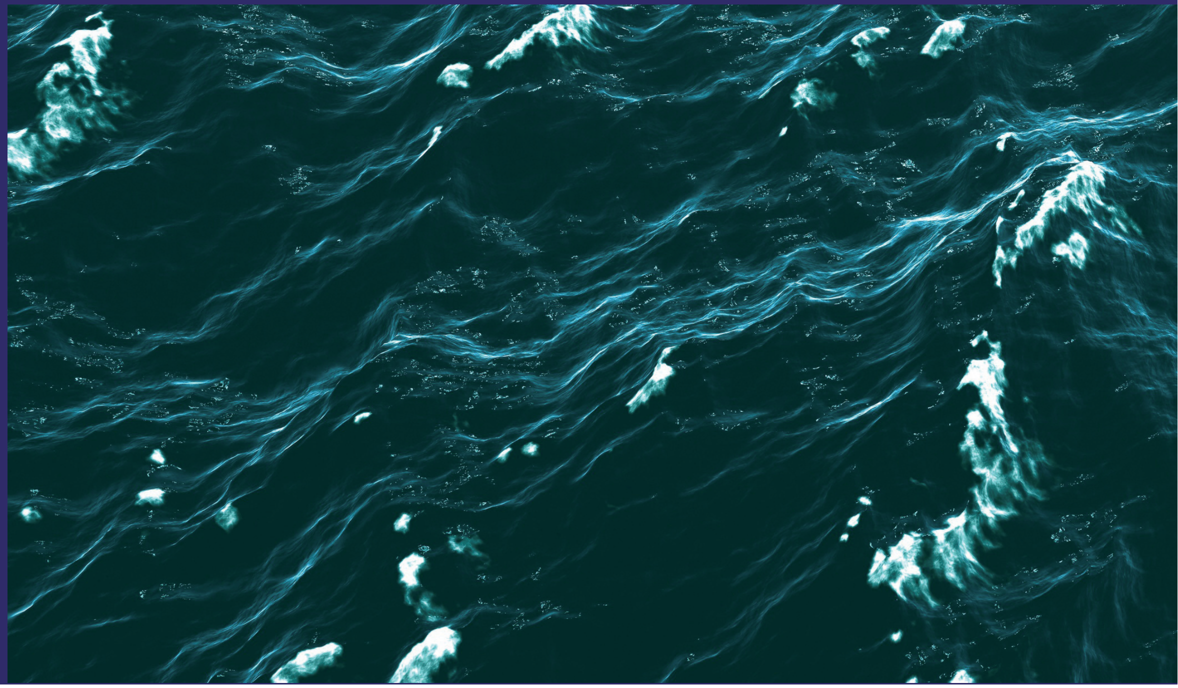
Properties of magnetic flows are described using high dimensional configuration spaces. Arnold asymptotic ergodic invariant of magnetic lines is the basic example. Asymptotic invariants of magnetic lines are related to stable homotopy groups of spheres, are detected by Postnikov k-invariants. Anosov flows on 3D homology spheres are models of stable magnetic knots. Finite-type invariants of magnetic line are required to describe properties of turbulent flows and determine constraints in magnetic feld relaxation with free boundary.



2000 Steklov Institute, dissertation: "Embeddings of compacta, stable homotopy groups of spheres and singularities theory". 1989-- Pushkov Institute of Terrestrial Magnetism, Ionosphere and Radio Wave Propagation (IZMIRAN), Troitsk, Russia, Leading Researcher. 2015-- HSE Moscow Institute of Electronics and Mathematics (MIEM HSE), Professor.

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Finite-type invariants of magnetic lines

Petr M. Akhmet'ev

Contents

Introduction

V.I.Arnold formulated the following problem [[Arn], Problem 1984- 12]: "To transform the asymptotic ergodic definition of the Hopf invariant of divergence-free vector fields to the theory of S.P.Novikov which generalizes the Whitehead product of homotopy groups of spheres"'. In the paper we recall and simplify (a partial) solution of the problem from the [A4] and present new results, which generalize the problem to non-simply connected manifolds. In July 2008 at the International Conference on Differential Equations and Dynamical Systems (Suzdal') V.I. Arnol'd told that the solution of this problem would be practically interesting and could help to solve engineering problem in plasma dynamics.

In the first section we present an additional motivation of the Arnold Problem, which is based on mean magnetic field theory. We use geometrical considerations due to K.Moffatt and formulate properties of invariants in ideal MHD, which are asymptotic and ergodic properties.

Then we introduce an asymptotic ergodic invariant, which is called M-invariant. We present a simpler new proof (in part) that the M-invariant is ergodic. The M-invariant is a higher invariant, this means that for the magnetic field with closed magnetic lines the invariant is not a function of pairwise linking numbers of the magnetic lines. This property is based on the following fact: an arithmetic residue of the M-invariant for a triple of closed magnetic lines, which is a model of a link with even pairwise linking numbers, coincides with the Arf-invariant (about the Arf-invariant, or, the Rokhlin-Robertello invariant, see [G-M], [Co]).

The new results concern magnetic fields on closed 3-dimensional manifolds and use the M-invariant. The manifolds with magnetic field, that we consider are not, generally speaking, simply-connected. This manifold is assumed homogeneous and is a rational Poncaré sphere. One can try to transform results on the asymptotics and ergodicity of the M-invariant for the magnetic fields on the standard sphere S^3 to an arbitrary rational homology sphere Σ. To make this idea precise we generalize the Arf-invariants of classical semi-boundary links (including the Arf-Brown $\mathbb{Z}/8$-invariant) (see [G-M]) and we introduce a new Arf-invariant, called the hyperquaternionic Arf-invariant.

This generalization could clarify the relationship between the M-invariant and homotopy groups of spheres. It is well-known that the helicity invariant is a specification of the Hopf invariant, see [A-Kh] for details. The Hopf invariant determines the homotopy group $\pi_3(S^2)$, the stabilization of this homotopy group is denoted by Π_1. The group Π_1 contains the only non-trivial element with the Hopf invariant one.

The Arf-invariant describes the stable homotopy group Π_2 via the geometrical approach due to L.S.Pontrjagin. The Arf-Brown invariant describes the 2-torsion of the stable homotopy group Π_3, this result follows from V.A.Rokhlin's theorems. The hyperquaternionic Arf-invariant describes the 2-torsion of the stable homotopy group Π_7. The hyperquaternionic Arf-invariant is introduced in Appendix I. This invariant could be useful to generalized M-invariant and to estimate the complexity of the generalized M-invariant for links and knots in rational homology spheres Σ.

Are complicated constructions related to stable homotopy groups required? The author assumes that this is a way to introduce a well-presented statistics of magnetic lines complicity. Near magneto-static configuration values of invariants of magnetic lines have to minimized. By small alterations we get configurations with greater complicity. This implies that transformations to a magneto-static configuration require magnetic reconnection, which decrease magnetic energy. This could prove stability of several explicit magnetic configurations (let us call such configurations generalized Kamchatnov-Hopf magnetic solitons) at least, for magnetic fields in non-simply connected domain, using well-presented invariants of magnetic lines.

The present paper was presented at the conference "'Knots and Links in Fluid Flows. From helicity to knot energies"' April 27 - 30, 2015 Independent University, Moscow. Preliminary results were presented at the A.B.Sossinsky Topological Seminar in IMU September-October 2014. A preliminary result was presented at the conference on differential equations, organized by V.P.Leksin in Kolomna, June 2014, and at the conference "Nonlinear Equations and Complex Analysis" 2009-2012 Bannoe Lake.

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Chapter 1

MHD

1.1 The mean magnetic field equation

Let us consider, as in [R], the domain Ω in $\mathbb{R}^3$, which is compact for simplicity, with a conductive liquid. In Ω a velocity field $\mathbf{u}$ of the liquid and a magnetic field $\mathbf{B}$ are well-defined. Moreover, the following decomposition of the considered vector-fields into a mean part and a random part is well defined:

$$\mathbf{B} = \bar{\mathbf{B}} + \mathbf{B}'; \qquad \mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'.$$

Assume that the mean velocity field $\bar{\mathbf{u}}(t)$ is done, then the equation for the mean magnetic field is following:

$$\begin{array}{c} \mathrm{rot}(\eta \mathrm{rot}\bar{\mathbf{B}}) - \mathrm{rot}(\bar{\mathbf{u}} \times \bar{\mathbf{B}} + \mathbf{E}) + \frac{\partial \bar{\mathbf{B}}}{\partial t} = 0, \\ \mathbf{E} = \overline{\mathbf{B}' \times \mathbf{u}'}, \quad \mathrm{div}(\bar{\mathbf{B}}) = 0. \end{array} \tag{1.1}$$

The equation (1.1) is called the kinematic dynamo equation. Assuming $\eta = 0, \mathbf{E} = 0$ this equation means that the magnetic field is frozen-in.

Assume that the following equation is satisfied:

$$\mathbf{E} = \alpha \bar{\mathbf{B}} - \beta \mathrm{rot}(\bar{\mathbf{B}}). \tag{1.2}$$

Then, using the condition that α changes the sign with respect to the mirror symmetry and using additional simplifying assumptions we get:

$$\alpha \sim \overline{(\mathbf{u}', \mathrm{rot}(\mathbf{u}'))}, \tag{1.3}$$

where the function $(\mathbf{u}', \mathrm{rot}(\mathbf{u}'))$ is called the density of (a small-scaled) the hydrodynamic helicity. Denote the hydrodynamic helicity by $\chi_{\mathbf{u}'} = \int (\mathbf{u}', \mathrm{rot}(\mathbf{u}'))\mathbf{d\Omega}$.

Take the scalar product of the both sides of the equation (1.2) with the vector $\bar{\mathbf{B}}$, assuming for simplicity that $\eta = 0$, and take the integral over the domain Ω. We get, using $\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t}$, the equation, which describes the transport of the magnetic helicity $\chi_{\bar{\mathbf{B}}} = \int(\mathbf{A}, \bar{\mathbf{B}})d\Omega$:

$$\frac{d\chi_{\bar{\mathbf{B}}}}{dt} = 2\alpha \int (\bar{\mathbf{B}}, \bar{\mathbf{B}})d\Omega - 2\beta \int (\bar{\mathbf{B}}, \mathrm{rot}(\bar{\mathbf{B}}))d\Omega. \tag{1.4}$$

The integral $U_{\bar{\mathbf{B}}} = 2\int(\bar{\mathbf{B}}, \bar{\mathbf{B}})d\Omega$ is called the magnetic energy (of the mean field), the integral $\chi_{\mathrm{rot}\bar{\mathbf{B}}} = 2\int(\bar{\mathbf{B}}, \mathrm{rot}(\bar{\mathbf{B}}))d\Omega$ is called the current helicity (of the mean field).

1.1.1 Topological considerations concerning the transport equation of the magnetic helicity

In the paper [M] by K.Moffatt the equation (1.4) is discussed from the point of view of geometry of magnetic lines, see also [S-S] for another one application. Assume that a support of a magnetic field consists of a finite set of magnetic tubes, see [B-F]. This means that the magnetic fields $\bar{\mathbf{B}}$, $\mathbf{B}'$ are inside the tubes and are tangent to the surfaces of the tubes. Additionally, let us assume that the same collection of the tubes is a support of a velocity field $\mathbf{u}'$. With this assumption the vorticity field points along the central axis of the each tube.

The magnetic and hydrodynamic tubes may be defined as the following condition, called "‘force-free"’ condition is satisfied: $\mathrm{rot}\mathbf{u}' \sim \mathbf{u}'$, $\mathrm{rot}\mathbf{B}' \sim \mathbf{B}'$.

With the considered assumption it is not hard to prove, using the formula (1.3), that the mean magnetic field $\bar{\mathbf{B}}$ in the collection of tubes tends $|\alpha|$-exponentially to $+\infty$, if the absolute value of α is sufficiently large. This is called the α-effect.

The contribution of the second term in the right side of the equation (1.4) could be clarified using the Calugareanu formula, see [M-R], [A-K-K].

In principal, the asymptotic ergodic invariant M, which is introduced in the present paper, can be applied to get analogous results. Practically this is impossible, because the formula of M is extremely complicated. Probably, the equation (1.1) can be investigated using the simplest higher invariants, called quadratic helicity invariants, introduced in [A], and the q-monomial helicity, and but this problem is complicated and is not a subject of the paper.

In the paper [C] the problem of magnetic field relaxation is investigated. The author assumed that the third and the forth order topological invariants of magnetic fields, which are not extracted from the linking number of magnetic lines, can be applied to study the slowed down decay of the magnetic energy. The invariant M is of the asymptotic order 12, and is more complicated. But it is the simplest asymptotic invariant, which is not a function of the pairwise asymptotic linking numbers of components! This invariant can be applied, probably, in the considered problem as a constraint in magnetic field relaxation with free boundary. At least, closed to force-free field configurations, f.ex., closed to Kamchatnov-Hopf magnetic solitons, see [T-S-W-B], properties of magnetic lines are sufficiently simple and M-invariant (and other higher invariants of similar type) can be calculated. An example, using Lemma (4), shows that M is non-degenerates for Kamchatnov-Hopf solitons.

Chapter 2

Ergodic integrals

2.1 Asymptotic invariants

For simplicity assume that magnetic lines in a magnetic tube Ω are closed. The magnetic tube Ω is characterized by the magnetic helicity integral, this integral equals to the mean pairwise linking number of magnetic lines is the magnetic tube Ω, which is normalized by magnetic flows thought the collection of infinitesimal magnetic lines Ω.

The magnetic tube Ω is also characterized by various combinatorial invariants $I(L_1, L_2, L_3)$, which are calculated for various collections $\{L_1, L_2, L_3\}$ of k magnetic lines (we assume $k = 3$ for simplicity). In this case we may assume that lines of collections are inside the magnetic tubes $\Omega_1, \Omega_2, \Omega_3$ correspondingly, some magnetic tubes could coincide. In a particular interesting case we have the only magnetic tube, magnetic lines of the collections are inside of this tube.

What are required conditions for a combinatorial invariant I, which can be applied to describe magnetic fields? From the consideration above of the equation (1.4) we have to assume the following conditions:

• C1. The invariant I is of a finite-type invariant of an order t in the sense of V.A.Vassiliev.

