GEOMETRIC HIGHER GROUPOIDS AND CATEGORIES

KAI BEHREND AND EZRA GETZLER

ABSTRACT. In an enriched setting, we show that higher groupoids and higher categories form categories of fibrant objects, and that the nerve of a differential graded algebra is a higher category in the category of algebraic varieties.

This paper develops a general theory of higher groupoids in a category $\mathcal{V}$. We consider a small category $\mathcal{V}$ of **spaces**, together with a subcategory of **covers**, satisfying the following axioms:

(D1) $\mathcal{V}$ has finite limits;
(D2) the pullback of a cover is a cover;
(D3) if f is a cover and gf is a cover, then g is a cover.

These axioms are reminiscent of those for a category of smooth morphisms $\mathbf{P}$ of Toën and Vezzosi ([24], Assumption 1.3.2.11). A topos satisfies these axioms, with epimorphisms as covers; so do the category of schemes, with surjective étale morphisms, smooth epimorphisms, or faithfully flat morphisms as covers, and the category of Banach analytic spaces, with surjective submersions as covers. We call a category satisfying these axioms a **descent category**. We call a simplicial object in a descent category a **simplicial space**.

A finite simplicial set is a simplicial set with a finite number of *degenerate* simplices. Given a simplicial space X and a finite simplicial set T, let

$$\operatorname{Hom}(T, X)$$

be the space of simplicial morphisms from T to X; it is a finite limit in $\mathcal{V}$, and its existence is guaranteed by (D1).

Let $\Lambda^n_i \subset \Delta^n$ be the **horn**, consisting of the union of all but the ith face of the n-simplex:

$$\Lambda^n_i = \bigcup_{j \neq i} \partial_j \Delta^n .$$

A simplicial set X is the nerve of a groupoid precisely when the induced morphism

$$X_n \to \operatorname{Hom}(\Lambda^n_i, X)$$

is an isomorphism for $n > 1$.

On the other hand, given a simplicial abelian group A, the associated complex of normalized chains vanishes above degree k if and only if the morphism $A_n \to \operatorname{Hom}(\Lambda^n_i, A)$ is an isomorphism for $n > k$. Motivated by these examples, Duskin defined a k-groupoid to be a simplicial set X such that the morphism $X_n \to \operatorname{Hom}(\Lambda^n_i, X)$ is surjective for $n > 0$ and bijective for $n > k$. (See Duskin [10] and Glenn [15]. In their work, k-groupoids are called “k-dimensional hypergroupoids.”)

We are grateful to Nick Roszenblyum for remarking that the path space of a quasi-category X is $\operatorname{Hom}(\mathbb{\Delta}^1, X)$, and to Jesse Wolfson for many helpful discussions. The first author thanks Imperial College for its hospitality during the period when this paper was begun. The second author thanks the University of Geneva for its hospitality during the period when it was completed, and the Simons Foundation for support under a Collaboration Grant for Mathematicians.

In this paper, we generalize Duskin's theory of k-groupoids to descent categories: Pridham takes a similar approach in [21].

Definition. Let k be a natural number. A simplicial space X in a descent category $\mathcal{V}$ is a ***k*-groupoid** if, for each $0 \le i \le n$, the morphism

$$X_n \longrightarrow \mathrm{Hom}(\Lambda^n_i, X)$$

is a cover for $n > 0$, and an isomorphism for $n > k$.

Denote by $s_k\mathcal{V}$ the category of k-groupoids, with morphisms the simplicial morphisms of the underlying simplicial spaces. Thus, the category $s_0\mathcal{V}$ of 0-groupoids is equivalent to $\mathcal{V}$, while the category $s_1\mathcal{V}$ of 1-groupoids is equivalent to the category of Lie groupoids in $\mathcal{V}$, that is, groupoids such that the source and target maps are covers. (The equivalence is induced by mapping a Lie groupoid to its nerve.)

Definition. A morphism $f : X \to Y$ between k-groupoids is a **fibration** if, for each $n > 0$ and $0 \le i \le n$, the morphism

$$X_n \longrightarrow \mathrm{Hom}(\Lambda^n_i, X) \times_{\mathrm{Hom}(\Lambda^n_i, Y)} Y_n$$

is a cover. It is a **hypercover** if, for each $n \ge 0$, the morphism

$$X_n \longrightarrow \mathrm{Hom}(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y)} Y_n$$

is a cover. It is a **weak equivalence** if there is a k-groupoid P and hypercovers $p : P \to X$ and $q : P \to Y$ such that $f = qs$, where s is a section of p.

Every k-groupoid is **fibrant**: that is, the unique morphism with target the terminal object e is a fibration. Every hypercover is a fibration.

The following is the first main result of this paper: for the definition of a category of fibrant objects, see Definition 1.1.

Theorem. The category of k-groupoids $s_k\mathcal{V}$ is a category of fibrant objects.

We will prove the following more direct characterization of weak equivalences in Section 5.

Theorem. A morphism $f : X \to Y$ between k-groupoids is a **weak equivalence** if and only if, for each $n \ge 0$, the morphisms

$$X_n \times_{Y_n} Y_{n+1} \longrightarrow \mathrm{Hom}(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y)} \mathrm{Hom}(\Lambda^{n+1}_{n+1}, Y)$$

are covers.

Parallel to the theory of k-groupoids, there is a theory of k-categories, modeled on the theory of complete Segal spaces (Rezk [22]). In the case where $\mathcal{V}$ is the category of sets, these are truncated weak Kan complexes in the sense of Boardman and Vogt [3]. Weak Kan complexes were studied further by Joyal [17], who calls them quasi-categories, and by Lurie [19], who calls them ∞-categories.

The **thick *n*-simplex** is the simplicial set $\mathbb{A}^n = \mathrm{cosk}_0\,\Delta^n$. Just as Δ^n is the nerve of the category with objects $\{0, \dots, n\}$ and a single morphism from i to j if $i \le j$, $\mathbb{A}^n$ is the nerve of the groupoid $[\![n]\!]$ with objects $\{0, \dots, n\}$ and a single morphism from i to j for all i and j. In other words, just as the k-simplices of the n-simplex are monotone functions from $\{0, \dots, k\}$ to $\{0, \dots, n\}$, the k-simplices of the thick simplex are *all* functions from $\{0, \dots, k\}$ to $\{0, \dots, n\}$. What we call the thick simplex goes under several names in the literature: Rezk [22] denotes it $E(n)$, while Joyal and Tierney [18] use the notation $\Delta'[n]$.

Definition. Let k be a positive integer. A simplicial space X in a descent category $\mathcal{V}$ is a ***k*-category** if for each $0 < i < n$, the morphism

$$X_n \to \operatorname{Hom}(\Lambda^n_i, X),$$

is a cover for $n > 1$ and an isomorphism for $n > k$, and the morphism

$$\operatorname{Hom}(\mathbb{\Delta}^1, X) \to X_0$$

induced by the inclusion of a vertex $\Delta^0 \hookrightarrow \mathbb{\Delta}^1$ is a cover.

In a topos, where all epimorphisms are covers, the last condition is automatic, since these morphisms have the section $X_0 \to \operatorname{Hom}(\mathbb{\Delta}^1, X)$ induced by the projection from $\mathbb{\Delta}^1$ to Δ^0.

Associated to a k-category X is the simplicial space $\mathbb{G}(X)$, defined by

$$\mathbb{G}(X)_n = \operatorname{Hom}(\mathbb{\Delta}^n, X).$$

The formation of $\mathbb{G}(X)_n$, while appearing to involve an infinite limit, is actually isomorphic to a finite limit, since (see Lemma 6.2)

$$\operatorname{Hom}(\mathbb{\Delta}^n, X) \cong \operatorname{cosk}_{k+1} X_n = \operatorname{Hom}(\operatorname{sk}_{k+1} \mathbb{\Delta}^n, X),$$

and $\operatorname{sk}_{k+1} \mathbb{\Delta}^n$, the $(k+1)$-skeleton of $\mathbb{\Delta}^n$, is a finite simplicial complex.

The following theorem is useful in constructing examples of k-groupoids.

Theorem. If X is a k-category, $\mathbb{G}(X)$ is a k-groupoid.

In fact, k-categories also form a category of fibrant objects.

Definition. A morphism $f : X \to Y$ of k-categories is a **quasi-fibration** if for $0 < i < n$, the morphism

$$X_n \to \operatorname{Hom}(\Lambda^n_i, X) \times_{\operatorname{Hom}(\Lambda^n_i, Y)} Y_n$$

is a cover, and the morphism

$$\operatorname{Hom}(\mathbb{\Delta}^1, X) \to X_0 \times_{Y_0} \operatorname{Hom}(\mathbb{\Delta}^1, Y)$$

induced by the inclusion of a vertex $\Delta^0 \hookrightarrow \mathbb{\Delta}^1$ is a cover. It is a **hypercover** if, for each $n \geq 0$, the morphism

$$X_n \longrightarrow \operatorname{Hom}(\partial\Delta^n, X) \times_{\operatorname{Hom}(\partial\Delta^n, Y)} Y_n$$

is a cover. (This is the same definition as for k-groupoids, except that now X and Y are k-categories.) It is a **weak equivalence** if there is a k-category P and hypercovers $p : P \to X$ and $q : P \to Y$ such that $f = qs$, where s is a section of p.

Theorem.

i) The category of k-categories is a category of fibrant objects.

ii) The functor $\mathbb{G}$ is an exact functor: it takes quasi-fibrations to fibrations, pullbacks of quasi-fibrations to pullbacks, and hypercovers to hypercovers.

We also have the following more direct characterization of weak equivalences between k-categories, proved in Section 6. Recall that if S and T are simplicial sets, then their **join** $K \star L$ is the simplicial set

$$(S \star T)_k = S_k \sqcup T_k \sqcup \bigsqcup_{j=0}^{k-1} S_j \times T_{k-j-1}$$

Theorem. A morphism $f : X \to Y$ of k-categories is a weak equivalence if and only if the morphism

$$X_0 \times_{Y_0} \operatorname{Hom}(\mathbb{\Delta}^1, Y) \longrightarrow Y_0$$

is a cover, and the morphisms

$$X_n \times_{Y_n} \operatorname{Hom}(\mathbb{\Delta}^1 \star \Delta^{n-1}, Y) \\ \longrightarrow \operatorname{Hom}(\partial\Delta^n, X) \times_{\operatorname{Hom}(\partial\Delta^n, Y)} \operatorname{Hom}(\mathbb{\Delta}^1 \star \partial\Delta^{n-1} \cup \mathbb{\Lambda}^1_0 \star \Delta^{n-1}, Y)$$

are covers for $n > 0$.

In a finite-dimensional algebra or a Banach algebra, invertibility is an open condition. To formulate this property in our general setting, we need the notion of a regular descent category.

A morphism in a category is an effective epimorphism if it equals its own coimage. (We recall the definition of the coimage of a morphism in Section 2.)

Definition. A **subcanonical** descent category is a descent category such that every cover is an effective epimorphism.

Definition. A **regular** descent category is a subcanonical descent category with a subcategory of **regular** morphisms, satisfying the following axioms:

(R1) every cover is regular;
(R2) the pullback of a regular morphism is regular;
(R3) every regular morphism has a coimage, and its coimage is a cover.

All of the descent categories that we consider are regular. In the case of a topos, we take all of the morphisms to be regular. When $\mathcal{V}$ is the category of schemes with covers the surjective étale (respectively smooth or flat) morphisms, the regular morphisms are the the étale (respectively smooth or flat) morphisms. When $\mathcal{V}$ is the category of Banach analytic spaces with covers the surjective submersions, the regular morphisms are the submersions.

Definition. A k-category in a regular descent category $\mathcal{V}$ is **regular** if the morphism

$$\operatorname{Hom}(\mathbb{\Delta}^1, X) \to \operatorname{Hom}(\Delta^1, X) = X_1$$

is regular.

Theorem. Let $\mathcal{V}$ be a regular descent category, and let X be a regular k-category in $\mathcal{V}$. Then for all $n \geq 0$, the morphism

$$\operatorname{Hom}(\mathbb{\Delta}^n, X) \to \operatorname{Hom}(\Delta^n, X) = X_n$$

is regular. Let $\mathsf{G}(X)_n$ be the image of this morphism (that is, the codomain of its coimage). Then the spaces $\mathsf{G}(X)$ form a simplicial space, this simplicial space is a k-groupoid, and the induced morphism

$$\mathbb{G}(X) \to \mathsf{G}(X)$$

is a hypercover.

In fact, as shown by Joyal (Corollary 1.5, [17]), $\mathsf{G}(X)_n$ is the space of n-simplices of X such that for each inclusion $\Delta^1 \hookrightarrow \Delta^n$, the induced 1-simplex lies in $\mathsf{G}(X)_1$. The simplices of $\mathsf{G}(X)_1$ are called **quasi-invertible**.

In the case where $\mathcal{V}$ is the category of sets, this theorem relates two different k-groupoids associated to a k-category: the k-groupoid $\mathbb{G}(X)$ was introduced by Rezk [22] and further studied by Joyal and Tierney [18], while the k-groupoid $\mathsf{G}(X)$ was introduced by Joyal [17].

In the last section of this paper, we construct examples of k-groupoids associated to differential graded algebras over a field. Let A be a differential graded algebra such that A^i is finite-dimensional for all i. The **Maurer-Cartan locus** $\mathsf{MC}(A)$ of A is the affine variety

$$\mathsf{MC}(A) = Z(da + a^2) \subset A^1.$$

If K is a finite simplicial set, let $C^\bullet(K)$ be the differential graded algebra of normalized simplicial cochains on K. The **nerve** of A is the simplicial scheme

$$N_n A = \mathsf{MC}(C^\bullet(\Delta) \otimes A).$$

This simplicial scheme has also been discussed by Lurie [19].

Theorem. Let A be a differential graded algebra finite-dimensional in each degree and vanishing in degree $-k$ and below. The nerve $N_\bullet A$ of A is a regular k-category in the descent category of schemes (with surjective submersions as covers).

The k-groupoid $\mathbb{N}_\bullet A = \mathbb{G}(NA)$ is the simplicial scheme

$$\mathbb{N}_n A = \mathsf{MC}(C^\bullet(\mathbb{\Delta}^n) \otimes A).$$

We see that $\mathbb{N}_\bullet A$ and $\mathsf{G}(N_\bullet A)$ are k-groupoids, and that the simplicial morphism

$$\mathbb{N}_\bullet A \to \mathsf{G}(N_\bullet A)$$

is a hypercover. The statement that $\mathsf{G}(N_\bullet A)$ is a k-groupoid has also been proved by Benzeghli [2].

This theorem has an evident generalization to differential graded categories. It may also be generalized to differential graded Banach algebras, in which case the nerve is a k-category in the descent category of Banach analytic spaces. There is also a more refined version of the theorem in which the Maurer-Cartan locus is taken in the category of derived schemes; this will be the topic of a sequel to this paper.

1. CATEGORIES OF FIBRANT OBJECTS

Definition 1.1. A **category with weak equivalences** is a category $\mathcal{V}$ together with a subcategory $\mathcal{W} \subset \mathcal{V}$ containing all isomorphisms, such that whenever f and g are composable morphisms such that gf is a weak equivalence, then f is a weak equivalence if and only if g is.

Associated to a small category with weak equivalences is its simplicial localization $L(\mathcal{V}, \mathcal{W})$. This is a category enriched in simplicial sets, with the same objects as $\mathcal{V}$, which refines the usual localization. (In fact, the morphisms of the localization are the components of the simplicial sets of morphisms of $L(\mathcal{V}, \mathcal{W})$.) The simplicial localization was introduced by Dwyer and Kan [11, 12], and studied further in Dwyer and Kan [13], Weiss [25], and Cisinski [6]: one may even say that abstract homotopy theory is the study of simplicial localizations. The simplicial category of ***k*-stacks** is the simplicial localization of the category of k-groupoids.

Categories of fibrant objects, introduced by Brown [4], form a class of categories with weak equivalences for which the simplicial localization is quite tractable: the simplicial sets of morphisms between objects may be realized as nerves of certain categories of diagrams.

Definition 1.2. A **category of fibrant objects** $\mathcal{V}$ is a small category with weak equivalences $\mathcal{W}$ together with a subcategory $\mathcal{F} \subset \mathcal{V}$ of fibrations, satisfying the following axioms. Here, we refer to morphisms which are both fibrations and weak equivalences as **trivial fibrations**.

