off-diagonal entry of absolute value 1 in each column, A is already normalised, hence $\|\boldsymbol{x}_i\| = 1$ for all $i \in [m]$. Since $A = [\boldsymbol{x}_1^* \dots \boldsymbol{x}_{2d}^*]^\mathsf{T} [\boldsymbol{x}_1 \dots \boldsymbol{x}_{2d}]$ and the submatrix of A consisting of odd rows and columns is the $d \times d$ identity matrix, it follows that $\{\boldsymbol{x}_1, \boldsymbol{x}_3, \dots, \boldsymbol{x}_{2d-1}\}$ is a basis of

X and $\{\boldsymbol{x}_1^*, \boldsymbol{x}_3^*, \dots, \boldsymbol{x}_{2d-1}^*\}$ is a basis of X^*. Since $\langle \boldsymbol{x}_i^*, \boldsymbol{x}_1 \rangle = \langle \boldsymbol{x}_i^*, \boldsymbol{x}_2 \rangle = 0$ for all $i \geq 3$, it follows that

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in \bigcap_{j=2,\dots,d} \ker \boldsymbol{x}_{2j-1}^*,$$

which is a one-dimensional subspace of X. Therefore, $\boldsymbol{x}_1 = -\boldsymbol{x}_2$. Similarly, $\boldsymbol{x}_{2j-1} = -\boldsymbol{x}_{2j}$ for all $j \in [d]$. This proves the final statement of the theorem. $\square$

7. Tight and almost tight upper bounds for $\mathscr{C}_k(X)$

We now consider the k-collapsing condition without any balancing condition. As in the previous section we solve a convex optimisation problem. This case is more complicated and our results are only partial.

Recall (Lemma 37) that the existence of a d-dimensional Minkowski space that contains m vectors of norm ≥ 1 that satisfy the k-collapsing condition is equivalent to the existence of a normalised $m \times m$ k-collapsing matrix $A = [a_{i,j}]$ of rank at most d. Thus A has to satisfy

$$a_{i,i} = 1 \quad \forall i \in [m], \tag{27}$$

$$|a_{i,j}| \leq 1 \quad \forall i, j \in [m], \tag{28}$$

$$\text{and} \quad -1 \leq \sum_{j \in I} a_{i,j} \leq 1 \quad \forall i \in [m], \forall I \in \binom{[m]}{k}. \tag{29}$$

We would like to determine $\max \sum_{i,j=1}^m a_{i,j}^2$ as a function of k and m where $2 \leq k \leq m-2$, given the constraints (27), (28) and (29). The constraints on each row of A are independent from each other. Consider then without loss of generality the last row

$$a_{m,1} \quad a_{m,2} \quad \dots \quad a_{m,m-1} \quad 1.$$

Write $x_i = a_{m,i}$ for $i \in [m-1]$. Because of symmetry in the objective function $\sum_{i=1}^{m-1} x_i^2$ and in the constraints (27) to (29), we assume without loss of generality that

$$x_1 \geq x_2 \geq \cdots \geq x_{m-1}. \tag{30}$$

Then (28) becomes

$$-1 \leq x_i \leq 1 \quad \forall i \in [m-1] \tag{31}$$

and (29) becomes the four inequalities

$$x_1 + x_2 + \cdots + x_k \leq 1 \tag{32}$$

$$-1 \leq x_{m-k} + x_{m-k+1} + \cdots + x_{m-1} \tag{33}$$

$$x_1 + x_2 + \dots x_{k-1} \leq 0 \tag{34}$$

$$-2 \leq x_{m-k+1} + x_{m-k+2} + \cdots + x_{m-1} \tag{35}$$

We next show that in the collection of inequalities (30)–(35), the inequalities (31), (32) and (35) are redundant.

First note that (31) and (34) together imply (32); and (31) and (33) together imply (35). Next, (30), (33) and (34) together imply (31) as follows. By (30) and (34), $x_{k+1} \leq x_k \leq x_{k-1} \leq 0$. By (30) and (33), and since $k \leq m-2$,

$$-1 \leq x_{m-k} + \cdots + x_{m-1} \leq x_2 + \cdots + x_{k+1},$$

hence

$$-x_2 - \cdots - x_{k-1} \leq 1 + x_k + x_{k+1} \leq 1.$$

Add this to (34) to obtain $x_1 \leq 1$, which by (30) is equivalent to the inequality on the right in (31). By (33), (30) and (34),

$$\begin{aligned} -1 &\leq (x_{m-k} + x_{m-k+1} + \cdots + x_{m-2}) + x_{m-1} \\ &\leq (x_1 + \cdots + x_{k-1}) + x_{m-1} \\ &\leq x_{m-1}, \end{aligned}$$

which by (30) is equivalent to the inequality on the left in (31).

Lemma 41. *Let $k, m \in \mathbb{N}$ such that $2 \leq k \leq m-2$. Then*

$$\max\left\{ \sum_{i=1}^{m-1} x_i^2 \,\middle|\, -1 \leq \sum_{i\in I} x_i \leq 1 \text{ for all } I \in \binom{[m]}{k}, \right.$$
$$\left. -2 \leq \sum_{i\in I} x_i \leq 0 \text{ for all } I \in \binom{[m-1]}{k-1} \right\}$$

$$\begin{cases} = \max\left\{ \dfrac{m-1}{k^2}, 1, \dfrac{(k-2)^2+m-2}{k^2} \right\} \quad \text{if } k < 2m/3, \\ \leq \max\left\{ \dfrac{m-1}{k^2}, 1, \dfrac{(k-2)^2+m-2}{k^2}, \dfrac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)} \right\} \\ \qquad \text{if } k \geq 2m/3, \end{cases}$$

$$\begin{cases} = \max\left\{ \dfrac{m-1}{4}, 1 \right\} & \text{if } k = 2, \\ = \dfrac{(k-2)^2+m-2}{k^2} & \text{if } 3 \leq k \leq \frac{m+2}{4}, \\ = 1 & \text{if } \frac{m+2}{4} \leq k < \frac{2m}{3},\ k \geq 3, \\ \leq \max\left\{ 1, \dfrac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)} \right\} & \text{if } k \geq 2m/3,\ k \geq 3. \end{cases}$$