• C2. The invariant I is characterized by a positive integer s, which is called the asymptotic denominator. Take a link (L_1, L_2, L_3), which is formed by central lines of disjoint magnetic tubes $\Omega_1, \Omega_2, \Omega_3$. Denote by (rL_1, rL_2, rL_3) the r-time spinning link, which is constructed from (L_1, L_2, L_3) by the r-fold spinning along the central line of the corresponding magnetic tubes $\Omega_1, \Omega_2, \Omega_3$. The following equation is satisfied:

$$r^{3s} I(L_1, L_2, L_3) = I(rL_1, rL_2, rL_3) + O(r^{3s-1}).$$

• C3. Assume we have two disjoint magnetic tubes Ω_2, Ω_3 and we have two parallel magnetic lines, which is a 2-component link (L_1, L_2) in Ω_2, and a magnetic line L_3 is a central line in Ω_3. Take a magnetic tube Ω_2^{tw}, which is obtained from the magnetic tube Ω_2 by a twist, $\Omega_2 \mapsto \Omega_2^{tw}$, $Tw(\Omega_2) = Tw(\Omega_2^{tw}) + const$. Take two parallel magnetic lines L_1^{tw}, L_2^{tw} in Ω_2^{tw}. Take the r-time spinning link (rL_1, rL_2), each commponent of this link is rotated along the central line $L_1 = L_2$ of Ω_2 r times. Take the r-time spinning link (rL_1^{tw}, rL_2^{tw}), each commponent of this link is rotated along the central line $L_1^{tw} = L_2^{tw}$ of Ω_2^{tw} r times. Take 3-component links (rL_1, rL_2, rL_3), $(rL_1^{tw}, rL_2^{tw}, rL_3)$. The following formula is satisfied:

$$I(rL_1, rL_2, rL_3) - I(rL_1^{tw}, rL_2^{tw}, rL_3) = O(r^{3s}).$$

• C4 (Condition-Definition). Assume that the invariant I is not a function of pairwise linking numbers of components of the link (for a 3-component link we get 3 pairwise linking numbers). In this case we say that I is a higher invariant.

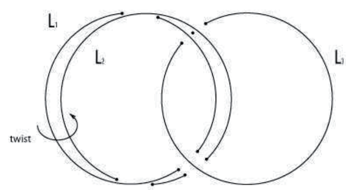

Figure 2.1: Condition $C3$.

2.2 Why are Higher Ergodic Invariants possible?

In [B-M] (Theorem 1) the authors prove that the asymptotic of Vassiliev invariants (see [P-S] for definition and properties) of magnetic lines is completely determined by the helicity of the vector field.

This means that the asymptotic limit of a Vassiliev order-q invariant v of knots for almost arbitrary pair of magnetic lines of $\mathbf{B}$ is calculated by $\alpha\chi^q(l)$, where $\chi(l)$ is the asymptotic value of the Gauss integral of $\mathbf{B}$ on the line l,

α is a constant not depended of $\mathbf{B}$. This result is generalized for a pairs of magnetic lines. The asymptotic q-Monomial helicity, defined in Section 2, is an example.

Higher asymptotic invariant M satisfies the Baader-Marché Theorem with $\alpha = 0$. M is of the order $q = 7$, the asymptotic denominator (the condition C3) of the invariant M is $3s = 12$.

The invariant M is well-defined by a complicated integral expression (4.5). An idea to express integrals of magnetic fields by similar integrals exists in [M-R], [B]. This idea was used by A.Ruzmaikin and the author in [A-R1], [A-R2]. The integral invariant of magnetic tubes, introduced in the above papers, is called the Sato-Levine invariant. The Sato-Levine invariant admits a natural extension, which is called the Generalized Sato-Levine invariant. This is an order 3 Vassiliev invariant, this invariant is well-defined for an arbitrary 2-component link in $\mathbb{R}^3$. The Generalized Sato-Levine invariant is not an asymptotic invariant of links. This is proved in the Appendix II.

The ergodic property of the main term of the expression is proved in Theorem 4. The extra terms satisfy a weaker condition of almost-ergodicity (introduced by the author). This means that the asymptotic values of the terms for generic $\mathbf{B}$, generally speaking, are multivalued. A priori M is an invariant of volume-preserved diffeomorphism of the domain in which $\mathbf{B}$ is frozen-in, this implies that the gauge of M is well-defined.

We conjecture that for magnetic lines of hyperbolic magnetic knots (generalized Kamchatnov-Hopf solitons in non-simply connected domain), which are generated by Anosov's almost geodesic flows [D-P], generalized ergodic integral invariants (analogs of M) are well-defined as single-valued integral invariants with stability properties. The present paper could be used towards the Conjecture.

Chapter 3

Momenta of the magnetic helicity

3.1 Configuration spaces

Definition 1. *Let* $\mathbf{B}$*,* $\mathrm{div}(\mathbf{B}) = 0$*, is a smooth magnetic field in* $\mathbb{R}^3$ *with a compact support* $\Omega \subset \mathbb{R}^3$*, which is a manifold with a boundary. We assume that the magnetic field* $\mathbf{B}$ *is tangent to the boundary* $\partial\Omega$ *and is not vanished inside* Ω*. In this case we say that the support* Ω *is called a magnetic tube. All the* C^∞*–magnetic fields in tubes are formed a space, equipped with* C^∞ *topology. We do not assume that magnetic lines of* $\mathbf{B}$ *are closed.*

The configuration space $K_{q,r}$ *is defined as following. Assume that a collection of* r *magnetic lines* $L_1, \dots, L_r$ *of the magnetic field* $\mathbf{B}$*, which is parametrized of the segments* $[0, T]$*, started at the prescribed points* $\{l_1, \dots, l_r\}$ *of the domain* Ω *correspondingly. The subcollection* $\{l_1; x_{1,1}, \dots, x_{1,q}\}$ *of the full collection consists of* q *points, which are on the first magnetic line* L_1 *of the magnetic field* $\mathbf{B}$*, the subcollection* $\{l_2; x_{2,1}, \dots, x_{2,q}\}$ *consists of* q *points, each point belongs to the second magnetic line* L_2 *of* $\mathbf{B}$*, e.t.c., the last subcollection* $\{l_r; x_{r,1}, \dots, x_{r,q}\}$ *of the full collection consists of* q *points, each point belongs to the* r*-th magnetic line* L_r *of* $\mathbf{B}$*. Each point* $x_{i,j}$*,* $1 \le i \le r$*,* $1 \le j \le q$ *is well-defined by the time-variable* $t_{i,j}$*,* $0 \le t_{i,j} \le T$*, which is the time of the evolution of the point* l_i *into the point* $x_{i,j}$ *by the magnetic flow.*

Let a real-valued functional $\bar{I} : \Omega \to \mathbb{R}$ *be well-defined. Let us say the functional* $\bar{I}$ *is of a finite-order, if it is defined as the average* $\bar{I} = I(l_1, l_2, \dots, l_r)$ *over the all collections* $\{l_1, \dots, l_r\}$ *of the asymptotic limits for* $T \to +\infty$ *(called Cesaro averages) of integrals* $\int f dx_{1,1} \dots dx_{r,q}$ *of a function* $f : K_{q,r} \to \mathbb{R}$ *over all finite collections* $\{l_1; x_{1,1}, \dots, x_{q,r}\} \in K_{q,r}$ *with fixed* $\{l_1, \dots, l_r\}$*.*

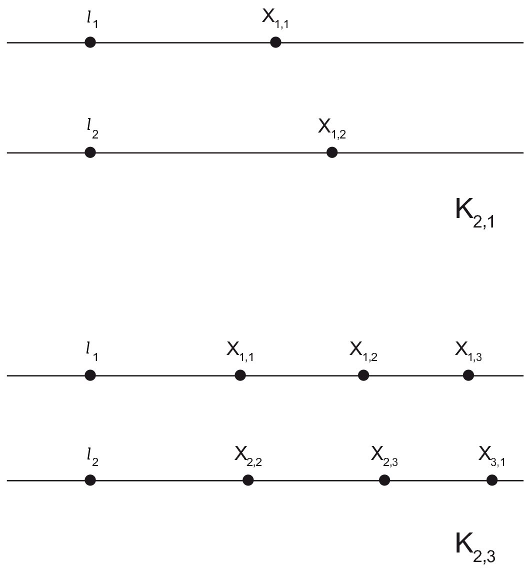

Figure 3.1: Configurations of configuration spaces $K_{2,1}, K_{2,3}$.

3.2 Asymptotic Hopf Invariant

Let us recall the definition of the asymptotic Hopf invariant by V.I. Arnol'd. Let $\{g^t : \Omega \to \Omega\}$ be the magnetic flow, which is determined by the magnetic field $\mathbf{B}$ with a support inside a ball $\Omega \subset \mathbb{R}^3$. Define the Gaussian linking number of the magnetic lines of $\mathbf{B}$ by the times T_1, T_2 correspondingly, issued from the points l_1, l_2 correspondingly, by the following formula:

$$\Lambda_{\mathbf{B}}(T_1, T_2; l_1, l_2) = \frac{1}{4\pi T_1 T_2} \int_0^{T_2} \int_0^{T_1} \frac{\langle \int \dot{x}_1(t_1), \dot{x}_2(t_2), x_1(t_1) - x_2(t_2) \rangle}{\| x_1(t_1) - x_2(t_2) \|^3} dt_1 dt_2, \quad (3.1)$$

$$\Lambda_{\mathbf{B}}(l_1, l_2) = \lim_{T_1, T_2 \to +\infty} \Lambda_{\mathbf{B}}(T_1, T_2; l_1, l_2), \quad (3.2)$$

where $x_i(t_i) = g^t(l_i)$ is the magnetic line, issued from the point l_i, $i = 1, 2$, and $\dot{x}_i(t_i) = \frac{d}{dt_i} g^{t_i} x_i$ are the corresponded velocity tangent vectors (= the vectors of the magnetic field).

(Lemma 2, [A-Kh], p. 158, Lemma 4.12.)

The limit $\Lambda_{\mathbf{B}}(x_1, x_2)$ exists almost everywhere on $\Omega \times \Omega \subset \mathbb{R}^3 \times \mathbb{R}^3$. The function $\Lambda_{\mathbf{B}}(x_1, x_2)$ belongs to L^1 (the absolute value is integrable) and the following equation $\int \Lambda_{\mathbf{B}}(x_1, x_2) dx_1 dx_2 = \chi_{\mathbf{B}}$ is well-defined, where the right side of the equation is given by the formula $\chi_{\mathbf{B}} = \int (\mathbf{A}, \mathbf{B}) dD$, $\mathrm{rot}\mathbf{A} = \mathbf{B}$.

The value is called the asymptotic Hopf invariant, or, the helicity integral. This invariant has the dimension $G^2 sm^4$, and the corresponding combinatorial invariant of links, the linking number, has the topological order 1.

Lemma 1. *For a generic magnetic field* $\mathbf{B}$ *with inside* Ω *the asymptotic Hopf invariant is an asymptotic functional in the sense of Definition* 1.

Proof of Lemma 4

Assume that $\Omega \subset \mathbb{R}^3$ be a magnetic tube (or a finite collection of magnetic tubes), which is the support of the magnetic field $\mathbf{B}$. Let us prove that the asymptotic Hopf invariant $\chi_{\mathbf{B}}$ is a local functional as in Definition 1. Let us define a configuration space $K_{2,1}(\mathbf{B})$, which is diffeomorphic to $\Omega \times \mathbb{R}^1 \times \Omega \times \mathbb{R}^1$. Let us define a mapping, which is called the evaluation mapping

$$F_1 : K_{2,1} \to \Omega \times \Omega, \tag{3.3}$$

by the following formula:

$$F_1(l_1, t_1, l_2, t_2) = (g^{t_1}(l_1), g^{t_2}(l_2)),$$

where g^t is the magnetic flow. Therefore, the space $K_{2,1}$ is well-defined as the configuration space of ordered pairs of points, where each point is on the corresponding magnetic line, which is issued from the point l_1, or l_2. The space $K_{2,1}$ is a particular example of a configuration space, see Definition 1.

On the configuration space $K_{2,1}$ the standard volume form $dK_{2,1} = d\Omega \wedge d\mathbb{R} \wedge d\Omega \wedge d\mathbb{R}$ is well-defined. Let us define an integral kernel as following:

$$\Gamma^{[1]} : K_{2,1} \to \mathbb{R}, \tag{3.4}$$

using the subintegral function in the Gauss integral (3.1) (the factor $(4\pi)^{-1}$ in the formula (6.29) provides the asymptotic linking number for two closed curves is an integer. The restriction of the function (6.29) to an arbitrary compact subspace in $K_{2,1}$ is integrable, and, moreover, belongs to the space L^2 (an integrable function with the integrable square). Let us consider two flows on the space $K_{2,1}$, which are commuted. Each flow is defined by the shift along the corresponding coordinate $\mathbb{R}_{x_1}$, or $\mathbb{R}_{x_2}$. By the Birkhoff Theorem (see [H]) the asymptotic Hopf invariant $\chi_{\mathbf{B}}$ is well-defined as the time average number of the function $\Gamma^{[1]}$ of this pair of the flows. Lemma 4 is proved.

Assume $\mathbf{B} \in \Omega$. Denote by $K_{r,q;T} \subset K_{r,q}$ a compact subspace in the configuration space, for which each time-variable coordinate belongs to the segment $[0, T]$. Let us formulate the definition of limiting tensor.

Definition 2. *Assume a function*

$$A : K_{r,q} \to \mathbb{R} \tag{3.5}$$

is integrable for each subspace $K_{r,q;T} \subset K_{r,q}$. Let us say that an integrable non-negative function

$$a^{[q]} : (\Omega^q)^r \to \mathbb{R}_+, \tag{3.6}$$

which, possibly, tends to $+\infty$, when a point in the origin tends to the thick diagonal $Diag \subset (\Omega^q)^r$, is called a limiting tensor for (3.5), if there exists $T_0 \geq 0$, such that for an arbitrary $T > T_0$ the function $a^{[q]} \circ F_q : K(r,q) \to (\Omega^q)^r \to \mathbb{R}_+$ (this function is integrable, because the function (3.6) is integrable) and $|A| : K_{r,q} \to \mathbb{R}_+$ satisfies the following equation:

$$\int |A| dK_{r,q;T} \leq \int a^{[q]} \circ F_q \quad dK_{r,q;T}. \tag{3.7}$$

In the diagram above the evaluation mapping is used:

$$K_{r,q} \quad \xrightarrow{F_q} \quad ((\Omega)^q)^r. \tag{3.8}$$

The evaluation mapping F_q for $q > 1$ is defined analogously to F_1, see the formula (3.9) below for the case $r = 2$. The evaluation mapping is used to investigate what's happening if we omit all the coordinates $\mathbb{R}_{1,i}, \mathbb{R}_{2,j}$ of points of the configuration spaces.