(F1) There exists a terminal object e in $\mathcal{V}$, and any morphism with target e is a fibration.

(F2) Pullbacks of fibrations are fibrations.

(F3) Pullbacks of trivial fibrations are trivial fibrations.

(F4) Every morphism $f : X \to Y$ has a factorization

$$\begin{tikzcd}
& P \arrow[dr, "q"] & \\
X \arrow[ur, "r"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

where r is a weak equivalence and q is a fibration.

An object X such that the morphism $X \to e$ is a fibration is called fibrant: Axiom (F1) states that every object is fibrant.

The reason for the importance of categories of fibrant objects is that they allow a simple realization of the simplicial localization $L(\mathcal{V}, \mathcal{W})$ solely in terms of the trivial fibrations. Namely, by a theorem of Cisinski [6, Proposition 3.23], the simplicial Hom-set $\underline{\mathrm{Hom}}(X, Y)$ of morphisms from X to Y in the simplicial localization of a category of fibrant objects is the nerve of the category whose objects are the spans

$$\begin{tikzcd}
& P \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & & Y
\end{tikzcd}$$

where f is a trivial fibration, and whose morphisms are commuting diagrams

$$\begin{tikzcd}
& P_0 \arrow[dl, "f_0"'] \arrow[dr, "g_0"] \arrow[dd, "h"'] & \\
X & & Y \\
& P_1 \arrow[ul, "f_1"] \arrow[ur, "g_1"'] &
\end{tikzcd}$$

(In the examples considered in this paper, in which the factorizations in the category of fibrant objects are functorial, this result already follows from the papers [11, 12].)

The following lemma is due to Brown; the idea behind the proof goes back to Serre's thesis (Chapître IV, Proposition 4 [23]).

Lemma 1.3. The weak equivalences of a category of fibrant objects are determined by the trivial fibrations: a morphism f is a weak equivalence if and only if it factorizes as a composition qs, where q is a trivial fibration and s is a section of a trivial fibration.

Proof. Let Y be an object of $\mathcal{V}$. The diagonal $Y \to Y \times Y$ has a factorization into a weak equivalence followed by a fibration:

$$Y \xrightarrow{\quad s \quad} PY \xrightarrow{\partial_0 \times \partial_1} Y \times Y .$$

The object PY is called a **path space** of Y.

Since Y is fibrant, the two projections from $Y \times Y$ to Y are fibrations, since they are pullbacks of the fibration $Y \to e$: it follows that the morphisms

$$\partial_0, \partial_1 : PY \longrightarrow Y$$

are fibrations as well. Since they are weak equivalences (by saturation of weak equivalences), they are actually trivial fibrations.

Given a morphism $f : X \to Y$, form the pullback

$$\begin{tikzcd}
P(f) \arrow[r, "\pi"] \arrow[d, "p(f)"'] & PY \arrow[d, "\partial_0"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

We see that the projection $p(f) : P(f) \to X$ is a trivial fibration, with section $s(f) : X \to P(f)$ induced by the morphisms $s : Y \to PY$ and $f : X \to Y$.

We may also express $P(f)$ as a pullback

$$\begin{tikzcd}
P(f) \arrow[r, "\pi"] \arrow[d, "p(f)\times q(f)"'] & PY \arrow[d, "\partial_0\times\partial_1"] \\
X \times Y \arrow[r, "f\times 1_Y"'] & Y \times Y
\end{tikzcd}$$

This shows that $p(f) \times q(f)$ is a fibration. Composing with the projection $X \times Y \to Y$, which is a fibration since X is fibrant, it follows that $q(f) : P(f) \to Y$ is a fibration. In this way, we obtain the desired factorization of f:

$$\begin{tikzcd}
 & P(f) \arrow[dr, "q(f)"] & \\
X \arrow[ur, "s(f)"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$ □

The proof of this lemma shows that Axiom (F4) is implied by the following special case:

(F4$*$) Each diagonal morphism $f : X \to X \times X$ has a factorization

$$\begin{tikzcd}
 & P \arrow[dr, "q"] & \\
X \arrow[ur, "r"] \arrow[rr, "f"'] & & X \times X
\end{tikzcd}$$

where r is a weak equivalence and q is a fibration.

2. DESCENT CATEGORIES

Recall the axioms for a descent category, which we stated in the introduction.

(D1) $\mathcal{V}$ has finite limits;

(D2) the pullback of a cover is a cover;

(D3) if f is a cover and gf is a cover, then g is a cover.

The covers in a descent category form a pre-topology on $\mathcal{V}$ (Grothendieck and Verdier [1]) with the special property that every cover consists of a single morphism: this class of pre-topologies will be sufficient for our purposes. Axiom (D3), which has no counterpart in the usual theory of Grothendieck topologies, plays a key role in this article.

The above axioms hold in the category of Kan complexes, with the trivial fibrations as covers. In the study of higher stacks, an additional axiom is sometimes assumed, that covers are closed under formation of retracts (c.f. Henriques [16]); we will not need this axiom here.

Example 2.1.

a) The category of schemes is a descent category, with surjective étale, smooth or flat morphisms as the covers.
b) The category of analytic spaces is a descent category, with surjective submersions as covers. A morphism $f : X \to Y$ of analytic spaces is a submersion if for every point $x \in X$, there is a neighbourhood U of x, a neighbourhood V of $f(x)$, and an isomorphism of analytic spaces $U \cong B \times V$ for which f is identified with projection to V, where B is an open ball in a complex vector space.
c) More generally, by Douady [7], the category of Banach analytic spaces is a descent category, again with surjective submersions as covers.

Example 2.2. A C^∞-ring (Dubuc [8]) is a real vector space R with operations

$$\rho_n : A(n) \times R^n \to R, \quad n \geq 0,$$

where $A(n) = C^\infty(\mathbb{R}^n, \mathbb{R})$. For every natural number n and n-tuple $(m_1, \dots, m_n)$, the following diagram must commute:

$$\begin{tikzcd}
A(n) \times A(m_1) \times \cdots \times A(m_n) \times R^{m_1} \times \cdots \times R^{m_n} \arrow[r, "A(n)\times\rho_{m_1}\times\cdots\times\rho_{m_n}"] \arrow[d] & A(n) \times R^n \arrow[d, "\rho_n"] \\
A(m_1 + \cdots + m_n) \times R^{m_1} \times \cdots \times R^{m_n} \arrow[r, "\rho_{m_1+\cdots+m_n}"'] & R
\end{tikzcd}$$

The category of C^∞-schemes is the opposite of the category of C^∞-rings. This category has finite limits, and contains the category of differentiable manifolds as a full subcategory. It is also a descent category, with covers the surjective submersions. The category of Lie groupoids in the category of C^∞-schemes is a natural generalization of the category of Lie groupoids in the usual sense: one of the results of this paper is that it is a category of fibrant objects.

The **kernel pair** of a morphism $f : X \to Y$ in a category with finite limits is the diagram

$$\begin{tikzcd}
X \times_Y X \arrow[r, shift left] \arrow[r, shift right] & X
\end{tikzcd}$$

The coequalizer p of the kernel pair of f, if it exists, is called the **coimage** of f:

$$\begin{tikzcd}
X \times_Y X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, "p"] \arrow[rr, bend left, "f"] & Z \arrow[r, dotted, "i"] & Y
\end{tikzcd}$$

The image of f is the morphism $i : Z \to Y$.

A morphism $f : X \to Y$ in a category $\mathcal{V}$ is an **effective epimorphism** if p equals f, in the sense that i is an isomorphism. One of the reasons for the importance of effective epimorphisms is that pullback along an effective epimorphism is conservative (reflects isomorphisms).

Definition 2.3. A descent category is **subcanonical** if covers are effective epimorphisms.

All of the descent categories which we have defined above have this property.

In the study of categories, regular categories play a special role: these are categories with finite limits in which pullbacks of effective epimorphisms are effective epimorphisms, and kernel pairs have coequalizers. Such categories share some basic properties with the category of sets: in particular, every morphism factors into an effective epimorphism followed by a monomorphism, and such a factorization is unique up to isomorphism.

Recall from the introduction that a regular descent category is a subcanonical descent category $\mathcal{V}$ together with a subcategory of **regular** morphisms satisfying the following axioms.

(R1) every cover is regular;
(R2) the pullback of a regular morphism is regular;
(R3) every regular morphisms has a coimage, and its coimage is a cover.

The following lemma is an example of the way in which a number of properties of regular categories, suitably reformulated, extend to regular descent categories.

Lemma 2.4. Let $\mathcal{V}$ be a regular descent category, and consider the factorization of a regular morphism $f : X \to Y$ into a cover $p : X \to Z$ followed by a morphism $i : Z \to Y$. Then i is a monomorphism.

Proof. The morphism

$$p \times_Y p : X \times_Y X \to Z \times_Y Z$$

is the composition of a pair of covers

$$X \times_Y X \xrightarrow{X \times_Y p} X \times_Y Z \xrightarrow{p \times_Y Z} Z \times_Y Z,$$

hence itself a cover. The two compositions $\pi_1 \circ (p \times_Y p)$ and $\pi_2 \circ (p \times_Y p)$ from $X \times_Y X$ to Z are equal. Since $p \times_Y p$ is a cover, it is an effective epimorphism, hence $\pi_1 = \pi_2 : Z \times_Y Z \to Z$. This implies that $i : Z \to Y$ is a monomorphism. ☐

3. k-GROUPOIDS

Fix a descent category $\mathcal{V}$. We refer to simplicial objects taking values in $\mathcal{V}$ as **simplicial spaces**. Denote the category of simplicial spaces by $s\mathcal{V}$.

Definition 3.1. Let T be a finite simplicial set, and let $S \hookrightarrow T$ be a simplicial subset. If $f : X \to Y$ is a morphism of simplicial spaces, define the space

$$\operatorname{Hom}(S \hookrightarrow T, f) = \operatorname{Hom}(S, X) \times_{\operatorname{Hom}(S,Y)} \operatorname{Hom}(T, Y).$$

This space parametrizes simplicial maps from T to Y with a lift to X along S.

Let $n \geq 0$ be a natural number. The **matching space** $\operatorname{Hom}(\partial\Delta^n, X)$ of a simplicial space X (also denoted $M_n(X)$) is the finite limit $\operatorname{Hom}(\partial\Delta^n, X)$, which represents simplicial morphisms from the boundary

$\partial\Delta^n$ of the n-simplex Δ^n to X. More generally, the matching space of a simplicial morphism $f : X \to Y$ between simplicial spaces is the finite limit

$$\mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, f) = \mathrm{Hom}(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y)} Y_n.$$

Definition 3.2. A simplicial morphism $f : X \to Y$ in $s\mathcal{V}$ is a **hypercover** if for all $n \geq 0$ the morphism

$$X_n \longrightarrow \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, f)$$

is a cover.

Lemma 3.3. Let T be a finite simplicial set, and let $S \hookrightarrow T$ be a simplicial subset. If $f : X \to Y$ is a hypercover, then the induced morphism

$$\mathrm{Hom}(T, X) \longrightarrow \mathrm{Hom}(S \hookrightarrow T, f)$$

is a cover.

Proof. There is a finite filtration of T

$$S = F_{-1}T \subset F_0T \subset F_1T \subset \cdots \subset T$$

satisfying the following conditions:

a) $T = \bigcup_\ell F_\ell T$;

b) there is a weakly monotone sequence n_ℓ, $\ell \geq 0$, and maps $x_\ell : \partial\Delta^{n_\ell} \longrightarrow F_\ell T$ and $y_\ell : \partial\Delta^{n_\ell} \longrightarrow F_{\ell-1}T$ such that the following diagram is a pushout:

$$\begin{tikzcd}
\partial\Delta^{n_\ell} \arrow[r, "y_\ell"] \arrow[d] & F_{\ell-1}T \arrow[d] \\
\Delta^{n_\ell} \arrow[r, "x_\ell"'] & F_\ell T
\end{tikzcd}$$

The morphism

$$\mathrm{Hom}(F_\ell T, X) \to \mathrm{Hom}(F_{\ell-1}T \hookrightarrow F_\ell T, f)$$

is a cover, since it is a pullback of the cover $X_{n_\ell} \to \mathrm{Hom}(\partial\Delta^{n_\ell} \hookrightarrow \Delta^{n_\ell}, f)$. ☐

Definition 3.4. Let k be a natural number. A simplicial space is a ***k*-groupoid** if the morphism

$$X_n \longrightarrow \mathrm{Hom}(\Lambda^n_i, X)$$

is a cover for all $n > 0$ and $0 \leq i \leq n$, and an isomorphism when $n > k$. Denote the category of k-groupoids by $s_k\mathcal{V}$.

Definition 3.5. A simplicial map $f : X \to Y$ in $s\mathcal{V}$ is a **fibration** if the morphism

$$X_n \longrightarrow \mathrm{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, f)$$

is a cover for all $n > 0$ and $0 \leq i \leq n$.

Our goal in the remainder of this section is to show that the k-groupoids in a descent category form a category of fibrant objects.

Theorem 3.6. With fibrations and hypercovers as fibrations and trivial fibrations, the category of k-groupoids $s_k\mathcal{V}$ is a category of fibrant objects.

The proof of Theorem 3.6 will consist of a sequence of lemmas; we also take the opportunity to derive some additional useful properties of fibrations and hypercovers along the way. Axiom (F1) is clear.

Definition 3.7. Let $m > 0$. An $\boldsymbol{m}$**-expansion** $S \hookrightarrow T$ (**expansion**, if $m = 1$) is a map of simplicial sets such that there exists a filtration

$$S = F_{-1}T \subset F_0T \subset F_1T \subset \cdots \subset T$$

satisfying the following conditions:

a) $T = \bigcup_\ell F_\ell T$;

b) there is a weakly monotone sequence $n_\ell \geq m$, $\ell \geq 0$, a sequence $0 \leq i_\ell \leq n_\ell$, and maps $x_\ell : \Delta^{n_\ell} \longrightarrow F_\ell T$ and $y_\ell : \Lambda^{n_\ell}_{i_\ell} \longrightarrow F_{\ell-1}T$ such that the following diagram is a pushout:

$$\begin{tikzcd}
\Lambda^{n_\ell}_{i_\ell} \arrow[r, "y_\ell"] \arrow[d] & F_{\ell-1}T \arrow[d] \\
\Delta^{n_\ell} \arrow[r, "x_\ell"'] & F_\ell T
\end{tikzcd}$$

Lemma 3.8. If $S \subset \Delta^n$ is the union of $0 < m \leq n$ faces of the n-simplex Δ^n, the inclusion $S \hookrightarrow \Delta^n$ is an m-expansion.

Proof. The proof is by induction on n: the initial step $n = 1$ is clear.

Enumerate the faces of Δ^n not in S:

$$\{\partial_{i_0}\Delta^n, \dots, \partial_{i_{n-m}}\Delta^n\},$$

where $0 \leq i_0 < \cdots < i_{n-m} \leq n$. Let

$$F_\ell\Delta^n = S \cup \bigcup_{j\leq\ell} \partial_{i_j}\Delta^n, \quad 0 \leq \ell \leq n-m.$$

By the induction hypothesis, we see that $F_{\ell-1}\Delta^n \cap \partial_{i_\ell}\Delta^n \hookrightarrow \partial_{i_\ell}\Delta^n$ is an m-expansion: on the one hand, each face of Δ^n contained in S contributes a face of $\partial_{i_\ell}\Delta^n$ to $F_{\ell-1}\Delta^n \cap \partial_{i_\ell}\Delta^n$, and hence $F_{\ell-1}\Delta^n \cap \partial_{i_\ell}\Delta^n$ contains at least m faces of $\partial_{i_\ell}\Delta^n$; on the other hand, $F_{\ell-1}\Delta^n \cap \partial_{i_\ell}\Delta^n$ does not contain the face $\partial_{i_{n-m}}\Delta^n \cap \partial_{i_\ell}\Delta^n$ of $\partial_{i_\ell}\Delta^n$. ☐

Lemma 3.9. Let T be a finite simplicial set, and let $S \hookrightarrow T$ be an m-expansion.

i) If X is a k-groupoid, the induced morphism

$$\mathrm{Hom}(T, X) \longrightarrow \mathrm{Hom}(S, X)$$

is a cover, and an isomorphism if $m > k$.

ii) If $f : X \to Y$ is a fibration of k-groupoids, the induced morphism

$$\mathrm{Hom}(T, X) \longrightarrow \mathrm{Hom}(S \hookrightarrow T, f)$$

is a cover, and an isomorphism if $m > k$.