Proof. Because of symmetry in the objective function $f(\boldsymbol{x}) = \sum_{i=1}^{m-1} x_i^2$ and in the constraints in the lemma, we may assume without loss of generality that (30) holds. By the discussion before the lemma, the constraints are equivalent to (30), (33) and (34). Thus we have to find the maximum of $f(\boldsymbol{x})$ over the set Δ of points $(x_1, \ldots, x_{m-1})$ that satisfy (30), (33) and (34). By the discussion before the lemma, (31) automatically holds, hence Δ is a polytope. Setting $x_i = -i/km$ for $i \in [m-1]$, we see that (30) and (34) are obviously satisfied with strict inequalities, and

$$\sum_{i=m-k}^{m-1} x_i = \frac{-1}{km}\sum_{i=m-k}^{m-1} i = \frac{-1}{km}\sum_{i=1}^{k}(m-i) = -1 + \frac{k(k+1)}{2km} > -1.$$

It follows that

$$\left(\frac{-1}{km}, \frac{-2}{km}, \dots, \frac{-(m-1)}{km}\right) \in \mathbb{R}^{m-1}$$

is an interior point of Δ. Since (30), (33) and (34) are m inequalities in total, it follows that Δ is a simplex. Since $f(\boldsymbol{x}) = \sum_{i=1}^{m-1} x_i^2$ is a convex function on $\mathbb{R}^{m-1}$, it attains its maximum at a vertex of Δ. Next we calculate the m vertices of Δ. We distinguish between the following three cases:

Case I. Equality in (30) and (33):

$$x_1 = \dots = x_{m-1} \quad \text{and} \quad -1 = x_{m-k} + \dots + x_{m-1}.$$

The vertex is

$$\boldsymbol{x} = \Big(\underbrace{\frac{-1}{k}, \dots, \frac{-1}{k}}_{m-1 \text{ times}}\Big),$$

and

$$f(\boldsymbol{x}) = \frac{m-1}{k^2}.$$

Case II. Equality in (30) and (34):

$$x_1 = \dots = x_{m-1} \quad \text{and} \quad x_1 + \dots + x_{k-1} = 0.$$

Then $\boldsymbol{x} = \boldsymbol{o}$ and $f(\boldsymbol{x}) = 0 < (m-1)/k^2$.

Case III. For some $t \in [m-2]$,

$$x_1 = \dots = x_t =: a \quad \text{and} \quad x_{t+1} = \dots = x_{m-1} =: b$$

and equality in (33) and (34): Equality in (33) gives that

$$\text{if } m-k \geq t+1 \quad \text{then} \quad b = \frac{-1}{k}; \tag{33a}$$

$$\text{if } m-k \leq t \quad \text{then} \quad (k-m+1+t)a + (m-1-t)b = -1. \tag{33b}$$

Independent of these two cases equality in (34) gives that

$$\text{if } k-1 \leq t \quad \text{then} \quad a = 0; \tag{34a}$$

$$\text{if } k-1 \geq t+1 \quad \text{then} \quad ta + (k-1-t)b = 0. \tag{34b}$$

This gives us the following four subcases:

Subcase III.i. If $k-1 \le t \le m-k-1$, then by (33a) and (34a),

$$\boldsymbol{x} = \Big(\underbrace{0,\dots,0}_{t\text{ times}}, \underbrace{\frac{-1}{k},\dots,\frac{-1}{k}}_{m-1-t\text{ times}}\Big)$$

and

$$f(\boldsymbol{x}) = \frac{m-1-t}{k^2} \le \frac{m-k}{k^2} < \frac{m-1}{k^2}.$$

This case occurs only if $2k \le m$.

Subcase III.ii. If $\max\{k-1, m-k\} \le t$, then by (33b) and (34a),

$$\boldsymbol{x} = \Big(\underbrace{0,\dots,0}_{t\text{ times}}, \underbrace{\frac{-1}{m-1-t},\dots,\frac{-1}{m-1-t}}_{m-1-t\text{ times}}\Big)$$

and

$$f(\boldsymbol{x}) = \frac{1}{m-1-t} \le 1,$$

with equality if $t = m-2$. This case always occurs.

Subcase III.iii. If $t \le \min\{k-2, m-k-1\}$ (which occurs only if $k \ge 3$), then by (33a) and (34b),

$$\boldsymbol{x} = \Big(\underbrace{\frac{k-1-t}{kt},\dots,\frac{k-1-t}{kt}}_{t\text{ times}}, \underbrace{\frac{-1}{k},\dots,\frac{-1}{k}}_{m-1-t\text{ times}}\Big)$$

and

$$\begin{aligned} f(\boldsymbol{x}) &= \frac{1}{k^2}\left(\frac{(k-1)^2}{t} - 2k + 1 + m\right) \\ &\le \frac{1}{k^2}\big((k-1)^2 - 2k + 1 + m\big) = \frac{(k-2)^2 + m - 2}{k^2} =: g(k,m). \end{aligned}$$

Note that $g(k,m) \ge \frac{m-1}{k^2}$ (equality iff $k = 3$). Also, $g(k,m) \le 1$ iff $k \ge (m+2)/4$.