The following statement is the Main Theorem of the section.

Theorem 1. *Let $I(l_1, x_{1,1}, \dots, x_{r,q})$ is defined as a polynomial $A(f_1, \dots, f_q)$ of functions $f_j \circ F_q(j) : K_{r,q}(l_j, x_{j,1}, \dots, x_{j,q}) \to \mathbb{R}$, $f_j : \Omega^q \to \mathbb{R}$, $F_q(j) : K_{r,q} \to \Omega^q$.*

–1. Assume that for an arbitrary j the function $f_j : \Omega^r \to \mathbb{R}$, $1 \le j \le q$ is integrable.

–2. Assume that there exists a limiting tensor (3.6) *for I in the sense of Definition* 2.

Then the asymptotic mean value $\bar{I}(l_1, \dots, l_r)$ of $I(l_1, x_{1,1}, \dots, x_{r,q})$ with respect to the coordinates $\mathbb{R}_{1,i}, \mathbb{R}_{2,j}$ is well-defined except, possibly, a subset of $\{l_1, \dots l_r\}$ in Ω^r of zero measure, the function $\bar{I}(l_1, \dots, l_r) : \Omega^r \to \mathbb{R}$ is integrable (belongs to L^1) and invariant with respect to the magnetic flow on Ω^r.

Proof of Theorem 1

We will prove the theorem in the case $r = 2$, this case is used for applications. In a general case the proof is analogous.

Let us define the configuration space $K_{2,q} = \Omega \times \mathbb{R}^k \times \Omega \times \mathbb{R}^k$ and the evaluation mapping:

$$F_q : K_{2,q} \to (\Omega)^q \times (\Omega)^q, \tag{3.9}$$

where (Ω^q) is the Cartesian product of q exemplars of a magnetic tube Ω (or, a finite number of magnetic tubes), by the formula:

$$F_q(l_1, t_{1,1} \dots t_{1,q}, l_2, t_{2,1} \dots t_{2,q}) =$$

$$(g^{t_{1,1}}(l_1), \dots g^{t_{1,q}}(l_1), g^{t_{2,1}}(l_2)), \dots g^{t_{2,q}}(l_2))$$

where g^t is the magnetic flow. The mapping F_q for $q = 1$ coincides with the mapping (3.3), defined above. Therefore, the configuration space $K_{2,q}$ is defined as the space of two ordered collections of q points, each collection on the corresponded magnetic line.

On the space $K_{2,q}$ a collection of $2q$ flows (each two flows are commuted) along the coordinates $\mathbb{R}_{1,i}, \mathbb{R}_{2,j}$, $1 \le i \le q$, $1 \le j \le q$ are well-defined. This coordinates are denoted by $t_{1,i}$, $t_{2,j}$, the images $g^{t_{1,i}}(x_1)$, $g^{t_{2,j}}(x_2)$ are denoted by $x_{1,i}(t_{1,i})$, $x_{2,j}(t_{2,j})$, or, briefly, by $x_{1,i}$, $x_{2,j}$. The standard volume form on $K_{2,q}$ is denoted by $dK_{2,q}$.

Prove Statement 1. This is a corollary of the Birkhoff Theorem followed the argument by V.I.Arnol'd as in Lemma 4.

Prove Statement 2. Let us consider the function (3.6) and apply for each of $2q$ flows on the space $(\Omega^2)^q$ the the Birkhoff Theorem by the induction.

In each step of the induction we get an integrable function on the Cartesian product of one less copies of Ω. At the last step of the induction we get an integrable function, denoted by $A : \Omega^2 \to \mathbb{R}_+$ as in (3.5), which is well-defined almost everywhere and which is a constant along each of the two magnetic flows on Ω^2.

The inequality (3.7) implies the following inequality:

$$a^{[2]}(l_1, l_2) \geq |I(l_1, l_2)|. \tag{3.10}$$

Because the left side of (3.10) is integrable, the right side is also integrable. Statement 2 is proved. Theorem 1 is proved.

3.3 q-Monomial helicities

Apply Theorem in the case of the q-power of the helicity density, defined by the equation (3.2).

Let us start to define the asymptotic q-monomial linking coefficient $\Lambda^{[q]}_{\mathbf{B}}$ and the asymptotic ergodic invariant $\chi^{[q]}_{\mathbf{B}}$, which is called q-helicity. Define q-linking coefficient $\Lambda^{[q]}_{\mathbf{B}}(T_1, T_2; l_1, l_2)$ of the magnetic lines of $\mathbf{B}$ by the times T_1, T_2, which are issued from the points l_1, l_2, as follows:

$$\Lambda^{[q]}_{\mathbf{B}}(T_1, T_2; l_1, l_2) = \tag{3.11}$$

$$\frac{1}{4^q \pi^q T_1^q T_2^q} \int_0^{T_1} \dots \int_0^{T_1} \int_0^{T_2} \dots \int_0^{T_2} \frac{\langle \dot{x}_{1,1}(t_{1,1}), \dot{x}_{2,1}(t_{2,1}), x_{1,1}(t_{1,1}) - x_{2,1}(t_{2,1}) \rangle}{\| x_{1,1}(t_{1,1}) - x_{2,1}(t_{2,1}) \|^3} \cdot \dots \cdot$$
$$\frac{\langle \dot{x}_{q,1}(t_{1,q}), \dot{x}_{q,2}(t_{2,q}), x_{1,q}(t_{1,q}) - x_{2,q}(t_{2,q}) \rangle}{\| x_{1,q}(t_{1,q}) - x_{2,q}(t_{2,q}) \|^3} dt_{1,1} \dots dt_{1,q} dt_{2,1} \dots dt_{2,q}.$$

In this integral we may put $T_1 = T_2 = T$, this gives a little simplification.

For almost arbitrary pair of initial points l_1, l_2 by the Arnol'd Theorem there exists a limit

$$\Lambda^{[q]}_{\mathbf{B}}(l_1, l_2) = \lim_{T \to +\infty} \Lambda^{[q]}_{\mathbf{B}}(T; l_1, l_2), \tag{3.12}$$

where the function in the right side of the equation is integrable on $\Omega \times \Omega$, see .

Let us define (at least a formal) functional on the space Ω, which generates the q–helicity by the formula:

$$\chi^{[q]}_{\mathbf{B}} = \int \Lambda^{[q]}_{\mathbf{B}}(l_1, l_2) d\Omega d\Omega, \tag{3.13}$$

where $\Lambda^{[q]}_{\mathbf{B}}(l_1, l_2)$ is defined in (3.12).

Theorem 2. *The integral at the right side of the formula* (3.13) *is well-defined.*

Corollary

The q-monomial helicity $\chi_{\mathbf{B}}^{[q]}$ depends continuously on a C^{∞}-small perturbation of magnetic fields in Ω. The proof is following: for an arbitrary $\varepsilon > 0$ there exists T, such that the term $\Lambda_{\mathbf{B}}^{[q]}(T; l_1, l_2)$ in the right side of the formula (3.12) is distinguished from the left side of the equation less then $\frac{\varepsilon}{2}$. The term $\Lambda_{\mathbf{B}}^{[q]}(T; l_1, l_2)$ depends smoothly on a perturbation of $\mathbf{B}$. This gives the required result. In [A2] is proved using Stokes lemma a modification of $\chi_{\mathbf{B}}^{[q]}$ in the case $q = 2$ depends Lipschitz continuity on a C^{∞}-small perturbation of $\mathbf{B}$.

Remark

The dimension of $\chi_{\mathbf{B}}^{[q]}$ is $Gs^{2q}sm^4$. The order of the corresponding topological invariant equals to q.

Let us define the configuration space $K_{2,q} = \Omega \times \mathbb{R}^k \times \Omega \times \mathbb{R}^k$. Let us denote

$$\Gamma^{[q]} : K_{2,q} \to \mathbb{R}. \tag{3.14}$$

by the integral kernel in (3.23). The function $\Gamma^{[q]}$ determines a functional (3.25) (probably, a formal functional). Define a smoothing of the subintegral kernel $\Gamma^{[q]}$, using a small finite parameter $\varepsilon > 0$, over this additional time-parameter the integration of $\Gamma^{[q]}$ as following. Each time-variable vary by additional small alterations in the interval $[-\varepsilon, +\varepsilon]$. Denote the result of the smoothing of $\Gamma^{[q]}$ by $K_{\varepsilon}(x_{1,1}, x_{2,1}, \dots, x_{1,q}, x_{2,q})$.

Replace the integral kernel $\Gamma^{[q]}$ into the smoothed kernel K_{ε} in the right side (below the limit of the equation (3.24). The mean value over the time-variables over the segment $[0, T]$ at the left side of the non-equality (3.7), which is used for $A^{[q]} = K_{\varepsilon}$, is the following:

$$\Lambda_{\mathbf{B}}^{(q)}(T_1, T_2; l_1, l_2; \varepsilon) = \tag{3.15}$$

$$\frac{1}{2^q \pi^q T_1^q T_2^q} \int_0^{T_1} \dots \int_0^{T_1} \int_0^{T_2} \dots \int_0^{T_2}$$

$$K_{\varepsilon}(x_{1,1}, x_{2,1}, \dots, x_{1,q}, x_{2,q}) dt_{1,1} \dots dt_{1,q} dt_{2,1} \dots dt_{2,q},$$

where the integral kernel K_{ε} is calculated by the formula:

$$K_{\varepsilon}(x_{1,1}, x_{2,1}, \dots, x_{1,q}, x_{2,q}) = \tag{3.16}$$

$$\varepsilon^{-q}\int_0^\varepsilon \dots \int_0^\varepsilon \int_0^\varepsilon \dots \int_0^\varepsilon \frac{\langle \dot{x}_{1,1}(t_{1,1}), \dot{x}_{2,1}(t_{2,1}), x_{1,1}(t_{1,1}) - x_{2,1}(t_{2,1})\rangle}{|x_{1,1}(t_{1,1}) - x_{2,1}(t_{2,1})|^3} \cdot \dots$$
$$\cdot \frac{\langle \dot{x}_{q,1}(t_{1,q}), \dot{x}_{q,2}(t_{2,q}), x_{1,q}(t_{1,q}) - x_{2,q}(t_{2,q})\rangle}{|x_{1,q}(t_{1,q}) - x_{2,q}(t_{2,q})|^3} dt_{1,1}\dots dt_{1,q}dt_{2,1}\dots dt_{2,q}.$$

Put is the right side of the non-equality (3.7) the expression $A^{[q]} = \Gamma^{[q]}$:

$$\int |\Gamma^{[q]}| dK_{2,q;T}. \tag{3.17}$$

Put in the same formula the expression $A^{[q]} = K_\varepsilon^{[q]}$:

$$\int |K_\varepsilon^{[q]}| dK_{2,q;T}. \tag{3.18}$$

Obviously, the integrals (3.17), (3.18) distinguishes by its absolute value by a real, which is non-depended of T and is arbitrary small, if $\varepsilon \to 0$. The difference of this two integrals are given by boundary conditions, when one of the parameter belongs to $\{0, T\}$. Therefore for the proof of the required statement is sufficient to construct a limiting tensor for $K_\varepsilon^{[q]}$ for an appropriate finite $\varepsilon > 0$.

The smoothed integral kernel (3.16) is equal to a product of the integrals as following:

$$K_\varepsilon(x_{1,1}, x_{2,1}, \dots, x_{1,q}, x_{2,q}) = \tag{3.19}$$
$$\varepsilon^{-q} \prod_{j=1}^{q} \int_0^\varepsilon \int_0^\varepsilon \frac{\langle \dot{x}_{1,j}(t_{1,j}), \dot{x}_{2,j}(t_{2,j}), x_{1,j}(t_{1,j}) - x_{2,j}(t_{2,j})\rangle}{|x_{1,j}(t_{1,j}) - x_{2,j}(t_{2,j})|^3} dt_{1,j}dt_{2,j}.$$

Let us apply the elementary non-equality between arithmetic and geometric mean values. Then we get:

$$K_\varepsilon(x_{1,1}, x_{2,1}, \dots, x_{1,q}, x_{2,q}) \le \tag{3.20}$$
$$\frac{1}{q\varepsilon^q} \sum_{j=1}^{q} (\int_0^\varepsilon \int_0^\varepsilon \left| \frac{\langle \dot{x}_{1,j}(t_{1,j}), \dot{x}_{2,j}(t_{2,j}), x_{1,j}(t_{1,j}) - x_{2,j}(t_{2,j})\rangle}{\|x_{1,j}(t_{1,j}) - x_{2,j}(t_{2,j})\|^3} \right| dt_{1,j}dt_{2,j})^q.$$

Let us fix an arbitrary small constant $1 >> \delta_0 > 0$, such that at the prescribed scale the magnetic field and its partial derivatives are of a small variation. Assume that points $x_{1,j}$, $x_{2,j}$ are distinguished not more then to a positive small constant δ_0, then the corresponding term in (3.20) is absolutely estimated by the following term

$$C \ln^q(\rho_{\mathbf{B}}(x_{1,j}, x_{2,j})), \tag{3.21}$$

where $\rho_{\mathbf{B}}(x_{1,j}, x_{2,j})$ is the distance from the point $x_{1,j}$ to the magnetic line, which contains the point $x_{2,j}$. The coefficient $C < 0$, in the case $q = 2s+1$, and $C > 0$, in the case $q = 2s$, depends of first and second order partial derivatives of $\mathbf{B}$ and of the prescribed constant δ_0.