Proof. The proof is by induction on the length of the filtration of T exhibiting $S \hookrightarrow T$ as an expansion. In the first case, the morphism $\mathrm{Hom}(F_\ell T, X) \to \mathrm{Hom}(F_{\ell-1}T, X)$ is a cover, since it is a pullback of the cover $X_{n_\ell} \to \mathrm{Hom}(\Lambda^{n_\ell}_{i_\ell}, X)$ (which is an isomorphism if $m > k$), and in the second case, the morphism

$$\mathrm{Hom}(F_\ell T, X) \to \mathrm{Hom}(F_{\ell-1}T \hookrightarrow S_\ell, f)$$

is a cover, since it is a pullback of the cover $X_{n_\ell} \to \mathrm{Hom}(\Lambda^{n_\ell}_{i_\ell} \hookrightarrow \Delta^n, f)$ (which is again an isomorphism if $m > k$). ☐

Corollary 3.10. If X is a k-groupoid, the face map $\partial_i : X_n \to X_{n-1}$ is a cover.

Lemma 3.11. If $f : X \to Y$ is a fibration of k-groupoids, then

$$X_n \longrightarrow \mathrm{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, f)$$

is an isomorphism for $n > k$.

Proof. We have the following commutative diagram, in which the square is a pullback:

$$\begin{tikzcd}
X_n \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathrm{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, f) \arrow[r] \arrow[d] & Y_n \arrow[d, "\gamma"] \\
& \mathrm{Hom}(\Lambda^n_i, X) \arrow[r] & \mathrm{Hom}(\Lambda^n_i, Y)
\end{tikzcd}$$

If $n > k$ and $0 \le i \le n$, β and γ are isomorphisms, and hence α is an isomorphism. ☐

Lemma 3.12. A hypercover $f : X \to Y$ of k-groupoids is a fibration.

Proof. For $n > 0$ and $0 \le i \le n$, we have the following commutative diagram, in which the square is a pullback:

$$\begin{tikzcd}
X_n \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, f) \arrow[r] \arrow[d] & X_{n-1} \arrow[d, "\gamma"] \\
& \mathrm{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, f) \arrow[r, "\delta"'] & \mathrm{Hom}(\partial\Delta^{n-1} \hookrightarrow \Delta^{n-1}, f)
\end{tikzcd} \tag{3.1}$$

If $n > 0$ and $0 \le i \le n$, then α and γ are covers, hence β is a cover. ☐

Lemma 3.13. Suppose the descent category $\mathcal{V}$ is subcanonical. If $f : X \to Y$ is a hypercover of k-groupoids, then $X_n \to \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, f)$ is an isomorphism for $n \ge k$.

Proof. Consider the diagram (3.1). If $n > k$, so that β is an isomorphism, we see that α is both a regular epimorphism and a monomorphism, and hence is an isomorphism.

To handle the remaining case, consider the diagram (3.1) with $n = k + 1$. We have already seen that all morphisms in the triangle forming the left side of the diagram are isomorphisms. But δ factors as the composition of the covers $\partial_i : X_{k+1} \to X_k$ and γ; hence, it is a cover. Since pullback along a cover in $\mathcal{V}$ reflects isomorphisms, we conclude that γ is an isomorphism. ☐

Next, we show that fibrations and hypercovers are closed under composition.

Lemma 3.14. If $f : X \to Y$ and $g : Y \to Z$ are hypercovers, then gf is a hypercover.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
X_n \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, f) \arrow[r] \arrow[d] & Y_n \arrow[d, "\gamma"] \\
& \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, gf) \arrow[r, "\delta"'] & \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, g)
\end{tikzcd} \tag{3.2}$$

in which the square is a pullback. Since α and γ are covers, it follows that β is a composition of two covers, and hence is itself a cover. ☐

Lemma 3.15. If $f : X \to Y$ and $g : Y \to Z$ are fibrations of k-groupoids, then gf is a fibration.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
X_n \arrow[r] \arrow[dr, "\alpha"'] & \mathrm{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, f) \arrow[r] \arrow[d] & Y_n \arrow[d, "\beta"] \\
& \mathrm{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, gf) \arrow[r] & \mathrm{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, g)
\end{tikzcd} \tag{3.3}$$

in which the square is a pullback. If $n > 0$ and $0 \leq i \leq n$, then β is a cover, implying that α is a composition of two covers, and hence itself a cover. ☐

Next, we prove Axioms (F2) and (F3).

Lemma 3.16. If $p : X \to Y$ is a hypercover and $f : Z \to Y$ is a morphism, the morphism q in the pullback diagram

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d, "q"'] & X \arrow[d, "p"] \\
Z \arrow[r, "f"'] & Y
\end{tikzcd}$$

is a hypercover.

Proof. In the pullback diagram

$$\begin{tikzcd}
X_n \times_{Y_n} Z_n \arrow[r] \arrow[d, "\alpha"'] & X_n \arrow[d, "\beta"] \\
\mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, q) \arrow[r] & \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, p)
\end{tikzcd}$$

the morphism α is a cover because β is. ☐

Lemma 3.17. If $p : X \to Y$ is a fibration of k-groupoids, and $f : Z \to Y$ is a morphism of k-groupoids, then $X \times_Y Z$ is a k-groupoid, and the morphism q in the pullback diagram

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d, "q"'] & X \arrow[d, "p"] \\
Z \arrow[r, "f"'] & Y
\end{tikzcd}$$

is a fibration.

Proof. Given $n > 0$ and $0 \leq i \leq n$, we have a pullback square

$$\begin{tikzcd}
X_n \times_{Y_n} Z_n \arrow[r] \arrow[d, "\alpha"'] & X_n \arrow[d, "\beta"] \\
\operatorname{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, q) \arrow[r] & \operatorname{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, p)
\end{tikzcd}$$

The morphism α is a cover because β is.

There is also a pullback square

$$\begin{tikzcd}
\operatorname{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, q) \arrow[r] \arrow[d, "\gamma"'] & Z_n \arrow[d] \\
\operatorname{Hom}(\Lambda^n_i, X \times_Y Z) \arrow[r] & \operatorname{Hom}(\Lambda^n_i, Z)
\end{tikzcd}$$

If Z is a k-groupoid, then γ is a cover, and an isomorphism if $n > k$. ☐

Next, we prove that $s\mathcal{V}$ is a descent category, with hypercovers as covers: that is, we show that hypercovers satisfy Axiom (D3).

Lemma 3.18. If $f : X \to Y$ and $g : Y \to Z$ are morphisms of simplicial spaces and f and gf are hypercovers, then g is a hypercover.

Proof. In diagram (3.2), α and β are covers. We will show that δ is a cover: applying Axiom (D3), it follows that γ is a cover.

For $-1 \leq j \leq n-1$, let

$$M_n(f,g,j) = \operatorname{Hom}(\operatorname{sk}_j \Delta^n, X) \times_{\operatorname{Hom}(\operatorname{sk}_j \Delta^n, Y)} \operatorname{Hom}(\partial^n\Delta \hookrightarrow \Delta, g),$$

where $\operatorname{sk}_j \Delta^n$, the j-skeleton of Δ^n, is the union of the j-simplices of Δ^n. The pullback square

$$\begin{tikzcd}
M_n(f,g,j) \arrow[r] \arrow[d] & \left(X_j\right)^{\binom{n+1}{j+1}} \arrow[d] \\
M_n(f,g,j-1) \arrow[r] & \operatorname{Hom}(\partial\Delta^j \hookrightarrow \Delta, f)^{\binom{n+1}{j+1}}
\end{tikzcd}$$

shows that the morphism $M_n(f,g,j) \to M_n(f,g,j-1)$ is a cover. Since

$$M_n(f,g,-1) \cong \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, g)$$

and

$$M_n(f,g,n-1) \cong \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, gf),$$

we see that the δ is a cover. □

In order to show that k-groupoids form a category of fibrant objects, we will need to construct path spaces. In fact, the proof requires iterated path spaces as well: it is convenient to organize these into a simplicial functor P_n. The proof of Theorem 3.6 actually only requires the functors P_1 and P_2 (and P_0, the identity functor).

Definition 3.19. Let $P_n : s\mathcal{V} \to s\mathcal{V}$ be the functor on simplicial spaces such that

$$(P_n X)_m = \mathrm{Hom}(\Delta^{m,n}, X),$$

where $\Delta^{m,n}$ is the prism $\Delta^m \times \Delta^n$.

The functor P_n is the space of maps from the n-simplex Δ^n to X; in particular, there is a natural isomorphism between $P_0 X$ and X, and $PX = P_1 X$ is a path space for X. Note that P_n preserves finite limits, and in particular, it preserves the terminal object e. Motivated by Brown's Lemma 1.3, we make the following definition.

Definition 3.20. A morphism $f : X \to Y$ of k-groupoids is a **weak equivalence** if the fibration

$$q(f) : P(f) \longrightarrow Y$$

is a hypercover, where $P(f) = X \times_Y P_1 Y$.

In the case of Kan complexes, this characterization of weak equivalences amounts to the vanishing of the relative simplicial homotopy groups. (A similar approach is taken, in the setting of simplicial sheaves, by Dugger and Isaksen [9].)

If T is a finite simplicial set and X is a simplicial space, denote by $P_T X$ the simplicial space

$$(P_T X)_n = \mathrm{Hom}(T, P_\bullet X_n) \cong \mathrm{Hom}(T \times \Delta^n, X).$$

The following theorem will be proved in the next section.

Theorem 3.21. The functor

$$P_\bullet : s\mathcal{V} \longrightarrow s^2\mathcal{V}$$

satisfies the following properties:

a) if $n \geq 0$ and $f : X \to Y$ is a fibration (respectively hypercover), the induced morphism

$$P_n X \longrightarrow P_{\partial\Delta^n} X \times_{P_{\partial\Delta^n} Y} Y_n$$

is a fibration (respectively hypercover);

b) if $f : X \to Y$ is a fibration, $n > 0$ and $0 \leq i \leq n$, the induced morphism

$$P_n X \longrightarrow P_{\Lambda^n_i} X \times_{P_{\Lambda^n_i} Y} Y_n$$

is a hypercover.

In particular, the functor P_1 satisfies the conditions for a (functorial) path space in a category of fibrant objects: the simplicial morphism $P_1X \to X \times X$ is a fibration, and the face maps $P_1X \to X$ are hypercovers. Lemma 1.3 now implies the following.

Lemma 3.22. Axiom (F4) holds in $s_k\mathcal{V}$.

Lemma 3.23. The weak equivalences form a subcategory of $s_k\mathcal{V}$.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be weak equivalences in $s_k\mathcal{V}$. Form the pullback

$$\begin{tikzcd}
P(g,f) \arrow[rr] \arrow[d] & & P_2Z \arrow[d, "\partial_0"] \\
P(f) \cong X \times_Y P_1Y \arrow[r, "f\times_Y P_1Y"'] & P_1Y \arrow[r, "P_1g"'] & P_1Z
\end{tikzcd}$$

In the following commutative diagram, the solid arrows are hypercovers:

$$\begin{tikzcd}
P(g,f) \arrow[r] \arrow[d] & P(gf) \times_X P(f) \arrow[r] & P(gf) \arrow[d, dashed] \\
P(g) \times_Y P(f) \arrow[r] & P(g) \arrow[r] & Z
\end{tikzcd}$$

The result now follows from Lemma 3.18. ☐

Lemma 3.24. If $f : X \to Y$ and $g : Y \to Z$ are morphisms of k-groupoids such that f and gf are weak equivalences, then g is a weak equivalence.

Proof. In the following commutative diagram, the solid arrows are hypercovers:

$$\begin{tikzcd}
P(g,f) \arrow[r] \arrow[d] & P(gf) \times_X P(f) \arrow[r] & P(gf) \arrow[d] \\
P(g) \times_Y P(f) \arrow[r] & P(g) \arrow[r, dashed] & Z
\end{tikzcd}$$

Again, the result follows from Lemma 3.18. ☐

Lemma 3.25. A fibration $f : X \to Y$ of k-groupoids is a weak equivalence if and only if it is a hypercover.

Proof. In the following commutative diagram, the solid arrows are hypercovers:

$$\begin{tikzcd}
P_1X \arrow[r] \arrow[d] & P(f) \arrow[d, dashed, "q(f)"] \\
X \arrow[r, dashed, "f"'] & Y
\end{tikzcd}$$

It follows by Lemma 3.18 that f is a hypercover if and only if $q(f)$ is. ☐

In order to complete the proof that $s_k\mathcal{V}$ is a category with weak equivalences, we need the following result, which is familiar in the case where $\mathcal{V}$ is a topos.

Lemma 3.26. If $f : X \to Y$ is a fibration of k-groupoids, and $g : Y \to Z$ and gf are hypercovers, then f is a hypercover.

Proof. The idea is to use the fact that $X_{n+1} \to \Lambda_{n+1,1}(f)$ is a cover in $\mathcal{V}$ in order to show that $X_n \to \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta, f)$ is a cover.

Define the fibred products

$$\begin{tikzcd}
V(f,g) \arrow[rr, "a"] \arrow[d, "b"'] & & X_{n+1} \arrow[d] \\
X_n \arrow[r, "s_0"'] & X_{n+1} \arrow[r] & \mathrm{Hom}(\Lambda^{n+1}_1 \hookrightarrow \Delta^n, gf)
\end{tikzcd}$$

$$\begin{tikzcd}
W(f,g) \arrow[rr, "\tilde{a}"] \arrow[d, "\tilde{b}"'] & & X_{n+1} \arrow[d] \\
X_n \arrow[r, "s_0"'] & X_{n+1} \arrow[r] & \mathrm{Hom}(\Lambda^{n+1}_0 \hookrightarrow \Delta^n, gf)
\end{tikzcd}$$

The spaces $V(f,g)$ and $W(f,g)$ are isomorphic: there is a morphism from $V(f,g)$ to $W(f,g)$, defined by the diagram

$$\begin{tikzcd}
V(f,g) \arrow[dr, dotted] \arrow[drrr, "a"] \arrow[ddr, "\partial_0 a"'] & & & \\
& W(f,g) \arrow[rr] \arrow[d] & & X_{n+1} \arrow[d] \\
& X_n \arrow[r, "s_0"'] & X_{n+1} \arrow[r] & \mathrm{Hom}(\Lambda^{n+1}_0 \hookrightarrow \Delta^n, gf)
\end{tikzcd}$$

Likewise, there is a morphism from $W(f,g)$ to $V(f,g)$, induced by the morphisms $\tilde{a} : V(f,g) \to X_{n+1}$ and $\partial_1\tilde{a} : V(f,g) \to X_n$. These morphisms between $V(f,g)$ and $W(f,g)$ are inverse to each other.

In this way, we see that the morphism $\partial_0 a : V_n(f,g) \to X_n$ is a cover: under the isomorphism $V(f,g) \cong W(f,g)$, it is identified with the morphism $\tilde{b} : V(f,g) \to X_n$, and this map is a pullback of a cover by Lemma 3.3, since gf is a hypercover.

Define the additional fibred products

$$\begin{tikzcd}
T(f,g) \arrow[rr] \arrow[d] & & Y_n \arrow[d] \\
X_n \arrow[r] & Y_n \arrow[r] & \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, g)
\end{tikzcd}$$

$$\begin{tikzcd}
U(f,g) \arrow[rrr] \arrow[d] & & & Y_{n+1} \arrow[d] \\
X_n \arrow[r, "s_0"'] & X_{n+1} \arrow[r] & Y_{n+1} \arrow[r] & \mathrm{Hom}(\Lambda^{n+1}_0 \hookrightarrow \Delta^n, g)
\end{tikzcd}$$

We have the following morphisms between the spaces $T(f,g)$, $U(f,g)$, and $V(f,g)$, each of which is a cover:

$$\begin{tikzcd}
T(f,g) \arrow[d, equal] & \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, f) \arrow[d, equal] \\
X_n \times_{\mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, g)} Y_n \arrow[r] & \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, gf) \times_{\mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, g)} Y_n
\end{tikzcd}$$

$$\begin{tikzcd}
U(f,g) \arrow[d, equal] & T(f,g) \arrow[d, equal] \\
X_n \times_{\mathrm{Hom}(\Lambda^{n+1}_1 \hookrightarrow \Delta^n, g)} Y_{n+1} \arrow[r] & X_n \times_{\mathrm{Hom}(\Lambda^{n+1}_1 \hookrightarrow \Delta^n, g)} \mathrm{Hom}(\partial\Delta^{n+1} \hookrightarrow \Delta^n, g)
\end{tikzcd}$$

$$\begin{tikzcd}
V(f,g) \arrow[d, equal] & U(f,g) \arrow[d, equal] \\
X_n \times_{\mathrm{Hom}(\Lambda^{n+1}_1 \hookrightarrow \Delta^n, gf)} X_{n+1} \arrow[r] & X_n \times_{\mathrm{Hom}(\Lambda^{n+1}_1 \hookrightarrow \Delta^n, gf)} \mathrm{Hom}(\Lambda^{n+1}_1 \hookrightarrow \Delta^n, f)
\end{tikzcd}$$

In this way, we obtain a diagram

$$\begin{tikzcd}
& V(f,g) \arrow[dr] & \\
X_n \arrow[ur] \arrow[rr, dashed] & & \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, f)
\end{tikzcd}$$

in which the solids arrows are covers, and hence the third arrow is as well. ☐

We can now complete the proof of Theorem 3.6.