Subcase III.iv. If $m-k \le t \le k-2$ (which occurs only if $2k \ge m+2$ and $k \ge 4$), then we solve (33b) and (34b) to obtain

$$a = \frac{k-1-t}{t+(m-1-k)(k-1)} \quad \text{and} \quad b = \frac{-t}{t+(m-1-k)(k-1)}.$$

This gives the vertex as

$$\boldsymbol{x} = \Big(\underbrace{\frac{k-1-t}{t+(m-1-k)(k-1)}}_{t\text{ times}}, \underbrace{\frac{-t}{t+(m-1-k)(k-1)}}_{m-1-t\text{ times}}\Big)$$

and

$$f(\boldsymbol{x}) = \frac{(m-2k+1)t^2 + (k-1)^2 t}{(t+(m-1-k)(k-1))^2} =: s_{k,m}(t).$$

We now determine

$$h(k,m) := \max\left\{s_{k,m}(t) \,\middle|\, t \in [m-k, k-2]\right\}.$$

Since this maximum could occur in the interior of the interval $[m-k,k-2]$, and the value of t where the maximum occurs might not be integral, we settle for determining the maximum of $s_{k,m}(t)$ over all real values of $t\in[m-k,k-2]$. Thus $h(k,m)$ will only be an upper bound for the maximum of $f(\boldsymbol{x})$ on the vertices of Δ falling under this subcase. A calculation shows that $s'_{k,m}(t)\geq 0$ iff

$$t\leq \frac{(k-1)^2(m-k-1)}{2(2k-m-1)(m-k-1)+k-1}=: t_0.$$

We next show that $m-k\leq t_0$ unless $k=4$ and $m=6$. A calculation shows that

$$m-k\leq t_0 \iff (k-1)^2\leq\left(\frac{1}{2}k^2-k+2\Big(m-\frac{3}{2}k\Big)^2\right)(m-k).$$

If $m\neq 3k/2$ then $(m-3k/2)^2\geq 1/4$, and since $m-k\geq 2$ we obtain

$$\begin{aligned}&\left(\frac{1}{2}k^2-k+2\Big(m-\frac{3}{2}k\Big)^2\right)(m-k)\\ &\geq\left(\frac{1}{2}k^2-k+\frac{1}{2}\right)\cdot 2=(k-1)^2,\end{aligned}$$

which gives $m-k\leq t_0$. Otherwise, $m=3k/2$ and

$$m-k\leq t_0 \iff (k-1)^2\leq\left(\frac{1}{2}k^2-k\right)\frac{k}{2}.$$

This holds if $k\geq 5$, but not if $k=4$. However, in that case $(k,m)=(4,6)$ and $m-k=k-2$.

Next we show that if $k\geq 2m/3$ then $t_0<k-2$, and if $k<2m/3$ then $t_0>k-2$. A calculation gives

$$t_0\leq k-2 \quad\iff\quad 0\leq (k-2)(m-k)(3k-2m)+2k-m-1.$$

Since $2k-m-1>0$, we obtain $t_0<k-2$ if $k\geq 2m/3$. Otherwise $3k-2m\leq -1$, and

$$\begin{aligned}&(k-2)(m-k)(3k-2m)+2k-m-1\\ &\leq -(k-2)(m-k)+2k-m-1=-(k-1)(m-k-1)<0.\end{aligned}$$

It follows that $t_0>k-2$ if $k<2m/3$.

In summary,

$$h(k,m)=\begin{cases}s_{k,m}(t_0) & \text{if } k\geq 2m/3 \text{ and } (k,m)\neq(4,6),\\ s_{k,m}(k-2) & \text{if } k<2m/3 \text{ or } (k,m)=(4,6).\end{cases}$$

We next show that $s_{k,m}(k-2)<1$, which means that this subcase is only relevant when $k\geq 2m/3$ and $(k,m)\neq(4,6)$. Since

$$s_{k,m}(k-2)=\frac{(m-2k+1)(k-2)^2+(k-1)^2(k-2)}{(k-2+(m-1-k)(k-1))^2},$$

a calculation shows that

$$s_{k,m}(k-2)<1 \iff m-2k<(k-1)^2\left((m-k)(m-k-1)-1\right),$$

which holds since $m-2k<0$ and $m-k\geq 2$. Finally we calculate

$$s_{k,m}(t_0)=\frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)}.$$

This concludes estimating f at the vertices of Δ. To summarise the above case analysis, we have obtained that

$$\max_{\boldsymbol{x}\in\Delta} f(\boldsymbol{x})=\max\left\{\frac{m-1}{k^2},1,\frac{(k-2)^2+m-2}{k^2}\right\}\quad\text{if } k<2m/3,$$

and

$$\begin{aligned}&\max_{\boldsymbol{x}\in\Delta} f(\boldsymbol{x})\\ &\leq\max\left\{\frac{m-1}{k^2},1,\frac{(k-2)^2+m-2}{k^2},\frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)}\right\}\end{aligned}$$

if $k\geq 2m/3$. The remaining claim of the lemma is now easily checked. □

Proof of Theorem 22. (1) Let $\sqrt{d}<k\leq(d+1)/2$. Suppose that there exist $m>2d(1+\frac{d-2k+1}{k^2-d})$ vectors of norm ≥ 1 satisfying the k-collapsing condition; equivalently, an $m\times m$ matrix A of rank $\leq d$ satisfying the k-collapsing condition. Since $m>2d$ we have $k<(m+2)/4$. Therefore, by Lemma 41 the sum of the squares of the entries in any row of A is $\leq 1+\frac{(k-2)^2+m-2}{k^2}=2+\frac{m-4k+2}{k^2}$. By Lemma 39,

$$d\geq\operatorname{rank}(A)\geq\frac{m^2}{m\left(2+\frac{m-4k+2}{k^2}\right)}=\frac{mk^2}{2k^2+m-4k+2}.$$

Solving for m (and taking note that $k>\sqrt{d}$) we obtain

$$m\leq\frac{2d(k-1)^2}{k^2-d},$$

contradicting the assumption on m. This shows that $\overline{\mathscr{C}}(k,d)\leq\frac{2d(k-1)^2}{k^2-d}$.