If the distance between points $x_{1,j}$, $x_{2,j}$ is greater then δ_0, the corresponding term in the expression (3.20) is absolutely estimated by a positive constant, which is not depended of δ_0 and of the components of $\mathbf{B}$. Putting the non-equalities (3.20), (3.21) into the expression (3.19). This gives the following absolute bound of the integral kernel (3.16) of the integral (3.15):

$$K_\varepsilon(x_{1,1}, x_{2,1}, \dots, x_{1,q}, x_{2,q}) \le \frac{C^q}{\varepsilon^q} \sum_{i=1}^{q} \ln^q(\rho_{\mathbf{B}}(x_{1,i}, x_{2,i})). \tag{3.22}$$

Define the limiting tensor by the formula:

$$\delta^{[q]}(x_1, x_2) = \frac{qC^q}{\varepsilon^q} \ln^q(\rho_{\mathbf{B}}(x_1, x_2)).$$

Because the integral

$$\int \ln^q(\rho_{\mathbf{B}}(x_1, x_2)) dx_1 dx_2$$

over an arbitrary compact domain in $\mathbb{R}^3(x_1) \times \mathbb{R}^3(x_2)$ exists, for an arbitrary $q \ge 1$ the function in the left side of (6.63) is integrable. The required estimation (3.7) for $A = K_\varepsilon^{[q]}$ follows from the equation (6.63): using this equation each term in the expression (3.20) is estimated. This proves Theorem 2.

Let us start to define the asymptotic q-monomial linking coefficient $\Lambda_{\mathbf{B}}^{[q]}$ and the asymptotic ergodic invariant $\chi_{\mathbf{B}}^{[q]}$, which is called q-helicity. Define q-linking coefficient $\Lambda_{\mathbf{B}}^{[q]}(T_1, T_2; l_1, l_2)$ of the magnetic lines of $\mathbf{B}$ by the times T_1, T_2, which are issued from the points l_1, l_2, as follows:

$$\Lambda_{\mathbf{B}}^{[q]}(T_1, T_2; l_1, l_2) = \tag{3.23}$$

$$\frac{1}{4^q \pi^q T_1^q T_2^q} \int_0^{T_1} \dots \int_0^{T_1} \int_0^{T_2} \dots \int_0^{T_2} \frac{\langle \dot{x}_{1,1}(t_{1,1}), \dot{x}_{2,1}(t_{2,1}), x_{1,1}(t_{1,1}) - x_{2,1}(t_{2,1}) \rangle}{\|x_{1,1}(t_{1,1}) - x_{2,1}(t_{2,1})\|^3} \cdot \dots \cdot$$
$$\frac{\langle \dot{x}_{q,1}(t_{1,q}), \dot{x}_{q,2}(t_{2,q}), x_{1,q}(t_{1,q}) - x_{2,q}(t_{2,q}) \rangle}{\|x_{1,q}(t_{1,q}) - x_{2,q}(t_{2,q})\|^3} dt_{1,1} \dots dt_{1,q} dt_{2,1} \dots dt_{2,q}.$$

In this integral we may put $T_1 = T_2 = T$, this gives a little simplification.

Below is proved (see Theorem 3.3) that for almost arbitrary pair of points l_1, l_2 there exists a limit

$$\Lambda_{\mathbf{B}}^{[q]}(l_1, l_2) = \lim_{T \to +\infty} \Lambda_{\mathbf{B}}^{[q]}(T; l_1, l_2), \tag{3.24}$$

where the function in the right side of the equation is integrable on $\Omega \times \Omega$.

The formula (3.24) determines the time-average of the corresponding integral kernel, which is denoted by $\Gamma^{[q]}$. In the particular case $q = 1$ the formula (3.23) coincides with (3.1).

Let us define (a formal) functional, which generates the q–helicity by the formula:

$$\chi_{\mathbf{B}}^{[q]} = \int \Lambda_{\mathbf{B}}^{[q]}(l_1, l_2) d\Omega d\Omega, \tag{3.25}$$

where $\Lambda_{\mathbf{B}}^{[q]}(l_1, l_2)$ is defined in (3.24).

Theorem 3. *The helicity density, given by* (3.2) *is is invariant with respect to volume-preserved diffeomorphisms of the domain* $\Omega \subset \mathbb{R}^3$. *In particular, the q-monomial helicities, given by* (3.13)*, are invariants in the ideal MHD.*

Let $D(s) : \mathbb{R}^3 \to \mathbb{R}^3$, $s \in [0, s_0]$ is a one-parameter family of volume-preserved diffeomorphisms with a compact support, which transforms the identity $Id : \mathbb{R}^3 \to \mathbb{R}^3$ to a diffeomorphism $D(s_0)$ (the existence of such a family of diffeomorphisms follows from the A.I.Shnirelman Theorem ([A-X], section 7, chapter IV). For an arbitrary s consider a function of T

$$\Lambda^{(q)}_{F(s)_*(\mathbf{B})}(T; l_1, l_2), \tag{3.26}$$

which is induced from $\Lambda_{\mathbf{B}}$ by a diffeomorphism F_s.

Lemma 2. *The function*

$$\frac{d}{ds}\Lambda_{F_{s,*}(\mathbf{B})}(l_1, l_2; T, s) \tag{3.27}$$

for an arbitrary T *is absolutely bounded by the positive function* $C(l_1, l_2)T^{-1}$*, where* $C : \Omega \times \Omega \to \mathbb{R}_+$ *is a non-negative integrable function, which is depended of components of* $\mathbf{B}$ *and of partial derivatives of components of* $\mathbf{B}$.

Proof of Lemma 2

An analogous statement is proved in [A2], statement 2.2.

Proof of Theorem 3

By the formula 3.27 we get a well-defined limit $\lim_{T\to+\infty}\frac{d}{ds}\Lambda_{F_{s,*}(\mathbf{B})}(l_1,l_2;T,s)$ of absolute integrable functions with respect to a measure, and this limit is equal to zero function. Therefore the function $\Lambda_{\mathbf{B}}(l_1,l_2)$, as an integrable function, is invariant with respect to the flow F_s. Therefore, $\Lambda_{\mathbf{B}}^{[q]}(l_1,l_2)$ is also invariant with respect to the flow F_s, because $\Lambda_{\mathbf{B}}^{[q]}(l_1,l_2)=(\Lambda_{\mathbf{B}}(l_1,l_2))^q$. The flow F_s preserves the volume, and, therefore, the integral (3.25). Theorem 3 is proved.

3.4 Possible applications

The most important application of magnetic helisity in MHD is related with the Arnol'd inequality:

$$\int \mathbf{B}^2 d\Omega \leq C\int(\mathbf{A},\mathbf{B})d\Omega,$$

where the constant $C>0$ depends only on geometrical properties of the domain Ω. In the right side of the equation we get the magnetic helicity integral $\chi_{\mathbf{B}}$, which is an invariant for the ideal MHD. In the left side we get the magnetic energy. Are analogous inequalities possible for higher magnetic energies in the left side of generalized inequalities with $\chi_{\mathbf{B}}^{[q]}$ instead of $\chi_{\mathbf{B}}$?

Generalizations are important for turbulence. For problems with magnetic field an approach, initially proposed by Kolmogorov A.N. [1941] is in progress. Expecting results relate with generalized Arnol'd inequalities and applications in MHD cascades [F-S].

Chapter 4

Higher invariants of magnetic lines

4.1 The M-Invariant for a triple of magnetic tubes

Consider a magnetic field $\mathbf{B} = \cup_i \mathbf{B}_i$ with a support into 3 magnetic tubes $\Omega_i, \quad i = 1,2,3$ correspondingly. Assume that inside the each magnetic tube a coordinate system $\Omega_i \cong D^2 \times S^1$ is fixed. Assume that this coordinate system corresponds with the standard volume form in $\mathbb{R}^3$ and the magnetic field $\mathbf{B}_i$ points strictly along the S^1–coordinate of the system. This assumption simplifies calculations and gives no loss of a generality.

The integral magnetic flow of $\mathbf{B}_i$ trough the cross-section of the magnetic tube Ω_i is denoted by Φ_i. The integral linking number $\int_{\Omega_i} (\mathbf{A}_j, \mathbf{B}_i) dx = \Phi_i \Phi_j lk(i,j)$ of magnetic tubes Ω_i, Ω_j is denoted by (i,j), $i,j = 1,2,3 \quad i \neq j$.

A multivalued function with the period (i,j), which is a restriction of the scalar branch of the vector-potential $\mathbf{A}_j$ on the magnetic tube Ω_i denote by $\varphi_{j,i} : \Omega_i \to \mathbb{R}$. The function $\varphi_{j,i}$ is well-defined up to an additive constant. Consider a function

$$\phi_1 = (3,1)\varphi_{2,1} - (1,2)\varphi_{3,1} : \Omega_1 \to \mathbb{R},$$

which is well-defined by means of multivalued functions $\varphi_{2,1}, \varphi_{3,1}$ up to an additive constant. To fix the constant, we assume that the following equation is satisfied: $\int_{\Omega_1} \phi_1 dx = 0$. Define the functions ϕ_2, ϕ_3 by analogous formula.

Define the vector

$$\mathbf{F} = (1,3)(2,3)\mathbf{A}_1 \times \mathbf{A}_2 + (2,1)(3,1)\mathbf{A}_2 \times \mathbf{A}_3 + (3,2)(1,2)\mathbf{A}_3 \times \mathbf{A}_1$$
$$-\phi_1 \mathbf{B}_1 (2,3) - \phi_2 \mathbf{B}_2 (3,1) - \phi_3 \mathbf{B}_3 (1,2).$$

Obviously, the equation $\mathrm{div}(\mathbf{F}) = 0$ is satisfied.

The vector-potential $\mathbf{G}$, $\mathrm{rot}\mathbf{G} = \mathbf{F}$ and the integral $\int_{\mathbb{R}^3}(\mathbf{G},\mathbf{F})dx$ are well-defined. This integral is modified into the required invariant of volume-preserved diffeomorphisms. This modification includes the following extra 10 terms:

$$e_{1,2,3} = -2(1,2)(2,3)(3,1)(\int_{\mathbb{R}^3} \langle \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \rangle dx)^2, \tag{4.1}$$

$$f_1 = -2(\int_{\Omega_1} \varphi_{2,1}^{var}(\mathbf{grad}\varphi_{3,1}^{var}, \mathbf{B}_1)d\Omega_1)(\int_{\mathbb{R}^3} \langle \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \rangle dx), \tag{4.2}$$

$$d_{1,1} = -(2,3)^2 \int \phi_1^2(\mathbf{A}_1, \mathbf{B}_1) d\Omega_1, \tag{4.3}$$

$$d_{1;3} = (2,3)(1,2) \int \phi_1^2(\mathbf{A}_3, \mathbf{B}_1) d\Omega_1. \tag{4.4}$$

In the formula (4.2) the terms $\varphi_{3,1}^{var}$, $\varphi_{2,1}^{var}$ are defined from $\varphi_{3,1}$, $\varphi_{2,1}$ correspondingly, see [A2]. The extra 6 terms are defined by cyclic permutation of the indexes $\{1, 2, 3\}$ in the formulas (4.2), (4.3), (4.4).

In [A2] the following result is proved.

Theorem 4. *The integral expression*

$$M(\mathbf{B}) = \int_{\mathbb{R}^3} (\mathbf{G},\mathbf{F})dx + e_{1,2,3} + \sum_{i=1,2,3} f_i + d_{i,i} + d_{i;i+2} \tag{4.5}$$

is invariant with respect to volume-preserved diffeomorphisms.

The invariant M is non-degenerated and is not expressed by the linking coefficients of the tubes

This example illustrates Theorem 4. Let us consider the field $\mathbf{B}$ decomposed into 3 tubes with the flows $\mathbf{fl}_1 = \mathbf{fl}_2 = \mathbf{fl}_3 = 1$. Let us consider the tubes Ω_1 and Ω_2, presented by the Whitehead link, and the tube Ω_3, such that the pairs of tubes (Ω_1, Ω_3) and (Ω_2, Ω_3) present Hopf links with the linking coefficients $+1$, (see Fig. 3).

Because $(1,2) = 0$ $(2,3) = (3,1) = 1$, the expression (4.5) is simplifyed:

$$\mathbf{F} = \mathbf{A}_1 \times \mathbf{A}_2 - \psi_2 \mathbf{B}_1 + \psi_1 \mathbf{B}_2,$$

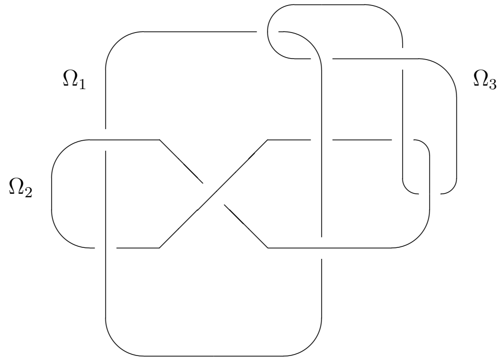

Figure 4.1: A link with non-trivial M-invariant.

$$M(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3) = \int [2\mathbf{G}\mathbf{F} - \psi_2^2(\mathbf{A}_1, \mathbf{B}_1) - \psi_1^2(\mathbf{A}_2, \mathbf{B}_2)] d\mathbb{R}^3. \qquad (4.6)$$

This equation coincides with the integral formula for Sato-Levine invariant presented in [A-R1], [A-R2]. Because Sato-Levine invariant for the Whitehead link is non-trivial [Co], we have $M \neq 0$. This proves that M is non-degenerated. If we change the pair of tubes Ω_1, Ω_2 to the trivial pair of tubes, keeping the pairs Ω_1, Ω_3, Ω_2, Ω_3 in the isotopy class of the Hopf link, the value M becomes trivial. This proves that the invariant M cannot be expressed from the linking numbers of the components.

4.2 The M-invariant of 3 closed magnetic lines

Assume that $\Phi_1 = \Phi_2 = \Phi_3 = 1$ and take a limit in the formula (4.5), when the thickness of magnetic tubes tends to zero. The result satisfies the following definition.

Definition

For an arbitrary 3-component link $\mathbf{L} \subset \mathbb{R}^3$ define a space (non-connected) $Conf^r(\mathbf{L}) = (\mathbf{L})^r$ as the Cartesian product of r copies of $\mathbf{L}$. The space

$Conf^r(\mathbf{L})$ is called the configuration space of the link $\mathbf{L}$.

