Applications of Lagrangian reduction to condensed matter

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Abstract

We consider a general approach for reduction procedure in chiral gauge models. We study two types of reductions: Lagrange-Poincaré and Euler-Poincaré reductions. We show that several interesting systems from Condensed Matter, like superfluid liquids and nematic liquid crystals also embedded in this general scheme.

Contents

1 Introduction

In condensed matter physics there are a lot of systems exhibiting common underlying properties. For example, they are determined by a Ginzburg-Landau equation with a

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multidimensional order parameter. These systems include uniaxial and biaxial nematics in the theory of liquid crystals, superfluid core in neutron stars, and superfluid helium ^{3}He.

One of the interesting features of this system is the existence of different thermodynamic states. Th approach based on the indification of thermodynamic phases with orbits of the potential of free energy was developed in Golo and Monastyrsky [1978]; Bogomolov and Monastyrsky [1987].

It is important to point out that, generally, these systems can be obtained by a reduction procedure developed in Castrillón-López and Ratiu [2003], Ellis, Gay-Balmaz, Holm, Ratiu [2011], Gay-Balmaz and Tronci [2010]. The goal of this paper is to show the effectiveness of this method that manages to unify these properties and techniques.

2 Lagrange-Poincaré versus Euler-Poincaré reduction in field theory

In this section we shall quickly review the general theory of Lagrange-Poincaré and Euler-Poincaré reduction for field theoretical problems. We shall emphasize the case of discrete symmetry groups since these appear often in applications.

Geometric setup. We start with a general construction. Let the manifold M be the parameter space of the theory and let $\Phi : G \times M \to M$ be a left transitive Lie group action. Usually, M is a particular orbit of the action of G on a bigger manifold. Selecting one particular orbit corresponds to choosing a particular phase of the physical system.

Choose an element $m_0 \in M$ and consider the isotropy subgroup $H := G_{m_0}$. We have the diffeomorphism $G/H \ni [g] := gH \stackrel{\sim}{\longmapsto} gm_0 \in M$, where H acts on G by right multiplication $R_h g := gh$ for all $h \in H$ and $g \in G$. We shall always identify M with G/H via this diffeomorphism.

Let $\pi_{X,\Sigma} : \Sigma := M \times X \ni (m, x) \mapsto x \in X$ be a trivial fiber bundle over the n-dimensional oriented manifold X. The theory described below generalizes to arbitrary locally trivial fiber bundles $\pi_{X,\Sigma} : \Sigma \to X$ but for the examples presented in this paper, it suffices to consider trivial bundles, where many of the formulas simplify. We shall fix in what follows a volume form μ on X, i.e., if $(x^1, \ldots, x^n)$ are local coordinates on the the open set $U \subset X$, then $\mu = a dx^1 \wedge \ldots \wedge dx^n$ for a smooth nowhere vanishing function $a \in C^\infty(U, \mathbb{R})$.

In general, a first order field theory is defined on the first jet bundle $J^1\Sigma \to \Sigma$ of the tangent bundle $T\Sigma \to \Sigma$ of a manifold Σ; as opposed to the tangent bundle $T\Sigma$, which is a vector bundle, the first jet bundle $J^1\Sigma$ is an affine bundle over Σ. However, for the trivial bundle $\pi_{X,\Sigma} : \Sigma \to X$, we have the identification $J^1_{(m,x)}\Sigma = L(T_xX, T_mM)$ of the fibers over $(m, x) \in M \times X$, where $L(T_xX, T_mM)$ is the vector space of linear maps from T_xX to T_mM, i.e., the first jet bundle $\pi_{M\times X, L(TX,TM)} : J^1\Sigma = L(TX, TM) \to \Sigma = M \times X$ coincides, in this case, with the vector bundle of linear bundle maps $L(TX, TM)$ from TX to TM over the base $M \times X$, namely, $L(T_xX, T_mM) \ni \lambda_{(m,x)} \mapsto (m, x) \in M \times X$. It is important in what follows to regard the jet bundle $J^1\Sigma$ also as a locally trivial fiber bundle over X, namely, $\pi_{X,L(TX,TM)} : L(TX,TM) \to X$ is given by $\pi_{X,L(TX,TM)}(\lambda_{x,m}) = \pi_{X,\Sigma}(m,x) = x$ for all $(m,x) \in \Sigma = M \times X$.