(2) In particular we obtain that $\overline{\mathscr{C}}(k,d)\leq 2d$ if

$$\frac{2d(k-1)^2}{k^2-d}<2d+1,$$

which is equivalent to $k\geq -2d+\sqrt{6d^2+3d+1}$. It remains to show that $\overline{\mathscr{C}}(k,d)\leq 2d$ if $(d+1)/2<k\leq 2d-\sqrt{d/2}$. Suppose that there exists an $m\times m$ matrix A of rank $\leq d$ satisfying the k-collapsing condition, where $m=2d+1$. It then follows from $k>(d+1)/2$ that $k>(m+2)/4$. If furthermore $k<2m/3$ then by Lemmas 39 and 41, $d\geq\operatorname{rank}(A)\geq\frac{m^2}{m(1+1)}$ and $m\leq 2d$, a contradiction. Therefore, $k\geq 2m/3$. We next show that

$$\frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)}<1,\tag{36}$$

which again gives the contradiction $m\leq 2d$ by Lemmas 39 and 41.

Consider $f(x) = (m-x-1)(2x-m)(m-x)$, $2m/3 \le x \le m-2$. Then $f'(x) = (4m-6x)(m-x-1)-2x+m < 0$, and it follows that the left-hand side of (36) increases with k. It is therefore sufficient to prove (36) for $k = 2d - \sqrt{d/2}$, that is

$$\frac{(2d-\sqrt{d/2}-1)^2}{4\sqrt{d/2}(2d-2\sqrt{d/2}-1)(\sqrt{d/2}+1)} < 1.$$

This is equivalent to $8d\sqrt{d/2} - 5d/2 - 6\sqrt{d/2} - 1 > 0$, which is easily seen to be true.

(3) Let $d \ge 3$ and $k > 2d - \sqrt{d/2}$. Suppose that there exists a k-collapsing $m \times m$ matrix of rank $\le d$, where $m > k + \frac{1+\sqrt{2d-3}}{2}$. As before, we aim to find a contradiction using Lemmas 39 and 41.

Writing $t = m-k$, we have $t > \frac{1+\sqrt{2d-3}}{2} > 1$. It follows that $d < 2t^2 - 2t + 2 < 2t^2$, hence $k > 2d - \sqrt{d/2} > 2d - t$ and $m = k+t \ge 2d+1$.

Since we may assume without loss of generality that

$$m = \left\lfloor k + \frac{1+\sqrt{2d-3}}{2} \right\rfloor + 1,$$

we have $m < k+2+\sqrt{d/2}$. We may conclude that $m < 4k-2$ if $3k > 4+\sqrt{d/2}$, which follows from $k > 2d - \sqrt{d/2}$. Therefore, $k > (m+2)/4$. By Lemma 41, if $k < 2m/3$ or

$$\frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)} \le 1,$$

then Lemma 39 would give $d \ge \frac{m^2}{m(1+1)}$ and $m \le 2d$, a contradiction. Therefore, without loss of generality, $k \ge 2m/3$ and

$$\frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)} > 1.$$

Lemma 39 now gives

$$d \ge \frac{m^2}{m\left(1 + \dfrac{(k-1)^2}{4(m-k-1)(m-k)(m-2k)}\right)} = \frac{m}{\left(1 + \dfrac{(k-1)^2}{4(t-1)t(k-t)}\right)},$$

which implies

$$k+t = m \le \left(1 + \frac{(k-1)^2}{4(t-1)t(k-t)}\right) d. \tag{37}$$

If we set $f(x) = \left(1 + \frac{(x-1)^2}{4(t-1)t(x-t)}\right) d - (x+t)$ for $x \ge 2d-t+1$, it follows (since $d < 2t^2$ and $t \ge 2$) that

$$\begin{aligned} f'(x) &= \frac{d}{4(t-1)t}\left(1 - \left(\frac{t-1}{x-t}\right)^2\right) - 1 \\ &< \frac{2t^2}{4(t-1)t} - 1 = \frac{2-t}{2(t-1)} \le 0, \end{aligned}$$

and f is strictly decreasing. It follows that since (37) holds for some $k \geq 2d-t+1$, it remains true if we substitute $2d-t+1$ into k, that is,

$$2d+1 \leq \left(1+\frac{(2d-t)^2}{4(t-1)t(2d-2t+1)}\right)d, \tag{38}$$

which is equivalent to

$$4(d+1)(t-1)t(2d-2t+1) \leq (2d-t)^2 d. \tag{39}$$

We next show that the opposite inequality holds, which gives the required contradiction. Since $t = m-k = \lfloor \frac{1+\sqrt{2d-3}}{2} \rfloor + 1$,

$$t-1 \leq \frac{1+\sqrt{2d-3}}{2} < t,$$

or equivalently,

$$2t^2-6t+6 \leq d \leq 2t^2-2t+1. \tag{40}$$

Note that

$$\begin{aligned} & 4(d+1)(t-1)t(2d-2t+1)-(2d-t)^2 d \\ = \quad & (t-1)^3(6t+4)+(t-1)^2-1 \\ & +(2t^2-2t+1-d)\big((2d-t-2)^2+12d-4t^2-4t\big). \end{aligned} \tag{41}$$

By (40), since $t \geq 2$,

$$12d-4t^2-4t \geq 12(2t^2-6t+6)-4t^2-4t = (5t-9)(4t-8) \geq 0,$$

hence

$$(2t^2-2t+1-d)((2d-t-2)^2+12d-4t^2-4t) \geq 0.$$

Substitute this into (41) to obtain

$$\begin{aligned} & 4(d+1)(t-1)t(2d-2t+1)-(2d-t)^2 d \\ & \geq (t-1)^3(6t+4)+(t-1)^2-1 \\ & > 0, \end{aligned}$$

which contradicts (39). $\square$

Proof of Theorem 23. Suppose that there exists an $m \times m$ k-collapsing matrix of rank $\leq d$. We first treat the case $k=2$. By Lemmas 39 and 41,

$$\frac{m^2}{m(1+\max\{1,(m-1)/4\})} \leq d.$$

If the maximum in the denominator equals 1 then $m \leq 2d$. Otherwise, $m \leq (1+(m-1)/4)d$ and it follows that $(1-d/4)m \leq 3d/4$. If $d<4$ then $m \leq 3d/(4-d)$. In particular, if $d=2$ then $m \leq 3$, and if $d=3$ then $m \leq 9$. This shows that $\overline{\mathscr{C}}(2,2)=4$ and $\overline{\mathscr{C}}(2,3) \leq 9$.