Let

$$F : Conf^r(\mathbf{L}) \to \mathbb{R} \tag{4.7}$$

be an integrable function on the configuration space.

Theorem 5. *Each term in the expression* (4.5) *is defined by the integral of a corresponding function on the configuration space of the link* $\mathbf{L}$*.*

Let us formulate an analogous definition for a function on the configuration space of magnetic lines.

4.2.1 Ergodic integrals and quasi-ergodic integrals

Let $\mathbf{B}$, $\mathrm{div}(\mathbf{B}) = 0$ be a smooth magnetic field in $\mathbb{R}^3$ with a support inside a finite collection of magnetic tubes $\Omega \subset \mathbb{R}^3$, the magnetic field $\mathbf{B}$ is tangent to the surface boundary of Ω and non-vanishes inside Ω.

Let us say that the function $F : K_{q,r} \to \mathbb{R}$ (see Definition 1) on configuration space determines an ergodic integral, if the following conditions are satisfied:

• -1. For almost an arbitrary point $\{l_1, \dots, l_r\} \in \Omega^r$ the mean value $\bar{F} : K_{q,r} \to \mathbb{R}$ (in the sense of Cesàro) of the function F with respect to position of points $\{x_{1,1}, \dots, x_{1,q}, \dots, x_{r,1}, \dots, x_{r,q}\}$ is well-defined. By definition $\bar{F}$ is induced by a function in the domain Ω^r with respect to the projection $\pi : K_{q,r} \to \Omega^r$, $\pi(z) = \{l_1, \dots, l_r\}$, $z \in K_{q,r}$; denote this function by $\bar{F} : \Omega^r \to \mathbb{R}$.

• -2. The function $\bar{F} : \Omega^r \to \mathbb{R}$ is locally integrable and is integrable.

The ergodic integral $I(\mathbf{B})$ is defined as the integral of the function $\bar{F}$ over the domain Ω^r.

Let us say that a function $F : K_{q,r} \to \mathbb{R}$ determines a quasi-ergodic integral, if a linear mapping (non-homogeneous) $X : K_{q,r} \to \mathbb{R}$ with respect to variables $\{t_{1,1}, \dots, t_{1,q}, \dots, t_{r,1}, \dots, t_{r,q}\}$ is well-defined (this mapping determines a relation between parameters $t_{i,j}$, $t_{i,j}(l_i) = x_{i,j}$) and, moreover, for an arbitrary $p \in \mathbb{R}$ the restriction of F to $X^{-1}(p) \subset K_{r,q}$ satisfy Conditions -1, -2; moreover, for an arbitrary $p > 0$ the integral $f(p) = \int \bar{F} d(X^{-1}(p))$ determines an absolute bounded function $f(p) : \mathbb{R}_+ \to \mathbb{R}, \quad p > 0$.

The quasi-ergodic integral $I(\mathbf{B})$ is defined as a mean value of the function $f(p)$ over $\mathbb{R}_+$. Generally speaking, this integral is multivalued and takes the value into a segment.

Additionally, if magnetic a lines, issued from the points $\{l_1,\dots,l_r\}$ are closed, the function $f:\mathbb{R}_+\to\mathbb{R}$ is periodic. In the case magnetic lines of $\mathbf{B}$ are closed, f is periodic and a value $I(\mathbf{B})$ is well-defined.

Theorem 6. *The terms $\int_{\mathbb{R}^3}(\mathbf{G},\mathbf{F})dx$, $e_{1,2,3}$, f_i in the formula of M, presented in* (5)*, are ergodic integrals. The terms $d_{i,i}$, $d_{i;i+2}$ in* (5) *are quasi-ergodic integrals.*

In the paper [A3] the following theorem is proved.

Theorem 7. *Assume that magnetic lines of* $\mathbf{B}$ *inside* Ω *are closed. Then the invariant M satisfy Condition $C1$ for $t=7$, Condition $C2$ for $s=12$, and Conditions $C3$, $C4$.*

Proof of Theorem 6.60

A particular proof of Theorem is in [A4] (Theorem 3.1,(1) and Lemma 4.1.). I present a simplification of the proof for the main term $\int_{\mathbb{R}^3}(\mathbf{G},\mathbf{F})dx$ with simple estimations of the integral.

Recall the definition of the term W of the integral W, $supp(\mathbf{B})=\Omega\subset\mathbb{R}^3$.

Coordinates of a point in $K_{3,4;2}$ are given by collections $\{l_1,t_{1,1},\dots t_{1,4},l_2,t_{2,1},\dots t_{2,4},l_3,t_{3,1},\dots t_{3,4};y_1,y_2\}$, where $l_i\in\Omega_i$, $t_{i,j}\in[0,T]\subset\mathbb{R}_{i,j}$, $j=1,2,3,4$, $y_1,y_2\in\mathbb{R}^3$.

Define the evolution mapping $F:K_{3,4;2}\to\Omega_1^4\times\Omega_2^4\times\Omega_3^4$ by the formula

$$F(l_1,t_{1,1},\dots t_{1,4},l_2,t_{2,1},\dots t_{2,4},l_3,t_{3,1},\dots t_{3,4})=$$

$$(g^{t_{1,1}}(l_1),\dots g^{t_{1,4}}(l_1),g^{t_{2,1}}(l_2),\dots g^{t_{2,4}}(l_2),g^{t_{3,1}}(l_3),\dots g^{t_{3,4}}(l_3),$$

where g^t is the magnetic flow of $\mathbf{B}$. From this formula the space $K_{3,4;2}$ is the configuration space of 17-points: $3(1+4)$ points $\{l_i,g^{t_{i,1}}(l_i),g^{t_{i,2}}(l_i),g^{t_{i,3}}(l_i),g^{t_{i,4}}(l_i)\}$ are on the magnetic lines, which are issued from l_i, $i=1,2,3$, and points $(y_1,y_2)\in(\mathbb{R}^3)^2$ are arbitrary. The standard volume form $dK_{3,4}$ on the space $K_{3,4;2}$ is well-defined.

The first step of the construction includes a definition of a function $W_{3,4;2}:K_{3,4;2}\to\mathbb{R}$, which is called the density function. The density function is not the lift of a function on Ω^3 by the projection $\pi:K_{3,4;2}\to\Omega^3$. The mean asymptotic value of the function $W_{3,4;2}$ over the coordinates $t_{i,j}$, which is well-defined almost everywhere, depends of the parameters $(l_1,l_2,l_3;y_1,y_2)$. The last second step of the construction is a construction of a limiting tensor, this proves that the integral of $W_{3,4;2}$ over $\Omega^3\times(\mathbb{R}^3)^2$ is well-defined.

Let us use the Gauss integral to calculate W in the following formulas:

$$(2,3)(3,1)^2(1,2)\gamma_{t_{1,1},t_{2,1},t_{2,2},t_{3,1}}(\vec{\alpha}_{1,2}(x_1,x_{2,1};y_1),\vec{\alpha}_{2,3}(x_{2,2},x_3;y_2)), \qquad (4.8)$$

$$(2,3)^2(1,2)^2\gamma_{t_{1,1},t_{1,2},t_{2,1},t_{2,2}}(\vec{\alpha}_{1,2}(x_{1,1},x_{2,1};y_1),\vec{\alpha}_{1,2}(x_{1,2},x_{2,2};y_2)). \qquad (4.9)$$

In this formula by $\gamma(\quad,\quad;\quad,\quad)$ is denoted the value of the kernel of the Gauss integral at a pair of corresponding magnetic vectors, the vectors of the pair depend of the variables $(x_1, x_{2,1}, x_{2,2}, x_3)$ and are attached to the points y_1, y_2 (each point is an arbitrary point the space $\mathbb{R}^3$) correspondingly, the vectors $\vec{\alpha}_{1,2}(x_1, x_{2,1}; y_1)$, $\vec{\alpha}_{2,3}(x_{2,2}, x_3; y_2)$ in (4.8) (for (4.9) the formulas are similar) are given by (4.12), (4.13). The terms (4.8), (4.9) are well-defined in the asymptotic limit of all the positions of the variables $x_{i,j}$. For short we take $x_1 = g^{t_{1,1}}(l_1)$, $x_3 = g^{t_{3,1}}(l_3)$. The integration over the variables $(y_1, y_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ is taken after the asymptotic limit.

Let us investigate the term (4.8) only, for the term (4.9) the proof is analogous. The coordinates $\{t_{1,1}, \dots t_{1,4}, t_{2,1}, \dots t_{2,4}, t_{3,1}, \dots t_{3,4}\}$ are divided into the following 2 groups of coordinates, the coordinates of the first group $\{t_{1,1}, t_{2,1}, t_{2,2}, t_{3,1}\}$ are re-denoted by $\{\tau_1, \tau_{2,1}, \tau_{2,2}, \tau_3\}$ correspondingly. The coordinates of the second group $\{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,2}, t_{3,3}, t_{3,4}\}$ are re-order as following: $\{t_{1,2}, t_{3,2}, t_{1,4}, t_{3,4}, t_{1,4}, t_{2,3}, t_{2,3}, t_{3,4}\}$ and are re-denoted by $\{\rho_{1,1}, \rho_{3,1}, \rho_{1,2}, \rho_{3,2}, \rho_{1,3}, \rho_{2,3}, \rho_{2,4}, \rho_{3,4}\}$ correspondingly.

Let us define the factors in the formula (4.8). Using the 4 points of the first group $x_1 = g^{\tau_1}(l_1), x_{2,1} = g^{\tau_{2,1}}(l_2), x_{2,2} = g^{\tau_{2,2}}(l_2), x_3 = g^{\tau_3}(l_3)$, define the integral kernel

$$\gamma_{\tau_1,\tau_{2,1},\tau_{2,2},\tau_3}(\vec{\alpha}_{1,2}(x_1,x_{2,1}),\vec{\alpha}_{2,3}(x_{2,2},x_3);y_1,y_2). \qquad (4.10)$$

Using the last 8 points of the second group $g^{\rho_{1,1}} = z_{1,1}, g^{\rho_{3,1}} = z_{3,1}, g^{\rho_{1,2}} = z_{1,2}, g^{\rho_{3,2}} = z_{3,2}, g^{\rho_{1,3}} = z_{1,3}, g^{\rho_{2,3}} = z_{2,3}, g^{\rho_{2,4}} = z_{2,4}, g^{\rho_{3,4}} = z_{3,4}$, define the integral kernel to calculate $(2,3)(3,1)^2(1,2)$ by obvious way, see [A-Kh] for the integral formula of the linking number. The product of the expressions gives (4.8).

Let us prove that for almost arbitrary collection $(l_1, l_2, l_3; y_1, y_2)$ there exists the asymptotic mean value of the expression (4.10) with respect to the variables $\{\tau_1, \tau_{2,1}, \tau_{2,2}, \tau_3\}$. Denote this asymptotic mean value by

$$\bar{\gamma}(l_1,l_2,l_3;y_1,y_2) \qquad (4.11)$$

The absolute value of coordinates of the vector-potential $\mathbf{A}(x_i; y)$ at an arbitrary point $x_i \in L$ is integrable with respect to the parameter $y \in \mathbb{R}^3$. This vector-potential determines the vector-functions

$$\vec{\alpha}_{1,2}(x_1,x_{2,1};y_1) = \mathbf{A}(x_1;y_1) \times \mathbf{A}(x_{2,1};y_1), \qquad (4.12)$$

$$\vec{\alpha}_{2,3}(x_3,x_{2,2};y_2) = \mathbf{A}(x_3;y_2) \times \mathbf{A}(x_{2,2};y_2). \qquad (4.13)$$

This vector-functions for arbitrary fixed y_1, y_2 are integrable with respect to the parameters $\{x_1, x_{2,1}, x_{2,2}, x_3\}$.

By the Birkhoff Theorem the vector-functions $\vec{\alpha}_{1,2}$, $\vec{\alpha}_{2,3}$ in (4.10) admit the asymptotic limits with respect to the first group coordinates. The mean vector-functions are denoted by $\bar{\vec{\alpha}}_{1,2}(l_1, l_2)(y_1)$, $\bar{\vec{\alpha}}_{2,3}(l_2, l_3)(y_2)$, this vector-functions depend formally of the points (l_1, l_2, l_3), but, in fact, depend of the triple of magnetic lines L_1, L_2, L_3 only, $l_i \in L_i$, $i = 1, 2, 3$.

The integral kernel (4.10) is calculated algebraically and the term (4.11) is well-defined for almost arbitrary $(l_1, l_2, l_3; y_1, y_2)$. Analogously, the integral kernel $W_{3,4;2}$, corresponded to (4.8), admits the mean value over all the variables $\{\tau_1, \tau_{2,1}, \tau_{2,2}, \tau_3; \rho_{1,1}, \rho_{3,1}, \rho_{1,2}, \rho_{3,2}, \rho_{1,3}, \rho_{2,3}, \rho_{2,4}, \rho_{3,4}\}$. Denote this mean value by

$$\bar{W}(l_1, l_2, l_3; y_1, y_2). \tag{4.14}$$

The product of pairwise asymptotic linking numbers is well-defined for almost arbitrary collections of magnetic lines (L_1, L_2, L_3). The vector (4.14) is well-defined and the first step of the construction is described.

Pass to the second step of the construction and prove that the integrals (4.11), (4.14) over $\Omega^3 \times (\mathbb{R}^3)^2$ are well-defined. Estimate the total term (4.8) by a limiting tensor (Definition 2), which is absolutely integrable over the configuration space $\Omega^3 \times (\mathbb{R}^3)^2$. Denote by $a(x_1, x_{2,1}, x_{2,2}, x_3)$ the absolute value of the term (4.10) (the value $+\infty$ is admitted) after the integration over the variables y_1, y_2. Assume firstly that the points $x_1, x_{2,1}, x_{2,2}, x_3$ belong to the triple of the segments of magnetic lines, which are pairwise close to each other. Denote by δ a small parameter, which is the distance of the segment on L_2 to the segments on $\{L_1, L_3\}$ (for short we assume that the segment on L_1 is closer that the segment on L_3 to the segment on L_2).