Lemma 3.27. If $f : X \to Y$ and $g : Y \to Z$ are morphisms of k-groupoids such that g and gf are weak equivalences, then f is a weak equivalence.

Proof. In the following commutative diagram, the solid arrows are hypercovers, while the dashed arrow is a fibration:

$$\begin{tikzcd}
P(g,f) \arrow[r] \arrow[d] & P(gf) \times_X P(f) \arrow[d, "P(gf)\times_X p(f)"] \\
P(g) \times_Y P(f) \arrow[d, dashed, "P(g)\times_Y p(f)"'] & P(gf) \arrow[d, "q(gf)"] \\
P(g) \arrow[r, "q(g)"'] & Z
\end{tikzcd}$$

It follows by Lemma 3.18 that the composition

$$\begin{tikzcd} P(g) \times_Y P(f) \arrow[r, dashed, "P(g)\times_Y q(f)"] & P(g) \arrow[r, "q(g)"] & Z \end{tikzcd}$$

is a hypercover. Lemma 3.26 implies that $P(g) \times_Y q(f)$ is a hypercover. In the following commutative diagram, the solid arrows are hypercovers, while the dashed arrow is a fibration:

$$\begin{tikzcd} P(g) \times_Y P(f) \arrow[r, "p(g)\times_Y P(f)"] \arrow[d, "P(g)\times_Y q(f)"'] & P(f) \arrow[d, dashed, "q(f)"] \\ P(g) \arrow[r, "p(g)"'] & Y \end{tikzcd}$$

Applying Lemma 3.18 one final time, we conclude that $q(f)$ is a hypercover, and hence that f is a weak equivalence. ☐

4. THE SIMPLICIAL RESOLUTION FOR k-GROUPOIDS

In this section, we prove Theorem 3.21. Consider the following subcomplexes of the prism $\Delta^{m,n}$:

$$\Lambda_i^{m,n} = (\Lambda_i^m \times \Delta^n) \cup (\Delta^m \times \partial\Delta^n) \qquad \tilde{\Lambda}_j^{m,n} = (\partial\Delta^m \times \Delta^n) \cup (\Delta^m \times \Lambda_j^n).$$

Moore has proved that the inclusions $\Lambda_i^{m,n} \hookrightarrow \Delta^{m,n}$ and $\tilde{\Lambda}_j^{m,n} \hookrightarrow \Delta^{m,n}$ are expansions. The following lemma is a refinement of his theorem.

Lemma 4.1. The inclusions $\Lambda_i^{m,n} \hookrightarrow \Delta^{m,n}$ and $\tilde{\Lambda}_j^{m,n} \hookrightarrow \Delta^{m,n}$ are m- and n-expansions respectively.

Proof. The proof is a modification of an argument of Cartan [5]. The proofs of the two parts are formally identical, and we will concentrate on the former.

An (m,n)-shuffle is a permutation π of $\{1,\dots,m+n\}$ such that

$$\pi(1) < \cdots < \pi(m) \text{ and } \pi(m+1) < \cdots < \pi(m+n).$$

The (m,n)-shuffles index the $\binom{m+n}{m}$ non-degenerate simplices of the prism $\Delta^{m,n}$: we denote the simplex labeled by a shuffle π by the same symbol π. Any simplex of dimension $m+n-1$ in $\Delta^{m,n}$ lies in at most two top-dimensional simplices.

The geometric realization of the simplex Δ^n is the convex hull of the vertices

$$v_i = (\underbrace{0,\dots,0}_{n-i \text{ times}},\underbrace{1,\dots,1}_{i \text{ times}}) \in \mathbb{R}^n.$$

Thus, the simplex is the convex set

$$\Delta^n = \{(t_1,\dots,t_n) \subset \mathbb{R}^n \mid 0 \le t_1 \le \cdots \le t_n \le 1\}.$$

Given sequences $0 < s_1 \cdots < s_m < 1$ and $0 < t_1 < \cdots < t_n < 1$ such that $s_i \neq t_j$, representing a pair of points in the interiors of Δ^m and Δ^n respectively, the union of these sequences determines a word of length $m+n$ in the letters s and t, with m letters s and n letters t, and hence an (m,n)-shuffle. The set of such points associated to a shuffle π is the interior of the geometric realization $|\pi| \subset |\Delta^{m,n}| \cong |\Delta^m| \times |\Delta^n|$.

Represent an (m,n)-shuffle π by the sequence of natural numbers

$$0 \le a_1(\pi) \le \cdots \le a_m(\pi) \le n,$$

in such a way that the associated shuffle has the form

$$t^{a_1}st^{a_2-a_1}s\dots t^{a_m-a_{m-1}}st^{n-a_m},$$

in other words,

$$0=s_0<\dots<s_j<t_{a_j+1}<\dots<t_{a_{j+1}}<s_{j+1}<\dots s_{m+1}=1.$$

We adopt the convention that $a_0=0$ and $a_{m+1}=n$.

Filter $\Delta^{m,n}$ by the subcomplexes

$$F_\ell\Delta^{m,n}=\Lambda^{m,n}_i\cup\bigcup_{\{\pi|b(\pi,i)\le\ell\}}\pi,$$

where

$$b(\pi,i)=\sum_{j=1}^{i}a_j(\pi)-\sum_{j=i+1}^{m}a_j(\pi).$$

The faces of a top-dimensional simplex π are as follows:

- the geometric realization of the face $\partial_{a_j+j-1}\pi$ is the intersection of the geometric realization of the simplex π with the hyperplane
$$t_{a_j}=s_j,$$
when $a_{j-1}<a_j$, and the hyperplane
$$s_{j-1}=s_j,$$
when $a_{j-1}=a_j$;
- the geometric realization of the face $\partial_{a_j+j}\pi$ is the intersection of the geometric realization of the simplex π with the hyperplane
$$s_j=t_{a_j+1},$$
when $a_j<a_{j+1}$, and the hyperplane
$$s_j=s_{j+1},$$
when $a_j=a_{j+1}$;
- when $a_j+j<k<a_{j+1}+j$, the geometric realization of the face $\partial_k\pi$ is the intersection of the geometric realization of the simplex π with the hyperplane
$$t_{k-j}=t_{k-j+1}.$$

We must show that at least one face of π does not lie in $F_{b(\pi,i)-1}\Delta^{m,n}$:

i) if $a_i(\pi)=a_{i+1}(\pi)$, the face $\partial_{a_i+i}\pi$ is not contained in $\Lambda^{m,n}_i$, nor in any top-dimensional simplex of $\Delta^{m,n}$ other than π;

ii) if $a_i(\pi)<a_{i+1}(\pi)$ and $i>0$, the face $\partial_{a_i+i}\pi$ is contained in the simplex $\tilde\pi$ with
$$a_j(\tilde\pi)=\begin{cases}a_j(\pi), & j<i,\\ a_j(\pi)+1, & j=i,\\ a_j(\pi), & j>i,\end{cases}$$
for which $b(\tilde\pi,i)=b(\pi,i)+1$;

iii) if $a_i(\pi) < a_{i+1}(\pi)$ and $i < m$, the face $\partial_{a_{i+1}+i-1}\pi$ is contained in the simplex $\tilde{\pi}$ with

$$a_j(\tilde{\pi}) = \begin{cases} a_j(\pi), & j < i+1, \\ a_j(\pi) - 1, & j = i+1, \\ a_j(\pi), & j > i+1, \end{cases}$$

for which $b(\tilde{\pi}, i) = b(\pi, i) + 1$.

By Lemma 3.8, the proof is completed by enumerating at least m faces of π which lie in either $\Lambda_i^{m,n}$ or a simplex $\tilde{\pi}$ for which $b(\tilde{\pi}, i) = b(\pi, i) - 1$:

i) For each $j < i$ with $a_j < a_{j+1}$, we obtain $a_{j+1} - a_j$ such faces as follows:
  a1) the $a_{j+1} - a_j - 1$ faces $\partial_\ell \pi$ with $a_j + j < \ell < a_{j+1} + j - 1$ lie in $\Lambda_i^{m,n}$;
  a2) the face $\partial_{a_{j+1}+j-1}\pi$ lies in the simplex $\tilde{\pi}$ with

$$a_j(\tilde{\pi}) = \begin{cases} a_k(\pi), & k < j+1, \\ a_k(\pi) - 1, & k = j+1, \\ a_k(\pi), & k > j+1, \end{cases}$$

  for which $b(\tilde{\pi}, i) = b(\pi, i) - 1$.

ii) For each $j > i$ with $a_j < a_{j+1}$, we obtain $a_{j+1} - a_j$ such faces as follows:
  b1) the $a_{j+1} - a_j - 1$ faces $\partial_\ell \pi$ with $a_j + j + 1 < \ell < a_{j+1} + j$ lie in $\Lambda_i^{m,n}$;
  b2) the face $\partial_{a_j+j+1}\pi$ lies in the simplex $\tilde{\pi}$ with

$$a_j(\tilde{\pi}) = \begin{cases} a_k(\pi), & k < j, \\ a_k(\pi) + 1, & k = j, \\ a_k(\pi), & k > j, \end{cases}$$

  for which $b(\tilde{\pi}, i) = b(\pi, i) - 1$.

iii) The $a_{i+1} - a_i - 1$ faces $\partial_\ell \pi$ with $a_i + i < \ell < a_{i+1} + i - 1$ lie in $\Lambda_i^{m,n}$.
iv) The face $\partial_0 \pi$ lies in $\Lambda_i^{m,n}$ unless $i = 0$ and $a_1 = 0$.
v) The face $\partial_{m+n}\pi$ lies in $\Lambda_i^{m,n}$ unless $i = m$ and $a_m = n$. ☐

Lemma 4.2. Let T be a finite simplicial set, and let $S \hookrightarrow T$ be a simplicial subset. Then

$$\Delta^m \times S \cup \Lambda_i^m \times T \hookrightarrow \Delta^m \times T$$

is an m-expansion, and

$$S \times \Delta^n \cup T \times \Lambda_j^n \hookrightarrow T \times \Delta^n$$

is an n-expansion.

Proof. We prove the first statement: the proof of the second is analogous.

Filter T by the simplicial subsets $F_\ell T = S \cup \mathrm{sk}_\ell T$. Let I_ℓ be the set of nondegenerate simplices in $T_\ell \setminus S_\ell$. There is a pushout square

$$\begin{tikzcd}
(\Lambda_i^{m,\ell})^{I_\ell} \arrow[r] \arrow[d] & \Delta^m \times F_{\ell-1}T \cup \Lambda_i^m \times T \arrow[d] \\
(\Delta^{m,\ell})^{I_\ell} \arrow[r] & \Delta^m \times F_\ell T \cup \Lambda_i^m \times T
\end{tikzcd}$$

and by Lemma 4.1, the vertical arrows of this diagram are m-expansions. Composing the m-expansions

$$\Delta^m \times F_{\ell-1}T \cup \Lambda^m_i \times T \hookrightarrow \Delta^m \times F_\ell T \cup \Lambda^m_i \times T$$

for $\ell \geq 0$, we obtain the result. ☐

Proof of Theorem 3.21. Let X be a k-groupoid. To show that $P_n X$ is a k-groupoid, we must show that for all $0 \leq i \leq m$, the morphism

$$P_n X_m \longrightarrow \operatorname{Hom}(\Lambda^m_i, P_n X)$$

is a cover, if $m > 0$, and an isomorphism, if $m > k$. This follows by Part i) of Lemma 3.9, since $\Lambda^{m,n}_i \hookrightarrow \Delta^{m,n}$ is an m-expansion.

If $f : X \to Y$ is a fibration, then for all $n \geq 0$, the simplicial morphism

$$P_n X \longrightarrow \operatorname{Hom}(\partial\Delta^n, P_\bullet X) \times_{\operatorname{Hom}(\partial\Delta^n, P_\bullet Y)} P_n Y$$

is a fibration since for all $m > 0$, the morphism $\Lambda^{m,n}_i \hookrightarrow \Delta^{m,n}$ is an expansion, and for all $n > 0$, the simplicial morphism

$$P_n X \longrightarrow \operatorname{Hom}(\Lambda^n_j, P_\bullet X) \times_{\operatorname{Hom}(\Lambda^n_j, P_\bullet Y)} P_n Y$$

is a cover since for all $m > 0$, the morphism $\tilde{\Lambda}^{m,n}_j \hookrightarrow \Delta^{m,n}$ is an expansion.

If $f : X \to Y$ is a hypercover, then for all $n \geq 0$, the simplicial morphism

$$P_n X \longrightarrow \operatorname{Hom}(\partial\Delta^n, P_\bullet X) \times_{\operatorname{Hom}(\partial\Delta^n, P_\bullet Y)} P_n Y$$

is a cover, by Lemma 3.3 applied to the inclusion of simplicial sets

$$(\partial\Delta^m \times \Delta^n) \cup (\Delta^m \times \Delta^n) \hookrightarrow \Delta^{m,n}.$$

☐

5. A CHARACTERIZATION OF WEAK EQUIVALENCES BETWEEN k-GROUPOIDS

A morphism $f : X \to Y$ of k-groupoids is a weak equivalence if and only if the morphism

$$P(f)_n \longrightarrow \operatorname{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, q(f))$$

is a cover for $n \geq 0$. When $n = 0$, this condition says that the morphism

$$X_0 \times_{Y_0} Y_1 \to Y_0$$

is a cover, which is a translation to the setting of simplicial spaces of the condition for a morphism between Kan complexes that the induced morphism of components $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ be surjective. For $n > 0$, it analogous to the condition for a morphism of Kan complexes $f : X \to Y$ that the relative homotopy groups $\pi_{n+1}(Y, X)$ (with arbitrary choice of basepoint) vanish.

The following theorem is analogous to Gabriel and Zisman's famous theorem on anodyne extensions [14, Chapter IV, Section 2].

Theorem 5.1. A morphism $f : X \to Y$ of k-groupoids is a weak equivalence if and only if the morphisms

$$\operatorname{Hom}(\Delta^n \hookrightarrow \Delta^{n+1}, f) \longrightarrow \operatorname{Hom}(\partial\Delta^n \hookrightarrow \Lambda^{n+1}_{n+1}, f) \tag{5.1}$$

are covers for $n \geq 0$.

Proof. We have

$$P(f)_n \cong \mathrm{Hom}(\Delta^n \hookrightarrow \Delta^{1,n}, f),$$

and

$$\mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, q(f)) \cong \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Lambda^{1,n}_1, f).$$

This shows that f is a weak equivalence if and only if the morphisms

$$\mathrm{Hom}(\Delta^n \hookrightarrow \Delta^{1,n}, f) \longrightarrow \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Lambda^{1,n}_1, f) \tag{5.2}$$

are covers for all $n \geq 0$.