Next assume that $k \geq 3$. Without loss of generality, $m=k+2>2d$. We aim for a contradiction. Clearly, $k=m-2>(m+2)/4$. If the maximum in Lemma 41 equals

1, Lemma 39 gives $m \leq 2d$, a contradiction. Therefore, $k \geq 2m/3$, the maximum in Lemma 41 equals

$$\frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)} = \frac{(m-3)^2}{8(m-4)} > 1, \tag{42}$$

and by Lemma 39,

$$\frac{m^2}{m\left(1+\frac{(m-3)^2}{8(m-4)}\right)} \leq d. \tag{43}$$

By (42), $m \geq 10$ and $k \geq 8$. Solving for m in (43) gives

$$m \leq \frac{d+16+2\sqrt{6d^2-38d+64}}{8-d}$$

if we assume $d < 8$. Since $k = m-2$, we obtain

$$k \leq \frac{3d+2\sqrt{6d^2-38d+64}}{8-d}.$$

Keeping in mind that $m = k+2 > 2d$ and $m \geq 10$, we obtain a contradiction if $d \leq 5$ (and $k \geq 3$); or if $d = 6$ and $k \geq 17$; or if $d = 7$ and $k \geq 41$. This proves the theorem. □

8. Upper bounds using the ranks of Hadamard powers of a matrix

The following lemma, used by Alon in [1, 2], bounds the ranks of the integral Hadamard powers of a square matrix from above in terms of the rank of the matrix. It can be used to change a matrix to one that is sufficiently close to the identity matrix so that Lemma 39 can give a good bound.

Lemma 42 (Alon [1, Lemma 9.2])**.** *Let $A = [a_{i,j}]$ be an $n \times n$ matrix of rank d (over any field), and let $p \geq 1$ be an integer. Then the rank of the p-th Hadamard power $A^{\odot p}$ satisfies*

$$\operatorname{rank}(A^{\odot p}) = \operatorname{rank}([a_{i,j}^p]) \leq \binom{p+d-1}{p}.$$

In order to use the above lemma in combination with Lemma 39 as before, we need to maximise $\sum_i x_i^{2p}$ on the simplex Δ from the proof of Lemma 41. Here we restrict the range of k to avoid the difficulties in Case III.iv in the proof of Lemma 41.

Lemma 43. *Let $p,k,m \in \mathbb{N}$ such that $2 \le k \le (m+1)/2$. Then*

$$\begin{aligned}\max\Bigg\{\sum_{i=1}^{m-1} x_i^{2p} \,\Bigg|\, & -1 \le \sum_{i\in I} x_i \le 1 \text{ for all } I \in \binom{[m]}{k},\\ & -2 \le \sum_{i\in I} x_i \le 0 \text{ for all } I \in \binom{[m-1]}{k-1}\Bigg\}\\ = & \begin{cases} \max\left\{1, \dfrac{m-1}{k^{2p}}\right\} & \text{if } k=2,\\ \max\left\{1, \dfrac{(k-2)^{2p}+m-2}{k^{2p}}\right\} & \text{if } k\ge 3.\end{cases}\end{aligned}$$

Proof. As in the proof of Lemma 41 we have to maximise the new objective function $f_p(\boldsymbol{x}) = \sum_{i=1}^{m-1} x_i^{2p}$ over the same simplex Δ defined by (30), (33) and (34) as before. Since f_p is convex, it is again sufficient to calculate the values of f_p on the vertices of Δ. Using the same case numbering as in the proof of Lemma 41, we obtain the following values:

Case I. $f_p(\boldsymbol{x}) = \dfrac{m-1}{k^{2p}}$.

Case II. $f_p(\boldsymbol{x}) = 0 < \dfrac{m-1}{k^{2p}}$.

Subcase III.i. $f_p(\boldsymbol{x}) = \frac{m-1-t}{k^{2p}} \le \frac{m-k}{k^{2p}} < \frac{m-1}{k^{2p}}$.

Subcase III.ii. $f_p(\boldsymbol{x}) = \frac{1}{(m-1-t)^{2p-1}} \le 1$ with equality iff $t = m-2$.

Subcase III.iii.

$$\begin{aligned} f_p(\boldsymbol{x}) &= \frac{1}{k^{2p}}\left(t\left(\left(\tfrac{k-1}{t}-1\right)^{2p}-1\right)+m-1\right) =: g_p(t)\\ &\le g_p(1) = \frac{1}{k^{2p}}\left((k-2)^{2p}+m-2\right)\end{aligned}$$

since $g_p(t)$ is decreasing for $0 < t < k-1$. This case occurs only if $k \ge 3$.

Subcase III.iv. The case $m-k \le t \le k-2$ occurs only if $2k \ge m+2$, which we have assumed to be false. □

Lemma 44. *If $p \in \mathbb{N}$ and $k > \binom{d+p-1}{p}^{\frac{1}{2p}}$ then*

$$\overline{\mathscr{C}}(k,d) < \max\left\{\frac{2k^{2p}\binom{d+p-1}{p}}{k^{2p}-\binom{d+p-1}{p}}, 2k-1\right\}.$$