Lemma 3. *Let $y_1 = y_2 = x_{1,1} = x_{1,2} = x_2 = x_3$, and $\omega > 0$ be a given positive (arbitrary small) number, constants $\delta_0 > \delta$ be arbitrary. Take an arbitrary non-degenerate δ-variation of the magnetic line L_1 and an arbitrary variation of the magnetic line L_3, which is estimated from above by δ and from below by δ_0. Then the absolute integral value of the term*

$$a(x_1, x_{2,1}, x_{2,2}, x_3) \tag{4.15}$$

over arbitrary ε–variations of variables $y_1, y_2, x_1, x_{2,2}, x_3$ (the variable $x_{1,1}$ is fixed) along the corresponding segments of magnetic lines is estimated by $C\delta^{-1-\omega}$, where the positive constant C depends only on ε. The constant ε depends on the norm of the 2-jets of $\mathbf{B}$ in Ω and depends no of δ.

Remark

By the results of [A4], one may replace $C\delta^{-1-\omega}$ by $C\log(\delta^{-1})$ in the lemma.

Proof of Lemma 3

To simplify the notation put $\varepsilon = 1$. The singularity in the configuration space is of the order r^{-10}, where r is the distance in $\mathbb{R}^3$ which corresponds to the parameter of deformation. This formal order includes the order -2 of the each magnetic dipole (4 dipoles), the order of the kernel in the Gauss integral, given by $\mathrm{dist}(y_1, y_2)^{-2}$. The integration of the term (4.15) is over the 6-dimensional domain of the variables y_1, y_2 and of a 3-dimensional domain, of the variables $x_{1,1}, x_{1,2}, x_2, x_3$. As the result, we get that the singularity of (4.15) is of the formal order -1.

After the deformation, described in the lemma, the term (4.15) is well-defined and integrable. To calculate this generic term, we integrate singular functions of the order r^{-6} (the coordinate r is the distance between the parameters $x_{1,1}, x_{1,2}$ on the line L_1) over 7-dimensional space. The integral is well-defined. A formal estimation (over the parameter δ) of the deformation of the singularity proves Lemma. 3.

Let us estimate W by absolute value using the lemma. Consider the cube with the edge of the length T in the configuration space, which is given by the parameter of the magnetic flow. The configuration space is a union of a finite number of small cubes. Let us define a limiting tensor of $W_{3,4;2}$ in each cube. Recall that the limiting tensor is absolutely integrable over the configuration space. and estimates the absolute value of $W_{3,4;2}$.

We start with cubes, called diagonal cubes, which are closed to top singularities, which are described in Lemma 3, up to parallel translations of all 4-points along the magnetic flow. The last cubes in the configuration space, called peripheral cubes, are defined analogously.

In each diagonal cube we get the estimation from Lemma 3. In an arbitrary peripheral cube estimations is more simple, and formally are given by the same formulas, δ_0 is not a small parameter. As the result we get that the expression (4.10) is estimated by a function of the order $\delta^{-1-\omega}$, where δ is the minimal pairwise distance between segments of magnetic lines (if there is a pair of close segments of magnetic line) and by a function of the order 1, if all the segments are pairwise non-closed.

By the Holder inequality we get:

$$\int fgdx \leq (\int |f|^q \, dx)^{\frac{1}{q}} (\int |g|^p \, dx)^{\frac{1}{p}}, \quad 1 < p < 2, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

In this inequality f is the limiting tensor for (4.10), g is the limiting tensor with logarithmic singularities for the term $(2,3)(3,1)^2(1,2)$, which is much simple. We use the denominator $p = 1 + \omega$ and a large denominator q. The function (4.14) is integrable and the main term W is given by an ergodic integral.

4.3 Examples of Magnetic Knots

We consider examples of magnetic knots with closed magnetic lines (or with magnetic lines on family of surfaces) inside compact (homogeneous) manifolds, for which M-invariant is non-vanished. The Examples I and II are generalizations with non-simply connected manifolds.

4.3.1 A one-parametric family of magnetic knots in S^3

Consider the standard singular fibration $S^3 \to S^2$ with 2 singular linked circles $S^1_1 \subset S^3$, $S^1_2 \subset S^3$, and with Hopf family of regular tori T_t, $t \in [1,2]$ between this two circles, T_1, T_2 are shrined into S^1_1 and S^1_2 correspondingly. Consider the Cartesian coordinate system (x,y,z) on $S^3 \setminus \{\infty\}$. The circle S^1_1 is the unite central circle on the plane (x,y). The circle S^1_2 is the standard vertical z-axis, $\infty \in S^1_2$, through the origin.

Define a real parameter r, $1 \leq r \leq 2$. Define a r-parameter family of magnetic knots Υ_r in S^3. Magnetic lines of Υ_r for each $t \in]1,2[$ are on T_t and wind 1 time along the S^1_1–parallel of T_t and r times along the S^1_2-meridian of T_t. For rational r, the magnetic knot Υ_r consists of closed lines. The magnetic knots Υ_{r_1}, Υ_{r_2}, in the case $r_1 \neq r_2$, are not equivalent with respect to volume-preserved diffeomorphisms of S^3. In the case $r = 1$ we get the standard Hopf fibration with fibers along the standard Hopf mapping $h : S^3 \to S^2$. For $r = 2$ we get a Kamchatnov-Hopf soliton.

The combinatorial formula of the invariant $M(\mathbf{L})$, in the case $\mathbf{L}$ is a 3-component link, is well-defined (probably, up to a sum with a polynomial $P((1,2),(2,3),(3,1))$, which depends on pairwise linking numbers of $\mathbf{L}$, see [A3]). In the case $(1,2) = (2,3) = (3,1)$, to keep asymptotic properties of M, assume that $\deg(P(k)) \leq 11$. We define $\dot{M} = M + P$, the invariant $\dot{M}$ is ergodic. Moreover, without lost of a generality we assume that $\dot{M}$ is trivial on a prescribed collection of the following 2 simplest links $\mathbf{L}_{1,1,1}$, $\mathbf{L}_{2,2,2}$.

The link $\mathbf{L}_{1,1,1}$ consists of 3 magnetic lines with pairwise linking number 1, each line is a fiber of the Hopf fibration $h : S^3 \to S^2$. By the construction, $\mathbf{L}(1) = \mathbf{L}_{1,1,1}$, where $\mathbf{L}(1)$ is the link, which is defined by an arbitrary ordered magnetic lines of the magnetic knot Υ_1.

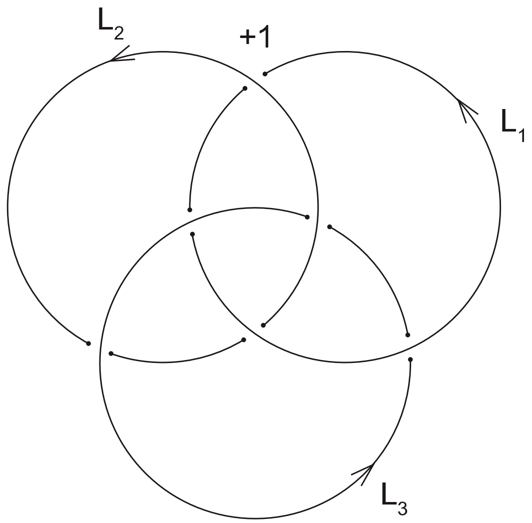

Figure 4.2: Link $\mathbf{L}(1) = \mathbf{L}_{1,1,1}$.

The link $\mathbf{L}_{2,2,2}$ is defined as following. Take the symmetric triangle with the unite edges on the plane. Take 3 circles L'_1, L'_2, L'_3 of the radius $\frac{1}{2}$ around its vertexes, which are tangent to each other in the centers of edges. Then take a small 3D deformation of $(L'_1, L'_2, L'_3) \to (L_1, L_2, L_3)$ in small neighborhoods of tangent points of the pairs (L'_1, L'_2), (L'_2, L'_3), (L'_3, L'_1); as the result we assume that the pairwise linking numbers of (L_1, L_2), (L_2, L_3), (L_3, L_1) are equal to $+2$.

Denote by $\mathbf{L}(2)$ a 3-component link, which is defined by an arbitrary ordered triple of generic magnetic lines of the magnetic knot Υ_2 It is not difficult to prove that pairwise linking numbers of $\mathbf{L}(2)$ and $\mathbf{L}_{2,2,2}$ coincide.

By the construction $\mathbf{L}_{2,2,2}$ is distinguished from $\mathbf{L}(2)$ by the commutator of 3-components (or, equivalently, by the Δ-moves of 3 components; for Δ-moves see Appendix II). By the following lemma and the combinato-

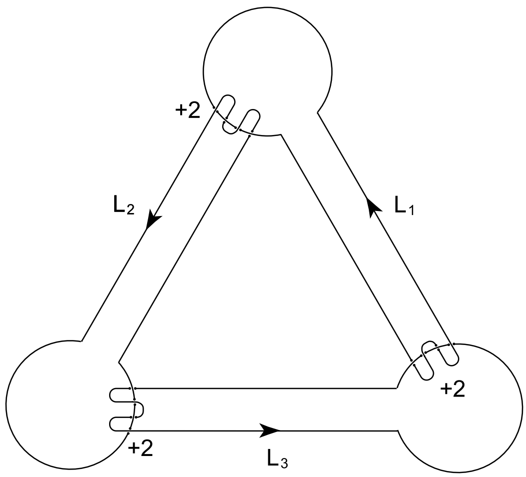

Figure 4.3: Link $\mathbf{L}_{2,2,2} = Hopf^{-}(2,2,2)$.

rial formula of M from [[A3](17)], the value $M(\mathbf{L}(2))$ is distinguished from $M(\mathbf{L}_{2,2,2})$ by a non-zero integer.

Lemma 4. *Let $\mathbf{L} = (L_1 \cup L_2 \cup L_3)$ be an arbitrary 3-component link for which the pairwise linking coefficients $(1,2), (2,3), (3,1)$ are even. Let $\mathbf{L}' = (L'_1 \cup L'_2 \cup L'_3)$ be the 3-component link, which is the result of a Δ–move of $\mathbf{L}$ with 3 different components.*

The parity of the coefficients $C_2(\mathbf{L})$, $C_2(\mathbf{L}')$ of the Conway polynomial are distinguished, and the invariants $Arf(\mathbf{L})$, $Arf(\mathbf{L}')$ are distinguished.

Remark 1. *The invariant $Arf(\mathbf{L})$ is well-defined in a less restrictive case, when all the pairwise linking numbers of $\mathbf{L}$ are odd.*

Proof of Lemma 4

For a link $\mathbf{L} = (L_1 \cup L_2 \cup L_3)$ which satisfies the lemma, the equation $\mu^2_{123}(\mathbf{L}) \equiv C_2(\mathbf{L}) \pmod{GCD(1,2),(2,3),(3,1)}$ is proved in [Me] (in the case $(1,2)^2 + (2,3)^2 + (3,1)^2 = 0$ the integer equation $\mu^2_{123}(\mathbf{L}) \equiv C_2(\mathbf{L})$ is proved in [Co],Theorem 5.1). The equation $Arf(\mathbf{L}) \equiv \mu_{123}(\mathbf{L}) \pmod 2$ is proved using the Gauss diagrams as in [M-P]. Arf-invariant satisfy the lemma. Lemma 4

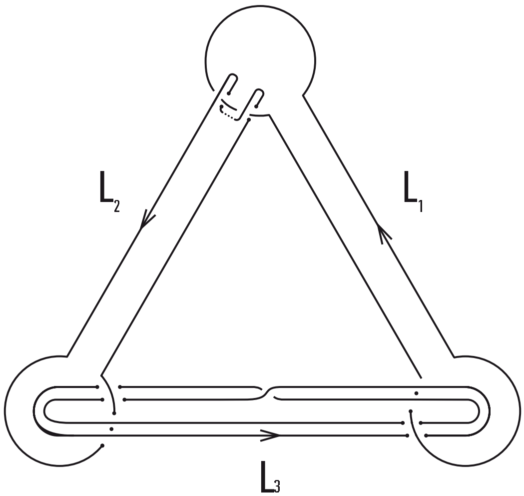

Figure 4.4: Link $\mathbf{L}(2) \neq \mathbf{L}_{2,2,2}$.

is proved.

The invariant M is ergodic, therefore $M(\mathbf{L}(r))$ is continuously varied from $M(\mathbf{L}(1)) = 0$ to $M(\mathbf{L}(2)) \neq 0$, $1 \leq r \leq 2$.

4.3.2 Examples of magnetic knots in rational homology spheres

Example I

In the group of unit quaternions $S\mathbb{H}$ consider the subgroup of integer quaternions $\mathbf{Q} \subset S\mathbb{H}$

$$\{\mathbf{i}, \mathbf{j}, \mathbf{k} \quad | \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\}.$$

Consider the standard (right) action $\mathbf{Q} \times S^3 \to S^3$, which is well-defined because of the diffeomorphism $S\mathbb{H} \cong S^3$. Consider the 2-sheeted covering $S\mathbb{H} \to SO(3)$, the image of the subgroup $\mathbf{Q} \subset S\mathbb{H}$ is the Klein subgroup $\mathbf{K} \subset SO(3)$, $\mathbf{K} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. The Klein group acts on S^2, this action is induced by the standard projection $SO(3) \to S^2$, the action has 6 fixed points, which are the intersection points of the standard unite sphere $S^2 \subset \mathbb{R}^3$ with the coordinate axis. The elements of $\mathbf{K}$ acts on S^2 by rotations trough the angle π with respect to the corresponding coordinate axis.

The following commutative diagram of groups

$$\begin{array}{ccc} \mathbf{Q} \times S^3 & \to & S^3/\mathbf{Q} \\ \downarrow & & \downarrow \\ \mathbf{K} \times S^2 & \to & S^2/\mathbf{K}, \end{array} \tag{4.16}$$

is well-defined. In this diagram horizontal maps are projections onto the orbits of the action, the left vertical mapping is the Cartesian product of the projection $\mathbf{Q} \to \mathbf{K}$ and the composition $S^3 \cong S\mathbb{H} \to SO(3) \to S^2$, which coincides with the standard Hopf fibration, the right vertical mapping is induced from the left vertical mapping by the projection onto the orbits.

The magnetic knot in $S^3/\mathbf{Q}$ with closed magnetic lines is well-defined by fibers of the right vertical mapping in the diagram.