Suppose that the morphism (5.1) is a cover for $n \geq 0$; we show that (5.2) is a cover for $n \geq 0$. For $0 \leq i \leq n$, let $\Delta^{n+1}_i \subset \Delta^{1,n}$ be the simplex whose vertices are

$$\{(0,0), \dots, (0,i), (1,i), \dots, (1,n)\}.$$

Observe that

$$\Delta^{n+1}_{i-1} \cap \Delta^{n+1}_i = \partial_i\Delta^{n+1}_{i-1} = \partial_i\Delta^{n+1}_i.$$

Filter the prism:

$$F_i\Delta^{1,n} = \Lambda^{1,n}_1 \cup \Delta^{n+1}_0 \cup \cdots \cup \Delta^{n+1}_i.$$

If $i < n$, there is a pullback diagram

$$\begin{tikzcd}
\mathrm{Hom}(\partial\Delta^n \hookrightarrow F_i\Delta^{1,n}, f) \arrow[r] \arrow[d] & Y_{n+1} \arrow[d, "\partial_i"] \\
\mathrm{Hom}(\partial\Delta^n \hookrightarrow F_{i-1}\Delta^{1,n}, f) \arrow[r] & Y_n
\end{tikzcd}$$

The vertical morphisms are covers by part i) of Lemma 3.9: composing them for $0 \leq i < n$, we see that the morphism

$$\mathrm{Hom}(\partial\Delta^n \hookrightarrow F_{n-1}\Delta^{1,n}, f) \longrightarrow \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Lambda^{1,n}_1, f)$$

is a cover.

There is also a pullback diagram

$$\begin{tikzcd}
\mathrm{Hom}(\Delta^n \hookrightarrow \Delta^{1,n}, f) \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^n \hookrightarrow \Delta^{n+1}, f) \arrow[d] \\
\mathrm{Hom}(\partial\Delta^n \hookrightarrow F_{n-1}\Delta^{1,n}, f) \arrow[r] & \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Lambda^{n+1}_{n+1}, f)
\end{tikzcd}$$

The right-hand vertical morphism is a cover by hypothesis, and hence the left-hand vertical morphism, namely (5.2), is also a cover.

Now, suppose that (5.2) is a cover for $n \geq 0$; we show that (5.1) is a cover for $n \geq 0$. There is a map from $\Delta^{1,n}$ to Δ^{n+1}, which takes the vertex $(0,i)$ to i, and the vertices $(1,i)$ to $n+1$. This map takes the simplicial subset $\Lambda^{1,n}_1 \subset \Delta^{1,n}$ to the horn $\Lambda^{n+1}_{n+1} \subset \Delta^{n+1}$, and induces a pullback square

$$\begin{tikzcd}
\mathrm{Hom}(\Delta^n \hookrightarrow \Delta^{n+1}, f) \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^n \hookrightarrow \Delta^{1,n}, f) \arrow[d] \\
\mathrm{Hom}(\partial\Delta^n \hookrightarrow \Lambda^{n+1}_{n+1}, f) \arrow[r] & \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Lambda^{1,n}_1, f)
\end{tikzcd}$$

It follows that (5.1) is a cover for $n \geq 0$. $\square$

6. k-CATEGORIES

In this section, we study a class of simplicial spaces bearing the same relationship to k-groupoids as categories bear to groupoids. The definition of k-categories is inspired by Rezk's definition of a complete Segal space [22].

Recall that the thick 1-simplex $\mathbb{\Delta}^1$ is the nerve of the groupoid $[\![1]\!]$ with objects $\{0, 1\}$ and a single morphism between any pair of objects.

Definition 6.1. Let $k > 0$. A ***k*-category** in a descent category $\mathcal{V}$ is a simplicial space X such that

1) if $0 < i < n$, the morphism
$$X_n \to \mathrm{Hom}(\Lambda^n_i, X)$$
is a cover, and an isomorphism if $n > k$;
2) if $i \in \{0, 1\}$, the morphism
$$\mathrm{Hom}(\mathbb{\Delta}^1, X) \to \mathrm{Hom}(\Lambda^1_i, X) \cong X_0$$
is a cover.

There is an involution permuting the two vertices of $\mathbb{\Delta}^1$. Thus, in the second axiom above, it suffices to consider one of the two morphisms $\mathrm{Hom}(\mathbb{\Delta}^1, X) \to \mathrm{Hom}(\Lambda^1_i, X)$, since they are isomorphic.

Lemma 6.2. A k-category X is $k + 1$-coskeletal, that is, for every $n \geq 0$,

$$X_n \cong \operatorname{cosk}_{k+1} X_n = \mathrm{Hom}(\operatorname{sk}_{k+1} \Delta^n, X).$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
& \mathrm{Hom}(\partial\Delta^n, X) \arrow[dr, "\beta_n"] & \\
X_n \arrow[ur, "\alpha_n"] \arrow[rr, "\gamma_n"'] & & \mathrm{Hom}(\Lambda^n_{n-1}, X)
\end{tikzcd}$$

If $n > k$, the morphism γ_n is an isomorphism, and hence β_n is a split epimorphism.

If furthermore $n > k + 1$, the upper horizontal morphism of the pullback square

$$\begin{tikzcd}
\mathrm{Hom}(\partial\Delta^n, X) \arrow[r] \arrow[d, "\beta_n"'] & X_{n-1} \arrow[d, "\alpha_{n-1}"] \\
\mathrm{Hom}(\Lambda^n_{n-1}, X) \arrow[r, "\partial_{n-1}"'] & \mathrm{Hom}(\partial\Delta^{n-1}, X)
\end{tikzcd}$$

factors into a composition

$$\operatorname{Hom}(\partial\Delta^n, X) \xrightarrow{\beta_n} \operatorname{Hom}(\Lambda^n_{n-1}, X) \xrightarrow{\beta_{n-1}\partial_{n-1}} \operatorname{Hom}(\Lambda^{n-1}_{n-2}, X) \xrightarrow{\gamma_{n-1}^{-1}} X_{n-1}$$

and hence, by universality of the pullback square, the morphism β_n is a monomorphism. Since this morphism is also a split epimorphism, it follows that β_n is an isomorphism. We conclude that α_n is an isomorphism.

The pullback square

$$\begin{tikzcd}
\operatorname{cosk}_\ell X_n \arrow[r] \arrow[d] & (X_\ell)^{\binom{n+1}{\ell+1}} \arrow[d, "\alpha_\ell^{\binom{n+1}{\ell+1}}"] \\
\operatorname{cosk}_{\ell-1} X_n \arrow[r] & \operatorname{Hom}(\partial\Delta^\ell, X)^{\binom{n+1}{\ell+1}}
\end{tikzcd}$$

shows that the morphism $\operatorname{cosk}_\ell X_n \operatorname{cosk}_{\ell-1} X_n$ is an isomorphism if $\ell > k+1$. The lemma follows by downward induction in ℓ, since $X_n \cong \operatorname{cosk}_n X_n$. ☐

If T is a finite simplicial set, form the coend

$$T \times_\Delta \mathbb{\Delta} = \int^{n\in\Delta} T_n \times \mathbb{\Delta}^n.$$

(This is denoted $k_!T$ by Joyal and Tierney [18].) As examples of this construction, we have the thick horns

$$\mathbb{\Lambda}^n_i = \Lambda^n_i \times_\Delta \mathbb{\Delta} \subset \mathbb{\Delta}^n$$

and the thick boundary

$$\partial\mathbb{\Delta}^n = \Delta^n \times_\Delta \mathbb{\Delta} \subset \mathbb{\Delta}^n$$

Of course, $\mathbb{\Lambda}^1_i \cong \Lambda^1_i$, and $\partial\mathbb{\Delta}^1 \cong \partial\Delta^1$.

Inner expansions play the same role in the theory of k-categories that expansions play in the theory of k-groupoids.

Definition 6.3. An **inner m-expansion** (inner expansion, if $m = 1$) is a map of simplicial sets such that there exists a filtration

$$S = F_{-1}T \subset F_0T \subset F_1T \subset \cdots \subset T$$

satisfying the following conditions:

1) $T = \bigcup_\ell F_\ell T$;
2) there is a weakly monotone sequence $n_\ell \geq m$, a sequence $0 < i_\ell < n_\ell$, and maps $x_\ell : \Delta^{n_\ell} \longrightarrow F_\ell T$ and $y_\ell : \Lambda^{n_\ell}_{i_\ell} \longrightarrow F_{\ell-1}T$ such that the following diagram is a pushout:

$$\begin{tikzcd}
\Lambda^{n_\ell}_{i_\ell} \arrow[r, "y_\ell"] \arrow[d] & F_{\ell-1}T \arrow[d] \\
\Delta^{n_\ell} \arrow[r, "x_\ell"'] & F_\ell T
\end{tikzcd}$$

It is not hard to see that inner n-expansions form a category.

Lemma 6.4. If $0 < i < n$, the inclusion $\mathbb{\Lambda}^n_i \cup \Delta^n \hookrightarrow \mathbb{\Delta}^n$ is an inner n-expansion.

Proof. The k-simplices of $\mathbb{\Delta}^n$ have the form $(i_0, \dots, i_k)$, where $i_0, \dots, i_k \in \{0, \dots, n\}$; a k-simplex is nondegenerate if $i_{j-1} \neq i_j$ for $1 \leq j \leq k$.

Let $Q_{k,m}$, $0 \leq m < k - i$ be the set of non-degenerate k-simplices $s = (i_0 \dots i_k)$ of $\mathbb{\Delta}^n$ which satisfy the following conditions:

a) s is not contained in $\mathbb{\Lambda}^n_i \cup \Delta^n$;
b) $i_{j-1} = i_{j+1}$ for $i \leq j < i + m$;
c) $i_{i+m} = i$;
d) $i_{i+m-1} \neq i_{i+m+1}$.

For example, if $n = 2$ and $i = 1$, then $Q_{2,0} = \{(2, 1, 0)\}$,

$$Q_{3,1} = \{(1, 0, 1, 2), (1, 2, 1, 0)\},$$

and

$$Q_{3,0} = \{(0, 1, 2, 0), (0, 1, 2, 1), (2, 1, 0, 1), (2, 1, 0, 2)\}.$$

Let R_k be the set of non-degenerate k-simplices which do not lie in $\mathbb{\Lambda}^n_i \cup \Delta^n$, nor in any of the sets $Q_{k,m}$.

The simplicial set $\mathbb{\Delta}^n$ is obtained from $\mathbb{\Lambda}^n_i \cup \Delta^n$ by inner expansions along the simplices of type $Q_{k,m}$ in order first of increasing k, then of decreasing m. (The order in which the simplices are adjoined within the sets $Q_{k,m}$ is unimportant.)

To prove this, consider a simplex $s = (i_0, \dots, i_k)$ in R_k. There is a unique natural number $0 \leq m_s < k - i$ such that the simplex

$$\tilde{s} = (i_0, \dots, i_{i+m_s-1}, i, i_{i+m_s}, \dots, i_k)$$

has type Q_{k+1,m_s}. In fact, m_s is either 0 or the largest positive number m satisfying the following conditions:

a) $i_{j-1} = i_{j+1}$ for $i \leq j < i + m$;
b) $i_{i+m-2} = i$;
c) $i_{i+m-1} \neq i$.

The simplex $\tilde{s}$ is non-degenerate: i_{i+m_s-1} does not equal i by hypothesis, while i_{i+m_s} does not equal i by the maximality of m_s. It is easily seen that $\tilde{s}$ has type Q_{k+1,m_s}.

We see that $s = \partial_{i+m_s}\tilde{s}$ is an inner face of $\tilde{s}$. The faces $\partial_j \tilde{s}$, $j < i$, are either degenerate, lie in $\mathbb{\Lambda}^n_i \cup \Delta^n$, or lie in Q_{k,m_s-1}. The faces $\partial_j \tilde{s}$, $j > i$, are either degenerate, lie in $\mathbb{\Lambda}^n_i \cup \Delta^n$, or lie in the boundary of simplex in $Q_{k+1,m}$, $m > m_s$. ☐

Corollary 6.5. If $S \hookrightarrow T$ is an inner n-expansion of finite simplicial sets, then

$$S \times_\Delta \mathbb{\Delta} \cup T \hookrightarrow T \times_\Delta \mathbb{\Delta}$$

is an inner n-expansion.

Proof. The proof is by induction on the length of the filtration

$$S = F_{-1}T \subset F_0T \subset F_1T \subset \cdots \subset T$$

exhibiting $S \hookrightarrow T$ as an inner n-expansion. We see that there is a pushout square

$$\begin{tikzcd}
\mathbb{\Lambda}^{n_\ell} \cup \Delta^{n_\ell} \arrow[r] \arrow[d] & \big(F_{\ell-1}T \times_\Delta \mathbb{\Delta}\big) \cup T \arrow[d, "(y_\ell \times_\Delta \mathbb{\Delta}) \cup x_\ell"] \\
\mathbb{\Delta}^{n_\ell} \arrow[r] & \big(F_\ell T \times_\Delta \mathbb{\Delta}\big) \cup T
\end{tikzcd}$$

It follows that $\big(F_\ell T \times_\Delta \mathbb{\Delta}\big) \cup T \hookrightarrow \big(F_{\ell-1} T \times_\Delta \mathbb{\Delta}\big) \cup T$ is an n_ℓ-expansion, where $n_\ell \geq n$. Since the inner n-expansions are closed under composition, the result follows. $\square$

Corollary 6.6. If $S \hookrightarrow T$ is an m-expansion of finite simplicial sets, where $m > 1$, then

$$S \times_\Delta \mathbb{\Delta} \hookrightarrow T \times_\Delta \mathbb{\Delta}$$

is an inner m-expansion.

Proof. The proof is by induction on the number of nondegenerate simplices in $T \setminus S$. For the induction step, it suffices to prove that if $n > 1$ and $0 \leq i \leq n$, the inclusion $\mathbb{\Lambda}^n_i \hookrightarrow \mathbb{\Delta}^n$ is an inner n-expansion.

The action of the symmetric group S_{n+1} on the simplicial set $\mathbb{\Delta}^n$ induces a transitive permutation of the subcomplexes $\mathbb{\Lambda}^n_i$. Thus, it suffices to establish the result when $i = 1$. But in this case, the inclusion $\mathbb{\Lambda}^n_1 \hookrightarrow \mathbb{\Lambda}^n_1 \cup \Delta^n$ is an inner n-expansion, and the result follows from Lemma 6.4. $\square$

We will also need some results involving the simplicial set $\mathbb{\Delta}^1$. This simplicial set has two nondegenerate simplices of dimension k, which we denote by

$$\mathbb{k} = (0, 1, \dots) \qquad\qquad \mathbb{k}^* = (1, 0, \dots).$$

Let $\mathbb{k}^\circ$ be the mirror of $\mathbb{k}$:

$$\mathbb{k}^\circ = (\dots, 1, 0) = \begin{cases} \mathbb{k} & k \text{ even} \\ \mathbb{k}^* & k \text{ odd} \end{cases}.$$

In particular, the simplicial subset $\mathbb{\Lambda}^1_1 \hookrightarrow \mathbb{\Delta}^1$ may be identified with the vertex $\mathbb{0} = (0)$.

Lemma 6.7. The inclusion

$$\partial\Delta^n \times \mathbb{\Delta}^1 \cup \Delta^n \times \mathbb{\Lambda}^1_1 \hookrightarrow \Delta^n \times \mathbb{\Delta}^1$$

is an expansion, and an inner expansion if $n > 0$.

Proof. The expansion $\mathbb{\Lambda}^1_1 = \mathbb{0} \hookrightarrow \mathbb{\Delta}^1$ is obtained by successively adjoining the simplices $\mathbb{1}$, $\mathbb{2}$, ...

The product $\Delta^n \times \mathbb{\Delta}^1$ is isomorphic to the iterated join of $n + 1$ copies of $\mathbb{\Delta}^1$. Indeed, a k-simplex of $\Delta^n \times \mathbb{\Delta}^1$ may be identified with a pair consisting of a k-simplex $0^{a_0} \dots n^{a_n}$ of Δ^n, where $a_0 + \dots + a_n = k + 1$, and a k-simplex $(i_0, \dots, i_k)$ of $\mathbb{\Delta}^1$. We may think of this k-simplex as a sequence of simplices $(\sigma_0, \dots, \sigma_n)$, where σ_i is an $(a_i - 1)$-simplex of $\mathbb{\Delta}^1$ if $a_i > 0$, and is absent if $a_i = 0$. Such a simplex is degenerate precisely when one of the σ_i is degenerate. Denote the simplex $(i_0, \dots, i_k) \times 0^{a_0} \dots n^{a_n}$ by $[\sigma_0; \dots; \sigma_n]$.

The simplicial subset $\partial\Delta^n \times \mathbb{\Delta}^1 \cup \Delta^n \times \mathbb{\Lambda}^1_1 \subset \Delta^n \times \mathbb{\Delta}^1$ is the union of the simplex $[\mathbb{0}; \dots; \mathbb{0}]$, the simplices $[\sigma_0; \dots; \sigma_{i-1}; ; \sigma_{i+1}; \dots; \sigma_n]$, and their faces.