Proof. Suppose that there exists a k-collapsing $m\times m$ matrix $A = [a_{i,j}]$ of rank at most d, where $m = \overline{\mathscr{C}}(k,d)$. Without loss of generality, $m \ge 2k-1$. By Lemma 43, for any row $i \in [m]$ of $A^{\odot 2p}$,

$$\sum_{j=1}^{m} a_{i,j}^{2p} < 2 + \frac{m}{k^{2p}}$$

and by Lemmas 39 and 42,

$$\binom{p+d-1}{p} \geq \operatorname{rank}([a_{i,j}^{2p}]) > \frac{m^2}{m\left(2+\frac{m}{k^{2p}}\right)},$$

from which follows

$$m < \frac{2k^{2p}\binom{d+p-1}{p}}{k^{2p}-\binom{d+p-1}{p}}. \qquad \square$$

Proof of Theorem 21. This is just a calculation from Lemma 44. Since

$$\frac{k^{2p}}{\binom{d+p-1}{p}} > \frac{((p!)^{-1/2p}+\varepsilon)^{2p}d^p}{\binom{d+p-1}{p}} \xrightarrow{d\to\infty} (1+(p!)^{1/2p}\varepsilon)^{2p} > 1+2p(p!)^{1/2p}\varepsilon,$$

it follows that if d is sufficiently large depending on p and ε, then

$$\frac{k^{2p}}{\binom{d+p-1}{p}} > 1+p(p!)^{1/2p}\varepsilon =: 1+\delta,$$

where $\delta > 0$ depends only on p and ε. Then

$$\binom{d+p-1}{p}^{-1} - k^{-2p} > \left(1-\frac{1}{1+\delta}\right)\binom{d+p-1}{p},$$

and by Lemma 44, (since $\overline{\mathscr{C}}(k,d) \geq 2d \geq 2k^2 > 2k-1$)

$$\overline{\mathscr{C}}(k,d) < \frac{2}{\binom{d+p-1}{p}^{-1}-k^{-2p}} < \frac{2\binom{d+p-1}{p}}{1-(1+\delta)^{-1}} < \frac{2(2d)^p}{(1-(1+\delta)^{-1})p!}$$

if we assume $d > p$. $\square$

Lemma 45. *Let $n > k \geq 1$ be integers and $\varepsilon = k/n$. Then*

$$\binom{n}{k} < \frac{(\varepsilon^{-\varepsilon}(1-\varepsilon)^{-(1-\varepsilon)})^n}{\sqrt{2\pi\varepsilon(1-\varepsilon)n}}.$$

Proof. Substitute the Stirling formula in the form $m! = \mathrm{e}^{\delta_m}(\frac{m}{\mathrm{e}})^m\sqrt{2\pi m}$, where $\frac{1}{12m+1} < \delta_m < \frac{1}{12m}$ [27] into $\frac{n!}{k!(n-k)!}$ to obtain

$$\binom{n}{k} < \frac{(\varepsilon^{-\varepsilon}(1-\varepsilon)^{-(1-\varepsilon)})^n}{\sqrt{2\pi\varepsilon(1-\varepsilon)n}}\mathrm{e}^{\frac{1}{12n}-\frac{1}{12k+1}-\frac{1}{12(n-k)+1}}.$$

It is easily seen that $\frac{1}{a+b} < \frac{1}{a+1}+\frac{1}{b+1}$ for all $a,b \geq 1$. In particular, $\frac{1}{12n} < \frac{1}{12k+1} + \frac{1}{12(n-k)+1}$ and the lemma follows. $\square$

Proof of Theorem 20. The function $f(x) = (1+x)^{1/x}(1+1/x)$ is strictly decreasing on $(0,1]$ with $\lim_{x\to 0+} f(x) = \infty$ and $f(1) = 4$. Therefore, $\gamma_2 = 1$ and (γ_k) is strictly decreasing. Since $f(x) < \mathrm{e}\cdot(1+1/x)$, we have $f(\mathrm{e}/(k^2-\mathrm{e})) < k^2$ and $\gamma_k < \mathrm{e}/(k^2-\mathrm{e})$. Also, since

$$\frac{x}{x+1} = 1-\frac{1}{x+1} < \mathrm{e}^{-1/(x+1)},$$

it follows that $(1+1/x)^{x+1} > \mathrm{e}$. Set $x = k^2/\mathrm{e}$ to obtain that $f(\mathrm{e}/k^2) > k^2$ and $\mathrm{e}/k^2 < \gamma_k$.

Let $p := \lceil \gamma_k d \rceil$ and $\gamma := p/d$. Then $\gamma \geq \gamma_k$ and it follows that

$$(1+\gamma)^{1/\gamma}\Big(1+\frac{1}{\gamma}\Big) \leq k^2. \tag{44}$$

We estimate $\binom{p+d-1}{p}$ as follows:

$$\begin{aligned}\binom{p+d-1}{p} &= \binom{(1+\gamma)d-1}{\gamma d} = \frac{1}{1+\gamma}\binom{(1+\gamma)d}{\gamma d} \\ &< \frac{\big((1+1/\gamma)^{\gamma}(1+\gamma)\big)^d}{\sqrt{2\pi\gamma(1+\gamma)d}} \quad \text{by Lemma 45} \\ &\leq \frac{k^{2\gamma d}}{\sqrt{2\pi\gamma(1+\gamma)d}} \quad \text{by (44)} \\ &= \frac{k^{2p}}{\sqrt{2\pi\gamma(1+\gamma)d}}.\end{aligned}$$

In particular, $\binom{p+d-1}{p} < k^{2p}$ since

$$\sqrt{2\pi\gamma(1+\gamma)d} > \sqrt{2\pi\gamma d} = \sqrt{2\pi p} \geq \sqrt{2\pi} > 1.$$

By Lemma 44, either $\overline{\mathscr{C}}(k,d) < 2k-1$ or

$$\begin{aligned}\overline{\mathscr{C}}(k,d) &< \frac{2k^{2p}k^{2p}}{\sqrt{2\pi\gamma(1+\gamma)d}\left(k^{2p} - \frac{k^{2p}}{\sqrt{2\pi\gamma(1+\gamma)d}}\right)} \\ &= \frac{2k^{2p}}{\sqrt{2\pi\gamma(1+\gamma)d}-1}.\end{aligned}$$