We suppose that the field theory is described by a Lagrangian density $\mathcal{L}\mu : J^1\Sigma \to \Lambda^n X$, where $\Lambda^n X$ is the bundle of n-forms on X, i.e., the sections of the line bundle $\Lambda^n X \to X$ are n-forms whose local expression is $f dx^1 \wedge \ldots \wedge dx^n$ for an arbitrary smooth function $f \in C^\infty(U,\mathbb{R})$, μ is a fixed volume form on X (a basis of the space of sections of $\Lambda^n X \to X$), and $\mathcal{L} : J^1\Sigma \to \mathbb{R}$ is a smooth function, i.e., we have

$$\mathcal{L} = \mathcal{L}(T_x m) = \mathcal{L}(m^i, \partial_j m^k)$$

for a smooth map $m : X \to M$ which in local coordinates is written as $(x^1, \ldots, x^n) \mapsto (m^1(x^1,\ldots,x^n), \ldots, m^p(x^1,\ldots,x^n))$, $p := \dim M$, and $\partial_j := \partial/\partial x^j$. Critical sections are given by solutions of the covariant Euler-Lagrange equations associated to $\mathcal{L}$.

Since the G-action is transitive on $M = G/H$, any section $m : X \to M$ can be written as $m(x) = \Phi_{g(x)}(m_0)$, where $g : X \to G$, i.e., g is a section of the trivial principal bundle $\pi_{X,P} : P = X \times G \to X$. By inserting this relation into the action functional

$$\int_X \mathcal{L}(T_x m)\mu,$$

and using the formula $T_x m(v_x) = (T_x g(v_x) g(x)^{-1})_M \left(\Phi_{g(x)} m_0\right)$, we can write the action functional in terms of g as

$$\int_X \mathcal{L}\left(\left(T_x g(v_x) g(x)^{-1}\right)_M \left(\Phi_{g(x)} m_0\right)\right)\mu.$$

This suggests the definition of the Lagrangian density $\mathfrak{L}_{m_0}\mu : J^1 P \to \Lambda^n X$ defined by

$$\mathfrak{L}_{m_0}(T_x g) := \mathcal{L}\left(\left(T_x g(v_x) g(x)^{-1}\right)_M \left(\Phi_{g(x)} m_0\right)\right).$$

This Lagrangian is G_{m_0}-invariant since

$$\mathfrak{L}_{m_0}(TR_h \circ T_x g) = \mathfrak{L}_{m_0}(T_x g), \quad \text{for all} \quad h \in H.$$

Our goal is to find an explicit the relation between the Euler-Lagrange equations for $\mathfrak{L}_{m_0}$ and $\mathcal{L}$ as well as to deduce another simpler equivalent formulation of these equations.

To do this, we fix a H-invariant Lagrangian density $\mathfrak{L}\mu$ defined on $J^1 P$ and carry out two reductions processes for $\mathfrak{L}$. The first one is a covariant Lagrange-Poincaré reduction, the second one is a covariant Euler-Poincaré reduction with parameters. This corresponds to two realizations of the quotient space $J^1 P/H$.

Lagrange-Poincaré approach. The Lagrange-Poincaré version corresponds to the vector bundle isomorphism

$$\beta_{\mathcal{A}} : L(TX,TG)/H \to L(TX,TM) \times_\Sigma L(TX,\widetilde{\mathfrak{h}}) = L(TX,TM) \oplus L(TX,\widetilde{\mathfrak{h}})$$

over $\Sigma := P/H = M \times X$, where $\widetilde{\mathfrak{h}} := G \times_H \mathfrak{h} \to M$ is the adjoint bundle. The notation used above is the standard one. The adjoint bundle $G \times_H \mathfrak{h} := (G \times \mathfrak{h})/H \to G/H$

is a Lie algebra bundle with fiber $\mathfrak{h}$, where the right H-action on $G \times \mathfrak{h}$ is given by $(g, \eta) \cdot h := (gh, \mathrm{Ad}_{h^{-1}} \eta)$ for all $h \in H$, $g \in G$, and $\eta \in \mathfrak{h}$. On the right hand side, $L(TX, TM) \times_\Sigma L(TX, \widetilde{\mathfrak{h}})$ denotes the vector bundle whose fiber at $(m, x) \in M \times X$ is $L(T_xX, T_mM) \oplus L(T_xX, \widetilde{\mathfrak{h}}_m)$; note that in both factors we have the same (m, x).

This vector bundle isomorphism $\beta_{\mathcal{A}}$ depends on the choice of a principal connection $\mathcal{A} \in \Omega^1(P, \mathfrak{h})$ on the principal bundle $P = G \times X \to \Sigma = P/H = (G/H) \times X = M \times X$. Note that we can choose here $\mathcal{A}$ as a principal connection on the right principal H-bundle $\pi : G \to M = G/H$, i.e., $\mathcal{A} \in \Omega^1(G, \mathfrak{h})$. Denoting by $l_{(x,g)}$ a linear map in $L(T_xX, T_gG)$, the isomorphism restricted to the fiber at (m, x) is given by (see Castrillón-López and Ratiu [2003], Ellis, Gay-Balmaz, Holm, Ratiu [2011])

$$\begin{aligned}\beta_{\mathcal{A}}([l_{(x,g)}]_H) := \left(T_g\pi(l_{(x,g)}(\_)), \left[g, \mathcal{A}(l_{(g,x)}(\_))\right]_H\right) = \left((l_{(x,g)}(\_)g^{-1})_M(m), \left[g, \mathcal{A}(l_{(g,x)}(\_))\right]_H\right)\\ \in L(T_xX, T_mM) \times L(T_xX, \widetilde{\mathfrak{h}}_m), \quad m = \pi(g).\end{aligned}$$