A generalization of Example I

The diagram (6.59) is included into the following diagram:

$$\begin{array}{ccc} \Sigma \times S^3 & \to & S^3/\Sigma \\ \downarrow & & \downarrow \\ \mathbb{I} \times S^2 & \to & S^2/\mathbb{I}. \end{array} \tag{4.17}$$

In this diagram $\mathbf{Q} \subset \Sigma$ is the Poincaré extension of the index 15 of the integer quaternions to the fundamental group of the integer homology sphere, $\mathbf{K} \subset \mathbb{I}$ is the extension of the Klein group to the icosahedron group, the lower horisonatal mapping of the diagram is a free action, the bottom mapping is the semi-free action. By the Poincar'e-Klein uniformization [K], the icosahedron group $\mathbb{I}$ is covered by the modular group $PSL(2,\mathbb{Z})$, which acts on the half-plane.

Below in the diagram (6.2) a quadratic extension $\mathbf{Q} \subset \aleph$ is well-defined. The quadratic extension $\mathbf{Q} \subset \aleph$ is mapped into a quadratic extension $\mathbf{K} \subset \mathbf{D}$ by the projection onto the factorgroup, where $\mathbf{D}$ is the dihedral group of the order 8.

The inclusion $\mathbf{K} \subset \mathbb{I}$ admits no extension of the quadratic extension $\mathbf{K} \subset \mathbf{D}$ of the subgroup to a quadratic extension of the group $\mathbb{I}$. The minimal infinite-order extension $\mathbb{I} \subset \Upsilon$ is well-defined, where Υ is a Kleinian group, which acts conform on $\hat{\mathbb{C}}$, and this action extends the action of the Fuchsian group. The group Υ is covered by a group, which acts conform on the half-plane.

Conjecture

The Poincarè-Klein uniformization mapping translates geodesic flows on orbifolds with negative scalar curvature (see [D-P]) into non-homogenouse generalized Kamchatnov-Hopf solitons in rational homology sphere. M invariant for a magnetic knot, which is C^2–closed to generalized Kamchatnov-Hopf soliton is well-defined end explicitly calculated.

Example II

The standard Hopf fibration $h : S^3 \to S^2$, is given by the formula $\{(z_1,z_2)\}, |z_1|^2+|z_2|^2=1$,

$$h : (z_1,z_2) \mapsto \frac{z_1}{z_2}.$$

The conjugated Hopf fibration $\bar{h} : S^3 \to S^2$ is given by the formula

$$\bar{h} : (z_1,z_2) \mapsto \frac{\bar{z}_1}{z_2}.$$

The following diagram

$$\begin{array}{ccc} \mathbf{Q} \times S^3 & \to & S^3/\mathbf{Q} \\ \downarrow & & \downarrow \\ \mathbb{Z}/2 \times S^2 & \to & \mathbb{RP}^2, \end{array}$$

is well-defined, where $\mathbf{Q} \to \mathbb{Z}/2$ is the epimorphism with the generator $\mathbf{i}$ is the kernel, $\mathbb{Z}/2 \times S^2 \to \mathbb{RP}^2$ is the projection of the antipodal involution, see [S].

Define the magnetic knot on $S^3/\mathbf{Q}$ by the fibers of $\bar{h}$. An arbitrary magnetic line $L \subset S^3/\mathbf{Q}$ of the magnetic knot is not an non-oriented boundary. The involution $[\mathbf{j}] : S^3/[\mathbf{i}] \to S^3/[\mathbf{i}]$ inverse the magnetic lines, the Seifert surface of the magnetic potential in $S^3/\mathbf{Q}$ is non-oriented. For a non-oriented Seifert surface the Arf-Brown invariant mod 8 is well-defined. The magnetic knot is investigated in [Z].

The group $\mathbf{Q}$, which is the fundamental group of the rational homology sphere $S^3/\mathbf{Q}$ admits a quadratic extension $\mathbf{Q} \subset \aleph$, which is defined below by (6.2). By this extension the image of the generator $\mathbf{i} \in \mathbf{Q}$ in $\aleph$ belongs to the commutant $[\aleph,\aleph] \subset \aleph$. A Seifert surface for L is well-defined as a surface with a prescribed normal bundle structure (see below the definition of this structure in Theorem 8) with a control to the Eilenberg-MacLane space $K(\aleph,1)$.

The simplest example of magnetic knot which corresponds to this extension is given by a double diffeomerphic copies of $S^3/\mathbf{Q}$, defined by Example II, the orientation of the second copy is opposite. The two copy are permuted by an element in $\aleph \setminus \mathbf{Q}$. For Seifert surfaces with prescribed normal bungle structures the hyperquaternionic Arf-invariant is well-defined as an integer (mod 16). A description of the normal bundle structure of the Seifert surfaces for Example II is given in the Appendix.

Examples I, II of magnetic knots on $S^3/\mathbf{Q}$ assume that asymptotic ergodic M-invariant is generalized for magnetic knots in rational homology spheres. The parity of C_2-coefficient of the Conway polynomial for classical links in $\mathbb{R}^3$ corresponds to the Arf-invariant. In the next section we determine a group W, which is called the Witt group of hyperquaternionic forms. The reason to introduce the hyperquaternionic Arf-invariant is clarify by the following diagram:

$$\begin{array}{ccc}
C_2 \ of\ the\ Conway\ polynomial & \longrightarrow & \begin{array}{c} Arf\ invariant \\ of\ classical\ links \end{array} \\
\downarrow & & \downarrow \\
? & \longrightarrow & \begin{array}{c} hyperquaternionic\ Arf - invariant \\ of\ links\ in\ S^3/\mathbf{Q}. \end{array}
\end{array}$$

In the diagram by ? is denoted a hypothetic integer-valued finite-type invariant of links in rational homology spheres, which determines asymptotic ergodic invariants.

4.3.3 Conjecture

M is generalised for 3-component oriented links in non-simply connected rational homology sphere and the order (in the sense of V.A.Vassiliev) of the generalized invariant M depends on the fundamental group π of the homology sphere and for $\pi \cong \mathbb{Z}/p^n$ is $O(n)$.

Chapter 5

Conclusion

V.I.Arnol'd formulated the problem [[Arn], Problem 1984-12]: "To transform asymptotic ergodic definition of the Hopf invariant of divergence-free vector fields to the theory of S.P.Novikov, which generalize the Whitehead product of homotopy groups of spheres"'.

Algebraic commutators, which are used to define the higher invariants of $3D$-links, are particular Whitehead products in homotopy groups of spheres. M-invariant is a special generalized Whitehead product, which admits asymptotic and ergodic property. To keep additional symmetry of magnetic fields we have to apply the M-invariant for links in various homogeneous manifolds, which are rational homology spheres. For links in the standard sphere M-invariant is related with the Arf-invariant in the stable homotopy group Π_2. Hypothetic modifications of M-invariant for links in $S^3/\mathbf{Q}$ are associated with Arf-Brown invariant in the stable homotopy group Π_3, and with hyperquaternionic Arf-invariant in the stable homotopy group Π_7. Stable homotopy groups are detected by Postnikov k-invariants. The constructions give a solution (in part) of the Arnol'd Problem.

A first-order commutators of component of links is the linking number. A high-order momenta of linking numbers for magnetic lines are called q-monomial helicities, is introduced in Theorems 2 and 3. In the case $q = 1$ we get the well-known magnetic helicity, see, for example, [A-Kh]. In the case $q = 1$ we get the quadratic helicity, defined in [A2]. The quadratic helicity is the dispersion of the magnetic helicity density. In the cases $q \geq 3$ we get higher momenta of the magnetic helicity. In [Y-H] the authors observed that the flux function is an action in the Hamiltonian formulation of the field line equations (see also [A-Kh] Ch. IV, paragraph 8). This shows that quadratic helicity contains statistical meaning, and is related with extremal of the magnetic energy.

The construction of q-monomial helicity is a straightforward application of Theorem 1. The theorem itself gives much more possibilities to define higher topological invariants of magnetic fields.

The proof of Theorem 2 contains a new idea: smoothing of the integral kernel in the Gauss integral, given by (3.16). The proof of Theorem 3, based on Lemma 2, also contains a new idea: to replace short paths of an open long magnetic line by pairs monopole–antimonopole as in [A2], formula (7). This means that instead of a short path, joined the ends of a magnetic line, the dipole, associated with the end points of the line, is introduced. A short path is replaced by the collection of the magnetic lines of the dipole. This trick gives a smoothing of end-singularities of a corresponding segment of a magnetic line. There in no explicit integral formula for q-monomial magnetic helicity, for $q \geq 2$.

Using an ergodic definition in [A2] is proved that the quadratic helicity depends continuously of magnetic fields $\mathbf{B}$ inside a finite collection of magnetic tubes with respect to C^2 topology (but not with respect to C^1 topology, see [Kudr]). This gives applications for the induction equation with the α-term and with the diffusion term.

Chapter 6

Appendix

6.1 Hyperquaternionic Arf-invariant

6.1.1 Arf-invariants of immersed surfaces

Consider an immersion $\varphi : M^2 \looparrowright \mathbb{R}^3$ of a closed, generally speaking, non-oriented surface into $\mathbb{R}^3$. The immersion φ up to regular cobordism represents an element of the group denoted by $Imm^{sf}(2,1)$. In this section we use notations from preprints by O.D.Frolkina and the author (2016).

The Arf-Brown invariant is an isomorphism

$$\Theta : Imm^{sf}(2,1) \to \mathbb{Z}/8.$$

Denote $Imm^{sf}(2,1)$ by V for short (an algebraic definition of $\Theta : V \cong \mathbb{Z}/8$, using $\mathbb{Z}/4$-quadratic forms, is in [G-M]). If M^2 is an orientable surface, the element $\Theta([\varphi])$ belongs to the subgroup $\mathbb{Z}/2 \subset \mathbb{Z}/8$. In this case the element $\frac{\Theta([\varphi])}{4} \pmod 2$ is called the Arf-invariant of $[\varphi]$.

Let K^3 be a closed oriented 3-dimensional manifold. Assume that a trivialization of the tangent bundle $\Psi : T(K^3) \cong 3\varepsilon$ is fixed. Assume that an immersion $\varphi : M^2 \looparrowright \mathbb{R}^3$ of a closed surface is given. The immersion φ represents an element $[\varphi]$ in the group V and the Arf-Brown invariant $\Theta([\varphi])$ is well-defined.

In the case, when M^2 is a surface with a boundary, assume that each component of the immersed curve $\varphi(\partial M^2)$ has the trivial stable Hopf invariant (= an even self-linking number). In this case the Arf-Brown invariant $\Theta([\varphi])$ is well-defined.

6.1.2 Group $\aleph$ of the order 16

Consider the cyclic group C_8 of the order 8, $C_8 = \{\exp\left(\frac{k\pi\mathbf{i}}{4}\right) \quad |k \in \mathbb{Z}/8\}$. Denote by $\theta : C_8 \to C_8$, $\theta : S \mapsto S^3$, $S \in C_8$ the cubing automorphism. Let us define a group $\aleph$ of the order 16, by attaching an element T of the order 2 by the equation $TST = S^3$, see for details [C-M], Ch.1 1.8. The following short exact sequence:

$$0 \to C_8 \to \aleph \to \mathbb{Z}/2 \to 0 \tag{6.1}$$

is well-defined. In this sequence the left mapping is the inclusion on the subgroup, the right mapping is the projection onto. Denote by $T\mathbf{j} \in C_8$ a generator of the subgroup $C_8 \subset \aleph$; denote by $\mathbf{j} \in \aleph$ the element $T(T\mathbf{j})$; denote by $\mathbf{k} \in \aleph$ the element $T\mathbf{j}T$; denote by $-1 \in C_8 \subset \aleph$ the element $(T\mathbf{j})^4$, denote by $-\mathbf{i}$ the element $(T\mathbf{j})^2 = \mathbf{kj}$.

Define the following short exact sequence

$$0 \to \mathbf{Q} \to \aleph \to \mathbb{Z}/2 \to 0, \tag{6.2}$$

where $\mathbf{Q}$ is the integer quaternions subgroup. The group $\mathbf{Q}$ is of the order 8, this group admits the following standard corepresentation:

$$\{\mathbf{i},\mathbf{j},\mathbf{k} \quad | \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\},$$

which corresponds to the notations of the generators.

Representation $\Phi : \aleph \to S\mathbb{O}(4)$

Define a $S\mathbb{O}(4)$–representation $\Phi : \aleph \to S\mathbb{O}(4)$ by the following matrices:

$$\Phi(T) = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix} \qquad \Phi(T\mathbf{j}) = \begin{matrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \tag{6.3}$$

The elements $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are given by the following matrices:

$$\Phi(\mathbf{i}) = \begin{matrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{matrix} \qquad \Phi(\mathbf{j}) = \begin{matrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{matrix} \qquad \Phi(\mathbf{k}) = \begin{matrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \tag{6.4}$$

The representation $\phi = \Phi|_{\mathbf{Q}} : \mathbf{Q} \to S\mathbb{O}(4)$ is equivalent to the standard representation $\mathbf{Q} \to S\mathbb{H} \subset S\mathbb{O}(4)$.

The octahedral extension $\aleph \subset \Upsilon$ of the index 3

Let us unify short exact sequences (6.1), (6.2) into the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & \mathbf{K} & \subset & \mathbf{D} & \to & \mathbb{Z}/2 & \to & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \to & \mathbf{Q} & \subset & \aleph & \to & \mathbb{Z}/2 & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathbb{Z}/2 & \cong & \mathbb{Z}/2 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \qquad (6.5)$$

In this diagram by $\mathbf{D}$ is denoted the dihedral group of the order 8, the projection $\aleph \to \mathbf{D}$ extends the reduction $C_8 \to C_4$ of the cyclic subgroup modulo 4, $\mathbf{K} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \subset \mathbf{D}$ is the Kleinian group, $\mathbf{Q} \to \mathbf{K}$ is the natural epimorphism, which is the projection onto the central quotient $\{\pm 1\} \subset \mathbf{Q}$. The group $\mathbf{K}$ is equipped with the representation $\mathbf{K} \to S\mathbb{O}(3)$, the image of the corresponding element $[\mathbf{i}]$, $[\mathbf{j}]$, $[\mathbf{k}]$ is the rotation trough the angle π with respect to the axis, which is perpendicular to the coordinate plane $P_{\mathbf{i}}, P_{\mathbf{j}}, P_{\mathbf{k}}$ in $\mathbb{R}^3$ correspondingly.