Let $S_{k,\ell,m}$ be the set of k-simplices in $\Delta^n \times \mathbb{\Delta}^1$ of the form

$$[\mathbb{0}; \dots; \mathbb{0}; \mathbb{m}; \sigma_{n-\ell+1}; \dots; \sigma_n],$$

if $\ell < n$, and of the form

$$[\mathbb{m}^\circ; \sigma_1; \dots; \sigma_n]$$

if $\ell = n$. The successive expansions of $\partial\Delta^n \times \mathbb{\Delta}^1 \cup \Delta^n \times \mathbb{\Lambda}^1_1$ along the simplices of $S_{k,\ell,m}$, in order first of ascending k, next of ascending ℓ (between 0 and n), and lastly of ascending m (between 1 and $k - n$), exhibit the inclusion

$$\partial\Delta^n \times \mathbb{\Delta}^1 \cup \Delta^n \times \mathbb{\Lambda}^1_1 \hookrightarrow \Delta^n \times \mathbb{\Delta}^1$$

as an inner expansion. $\square$

Corollary 6.8. A k-groupoid is a k-category.

Proof. This follows from Lemma 3.9 and the special case of the lemma where $n = 1$. □

Corollary 6.9. If $S \subset T$ is a simplicial subset containing the vertices of T, then the inclusion

$$S \times \mathbb{\Delta}^1 \cup T \times \mathbb{\Lambda}^1_1 \hookrightarrow T \times \mathbb{\Delta}^1$$

is an inner expansion.

The following definition is modeled on Joyal's definition of quasi-fibrations between quasi-categories [17].

Definition 6.10. A **quasi-fibration** $f : X \to Y$ of k-categories is a morphism of the underlying simplicial spaces such that

1) if $0 < i < n$, the morphism
$$X_n \longrightarrow \operatorname{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, f)$$
is a cover;
2) if $i \in \{0, 1\}$, the morphism
$$\operatorname{Hom}(\mathbb{\Delta}^1, X) \longrightarrow \operatorname{Hom}(\Delta^0 \hookrightarrow \mathbb{\Delta}^1, f) = X_0 \times_{Y_0} \operatorname{Hom}(\mathbb{\Delta}^1, Y)$$
is a cover.

Clearly, the morphism from a k-category X to the terminal simplicial space e is a quasi-fibration.

The proof of the following lemma is the same as that of Lemma 3.9. Note that $\operatorname{Hom}(S \hookrightarrow T, f)$ is isomorphic to $\operatorname{Hom}(\operatorname{sk}_{k+1} S \hookrightarrow \operatorname{sk}_{k+1} T, f)$ by Lemma 6.2; this is important, since $\operatorname{Hom}(S \hookrightarrow T, f)$ is only defined *a priori* when T is a finite simplicial set.

Lemma 6.11. Let T be a simplicial set such that $\operatorname{sk}_n T$ is finite for all n.

i) Let $i : S \hookrightarrow T$ be an inner expansion, and let $f : X \to Y$ be a quasi-fibration of k-categories. Then the morphism
$$\operatorname{Hom}(T, X) \longrightarrow \operatorname{Hom}(S \hookrightarrow T, f)$$
is a cover.

ii) Let $i : S \hookrightarrow T$ be an inclusion, and let $f : X \to Y$ be a hypercover of k-categories. Then the morphism
$$\operatorname{Hom}(T, X) \longrightarrow \operatorname{Hom}(S \hookrightarrow T, f)$$
is a cover.

We now introduce a functor $X \mapsto \mathbb{G}(X)$ from k-categories to k-groupoids, which may be interpreted as the k-groupoid of quasi-invertible morphisms in X.

Theorem 6.12.

i) If X is a k-category, then the simplicial space
$$\mathbb{G}(X)_n = \operatorname{Hom}(\mathbb{\Delta}^n, X)$$
is a k-groupoid.

ii) If $f : X \to Y$ is a quasi-fibration of k-categories, then

$$\mathbb{G}(f) : \mathbb{G}(X) \to \mathbb{G}(Y)$$

is a fibration of k-groupoids.

iii) If $f : X \to Y$ is a hypercover of k-categories, then

$$\mathbb{G}(f) : \mathbb{G}(X) \to \mathbb{G}(Y)$$

is a hypercover of k-groupoids.

Proof. To prove Part i), we must show that the morphism

$$\mathbb{G}(X)_n \longrightarrow \mathrm{Hom}(\Lambda^n_i, \mathbb{G}(X)),$$

or equivalently, the morphism

$$\mathrm{Hom}(\mathbb{\Delta}^n, X) \longrightarrow \mathrm{Hom}(\mathbb{\Lambda}^n_i, X),$$

is a cover for all $n > 0$, and for $0 \leq i \leq n$, and an isomorphism for $n > k$. For $n = 1$, this is part of the definition of a quasi-fibration, and for $n > 1$, it is a consequence of Corollary 6.6.

The proof of Part ii) is similar, since if $f : X \to Y$ is a quasi-fibration of k-categories, then the morphism

$$\mathrm{Hom}(\mathbb{\Delta}^n, X) \longrightarrow \mathrm{Hom}(\mathbb{\Lambda}^n_i \hookrightarrow \mathbb{\Delta}^n, f),$$

is a cover for all $n > 0$, and for $0 \leq i \leq n$, by the same argument.

To prove Part iii), we must show that if $f : X \to Y$ is a hypercover, the morphism

$$\mathbb{G}(X)_n \longrightarrow \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, \mathbb{G}(f)),$$

or equivalently, the morphism

$$\mathrm{Hom}(\mathbb{\Delta}^n, X) \longrightarrow \mathrm{Hom}(\partial\mathbb{\Delta}^n \hookrightarrow \mathbb{\Delta}^n, f),$$

is a cover for all $n \geq 0$: this follows from Lemma 3.3, applied to the inclusion of simplicial sets $\partial\mathbb{\Delta}^n \hookrightarrow \mathbb{\Delta}^n$. ☐

It is clear that $\mathbb{G}$ takes pullbacks to pullbacks. We will show that k-categories form a category of fibrant objects, and that $\mathbb{G}$ is an exact functor from this category to the category of k-groupoids.

The main step which remains in the proof that k-categories form a category of fibrant objects is the construction of a simplicial resolution for k-categories. We use the following refinement of Lemma 4.2, which was already implicit in the proof of Lemma 4.1.

Lemma 6.13. Let T be a finite simplicial set, and let $S \hookrightarrow T$ be a simplicial subset. Then the morphism

$$\Delta^m \times S \cup \Lambda^m_i \times T \hookrightarrow \Delta^m \times T, \quad 0 < i < m,$$

is an inner m-expansion, and the morphism

$$S \times \Delta^n \cup T \times \Lambda^n_j \hookrightarrow T \times \Delta^n, \quad 0 < j < n,$$

is an inner n-expansion.

Definition 6.14. Define $\mathbb{P}_n X$ to be the simplicial space

$$(\mathbb{P}_n X)_m = \mathrm{Hom}(\Delta^m \times \mathbb{\Delta}^n, X).$$

Theorem 6.15. The functor $\mathbb{P}_\bullet X$ is a simplicial resolution.

Proof. Let $f : X \to Y$ be a quasi-fibration. By Lemma 6.13, the inclusion

$$\Lambda^m_i \times \mathbb{\Delta}^n \cup \Delta^m \times \partial\mathbb{\Delta}^n \hookrightarrow \Delta^m \times \mathbb{\Delta}^n$$

is an inner expansion for $0 < i < m$. Applying Lemma 6.11, we conclude that the morphism

$$\mathrm{Hom}(\Delta^m \times \mathbb{\Delta}^n, X) \longrightarrow \mathrm{Hom}(\Lambda^m_i \times \mathbb{\Delta}^n \cup \Delta^m \times \partial\mathbb{\Delta}^n \hookrightarrow \Delta^m \times \mathbb{\Delta}^n, f)$$

is a cover.

By Corollary 6.9, the inclusion

$$\mathbb{\Delta}^1 \times \partial\mathbb{\Delta}^n \cup \mathbb{\Lambda}^1_1 \times \mathbb{\Delta}^n \hookrightarrow \mathbb{\Delta}^1 \times \mathbb{\Delta}^n$$

is an inner expansion for $n > 0$. It follows by Lemma 6.11 that the morphism

$$\mathrm{Hom}(\mathbb{\Delta}^1 \times \mathbb{\Delta}^n, X) \longrightarrow \mathrm{Hom}(\mathbb{\Delta}^1 \times \partial\mathbb{\Delta}^n \cup \mathbb{\Lambda}^1_1 \times \mathbb{\Delta}^n \hookrightarrow \mathbb{\Delta}^1 \times \mathbb{\Delta}^n, f)$$

is a cover for $n > 0$. Together, these two results show that the simplicial morphism

$$\mathbb{P}_n X \longrightarrow \mathbb{P}_{\partial\Delta^n} X \times_{P_{\partial\Delta^n} Y} \mathbb{P}_n Y$$

is a quasi-fibration for $n > 0$.

By Corollary 6.6 and Lemma 6.13, the inclusion

$$\partial\Delta^m \times \mathbb{\Delta}^n \cup \Delta^m \times \mathbb{\Lambda}^n_j \hookrightarrow \Delta^m \times \mathbb{\Delta}^n$$

is an inner expansion for $n > 1$ and $0 \leq j \leq n$. It follows that the morphism

$$\mathrm{Hom}(\Delta^m \times \mathbb{\Delta}^n, X) \longrightarrow \mathrm{Hom}(\partial\Delta^m \times \mathbb{\Delta}^n \cup \Delta^m \times \mathbb{\Lambda}^n_j \hookrightarrow \Delta^m \times \mathbb{\Delta}^n, f)$$

is a cover, and hence that the simplicial morphism

$$\mathbb{P}_n X \longrightarrow \mathbb{P}_{\Lambda^n_i} X \times_{\mathbb{P}_{\Lambda^n_i} Y} \mathbb{P}_n Y$$

is a hypercover for $n > 1$.

Let $f : X \to Y$ be a hypercover. Applying Lemma 3.3, we see that the morphism

$$\mathrm{Hom}(\Delta^m \times \mathbb{\Delta}^n, X) \longrightarrow \mathrm{Hom}(\partial\Delta^m \times \mathbb{\Delta}^n \cup \Delta^m \times \partial\mathbb{\Delta}^n \hookrightarrow \Delta^m \times \mathbb{\Delta}^n, f)$$

is a cover for $n > 0$, and hence the simplicial morphism

$$\mathbb{P}_n X \longrightarrow \mathbb{P}_{\partial\Delta^n} X \times_{P_{\partial\Delta^n} Y} \mathbb{P}_n Y$$

is a hypercover for $n > 0$. ☐

The following lemma is the analogue of Lemma 3.26 for k-categories.

Lemma 6.16. If $f : X \to Y$ is a fibration of k-categories, and $g : Y \to Z$ and gf are hypercovers, then f is a hypercover.

Proof. The proof of Lemma 3.26 extends to this setting as well. Indeed, the proof contained there establishes that the morphism $X_n \to \mathrm{Hom}(\partial\Delta^n \hookrightarrow \Delta^n, f)$ is a cover for $n > 0$. It remains to show that $f_0 : X_0 \to Y_0$ is a cover, which follows from Lemma 3.26 applied to the morphisms $\mathbb{G}(f)$ and $\mathbb{G}(g)$. ☐

With these results in hand, we may easily adapt the proof of Theorem 3.6 to prove the following result.

Theorem 6.17. The category of k-categories is a category of fibrant objects.

The following corollary is immediately implied by Lemma 1.3 ("Brown's Lemma").

Corollary 6.18. If $f : X \to Y$ is a weak equivalence of k-categories, then

$$\mathbb{G}(f) : \mathbb{G}(X) \longrightarrow \mathbb{G}(Y)$$

is a weak equivalence of k-groupoids.

We have the following analogue of Theorem 5.1.

Theorem 6.19. A morphism $f : X \to Y$ of k-categories is a weak equivalence if and only if the morphism

$$X_0 \times_{Y_0} \operatorname{Hom}(\mathbb{\Delta}^1, Y) \longrightarrow Y_0$$

is a cover, and the morphisms

$$\operatorname{Hom}(\Delta^n \hookrightarrow \mathbb{\Delta}^1 \star \Delta^{n-1}, f) \longrightarrow \operatorname{Hom}(\partial\Delta^n \hookrightarrow \mathbb{\Delta}^1 \star \partial\Delta^{n-1} \cup \mathbb{\Lambda}^1_0 \star \Delta^{n-1}, f)$$

are covers for $n \geq 0$.

Proof. The morphism f is a weak equivalence if and only if the morphisms

$$\operatorname{Hom}(\Delta^n \hookrightarrow \Delta^n \times \mathbb{\Delta}^1, f) \longrightarrow \operatorname{Hom}(\partial\Delta^n \times \mathbb{\Lambda}^1_1 \hookrightarrow \partial\Delta^n \times \mathbb{\Delta}^1 \cup \Delta^n \times \mathbb{\Lambda}^1_1, f) \tag{6.1}$$

are covers for all $n \geq 0$. For $n = 0$, this is the first hypothesis of the theorem. Thus, from now on, we take $n > 0$.

We have seen in Lemma 6.7 that the simplicial set $\Delta^n \times \mathbb{\Delta}^1$ is an inner expansion of $\partial\Delta^n \times \mathbb{\Delta}^1 \cup \Delta^n \times \mathbb{\Lambda}^1_1$, by the successive adjunction of the simplices $[\mathbb{0}; \dots; \mathbb{0}; \mathbb{m}; \sigma_{n-\ell+1}; \dots; \sigma_n]$ and

$$[\mathbb{m}^\circ; \sigma_1; \dots; \sigma_n].$$

Of these simplices, only one, namely $[\mathbb{1}^*; \mathbb{0}^*; \dots; \mathbb{0}^*] \in S_{n+1,n,1}$, has a face in the simplicial subset $\Delta^n \times \mathbb{\Lambda}^1_1 \subset \Delta^n \times \mathbb{\Delta}^1$. Thus, the morphism (6.1) factors into a sequence of horn-filler morphisms indexed by this sequence of simplices, all of which are seen to be covers, except possibly the one corresponding to the simplex $[\mathbb{1}^*; \mathbb{0}^*; \dots; \mathbb{0}^*]$. But the morphism corresponding to this simplex is a cover under the hypotheses of the theorem.

Now suppose that (6.1) is a cover for $n > 0$. The map

$$0^{a_0} \dots n^{a_k} \times i_0 \dots i_k \mapsto 0^{a_0} \dots n^{a_k} \times i_0 \dots i_{a_0-1} 0 \dots 0$$

from $\Delta^n \times \mathbb{\Delta}^1$ to $\mathbb{\Delta}^1 \star \Delta^{n-1}$ takes $\partial\Delta^n \times \mathbb{\Delta}^1 \cup \Delta^n \times \mathbb{\Lambda}^1_1$ to $\mathbb{\Delta}^1 \star \partial\Delta^{n-1} \cup \mathbb{\Lambda}^1_0 \star \Delta^{n-1}$ and induces a pullback square

$$\begin{tikzcd}
\operatorname{Hom}(\Delta^n \hookrightarrow \mathbb{\Delta}^1 \star \Delta^{n-1}, f) \arrow[r] \arrow[d] & \operatorname{Hom}(\Delta^n \hookrightarrow \Delta^n \times \mathbb{\Delta}^1, f) \arrow[d] \\
\operatorname{Hom}(\partial\Delta^n \hookrightarrow \mathbb{\Delta}^1 \star \partial\Delta^{n-1} \cup \mathbb{\Lambda}^1_0 \star \Delta^{n-1}, f) \arrow[r] & \operatorname{Hom}(\partial\Delta^n \hookrightarrow \partial\Delta^n \times \mathbb{\Delta}^1 \cup \Delta^n \times \mathbb{\Lambda}^1_1, f)
\end{tikzcd}$$

This completes the proof of the theorem. ☐

7. REGULAR k-CATEGORIES

If $\mathcal{V}$ is a regular descent category, it is natural to single out the following class of k-categories.