So, if $g : X \to G$ is a given section and $m := \pi \circ g : X \to M$, the reduced section reads

$$\begin{aligned}\beta_{\mathcal{A}}\left([T_xg]_H\right) = (T\pi \circ T_xg, [g(x), \mathcal{A}(T_xg(\_))]_H) =: (T_xm, \sigma(x))\\ \in L(T_xX, T_{m(x)}M) \times L(T_xX, \widetilde{\mathfrak{h}}_{m(x)}).\end{aligned}$$

From the given right H-invariant Lagrangian density $\mathfrak{L}\mu : J^1(X \times G) \to \Lambda^n(X)$, we get the reduced Lagrangian denisity $\ell\mu : L(TX, TM) \times_\Sigma L(TX, \widetilde{\mathfrak{h}}) \to \Lambda^n(X)$, $\ell = \ell(T_xm, \sigma(x))$, obtained via the Lagrange-Poincaré process:

$$\mathfrak{L}(T_xg) = \ell\left(T\pi \circ T_xg, [g(x), \mathcal{A}(T_xg(\_))]_H\right). \tag{2.1}$$

The reduced Euler-Lagrange equations on the Lagrange-Poincaré side have been derived in Castrillón-López and Ratiu [2003], Ellis, Gay-Balmaz, Holm, Ratiu [2011]. They are obtained by computing the variations δm and $\delta^{\mathcal{A}}\sigma$ of the reduced variables $m : X \to M$ and the section $\sigma : X \to L(TX, \widetilde{\mathfrak{h}})$. Let $g : U \subset X \to G$ be defined on an open subset such that $\bar{U}$ is compact. Recall that the Euler-Lagrange equations are obtained by the critical action principle

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_U \mathfrak{L}(T_xg_\varepsilon)\mu = 0,$$

for all smooth variations $g_\varepsilon : U \subset X \to G$ such that $g_0 = g$ and $g_\varepsilon|_{\partial U} = g|_{\partial U}$.

The variation of m is such that $\delta m|_{\partial U} = 0$. The variation of σ is given by

$$\begin{aligned}\delta^{\mathcal{A}}\sigma(x) &= \left.\frac{D^{\mathcal{A}}}{D\varepsilon}\right|_{\varepsilon=0} [g_\varepsilon(x), \mathcal{A}(T_xg_\varepsilon(\_)]_H\\ &= \nabla^{\mathcal{A}}\eta(x) + [\eta(x), \sigma(x)] + m^*\left(\mathbf{i}_{\delta m}\widetilde{\mathcal{B}}\right)(x) \in L\left(T_xX, \widetilde{\mathfrak{h}}_{m(x)}\right),\end{aligned}$$

where $\widetilde{\mathcal{B}} \in \Omega^2(M, \widetilde{\mathfrak{h}})$ is the reduced curvature on the base associated to the connection $\mathcal{A} \in \Omega^1(G, \mathfrak{h})$, $\eta(x) = [g(x), \mathcal{A}(\delta g(x))]_H \in \widetilde{\mathfrak{h}}_{m(x)}$ is arbitrary such that $\eta|_{\partial U} = 0$, and $[\eta(x), \sigma(x)]$ is the Lie bracket in the fiber $\widetilde{\mathfrak{h}}_{m(x)}$ of the adjoint bundle.

The partial derivatives of the reduced Lagrangian $\ell : L(TX, TM) \times_\Sigma L(TX, \widetilde{\mathfrak{h}}) \to \mathbb{R}$ are denoted by

$$\frac{\delta \ell}{\delta m}(x) \in T^*_{m(x)}M, \quad \frac{\delta \ell}{\delta\, Tm}(x) \in L\left(T^*_x X, T^*_{m(x)}M\right), \quad \frac{\delta \ell}{\delta \sigma}(x) \in L\left(T^*_x X, \widetilde{\mathfrak{h}}^*_{m(x)}\right).$$

The first two partial derivatives are fiber derivatives, the third one is the horizontal partial derivative defined with the help of the connection $\mathcal{A}$ and a covariant derivative on M (see Castrillón-López and Ratiu [2003], [Ellis, Gay-Balmaz, Holm, Ratiu, 2011, equation (3.22)] with a sign change in the first equation because in these papers H acts on the left whereas here the H-action is on the right). The resulting reduced Euler-Lagrange equations, i.e., the Lagrange-Poincaré equations, are

$$\operatorname{div}^{\mathcal{A}} \frac{\delta \ell}{\delta \sigma} + \operatorname{ad}^*_\sigma \frac{\delta \ell}{\delta \sigma} = 0, \qquad \frac{\delta \ell}{\delta m} - \operatorname{div} \frac{\delta \ell}{\delta\, Tm} = \left\langle \frac{\delta \ell}{\delta \sigma}, \mathbf{i}_{Tm} \widetilde{\mathcal{B}} \right\rangle, \tag{2.2}$$