This gives

$$\overline{\mathscr{C}}(k,d) < \max\left\{\frac{2}{\sqrt{2\pi}-1}k^{2p}, 2k-1\right\} < 1.33k^{2\gamma_k d+2}.$$

We now assume that $k < \sqrt{d}$. Then $\overline{\mathscr{C}}(k,d) \geq 2d > 2k-2$ and

$$\begin{aligned}\overline{\mathscr{C}}(k,d) &< \frac{2k^{2p}}{\sqrt{2\pi\gamma(1+\gamma)d}-1} \\ &< \frac{2k^{2\gamma_k+2}}{\sqrt{2\pi\gamma_k d}-1} \\ &< \frac{2k^{2\gamma_k+2}}{\sqrt{2\pi(\mathrm{e}/k^2)d}-1}.\end{aligned}$$

Since $\sqrt{2\pi\mathrm{e}} > 3$ and $d/k^2 > 1$, it follows that $\sqrt{2\pi(\mathrm{e}/k^2)d}-1 > 2\sqrt{d/k^2}$ and $\overline{\mathscr{C}}(k,d) < k^{3+2\gamma_k d}/\sqrt{d}$. $\square$

9. LOWER BOUNDS

Lemma 46. *Let $k \geq 2$. Suppose there exist at least m unit vectors $\boldsymbol{u}_i \in \ell_2^{d-1}$ such that*

$$\left|\left\langle \boldsymbol{u}_i, \boldsymbol{u}_j \right\rangle\right| \leq \frac{1}{2k+1} \quad \textit{for all distinct } i, j.$$

Then there exists a d-dimensional Minkowski space X^d such that $\mathscr{C}_k(X^d) \geq m$. If $k \geq 3$ or if $k = 2$ and $-1/5 \leq \left\langle \boldsymbol{u}_i, \boldsymbol{u}_j \right\rangle < 1/5$ for all distinct i, j, then X^d can be chosen to be strictly convex and C^∞.

Proof. The construction is similar to the construction in [13] of a strictly convex d-dimensional space X^d such that $C_2(X^d) \geq 1.02^d$. The main difference is that we define the unit ball as an intersection of half spaces instead of a convex hull of a finite set of points.

Consider ℓ_2^{d-1} to be a hyperplane of ℓ_2^d with unit normal $\boldsymbol{e}$. Let $\boldsymbol{x}_i = \boldsymbol{u}_i + \boldsymbol{e}$ and $\boldsymbol{y}_i = (1 + \frac{1}{2k})\boldsymbol{u}_i - \frac{1}{2k}\boldsymbol{e}$ for each $i \in [m]$. Let

$$B := \left\{ \boldsymbol{x} \in \ell_2^d \,\middle|\, |\langle \boldsymbol{x}, \boldsymbol{y}_i \rangle| \leq 1 \text{ for all } i \in [m] \right\}.$$

If $\operatorname{span}(\{\boldsymbol{y}_i\}) = \mathbb{R}^d$ then B is bounded and the unit ball of some norm $\|\cdot\|_B$. Otherwise $\{\boldsymbol{y}_i\}$ spans a hyperplane with normal $\boldsymbol{e}'$, say. In this case B as defined above is unbounded, so we have to modify it. Before doing that, we show that $\boldsymbol{x}_i \in \partial B$ and

$$\sum_{i \in I} \boldsymbol{x}_i \in B \quad \text{for all } I \in \binom{[m]}{k}.$$

Let $i, j \in [m]$. Then

$$\left\langle \boldsymbol{x}_i, \boldsymbol{y}_j \right\rangle = \left(1 + \frac{1}{2k}\right) \left\langle \boldsymbol{u}_i, \boldsymbol{u}_j \right\rangle - \frac{1}{2k}. \tag{45}$$

In particular, $\langle \boldsymbol{x}_i, \boldsymbol{y}_i \rangle = 1$, and since

$$-\frac{1}{2k+1} \leq \left\langle \boldsymbol{u}_i, \boldsymbol{u}_j \right\rangle \leq \frac{1}{2k+1} \quad \text{for distinct } i, j, \tag{46}$$

we obtain $-1 < -\frac{1}{k} \leq \left\langle \boldsymbol{x}_i, \boldsymbol{y}_j \right\rangle \leq 0 < 1$, and it follows that $\boldsymbol{x}_i \in \partial B$.

Next let $I \in \binom{[m]}{k}$ and $i \in [m]$. We distinguish between two cases, depending on whether $i \in I$ or not.

If $i \notin I$, then by (45),

$$\left\langle \sum_{j \in I} \boldsymbol{x}_j, \boldsymbol{y}_i \right\rangle = \left(1 + \frac{1}{2k}\right) \sum_{j \in I} \left\langle \boldsymbol{u}_i, \boldsymbol{u}_j \right\rangle - \frac{1}{2},$$

and by (46),

$$-1 < \frac{-1}{2k} - \frac{1}{2} \leq \left\langle \sum_{j \in I} \boldsymbol{x}_j, \boldsymbol{y}_i \right\rangle \leq \frac{1}{2k} - \frac{1}{2} < 0.$$

If $i \in I$, then again by (45),

$$\left\langle \sum_{j \in I} \boldsymbol{x}_j, \boldsymbol{y}_i \right\rangle = \left(1 + \frac{1}{2k}\right) \sum_{j \in I \setminus i} \left\langle \boldsymbol{u}_i, \boldsymbol{u}_j \right\rangle + \frac{1}{2} + \frac{1}{2k},$$

and by (46),

$$0 < \frac{1}{k} \leq \Big\langle \sum_{j\in I} \boldsymbol{x}_j, \boldsymbol{y}_i \Big\rangle \leq 1.$$