Definition 7.1. A **regular** k-category is a k-category X such that the morphism

$$\operatorname{Hom}(\mathbb{\Delta}^1, X) \longrightarrow \operatorname{Hom}(\Delta^1, X) \cong X_1$$

induced by the inclusion $\Delta^1 \hookrightarrow \mathbb{\Delta}^1$ is regular.

Since $\Delta^1 \hookrightarrow \mathbb{\Delta}^1$ is an expansion, every k-groupoid is a regular k-category.

Proposition 7.2. If X is a regular k-category, then for all $n \geq 0$, the morphism

$$\operatorname{Hom}(\mathbb{\Delta}^n, X) \longrightarrow \operatorname{Hom}(\Delta^n, X) \cong X_n$$

induced by the inclusion $\Delta^n \hookrightarrow \mathbb{\Delta}^n$ is regular.

Proof. Let $\mathsf{T}^n_i \subset \Delta^n$ be the union of the 1-simplices

$$(j-1, j), \quad 1 \leq j \leq i.$$

For $k > 0$, let Q_k be the set of k-simplices of Δ^n such that $i_1 = i_0 + 1$. In particular, Q_1 is the set of 1-simplices in T^n_n.

Let $k > 1$. Given a simplex $(i_0, \dots, i_k) \in Q_k$, the faces $\partial_j(i_0, \dots, i_k)$ lie in Q_{k-1} for $j > 1$, while $\partial_0(i_0, \dots, i_k)$ either lies in Q_{k-1}, if $i_2 = i_1 + 1$, or equals $\partial_1(i_1, i_1 + 1, i_2, \dots, i_k)$ if $i_2 > i_1 + 1$.

On the other hand, $\partial_1(i_0, \dots, i_k)$ lies neither in Q_{k-1} nor is it a face of any simplex $(i'_0, \dots, i'_k) \in Q_k$ with $i'_0 + \dots + i'_k > i_0 + \dots + i_k$. This shows that the inclusion $\mathsf{T}^n_n \hookrightarrow \Delta^n$ is an inner expansion, in which the simplices of Q_k are attached in order of increasing $k \geq 2$, and for fixed k, in order of decreasing $i_0 + \dots + i_k$.

Let $\mathbb{T}^n_i = (\mathsf{T}^n_i \otimes_\Delta \mathbb{\Delta}) \cup \Delta^n \subset \mathbb{\Delta}^n$. By Lemma 6.5, $\mathbb{T}^n_n \hookrightarrow \mathbb{\Delta}^n$ is an inner expansion. Hence the morphism

$$\operatorname{Hom}(\mathbb{\Delta}^n, X) \longrightarrow \operatorname{Hom}(\mathbb{T}^n_n, X)$$

is a cover, and hence regular. For each $1 \leq i \leq n$, the morphism

$$\operatorname{Hom}(\mathbb{T}^n_i \Delta^n, X) \longrightarrow \operatorname{Hom}(\mathbb{T}^n_{i-1} \Delta^n, X)$$

is regular, since it may be realized as the pullback of a regular morphism:

$$\begin{tikzcd}
\operatorname{Hom}(\mathbb{T}^n_i, X) \arrow[r] \arrow[d] & \operatorname{Hom}(\mathbb{\Delta}^1, X) \arrow[d] \\
\operatorname{Hom}(\mathbb{T}^n_{i-1}, X) \arrow[r] & \operatorname{Hom}(\Delta^1, X)
\end{tikzcd}$$

This completes the proof of the theorem, since $\mathbb{T}^n_0 = \Delta^n$, and the composition of regular morphisms is regular. ☐

Let $\mathsf{G}(X)_n$ be the image of the regular morphism $\mathbb{G}(X)_n \to X_n$. The spaces $\mathsf{G}(X)_n$ form a simplicial space, and for each n, the morphism $\mathbb{G}(X)_n \to \mathsf{G}(X)_n$ (coimage of $\mathbb{G}(X)_n \to X_n$) is a cover. We call $\mathsf{G}(X)_1$ the space of **quasi-invertible** morphisms.

It follows from the proof of Theorem 7.2 that $\mathsf{G}(X)_n$ is the image of the morphism

$$\operatorname{Hom}(\mathsf{T}^n_n, \mathbb{G}(X)) \times_{\operatorname{Hom}(\mathsf{T}^n_n, X)} X_n \longrightarrow X_n.$$

Lemma 7.3. $\mathbb{G}(\mathbb{G}(X)) \cong \mathbb{G}(\mathsf{G}(X)) \cong \mathbb{G}(X)$

Proof. In order to prove that $\mathbb{G}(\mathbb{G}(X))$ is isomorphic to $\mathbb{G}(X)$, it suffices to show that for all $k, n \geq 0$,

$$\operatorname{Hom}(\Delta^k, \mathbb{\Delta}^n) \cong \operatorname{Hom}(\mathbb{\Delta}^k, \mathbb{\Delta}^n).$$

Since $\mathbb{\Delta}^k$ is the nerve of the groupoid $\llbracket k \rrbracket$, we see that $\operatorname{Hom}(\mathbb{\Delta}^k, \mathbb{\Delta}^n)$ may be identified with the set of functors from $\llbracket k \rrbracket$ to $\llbracket n \rrbracket$. But a functor from $\llbracket k \rrbracket$ to $\llbracket n \rrbracket$ determines, and is determined by, a functor from $[k]$ to $\llbracket n \rrbracket$, i.e. by a k-simplex of the nerve $\mathbb{\Delta}^n = N_\bullet \llbracket n \rrbracket$ of $\llbracket n \rrbracket$.

Applying the functor $\mathbb{G}_n$ to the composition of morphisms

$$\mathbb{G}(X) \to \mathsf{G}(X) \to X,$$

we obtain a factorization of the identity map of $\mathbb{G}(X)_n$:

$$\mathbb{G}(\mathbb{G}(X))_n \cong \mathbb{G}(X)_n \to \mathbb{G}(\mathsf{G}(X))_n \to \mathbb{G}(X)_n.$$

Since the functor $\mathbb{G}_n$ is a limit, it preserves monomorphisms. Thus the morphism from $\mathbb{G}(\mathsf{G}(X))_n$ to $\mathbb{G}(X)_n$ is a monomorphism, and since it has a section, an isomorphism. ☐

The statement and proof of the following lemma are similar to those of Lemma 6.4.

Lemma 7.4. The inclusion $\partial\mathbb{\Delta}^n \cup \Delta^n \hookrightarrow \mathbb{\Delta}^n$ is an expansion.

Proof. Let $Q_{k,m}$, $0 \leq m < n$ be the set of non-degenerate k-simplices $s = (i_0 \dots i_k)$ of $\mathbb{\Delta}^n$ which satisfy the following conditions:

a) s is not contained in $\partial\mathbb{\Delta}^n \cup \Delta^n$;
b) $i_j = j$ for $i \leq j \leq m$;
c) $\{i_{m+1}, \dots, i_n\} = \{m, \dots, n\}$.

Let Q_k be the union of the sets $Q_{k,m}$.

The simplicial set $\mathbb{\Delta}^n$ is obtained from $\mathbb{\Lambda}^n_i \cup \Delta^n$ by inner expansions along the simplices of type $Q_{k,m}$ in order first of increasing k, then of decreasing m. (The order in which the simplices are adjoined within the sets $Q_{k,m}$ is unimportant.)

Given a non-degenerate simplex $s = (i_0, \dots, i_k)$ which does not lie in the union of $\partial\mathbb{\Delta}^n \cup \Delta^n$ and Q_k, let m be the largest integer such that $i_j = j$ for $j < m$. Thus

$$s = (0, \dots, m-1, i_m, \dots, i_k),$$

and $i_m \neq m$. The infimum ℓ of the set $\{i_m, \dots, i_k\}$ equals m: it cannot be any larger, or the simplex would lie in $\partial\mathbb{\Delta}^n$, and it cannot be any smaller, or the simplex would lie in Q_k. Define the simplex

$$\tilde{s} = (0, \dots, m, i_m, \dots, i_k)$$

in $Q_{k+1,m}$. We have $s = \partial_m \tilde{s}$.

If m occurs more than once in the sequence $\{i_m, \dots, i_k\}$, then the remaining faces of the simplex $\tilde{s}$ are either degenerate, or lie in the union of $\partial\mathbb{\Delta}^n \cup \Delta^n$ and Q_k. If m occurs just once in this sequence, say $i_\ell = m$, then all faces of the simplex $\tilde{s}$ other than $s = \partial_m \tilde{s}$ and $\partial_{\ell+1}\tilde{s}$ are either degenerate, or lie in the union of $\partial\mathbb{\Delta}^n \cup \Delta^n$ and Q_k, while $\partial_{\ell+1}\tilde{s}$ is a face of a simplex of type $Q_{k+1,m'}$, where $m' > m$. ☐

This lemma implies that the natural morphism $\mathbb{G}(X) \to X$ is a hypercover when X is a k-groupoid, even if the descent category is not assumed to be regular.

The following theorem is related to results of Rezk [22] and Joyal and Tierney [18].

Theorem 7.5. If X is a regular k-category, then $\mathsf{G}(X)$ is a k-groupoid, and the induced morphism

$$\mathbb{G}(X) \longrightarrow \mathsf{G}(X)$$

is a hypercover.

Proof. For $n > 0$, consider the assertions
A$_n$: for all $0 \leq i \leq n$, the morphism $\mathsf{G}(X)_n \to \operatorname{Hom}(\Lambda^n_i, \mathsf{G}(X))$ is a cover; and
B$_n$: for all $0 \leq i \leq n$, the morphism

$$\mathbb{G}(X)_n \longrightarrow \operatorname{Hom}(\Lambda^n_i \to \Delta^n, \mathbb{G}(X) \to \mathsf{G}(X))$$

is a cover. These imply that $\mathsf{G}(X)$ is a k-groupoid.

Let us demonstrate A$_1$. In the commuting diagram

$$\begin{tikzcd}
& \mathbb{G}(X)_1 \arrow[dl] \arrow[dr] & \\
\mathsf{G}(X)_1 \arrow[rr, dotted] & & \mathbb{G}(X)_0 \cong X_0
\end{tikzcd}$$

the solid arrows are covers, hence by Axiom (D3), the bottom arrow is a cover.

Consider the commuting diagram

$$\begin{tikzcd}
& \mathbb{G}(X)_n \times_{\operatorname{Hom}(\Lambda^n_i, \mathsf{G}(X))} \mathsf{G}(X)_n \arrow[dl, dotted] \arrow[dr] & \\
\mathbb{G}(X)_n \arrow[rr, dotted] & & \operatorname{Hom}(\Lambda^n_i \to \Delta^n, \mathbb{G}(X) \to \mathsf{G}(X))
\end{tikzcd}$$

in which the solid arrow is a cover. If A$_n$ holds, the left-hand arrow is a cover, and hence by Axiom (D3), so is the bottom arrow, establishing B$_n$.

Suppose that T is a finite simplicial set and $S \hookrightarrow T$ is an expansion obtained by attaching simplices of dimension at most $n-1$ to S. Suppose that B$_{n-1}$ holds. Then the same proof as for Lemma 3.9 shows that the morphism

$$\operatorname{Hom}(T, \mathbb{G}(X)) \to \operatorname{Hom}(S \hookrightarrow T, \mathbb{G}(X) \to \mathsf{G}(X))$$

is a cover. Applying this argument to the expansion $\Delta^0 \hookrightarrow \Lambda^n_i$ shows that

$$\operatorname{Hom}(\Lambda^n_i, \mathbb{G}(X)) \longrightarrow \operatorname{Hom}(\Lambda^n_i, \mathsf{G}(X))$$

is a cover. In the commuting diagram

$$\begin{tikzcd}
\mathbb{G}(X)_n \arrow[r] \arrow[d] & \operatorname{Hom}(\Lambda^n_i, \mathbb{G}(X)) \arrow[d] \\
\mathsf{G}(X)_n \arrow[r, dotted] & \operatorname{Hom}(\Lambda^n_i, \mathsf{G}(X))
\end{tikzcd}$$

the solid arrows are covers, hence by Axiom (D3), so is the bottom arrow, establishing A$_n$.

Now that we know that $\mathsf{G}(X)$ is a k-groupoid, it follows from Lemma 7.4 that $\mathbb{G}(X) \to \mathsf{G}(X)$ is a hypercover. $\square$

8. THE NERVE OF A DIFFERENTIAL GRADED ALGEBRA

In this final section, we give an application of the formalism developed in this paper to the study of the nerve of a differential graded algebra A over a field $\mathbb{K}$. There are different variants of this construction: we give the simplest, in which the differential graded algebra A is finite-dimensional in each dimension and concentrated in degrees $> -k$. Working in the descent category of schemes of finite type, with surjective smooth morphisms (respectively smooth morphisms) as covers (respectively regular morphisms), we will show that the nerve of A is a regular k-category.

In the special case that $A = M_N(\mathbb{K})$ is the algebra of $N \times N$ square matrices, our construction produces the nerve of the monoid $\operatorname{End}(\mathbb{K}^N)$: the associated 1-groupoid $\mathsf{G}(N_\bullet A)$ is the nerve of the algebraic group $\operatorname{GL}(N)$. If V is a perfect complex of amplitude k, then $\mathbb{G}(N_\bullet \operatorname{End}(V))$ is the k-groupoid of quasi-automorphisms of V. A straightforward generalization of this construction from differential graded algebras to differential graded categories yields the stack of perfect complexes: in a sequel to this paper, we show how this gives a new construction of the derived stack of perfect complexes of Toën and Vezzosi [24].

Let A be a differential graded algebra over a field $\mathbb{K}$, with differential $d : A^\bullet \to A^{\bullet+1}$. The curvature map is the quadratic polynomial

$$\Phi(\mu) = d\mu + \mu^2 : A^1 \to A^2.$$

The Maurer-Cartan locus $\mathsf{MC}(A) = V(\Phi) \subset A^1$ is the zero locus of Φ.

The graded commutator of elements $a \in A^i$ and $b \in A^j$ is defined by the formula

$$[a, b] = ab - (-1)^{ij} ba \in A^{i+j}.$$

In particular, if $\mu \in A^1$, then

$$[\mu, a] = \mu a - (-1)^i a\mu \in A^{i+1}.$$

If μ lies in the Maurer-Cartan locus, the operator $d_\mu : a \mapsto da + [\mu, a]$ is a differential.

Given μ and ν lying in the Maurer-Cartan locus of $A^\bullet$, define a differential $d_{\mu,\nu}$ on the graded vector space underlying A by the formula

$$A^i \ni a \mapsto d_{\mu,\nu} a = da + \mu a - (-1)^i a\nu \in A^{i+1}.$$

Let $C^\bullet(\Delta^n)$ be the differential graded algebra of normalized simplicial cochains on the n-simplex Δ^n (with coefficients in the field $\mathbb{K}$): this algebra is finite-dimensional, of dimension $\binom{n+1}{i+1}$ in degree i. An element $a \in C^\bullet(\Delta^n) \otimes A^\bullet$ corresponds to a collection of elements

$$(a_{i_0 \dots i_k} \in A^{i-k} \mid 0 \le i_0 < \dots < i_k \le n),$$

where $a_{i_0 \dots i_k}$ is the evaluation of the cochain a on the face of the simplex Δ^n with vertices $\{i_0, \dots, i_k\}$.

The differential on the differential graded algebra $C^\bullet(\Delta^n) \otimes A$ is the sum of the simplicial differential on $C^\bullet(\Delta^n) \otimes A$ and the internal differential of A:

$$(\delta a)_{i_0 \dots i_k} = \sum_{\ell=0}^{k} (-1)^\ell a_{i_0 \dots \widehat{\imath}_{i_\ell} \dots i_k} + (-1)^k d(a_{i_0 \dots i_k}).$$

The product of $C^\bullet(\Delta^n) \otimes A$ combines the Alexander-Whitney product on simplicial cochains with the product on A: if a has total degree j, then

$$(a \cup b)_{i_0 \dots i_k} = \sum_{\ell=0}^{k} (-1)^{(j-\ell)(k-\ell)} a_{i_0 \dots i_\ell} b_{i_\ell \dots i_k}.$$

The **nerve** of a differential graded algebra A is the simplicial scheme $N_\bullet A$ such that $N_n A$ is the Maurer-Cartan locus of $C^\bullet(\Delta^n) \otimes A$:

$$N_n A = \mathsf{MC}(C^\bullet(\Delta^n) \otimes A).$$

If T is a finite simplicial set, the Yoneda lemma implies that the scheme of morphisms from T to $N_\bullet A$ is the Maurer-Cartan set of the differential graded algebra $C^\bullet(T) \otimes A$.