where $\operatorname{div}^{\mathcal{A}} : \Gamma\left(\pi_{X, L\left(T^*X, \widetilde{\mathfrak{h}}^*\right)}\right) \to \Gamma(\pi_{X, \widetilde{\mathfrak{h}}^*})$ is the covariant divergence associated to $\mathcal{A}$, defined as minus the adjoint differential operator to the covariant derivative $\nabla^{\mathcal{A}}$, and $\operatorname{div} : \Gamma\left(\pi_{X, L(T^*X, T^*M)}\right) \to \Gamma(\pi_{X, T^*M})$ is the divergence operator associated to a covariant derivative on M (see Ellis, Gay-Balmaz, Holm, Ratiu [2011]). Here $\pi_{X, L\left(T^*X, \widetilde{\mathfrak{h}}^*\right)} : L\left(T^*X, \widetilde{\mathfrak{h}}^*\right) \to X$, $\pi_{X, \widetilde{\mathfrak{h}}^*} : \widetilde{\mathfrak{h}}^* \to X$, $\pi_{X, L(T^*X, T^*M)} : L(T^*X, T^*M) \to X$, and $\pi_{X, T^*M} : T^*M \to X$ are the locally trivial fiber bundles obtained by composing the locally trivial bundles given by the total spaces whose natural base is $M \times X$ with the projection $M \times X \to X$; Γ of a projection denotes the space of section of that projection. We summarize the considerations above in the following statement.

Theorem 2.1 *Given is a right H-invariant Lagrangian $\mathfrak{L} : J^1P \to \mathbb{R}$, where $P = G \times X \to X$ is a trivial principal bundle and H is a closed subgroup of G. Define the reduced Lagrangian $\ell : L(TX, TM) \times_\Sigma L(TX, \widetilde{\mathfrak{h}}) \to \mathbb{R}$ obtained by Lagrange-Poincaré reduction; see* (2.1)*. Then the Euler-Lagrange equations for $\mathfrak{L}$ are equivalent to the Lagrange-Poincaré equations* (2.2) *for ℓ for a smooth section $\sigma : U \subset X \to L(TX, \widetilde{\mathfrak{h}})$ and a smooth function $m : U \subset X \to M$.*

Lagrange-Poincaré equations for H discrete. Assume now that H is a closed discrete subgroup of G. Then $\mathfrak{h} = \{0\}$, $\widetilde{\mathfrak{h}}$ is the vector bundle with zero dimensional fiber and base M, $\mathcal{A} = 0$, and hence the vector bundle isomorphism $\beta_{\mathcal{A}}$ becomes canonical, $\beta : L(TX, TG)/H \to L(TX, TM)$, the source and target spaces viewed as a vector bundles over $\Sigma = M \times X$, and it is given by

$$\beta([l_{(x,g)}]_H) := T_g\pi(l_{(x,g)}(\_)) = (l_{(x,g)}(\_)g^{-1})_M(m) \in L(T_xX, T_mM).$$

So, if $g : X \to G$ is a given section and $m := \pi \circ g : X \to M$, the reduced section becomes

$$\beta\left([T_xg]_H\right) = T\pi \circ T_xg = T_xm \in L(T_xX, T_{m(x)}M).$$

The reduced Lagrangian $\ell : L(TX, TM) \to \mathbb{R}$ yields the Lagrange-Poincaré equations (2.2) which in this case become

$$\frac{\delta \ell}{\delta m} - \operatorname{div} \frac{\delta \ell}{\delta\, Tm} = 0. \tag{2.3}$$

Remark 2.2 It is instructive to consider in more detail the isomorphism β in the case of a discrete subgroup. In this case, the kernel of the tangent map is zero, so that at any $g \in G$ we have the isomorphism $T_g\pi : T_gG \to T_{[g]}(G/H)$ which implies that

$$\beta : (TG)/H \to T(G/H), \quad [v_g]_H \mapsto T\pi(v_g)$$

is a vector bundle isomorphism covering the identity on G/H. Indeed, if $T_g\pi(v_g) = T_{\bar{g}}\pi(w_{\bar{g}})$, then necessarily $\bar{g} = gh$ for $h \in H$, and we can write $T_g\pi(v_g) = T_{\bar{g}}\pi(w_{\bar{g}}) = T_g\pi(w_{\bar{g}}h^{-1})$, so that $v_g = w_{\bar{g}}h^{-1}$, since $\ker T_g\pi = \{0\}$. This proves that $[v_g]_H = [w_{\bar{g}}]_H$. The same argument shows that

$$\beta : (L(TX, TG))/H \to L(TX, T(G/H)), \quad [l_{(x,g)}]_H \mapsto T\pi \circ l_{(x,g)}$$