In both cases we have $\left|\left\langle \sum_{j\in I} \boldsymbol{x}_j, \boldsymbol{y}_i \right\rangle\right| \leq 1$ for all i, and it follows that $\sum_{j\in I} \boldsymbol{x}_j \in B$ for all I. If $\operatorname{span}(\{\boldsymbol{y}_i\}) = \mathbb{R}^d$, then we have shown that B is the unit ball of a norm $\|\cdot\|_B$ such that $\{\boldsymbol{x}_i\}$ is a k-collapsing family of unit vectors in $\mathbb{R}^d, \|\cdot\|_B)$. In the case where $\operatorname{span}(\{\boldsymbol{y}_i\})$ is a hyperplane with normal $\boldsymbol{e}'$, we choose $\lambda > 0$ sufficiently large so that $|\langle \boldsymbol{x}_i, \boldsymbol{e}'\rangle| < \lambda$ for all i and $|\langle \sum_{i\in I} \boldsymbol{x}_i, \boldsymbol{e}'\rangle| < \lambda$ for all $I \in \binom{[m]}{k}$, and define the required unit ball to be

$$B := \left\{ \boldsymbol{x} \in \ell_2^d \,\middle|\, |\langle \boldsymbol{x}, \boldsymbol{y}_i\rangle| \leq 1 \text{ for all } i \in [m] \text{ and } \left|\left\langle \boldsymbol{x}, \boldsymbol{e}' \right\rangle\right| \leq \lambda \right\}.$$

If $k \geq 3$ or if $k = 2$ and $\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle < 1/5$ for distinct i, j, then $\left|\left\langle \sum_{j\in I} \boldsymbol{x}_j, \boldsymbol{y}_i \right\rangle\right| < 1$ for all i. Therefore, $\sum_{j\in I} \boldsymbol{x}_j \in \operatorname{int} B$ for all I. Also note that no $\boldsymbol{x}_j$, $j \neq i$ is on any of the hyperplanes

$$\left\{ \boldsymbol{x} \in \ell_2^d \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{y}_i\rangle = \pm 1 \right\} \text{ or } \left\{ \boldsymbol{x} \in \ell_2^d \,\middle|\, \left\langle \boldsymbol{x}, \boldsymbol{e}' \right\rangle = \pm\lambda \right\}.$$

Then a strictly convex and C^∞ norm can be found with unit ball between $\operatorname{conv}\{\boldsymbol{x}_i\}$ and B [14]. ☐

For a detailed proof of the following lemma, see [35]. It uses a greedy construction.

Lemma 47. *Let $\delta > 0$. For sufficiently large d depending on δ, there exist $m \geq \left(1 + \frac{\delta^2}{2}\right)^d$ unit vectors $\boldsymbol{u}_i$ in ℓ_2^{d-1} such that $\left|\left\langle \boldsymbol{u}_i, \boldsymbol{u}_j \right\rangle\right| < \delta$ for all distinct i, j.*

Proof of Theorem 29. This theorem follows immediately from Lemmas 46 and 47. ☐

The following construction was explained to the author by Noga Alon (personal communication).

Lemma 48. *Let q be a prime power and $s \in \mathbb{N}$ with $s < q$. Then there exist q^{s+1} unit vectors in $\ell_2^{q^2-q}$ such that the inner product of any two vectors is in the interval $[-\frac{1}{q-1}, \frac{s-1}{q-1}]$.*

Proof. Let $\mathscr{P}_s$ be the collection of polynomials over the field of q elements of degree at most s:

$$\mathscr{P}_s = \left\{ p \in \mathbb{F}_q[x] \,\middle|\, \deg(p) \leq s \right\}.$$

Then $|\mathscr{P}_s| = q^{s+1}$. For each $p \in \mathscr{P}_s$ define a real $q \times q$ matrix $M(p)$ by

$$M(p)_{i,j} = \begin{cases} 1 & \text{if } p(i) = j, \\ -\frac{1}{q-1} & \text{if } p(i) \neq j. \end{cases}$$

These matrices are in the q^2-dimensional vector space of all real $q \times q$ matrices with inner product $\langle A, B\rangle = \sum_{i=1}^q \sum_{j=1}^q a_{i,j} b_{i,j}$.

Note that $M(p_1) = M(p_2)$ iff $p_1(x) = p_2(x)$ for all $x \in \mathbb{F}_q$. It follows that if $s < q$ then all $M(p)$ ($p \in \mathscr{P}_s$) are distinct. Otherwise $M(p_1) = M(p_2)$ for some $p_1, p_2 \in \mathbb{F}_q[x]$ with $p_1 \neq p_2$, and then $p_1 - p_2$ would have $q > s \geq \deg(p_1 - p_2)$ roots, implying that $p_1 - p_2$ is the zero polynomial. This also shows that two distinct polynomials from $\mathscr{P}_s$ are equal at at most s points.

Let $p_1, p_2 \in \mathscr{P}_s$ with p_1 and p_2 not necessarily distinct. Let c denote the number of points where p_1 and p_2 coincide. Then

$$\begin{aligned}\langle M(p_1), M(p_2)\rangle &= c - 2(q-c)\frac{1}{q-1} + (q^2 - 2q + c)\frac{1}{(q-1)^2}\\ &= (c-1)\left(\frac{q}{q-1}\right)^2.\end{aligned}$$

If $p_1 \neq p_2$, then $0 \leq c \leq s$ and

$$-\left(\frac{q}{q-1}\right)^2 \leq \langle M(p_1), M(p_2)\rangle \leq (s-1)\left(\frac{q}{q-1}\right)^2.$$

On the other hand, since a polynomial coincides with itself at exactly q points, $\langle M(p), M(p)\rangle = \frac{q^2}{q-1}$. Thus $\frac{\sqrt{q-1}}{q}M(p)$ has norm 1, and inner products of distinct $\frac{\sqrt{q-1}}{q}M(p)$ lie in $[\frac{-1}{q-1}, \frac{s-1}{q-1}]$. Since each column of each $M(p)$ sums to 0, the $M(p)$ lie in a $(q^2 - q)$-dimensional subspace of the space of $q \times q$ matrices. $\square$

Proof of Theorem 30. Set $s = c + 1$ in Lemma 48 and then apply Lemma 46. $\square$

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