A simplex $\mathbb{\mu} \in N_n A$ consists of a collection of elements of A

$$\mathbb{\mu} = \bigl\{ \mu_{i_0 \dots i_k} \in A^{1-k} \mid 0 \le i_0 < \dots < i_k \le n \bigr\},$$

such that the following Maurer-Cartan equations hold: for

$$0 \le i_0 < \dots < i_k \le n,$$

we have

$$(-1)^k \, (d\mathbb{\mu} + \mathbb{\mu}^2)_{i_0 \dots i_k} = d\mu_{i_0 \dots i_k} + \sum_{\ell=0}^{k} (-1)^{k-\ell} \, \mu_{i_0 \dots \widehat{\imath_\ell} \dots i_k} + \sum_{\ell=0}^{k} (-1)^{k\ell} \, \mu_{i_0 \dots i_\ell} \mu_{i_\ell \dots i_k} = 0.$$

The components μ_i and μ_{ij} play a special role in the Maurer-Cartan equation. The components μ_i are Maurer-Cartan elements of A, and determine differentials $d_{ij} : A^\bullet \to A^{\bullet+1}$ by the formula

$$d_{ij} a = da + \mu_i a - (-1)^{|a|} a \mu_j .$$

In terms of the translate $f_{ij} = 1 + \mu_{ij}$ of the coefficient μ_{ij}, the Maurer-Cartan equation for μ_{ij} becomes

$$d_{ij} f_{ij} = 0.$$

The Maurer-Cartan equation for μ_{ijk} may be rewritten

$$d_{ik} \mu_{ijk} + f_{ij} f_{jk} - f_{ik} = 0.$$

In other words, μ_{ijk} is a homotopy between $f_{ij} f_{jk}$ and f_{ik}. For $n > 2$, the Maurer-Cartan equation becomes

$$d_{i_0 i_k} \mu_{i_0 \dots i_k} + \sum_{\ell=1}^{k-1} (-1)^{k-\ell} \, \mu_{i_0 \dots \widehat{\imath_\ell} \dots i_k} \\ + (-1)^k \, f_{i_0 i_1} f_{i_1 \dots i_k} + \mu_{i_0 \dots i_{k-1}} \mu_{i_{k-1} i_k} + \sum_{\ell=2}^{k-2} (-1)^{k\ell} \, \mu_{i_0 \dots i_\ell} \mu_{i_\ell \dots i_k} = 0.$$

The following is the main result of this section.

Theorem 8.1. Let A be a differential graded algebra such that A^i is finite-dimensional for $i \le 1$, and vanishes for $i \le -k$. Then $N_\bullet A$ is a regular k-category.

Proof. The proof divides into three parts.

1) If $0 < i < n$, the morphism $N_n A \to \operatorname{Hom}(\Lambda^n_i, N_\bullet A)$ is a smooth epimorphism, and an isomorphism if $n > k$.
2) The morphisms $\operatorname{Hom}(\mathbb{\Delta}^1, N_\bullet A) \to \mathsf{MC}(A)$ are smooth.
3) The morphism $\operatorname{Hom}(\mathbb{\Delta}^1, N_\bullet A) \to N_1 A$ is smooth.

Part 1) is established by rearranging the Maurer-Cartan equations for $\mu_{0\dots n}$ and $\mu_{0\dots\widehat{\imath}\dots n}$ to give a natural isomorphism $N_nA \cong \operatorname{Hom}(\Lambda^n_i, N_\bullet A) \times A^{1-n}$:

$$\begin{aligned} \mu_{0\dots n} &= x \in A^{1-n} \\ \mu_{0\dots\widehat{\imath}\dots n} &= -(-1)^{n-i} d_{0n}x - (-1)^i f_{01}\mu_{1\dots n} - (-1)^{n-i}\mu_{0\dots n-1}f_{n-1,n} \\ &\qquad - \sum_{\ell\notin\{0,i,n\}} (-1)^{\ell-i}\mu_{0\dots\widehat{\ell}\dots n} - \sum_{\ell=2}^{n-2}(-1)^{n\ell-n+i}\mu_{0\dots\ell}\mu_{\ell\dots n} \in A^{2-n}. \end{aligned}$$

The case $n = 2$ is slightly special:

$$\begin{aligned} \mu_{012} &= x \in A^{-1} \\ \mu_{02} &= dx + \mu_0 x + x\mu_2 + f_{01}f_{12} - 1 \in A^0. \end{aligned}$$

To establish Parts 2) and 3), we will use an alternative representation of the algebra $C^\bullet(\mathbb{\Delta}^1) \otimes A$ in terms of 2×2 matrices with coefficients in $A[u]$, where u is a formal variable of degree 2.

Associate to a differential graded algebra A the auxilliary differential graded algebra UA, such that UA^k is the space of 2×2 matrices

$$UA^k = \left\{ \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} \middle| \alpha_{ij} \in A^{k+i-j}[u] \right\}.$$

Composition is the usual matrix product. Let $d : UA \to UA$ be the differential given by the formula

$$(da)_{ij} = (-1)^i\, d\bigl(\alpha_{ij}\bigr).$$

Let $VA \subset UA$ be the differential graded subalgebra

$$VA = \left\{ \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} \in UA \middle| \alpha_{10}(0) = 0 \right\}.$$

In other words, the bottom left entry α_{10} of the matrix has vanishing constant term. Let $a_0 \in VA$ be the element

$$a_0 = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}.$$

The following lemma is a straightforward calculation.

Lemma 8.2. The map from $C^\bullet(\mathbb{\Delta}^1) \otimes A$ to VA given by the formula

$$x \mapsto \psi(x) = \begin{pmatrix} x_0 + ux_{010} + u^2x_{01010} + \dots & x_{01} + ux_{0101} + u^2x_{010101} + \dots \\ ux_{10} + u^2x_{1010} + \dots & -x_1 - ux_{101} - u^2x_{10101} - \dots \end{pmatrix}$$

is an isomorphism of differential graded algebras between $C^\bullet(\mathbb{\Delta}^1) \otimes A$ and VA with differential

$$\delta x = dx + [a_0, x].$$

Corollary 8.3. The morphism

$$\mathbb{\mu} \mapsto a(\mathbb{\mu}) = a_0 + \psi(\mathbb{\mu})$$

induces an isomorphism of schemes between $\mathbb{N}_1A = \mathsf{MC}(C^\bullet(\mathbb{\Delta}^1) \otimes A)$ and

$$Z(da + a^2 - u1) \subset VA^1.$$

A Maurer-Cartan element $\mathbb{\mu} = (\mu_0, \mu_1, \mu_{01})$ is quasi-invertible if

$$f = 1 + \mu_{01}$$

is quasi-invertible in A^0: that is, there exist elements $g \in A^0$ and h and $k \in A^{-1}$ such that

$$dh + [\mu_0, h] = fg - 1, \qquad\qquad dk + [\mu_1, k] = gf - 1.$$

The following result (with a different proof) is due to Markl [20].

Proposition 8.4. Every quasi-invertible point of $N_1 A$ may be lifted to a point of $\mathbb{N}_1 A$.

Proof. Consider the matrices

$$\alpha = \begin{pmatrix} \mu_0 & f \\ 0 & -\mu_1 \end{pmatrix} \in VA^1 \qquad\qquad \beta = \begin{pmatrix} h & h(fk - hf) \\ g & -k + g(fk - hf) \end{pmatrix} \in VA^{-1}$$

It is easily checked that $d\beta + [\alpha, \beta] = 1$. Let C_n be the nth Catalan number. The matrix

$$a = \alpha + u \sum_{n=0}^{\infty} (-u)^n C_n \, \beta^{2n+1} \in VA^1$$

solves the equation $da + a^2 = u1$, and corresponds to an element of $\mathbb{N}_1 A$ lifting $\mathbb{\mu} \in N_1 A$. (The sum defining a is finite, since the differential graded algebra $A^\bullet$ is bounded below.) ☐

The following lemma is our main tool in the proofs of Parts 2) and 3).

Lemma 8.5. Let A be a differential graded algebra such that A^1 is finite-dimensional. Let $h : A^\bullet \to A^{\bullet-1}$ be an operator on A satisfying the following conditions:

a) $hdh = h$ and $h^2 = 0$;

b) the image of $p = dh + hd$ is an ideal $I \subset A$.

Then the natural morphism $\mathsf{MC}(A) \to \mathsf{MC}(A/I)$ is smooth at $0 \in \mathsf{MC}(A)$.

Proof. Let U be the open neighbourhood of 0 in A^1 on which the determinant of the linear transformation

$$1 + h \operatorname{ad}(\mu) : A^1 \longrightarrow A^1$$

is nonzero. We will show that the projection $\mathsf{MC}(A) \to \mathsf{MC}(A/I)$ is smooth on the open subset $U \cap \mathsf{MC}(A)$.

There is an isomorphism between $\mathsf{MC}(A)$ and the variety

$$\mathcal{V} = Z(p\nu, (1-p)x, dhx - y, \Phi(\nu) + d_\nu x + x^2) \subset \mathcal{X} = \{(\nu, x, y) \in A^1 \times A^1 \times A^1\},$$

induced by the morphism taking $\mu \in A^1$ to $((1-p)\mu, p\mu, h\mu)$. Likewise, there is an isomorphism between $\mathsf{MC}(A/I)$ and the variety

$$Z(p\nu, (1-p)\Phi(\nu)) \subset \{\nu \in A^1\}.$$

It follows that the variety

$$\mathcal{W} = Z(p\nu, (1-dh)y, (1-p)\Phi(\nu)) \subset \{(\nu, y) \in A^1 \times A^1\}$$

is a trivial finite-dimensional vector bundle over $\mathsf{MC}(A/I)$, with fibre the image of $hd : A^0 \to A^0$, or equivalently, the image of $h : A^1 \to A^0$.

Denote the differentials of x and $y : \mathcal{X} \to A^1$ by ξ and $\eta \in \Omega_{\mathcal{X}} \otimes A^1$. Taking the differentials of the equations defining $\mathcal{V}$ with respect to x and y, we obtain the differentials

$$\omega_1 = (1-p)\xi \qquad\qquad \omega_2 = dh\xi - \eta \qquad\qquad \omega_3 = d\xi + \operatorname{ad}(\nu + x)\xi.$$

By the equation

$$(1 + h\operatorname{ad}(\nu + x))^{-1}\big(\omega_1 + \omega_2 + h\omega_3\big) = \xi - (1 + h\operatorname{ad}(\nu + x))^{-1}\eta,$$

we see that the projection from $U \cap \mathcal{V}$ to $\mathcal{W}$ is étale, proving the lemma. $\square$

We next prove Part 2). Let $b(\mathbb{\mu}) \in \mathit{UA}$ be the derivative of $a(\mathbb{\mu})$ with respect to u:

$$b(\mathbb{\mu}) = \begin{pmatrix} \mu_{010} + 2u\mu_{01010} + \dots & \mu_{0101} + 2u\mu_{010101} + \dots \\ 1 + \mu_{10} + 2u\mu_{1010} + \dots & -\mu_{101} - 2u\mu_{10101} - \dots \end{pmatrix}$$

We have the equation

$$d_{a(\mathbb{\mu})}b(\mathbb{\mu}) = 1.$$

Consider the projection $q_0 : \mathit{VA} \to \mathit{VA}$ given by the formula

$$q_0 \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} = \begin{pmatrix} \alpha_{00}(0) & 0 \\ 0 & 0 \end{pmatrix},$$

where $\alpha_{00}(0)$ is the constant term of $\alpha_{00} \in A[u]$. Let $p_0 = 1 - q_0$: this is the projection onto the two-sided differential ideal in VA

$$\begin{aligned} I &= \left\{ \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} \in VA \;\middle|\; \alpha_{00}(0) = 0 \right\} \\ &= \left\{ \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} \in UA \;\middle|\; \alpha_{00}(0) = \alpha_{10}(0) = 0 \right\}. \end{aligned}$$

The homotopy

$$h = b(\mathbb{\mu})d_{a(\mathbb{\mu})}b(\mathbb{\mu})\big[d_{a(\mathbb{\mu})}, b(\mathbb{\mu})p_0\big]$$

maps $\mathit{VA}^\bullet$ to $\mathit{VA}^\bullet$, and satisfies the hypotheses of Lemma 8.5, with respect to the differential $d_{a(\mathbb{\mu})}$: the projection p is given by the explicit formula

$$p = \big[d_{a(\mathbb{\mu})}, b(\mathbb{\mu})p_0\big] = p_0 + b(\mathbb{\mu})\big[d_{a(\mathbb{\mu})}, q_0\big],$$

and has the same image I as p_0. It follows that the morphism $\mathsf{MC}(C^\bullet(\mathbb{\Delta}^1) \otimes A) \to \mathsf{MC}(A)$ is smooth at $\mathbb{\mu}$. This proves Part 2).

Likewise, consider the projection $Q_0 : \mathit{VA} \to \mathit{VA}$ given by evaluation at $u = 0$, and let $P_0 = 1 - Q_0$. Applying Lemma 8.5 to the differential graded algebra VA, with differential $d_{a(\mathbb{\mu})}$, and with homotopy

$$H = b(\mathbb{\mu})d_{a(\mathbb{\mu})}b(\mathbb{\mu})\big[d_{a(\mathbb{\mu})}, b(\mathbb{\mu})P_0\big],$$

we see that the morphism $\mathsf{MC}(C^\bullet(\mathbb{\Delta}^1) \otimes A) \to \mathsf{MC}(C^\bullet(\Delta^1) \otimes A)$ is smooth at $\mathbb{\mu}$. This proves Part 3). $\square$

REFERENCES

[1] M. Artin, A. Grothendieck, and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas. Tome 2*, Lecture Notes in Mathematics, Vol. 270, Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4).

[2] B. Benzeghli, *Un schéma simplicial de Grothendieck-Pridham*, available at `arXiv:1303.4941`.

[3] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin, 1973.

[4] K. S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. Amer. Math. Soc. **186** (1973), 419–458.

[5] H. Cartan, *Sur le foncteur* $\mathrm{Hom}(X,Y)$ *en théorie simpliciale*, Séminaire Henri Cartan; 9e année: 1956/57. Quelques questions de topologie, Exposé no 3, Secrétariat mathématique, Paris, 1958.

[6] D.-C. Cisinski, *Invariance de la K-théorie par équivalences dérivées*, J. K-Theory **6** (2010), no. 3, 505–546.

[7] A. Douady, *Le problème des modules locaux pour les espaces* $\mathbf{C}$*-analytiques compacts*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 569–602 (1975) (French).

[8] E. J. Dubuc, C^∞*-schemes*, Amer. J. Math. **103** (1981), no. 4, 683–690.

[9] D. Dugger and D. C. Isaksen, *Weak equivalences of simplicial presheaves*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 97–113.

[10] J. Duskin, *Simplicial methods and the interpretation of "triple" cohomology*, Mem. Amer. Math. Soc. **3** (1975), no. issue 2, 163.

[11] W. G. Dwyer and D. M. Kan, *Simplicial localizations of categories*, J. Pure Appl. Algebra **17** (1980), no. 3, 267–284.

[12] ———, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35.

[13] ———, *Function complexes in homotopical algebra*, Topology **19** (1980), no. 4, 427–440.

[14] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967.

[15] P. G. Glenn, *Realization of cohomology classes in arbitrary exact categories*, J. Pure Appl. Algebra **25** (1982), no. 1, 33–105.

[16] A. Henriques, *Integrating L_∞-algebras*, Compos. Math. **144** (2008), no. 4, 1017–1045.

[17] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222.

[18] A. Joyal and M. Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326.

[19] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[20] M. Markl, *Ideal perturbation lemma*, Comm. Algebra **29** (2001), no. 11, 5209–5232.

[21] J. P. Pridham, *Presenting higher stacks as simplicial schemes*, Adv. Math. **238** (2013), 184–245.

[22] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007.

[23] J.-P. Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) **54** (1951), 425–505.

[24] B. Toën and G. Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902.

[25] M. Weiss, *Hammock localization in Waldhausen categories*, J. Pure Appl. Algebra **138** (1999), no. 2, 185–195.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA
E-mail address: `behrend@math.ubc.ca`

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY
E-mail address: `getzler@northwestern.edu`