is a vector bundle isomorphism covering the identity on $X \times G/H$. Indeed, if $T_g\pi \circ l_{(x,g)} = T\pi \circ l_{(\bar{x},\bar{g})}$, then $\bar{x} = x$, $\bar{g} = gh$, and $T_g\pi(l_{(x,g)}(v_x)) = T_{\bar{g}}\pi(l_{(x,\bar{g})}(v_x))$, for all $v_x \in T_xX$. So we can write $T_g\pi(l_{(x,g)}(v_x)) = T_g\pi(l_{(x,\bar{g})}(v_x)h^{-1})$, for all $v_x \in T_xX$, and hence $l_{(x,g)}(v_x) = l_{(x,\bar{g})}(v_x)h^{-1}$, for all $v_x \in T_xX$, since $\ker T_g\pi = \{0\}$. This proves that $l_{(x,g)} = l_{(x,\bar{g})}h^{-1}$ whence $[l_{(x,g)}]_H = [l_{(\bar{x},\bar{g})}]_H$. The inverse of β is given by

$$L(TX, TM) \ni \lambda_{(x,m)} \mapsto [\lambda_{(x,g)}]_H \in L(TX, TG)/H,$$

where $\lambda_{(x,g)} \in L(TX, TG)$ is such that $T\pi \circ \lambda_{(x,g)} = \lambda_{(x,m)}$. ♦

Euler-Poincaré approach. The Euler-Poincaré version corresponds to the vector bundle isomorphism

$$\bar{i}_{m_0} : L(TX, TG)/H \to L(TX, \mathfrak{g}) \times M$$

over $\Sigma = M \times X$, whose restriction to the fibers at (m, x) reads

$$\bar{i}_{m_0}\left([l_{(x,g)}]_H\right) = \left(TR_{g^{-1}}(l_{(x,g)}(\_)), \Phi_g(m_0)\right) =: (\xi_x, m) \in L(T_xX, \mathfrak{g}) \times \{m\},$$

where an element $m_0 \in M$ has been fixed. We refer to Gay-Balmaz and Tronci [2010] for the corresponding approach in classical mechanics. We note that a connection is not needed to write this isomorphism. If $g : X \to G$ is a given section, the formula for the reduced section given above becomes

$$\bar{i}_{m_0}([T_xg]_H) = (TR_{g(x)^{-1}} \circ T_xg, \Phi_{g(x)}(m_0)) =: (\xi(x), m(x)). \tag{2.4}$$

Note that by composing the two vector bundle isomorphisms $\beta_{\mathcal{A}}$ and $\bar{i}_{m_0}$ over $\Sigma = M \times X$, we get an isomorphism $L(TX, \mathfrak{g}) \times M \to L(TX, TM) \times L(TX, \widetilde{\mathfrak{h}})$, whose restriction to the fiber at $(m, x) \in \Sigma$ reads

$$L(T_xX, \mathfrak{g}) \times \{m\} \ni (\xi_x, m) \mapsto ((\xi_x(\_))_M(m), [g, \mathcal{A}(\xi_x(\_)g)]_H) \in L(T_xX, T_mM) \times L(T_xX, \widetilde{\mathfrak{h}}_m),$$

where $g \in G$ is arbitrary such that $\pi(g) = m$.

Given a H-invariant Lagrangian $\mathfrak{L}$ on J^1P, the associated reduced Lagrangian $l : L(TX, \mathfrak{g}) \times M \to \mathbb{R}$ obtained through the Euler-Poincaré process is

$$\mathfrak{L}(T_xg) = l\left(T_xg(\_)g(x)^{-1}, \Phi_{g(x)}m_0\right). \tag{2.5}$$

The partial derivatives are

$$\frac{\delta l}{\delta \xi}(x) \in L(T_x^*X, \mathfrak{g}^*), \quad \frac{\delta l}{\delta m}(x) \in T^*_{m(x)}M,$$

where $\xi : X \to L(TX, \mathfrak{g})$ is a section and $m : X \to M$ is a smooth function.

Given smooth variations $g_\varepsilon : U \subset X \to G$ such that $g_0 = g$ and $g_\varepsilon|_{\partial U} = g|_{\partial U}$, we get the following constrained variations for the reduced sections: if $c(t)$ is a smooth curve such that $c(0) = x$ and $\left.\frac{d}{dt}\right|_{t=0} c(t) = v_x$, we get

$$\begin{aligned}
\delta\xi(x)(v_x) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} T_x g_\varepsilon(v_x) g_\varepsilon(x)^{-1} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left.\frac{d}{dt}\right|_{t=0} g_\varepsilon(c(t)) g_\varepsilon(x)^{-1} \\
&= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g_\varepsilon(c(t)) g_\varepsilon(x)^{-1} = \left.\frac{d}{dt}\right|_{t=0} \delta g(c(t)) g(x)^{-1} - g(c(t)) g(x)^{-1} \delta g(x) g(x)^{-1} \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\delta g(c(t)) g(c(t))^{-1} - \delta g(x) g(c(t))^{-1} - g(c(t)) g(x)^{-1} \delta g(x) g(x)^{-1} \right) \\
&= \mathbf{d}(\delta g g^{-1})(x)(v_x) + [\delta g(x) g^{-1}(x), T_x g(v_x) g(x)^{-1}] \\
\delta m(x) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Phi_{g_\varepsilon(x)}(m_0) = \left(\delta g(x) g(x)^{-1}\right)_M (m(x)),
\end{aligned}$$

so we get the constrained variations

$$\delta\xi = \mathbf{d}\eta + [\eta, \xi], \quad \delta m = \eta_M(m),$$

where $\eta : U \subset X \to \mathfrak{g}$ is arbitrary with $\eta|_{\partial U} = 0$. We thus get

$$\begin{aligned}
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_U l(\xi_\varepsilon, m_\varepsilon)\mu &= \int_U \left\langle \frac{\delta l}{\delta \xi}, \mathbf{d}\eta + [\eta, \xi] \right\rangle + \left\langle \frac{\delta l}{\delta m}, \eta_M(m) \right\rangle \mu \\
&= \int_U \left\langle -\operatorname{div} \frac{\delta l}{\delta \xi} - \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \mathbf{J}\left(\frac{\delta l}{\delta m}\right), \eta \right\rangle,
\end{aligned}$$

where the pairings are, respectively, between $L(TX, \mathfrak{g})$ and $L(T^*X, \mathfrak{g}^*)$, TM and T^*M, $\mathfrak{g}$ and $\mathfrak{g}^*$; $\mathbf{J} : T^*M \to \mathfrak{g}^*$ is the standard equivariant momentum map of the cotangent lifted action given by $\langle \mathbf{J}(\alpha_m), \zeta \rangle = \langle \alpha_m, \zeta_M(m) \rangle$ for all $\alpha_m \in T_m^*M$, $\zeta \in \mathfrak{g}$.

We thus obtain the following result.

Theorem 2.3 *Given is a right H-invariant Lagrangian $\mathfrak{L} : J^1P \to \mathbb{R}$, where $P = G \times X \to X$ is a trivial principal bundle and H is a closed subgroup of G. Define the reduced Lagrangian $l : L(TX, \mathfrak{g}) \times M \to \mathbb{R}$ obtained by Euler-Poincaré reduction with parameter; see* (2.5). *Then the Euler-Lagrange equations for $\mathfrak{L}$ are equivalent to the Euler-Poincaré equations for l, namely,*

$$\operatorname{div} \frac{\delta l}{\delta \xi} + \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} = \mathbf{J}\left(\frac{\delta l}{\delta m}\right), \quad Tm = \xi_M(m) \tag{2.6}$$

for a smooth section $\xi : X \to L(TX, \mathfrak{g})$ and a smooth function $m : X \to M$.

The second equation can be explicitly written as

$$T_x m(v_x) = \left(\xi(x)(v_x)\right)_M (m(x)), \quad \text{for all} \quad v_x \in TX.$$

If H is closed and discrete, the Euler-Poincaré equations (2.6) for $l : L(TX, \mathfrak{g}) \times M \to \mathbb{R}$ do not simplify.

If we have the inclusion of closed subgroups $H_1 \subset H_2$ then we have $G/H_2 \to G/H_1$ as a locally trivial fiber bundle with fiber H_2/H_1 and structure group $N_{H_2}(H_1)/H_1$, where $N_{H_2}(H_1)$ is the normalizer of H_1 inH_2. This leads to a sequence of reductions from H_2 to H_1.

An example of such a sequence is the reduction from biaxial nematics with the group $H_1 = Z_2 \times Z_2$ to uniaxial nematics with the group $H_2 = Z_2 \times SO(2)$.

3 Applications to condensed matter

As discussed previously, for condensed matter theories the Lagrangian is defined on $L(TX, TM)$, i.e., on the first jet bundle of a trivial bundle $M \times X \to X$, so that $\mathcal{L} = \mathcal{L}(T_x m)$. The manifold M is assumed to be a homogeneous space, relative to the transitive action of a Lie group G, with isotropy group $H = G_{m_0}$ for some preferred element $m_0 \in M$. From $\mathcal{L}$ one can construct a Lagrangian $\mathfrak{L}$ on the trivial principal bundle $P = X \times G \to X$, by

$$\mathfrak{L}(T_x g) := \mathcal{L}\left(T_x(\Phi_{g(\_)} m_0)\right),$$

where $g : X \to G$, i.e., g is a section of the trivial principal bundle $P = X \times G \to X$.

Using the results of §2, we will show that the Euler-Lagrange equations for $\mathfrak{L}$ are equivalent to those for $\mathcal{L}$ by using Lagrange-Poincaré reduction. Then we use the equivalence with the Euler-Poincaré approach obtained above to write the equations in a simpler form.

Since $\mathfrak{L}$ is H-invariant, by fixing a connection $\mathcal{A} \in \Omega^1(G, \mathfrak{h})$, we get the Lagrange-Poincaré Lagrangian, that we now compute. Given a map $g : U \subset X \to G$, and defining $m := \pi \circ g : U \subset X \to M$, we have

$$\ell\left(T_x m, [g(x), \mathcal{A}(T_x g(\_))]_H\right) = \mathfrak{L}(T_x g) = \mathcal{L}\left(T_x(\Phi_{g(\_)} m_0)\right) = \mathcal{L}(T_x m).$$

This means that $\ell : L(TX, TM) \times_\Sigma L(TX, \widetilde{\mathfrak{h}}) \to \mathbb{R}$ does not depend on the second variable, so $\frac{\delta \ell}{\delta \sigma} = 0$, and $\ell = \mathcal{L}$. Thus, the second group of Lagrange-Poincaré equations in (2.2) are the Euler-Lagrange equations for $\mathcal{L}$ on $L(TX, TM)$.

We now compute the Euler-Poincaré reduced Lagrangian. We have

$$l(\xi(x), m(x)) = l\left(T_x g(\_) g(x)^{-1}, \Phi_{g(x)} m_0\right) = \mathfrak{L}(T_x g) = \mathcal{L}\left(T_x(\Phi_{g(\_)} m_0)\right) = \mathcal{L}(\xi(x)(m(x))_M).$$

From the above results, we know that the Euler-Lagrange equations for $\mathcal{L}$ are equivalent to the Euler-Poincaré equations

$$\operatorname{div} \frac{\delta l}{\delta \xi} + \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} = \mathbf{J}\left(\frac{\delta l}{\delta m}\right), \quad Tm = \xi_M(m).$$

We summarize these considerations.

Corollary 3.1 *The following equations are equivalent.*

(i) *The Euler-Lagrange equations for* $\mathcal{L} : L(TX, TM) \to \mathbb{R}$*, i.e.,*

$$\frac{\delta \mathcal{L}}{\delta m} - \operatorname{div} \frac{\delta \mathcal{L}}{\delta \, Tm} = 0,$$

where $m : X \to M$*.*

(ii) *The Euler-Poincaré equations for* $l : L(TX, \mathfrak{g}) \times M \to \mathbb{R}$*, i.e.,*

$$\operatorname{div} \frac{\delta l}{\delta \xi} + \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} = \mathbf{J}\left(\frac{\delta l}{\delta m}\right), \quad Tm = \xi_M(m), \tag{3.1}$$

where $\xi : X \to L(TX, \mathfrak{g})$ *is a smooth section*

We shall study next example using the following general approach. We determine from physical reasons the order parameter of the system. The Lagrangian of the system depends on A and ∇A and its form depends on the specific system under consideration. To find a solution of the Euler-Lagrange equations determined by this Lagrangian it is necessary

• to find thermodynamics phases of an appropriate system

• to determine the different types of textures as solutions of the Ginzburg–Landau (GL) equations.

To solve the first problem we use the technique of (Golo and Monastyrsky [1978],Bogomolov and Monastyrsky [1987],Monastyrsky [1993]), where thermodynamic phases are identified with an orbit containing a minimum of the potential of the free energy. To address the second problem, we fix the phase and so different textures can be found as solutions of the Ginzburg-Landau equations determined by this phase. It turns out that on a fixed orbit the GL equations are the Euler-Lagrange equations considered above, where the Lagrangian function is the gradient term in the GL equation which determines a metric on the orbit.

In order to show this equivalence we consider the one-dimensional textures in the A-phases of ^{3}He. Here the orbit M_A is isomorphic to the space $G/H = SO(3) \times S^2/Z_2$

Liquid Helium ^{3}He. (Golo, Monastyrsky, and Novikov [1979], Monastyrsky [1993]) Let $G = U(1) \times SO(3)_L \times SO(3)_R$ act on $A \in \mathfrak{gl}(3, \mathbb{C})$ by

$$\left(e^{\mathrm{i}\varphi}, R_1, R_2\right) \cdot A := e^{\mathrm{i}\varphi} R_1 A R_2^{-1}. \tag{3.2}$$

The Lagrangian $\mathcal{L} = \mathcal{L}(A, \nabla A)$ of the theory is defined on maps $A : X \to \mathfrak{gl}(3, \mathbb{C})$. Different orbits correspond to different phases. We choose M to be one of these orbits and find critical points of $\int_X \mathcal{L}(T_x A)\mu$ among maps $A : X \to M$ (and not $X \to \mathfrak{gl}(3, \mathbb{C})$).

This Lagrangian density is of the form

$$\mathcal{L}(A, \nabla A) = F_{dip}(A) + F_H(A) + F_{grad}(A, \nabla A) + U(A),$$

where $F_{grad}(A, \nabla A) + U(A)$ is the free energy and $F_{grad}(A, \nabla A)$ is its gradient term whereas $U(A)$ is a potential, $F_{dip}(A)$ is the dipole energy, and $F_H(A)$ is the energy of the magnetic field H. The general explicit formulas are (see [Monastyrsky, 1993, §5.2.1, §5.2.3]

$$F_{grad}(A,\nabla A) = \gamma_1 \left(\partial_k \overline{A_{pi}}\right)\left(\partial_k A_{pi}\right) + \gamma_2 \left(\partial_k \overline{A_{pi}}\right)\left(\partial_i A_{pk}\right) + \gamma_3 \left(\partial_k \overline{A_{pk}}\right)\left(\partial_i A_{pi}\right)$$

$$U(A) = \alpha \operatorname{Tr}(AA^*) + \beta_1 |\operatorname{Tr}(AA^T)| + \beta_2 \left[\operatorname{Tr}(AA^*)\right]^2 + \beta_3 \operatorname{Tr}\left[(A^*A)\overline{(A^*A)}\right] + \beta_4 \operatorname{Tr}\left[(AA^*)\right]^2 + \beta_5 \operatorname{Tr}\left[(AA^*)\overline{(AA^*)}\right]^2$$

$$F_H(A) = -\frac{1}{2}\chi_{ij} H_i H_j$$

where $\alpha, \beta_1, \ldots, \beta_5, \gamma_1, \gamma_2, \gamma_3$ are constants and χ_{ij} are the components of the tensor of magnetic susceptibility. Here $\star$ means the complex conjugacy and repeated indices imply summation.

The equilibrium of the superfluid phase is given by the minimum of the potential $U(A)$ under the condition that $F_{grad} = 0$. In a macroscopic description of textures in ^{3}He, the Lagrangian $\mathcal{L}$, F_{grad}, F_{dip}, and F_H each have their own characteristic length scales $L, L_{grad}, L_{dip}, L_H$. It is possible to study textures at different regimes: $L \ll L_{dip}$ and $L \gg L_{dip}$. It is easy to take into account additional interactions. It leads only to additional degeneration of the space of a parameter of order. Here we consider less generate case: $L \ll L_{dip}$ with $F_H = F_{dip} = 0$. The orbit M_A is determined by minimizing $U(A)$ and one finds that this orbit generates by the matrix

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & i & 0 \end{pmatrix} \in \mathfrak{gl}(3,\mathbb{C}).$$

In this case, the subgroup $SO(3)_L \times SO(3)_R$ acts transitively on M_A (i.e., the $U(1)$-part in (3.2) is conjugated to an element of $SO(3)_L \times SO(3)_R$. On the orbit M_A one takes only the gradient part of the Lagrangian density, i.e., $\mathcal{L}(A,\nabla A) = F_{grad}(A,\nabla A)$.

The reduced velocity $\xi = (\nabla g)g^{-1} = TR_{g(x)^{-1}} \circ T_x g$ of the general theory (see (2.4)) is given here by $\xi = (v,w) : X \to \mathfrak{so}(3)$, $v = (\partial_z R_1)R_1^{-1}$ and $w = R_2^{-1}(\partial_z R_2)$. The Euler-Poincaré Lagrangian $l(\xi, m)$ of the general theory (see (2.5)), is in this case given by

$$l(v,w,A) = I_{ab}(A) w_a w_b + \chi_{ab} v_a v_b$$

where chiral velocities v, w are: $v = i\partial_x R_1 {R_1}^{-1}$ and $w = i\partial_x R_2 {R_2}^{-1}$ in accordance with [Monastyrsky, 1993, formula (5.133)]. Thus, the Euler-Poincaré equations (3.1) read

$$\partial_z \frac{\delta l}{\delta v} + \operatorname{ad}^*_v \frac{\delta l}{\delta v} = \mathbf{J}_1\left(\frac{\delta l}{\delta A}\right), \qquad \partial_z \frac{\delta l}{\delta v} - \operatorname{ad}^*_w \frac{\delta l}{\delta w} = \mathbf{J}_2\left(\frac{\delta l}{\delta A}\right), \tag{3.3}$$

where $\mathbf{J}_1 : T^*M \to \mathfrak{so}^*(3)$ is the momentum map of the left action and $\mathbf{J}_2 : T^*M \to \mathfrak{so}^*(3)$ is the momentum map of the right action of $SO(3)$ on the orbit M, respectively.

4 Conclusion

The possibility of identifying condensed matter systems with multidimensional order parameters with different types of of gauge fields reductions opens new opporunities to find solutions in diverse problems in condensed matter:

1. One and two-dimensional textures in Liquid crystals and Superfluids,
2. Shapes of interfaces between different phases in Superfluid ^3He,
3. Phase transitions between biaxial and uniaxial nematics
4. Vortices in rotating neutron stars (Monastyrsky and Sasorov [2011]).

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