The group $\mathbf{D}$ is equipped with the representation $\tilde{\lambda} : \mathbf{D} \to \mathbb{O}(3)$, the element $[T] \in \mathbf{D}$, which is define as the image of the element $T \in \aleph$ by the projection $\aleph \to \mathbf{D}$, is represented by symmetry with respect to the plane, which is perpendicular to $P_{\mathbf{i}}$, along the bisector of the coordinate planes $P_{\mathbf{j}}$ and $P_{\mathbf{k}}$. The representation $\tilde{\lambda}|_{\mathbf{K}} = \lambda$ is defined such that the representation $\phi : \mathbf{Q} \to \mathbb{S}^3 \subset S\mathbb{O}(4)$ covers the representation λ by the projection $S^3 \to S\mathbb{O}(3)$.

The representation $\tilde{\lambda} : \aleph \to \mathbb{O}(4)$ is the quadratic extension of the representation λ by the standard quadratic extension $S\mathbb{O}(3) \subset \mathbb{O}(3)$.

Define the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathbb{Z}/3 \tilde{\times} \mathbf{K} & \subset & \mathbb{Z}/3 \tilde{\times} \mathbf{D} & \to & \mathbb{Z}/2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathbb{I} & \subset & \Upsilon & \to & \mathbb{Z}/2 & \to & 0
\end{array} \qquad (6.6)$$

The group $\mathbb{Z}/3 \tilde{\times} \mathbf{K}$ is a semi-direct product of the subgroups $\mathbf{D}$, $\mathbb{Z}/3$ in the icosahedron group. The subgroup $\mathbb{Z}/3$ permutes the images of quaternion units $[\mathbf{i}]$, $[\mathbf{j}]$, $[\mathbf{k}]$ in $\mathbf{K}$. The inclusion $\mathbb{Z}/3 \tilde{\times} \mathbf{K} \subset \mathbb{I}$ into the icosahedron group is of the index 5. The group Υ is the fundamental group of the homology Poincaré sphere. The inclusion $\mathbb{Z}/3\tilde{\times}\mathbf{D} \subset \Upsilon$ is the quadratic extension of the inclusion $\mathbb{Z}/3\tilde{\times}\mathbf{K} \subset \mathbb{I}$.

Lemma 5. *–1. Diagram* (6.6) *is well-defined and contains the diagram* (6.5) *as a subdiagram.*

–2. The groups Υ, $\mathbf{B}$ are equipped with representations $M : \Upsilon \to S\mathbb{O}(4)$, $\mu : \mathbb{Z}/3\tilde{\times}\mathbf{D} \to \mathbb{O}(3)$, the representations M, μ extend the representations Φ, $\tilde{\lambda}$ correspondingly.

6.1.3 The Witt group W of hyperquaternionic forms

Define the regular cobordism group of closed surfaces, the elements of W will be called hiperquaternionic forms. Denote this group by W, from algebraic point of view, W is a Witt group of special quadratic forms.

Define an epimorphism $\alpha : \aleph \to \mathbb{Z}/2 = \{\pm 1\}$ by the following formula: $T\mathbf{j}, T \in \aleph$, $\alpha(T\mathbf{j}) = -1, \alpha(T) = +1$. The kernel $Ker(\alpha)$ coincides with the dihedral subgroup $\mathbf{D} \subset \aleph$.

Define an epimorphism $\beta : \aleph \to \mathbb{Z}/2 = \{\pm 1\}$ by the following formula: $\beta(T\mathbf{j}) = -1, \beta(T) = -1$. The kernel $Ker(\beta)$ coincides with the quaternion subgroup $\mathbf{Q} \subset \aleph$.

Over the space $B\aleph = K(\aleph, 1)$ the canonical vector $S\mathbb{O}(4)$–bundle is well-defined, the structure group of the canonical bundle is defined by the representation $\Phi : \aleph \to S\mathbb{O}(4)$, denote this universal bundle by A. Denote by γ the line canonical bundle over $B\mathbb{Z}/2 \cong \mathbb{P}^{\infty}$. Denote by $\alpha : B\aleph \to B\mathbb{Z}/2$ the mapping of the classifing spaces, which is associated with the homomorphism α, denote by $\beta : B\aleph \to B\mathbb{Z}/2$ the mapping, which is associated with the homomorphism β.

A triple (M^2, η_M, Ξ_M) is called a hyperquaternionic form, where

• M^2 is a closed, generally speaking, non-orientable surface;

• $\eta_M =: M^2 \to B\aleph$ is a characteristic class, the composition $\alpha \circ \eta_M$ is denoted by $\eta_{\alpha;M} : M^2 \to B\mathbb{Z}/2$, the composition $\beta \circ \eta_M$ is denoted by $\eta_{\beta;M} : M^2 \to B\mathbb{Z}/2$;

• Ξ_M is the isomorphism $T(M) \oplus \eta_{\alpha;M}(\gamma) \oplus \eta^*_M(A) \oplus 3\eta_{\beta;M}(\gamma) \cong 10\varepsilon$, where by ε is the trivial line bundle.

In particular, by definition of Ξ_M, the characteristic class $\eta_{\alpha;M} + \eta_{\beta;M} : M^2 \to B\mathbb{Z}/2$ corresponds to the orientation homomorphism $H_1(M;\mathbb{Z}/2) \to \mathbb{Z}/2$, (denote $\eta_{\alpha;M} + \eta_{\beta;M} = \kappa_M : M^2 \to B\mathbb{Z}/2$, this characteristic class coincides with the characteristic Stiefel-Whitney class $w_1(M)$).

On a set of all hyperquaternionic form an additive operation by a disjoint union is well defined. The standard regular cobordism relation determines an equivalence relation of quadratic hyperquaternionic forms. The cobordism group up to this equivalence relation is denoted by W, this is the required Witt group.

Definition 3. *A hyperquaternionic form (M^2, η_M, Ξ_M), for which the characteristic mapping η takes values in the subspace $B\mathbf{Q} \subset B\aleph$, is called a quaternionic form.*

Theorem 8. *The group W contains a cyclic subgroup $P \subset W$ of the order 16, $P \cong \mathbb{Z}/16$.*

Definition 4. *Define a subgroup $W_{\mathbf{Q}} \subset W$ in the Witt group as the group, which is generated by quaternionic forms. Define the group $W^{\odot}_{\mathbf{Q}}$, which is called the Witt group of quaternion forms. The group $W^{\odot}_{\mathbf{Q}}$ is generated by quaternion forms, the regular cobordism relation for this group assumes the following additional property:*

- *the structure mapping on a cobordism manifold admits a prescribed reduction to a mapping with the image in the quaternion classifying subspace $B\mathbf{Q} \subset B\aleph$.*

By the construction, the canonical projection $p : W^{\odot}_{\mathbf{Q}} \to W_{\mathbf{Q}}$ is well-defined.

6.1.4 The Arf-Brown homomorphism the group $W_{\mathbf{Q}}$ onto the Witt group of $\mathbb{Z}/4$–quadratic forms

Denote by V the Witt group of $\mathbb{Z}/4$–quadratic forms with Arf-Brown invariants. This group is related with the Rokhlin's Signature Theorem, see [G-M]. The group V is the cyclic group of the order 8. Define the forgetful homomorphism

$$\rho^{\odot} : W^{\odot}_{\mathbf{Q}} \to V$$

from the Witt group of quaternionic forms into the Witt group of quadratic $\mathbb{Z}/4$–forms as following.

Let (M^2, η, Ξ) be a quaternionic form represented an element in $W^{\odot}_{\mathbf{Q}}$. Consider the standard 3-skeleton $S^3/\mathbf{Q} \subset B\mathbf{Q}$, which is represented by the standard quaternion lens space. The pull-back of the bundle A over $B\aleph$ with respect to the inclusion $S^3/\mathbb{I} \subset B\mathbf{Q} \to B\aleph$ is denoted by $A_{S^3/\mathbf{Q}}$. The canonical isomorphism $A_{S^3/\mathbf{Q}} \cong 4\varepsilon$ of the vector bundles over $S^3/\mathbf{Q}$ is well-defined. The pull-back isomorphism $\eta^*(A_{S^3/\mathbf{Q}}) \cong 4\varepsilon$, determines the isomorphism $\Xi_M : \nu_M \to 7\varepsilon \oplus \kappa$, where ν_M is the stable normal bundle over M^2. Define $\rho^{\odot}([(M^2, \eta_M, \Xi_M)]) \in V$ by the formula: $\rho^{\odot}(M^2, \eta_M, \Xi_M) = (M^2, \Xi_M)$, $[(M^2, \Xi_M)] \in V$.

Lemma 6. *The homomorphism* $\rho^{\odot}: W^{\odot}_{\mathbf{Q}} \to V$ *is decomposed as following:*

$$\rho^{\odot} = \rho \circ p : W^{\odot}_{\mathbf{Q}} \to W_{\mathbf{Q}} \to V,$$

where the homomorphism $\rho : W_{\mathbf{Q}} \to V$ *is well-defined and is an epimorphism onto the index* 2 *subgroup in* V *of elements of the order* 4.

Proof of Lemma 6

Consider the standard transfer homomorphism with respect to the subgroup $\mathbf{Q} \subset \aleph$, denote the transfer homomorphism by $! : W \to W^{\odot}_{\mathbf{Q}}$. The following lemma is required.

Lemma 7. *The image of the transfer homomorphism* $! : W \to W^{\odot}_{\mathbf{Q}}$ *is inside the kernel* $Ker \rho^{\odot}$.

Proof of Lemma 7

A given arbitrary hyperquaternionic form (M^2, η_M, Ξ_M), is represented by a connected surface. Take a geometrical stabilization of the surface M^2 by a connected sum with 2 mirror copies of Moebius bands, the generators of the bands are represented by the element T (we say that a band of the considered type is a T-band). Denote the result of the stabilization again by (M^2, η_M, Ξ_M). As the result, the surface M^2 is a connected sum of Moebius bands, which are represented by the elements $\mathbf{j}$, or by $\mathbf{k}$ (we say that a band of the considered type is a quaternion band).

Take the decomposition of M^2 into a connected sum of Moebius bands with the only T-band and several quaternion bands. By the transfer homomorphism M^2 is covered by (a non-oriented) surface $\tilde{M}^2$. A T-band in the decomposition of M^2 is transformed into a cylinder on $\tilde{M}^2$, the generator $\tilde{l} \subset \tilde{M}^2$ of the cylinder is a closed loop on corresponds to the double covering over the generator $l \subset M^2$ of the T-band, the Hopf invariant $h(\tilde{l}) \in \mathbb{Z}/2$ of $\tilde{l}$ loop is trivial. The each quaternion band on M^2 is covered by a pair of quaternion mirror-symmetric bands on $\tilde{M}^2$. This proves that the image $(M^2, \eta_M, \Xi_M)^!$ in V is trivial. Lemma 7 is proved.

The last part of the proof of Lemma 6 is following. Let (M^2, η_M, Ξ_M) represents an arbitrary element in $W_{\mathbf{Q}}$. Consider the manifold P^3 with boundary $\partial P^3 = M^2$, the manifold is equipped with a normal bundle structure (P^3, ζ_P, Ψ_P), this structure determines a boundary of the form (M^2, η_M, Ξ_M). Denote by $Q^2 \subset P^3$ a surface, which is defined as a dual surface to ζ_β. Obviously, there exists a closed characteristic surface, because $\zeta_{\beta;P}|_{\partial P^3}$ is null-homotopic. Then $(Q^2, \zeta_P|_Q, \Psi_P|_Q)$ determines an element $x \in W$, the transfer $x^!$ belongs to $Ker(\rho^{\otimes})$ by Lemma 7. By the construction, $\rho^{\otimes}[(M^2, \eta_M, \Xi_M)]$ coincides with $x^! = (Q^2, \zeta_P|_Q)^!$ in V. Lemma 6 is proved.

Proof of Theorem 8

Let us define a hyperquaternionic form (M^2, η_M, Ξ_M). Consider a pair of Moebious bands $(\mu_i, \partial) \subset M^2, i = 1, 2$, the generators of μ_1, μ_2 is represented by η_M into the elements TJ, T correspondingly. The connected sum $(\mu_1, \partial)\sharp(\mu_2, \partial)$ along the common boundary $\partial\mu_1 = \partial\mu_2$ coincides to M^2. The characteristic mapping η_M admits a reduction: $\eta_M : M^2 \to B\mathbf{D} \subset B\aleph$.

By the construction, M^2 contains a thin cylinder $C_J \subset M^2$, the (orientation preserved) loop $l_J \subset C_J$ which corresponds to the element $J \in \mathbf{D} \subset \aleph$ by η_M. The surface $M^2 \setminus C_J$ is diffeomorphic to the cylinder C_{-J}, this cylinder is a non-oriented cycle between the two copies of ∂C_J. Denote the segment of the cylinder C_{-J}, which is transversal to the central line of C_{-J} by $l_T \subset C_{-T}$. Extend the segment $l_T \subset C_{-T} \subset M^2$ by a closed loop on M^2 by a short path, which is transversal to l_J. This closed path is denoted by $l_T \subset M^2$. The closed path $l_{JT} \subset M^2$ are defined as the central path in $M^2 \setminus l_T$. The paths l_T, l_{JT} coincide with central lines the the Moebious bands μ_1, μ_2 on M^2.

The (orientation reversed) loop $l_T \subset M^2$ corresponds to the element $T \in \mathbf{D} \subset \aleph$ by η_M. The element $\eta_M(l_T^{-1} \circ l_J \circ l_T \circ l_J)$ is the trivial element in $\mathbf{D} \subset \aleph$, because $[T, J] = -1$. Informally speaking, the Klein bottle M^2 is the result of a non-oriented self-homology of l_J by l_T.

Describe a regular cobordism of $2(M^2, \eta_M, \Xi_M)$ into a form (L^2, η_L, Ξ_L), where L^2 is the Klein bottle, which is defined analogously to M^2. Take the orientation preserving loop $l_{-1} \subset C_{-1} \subset L^2$, which represents the element $J^2 = -1 \in \mathbf{D} \subset \aleph$ by η_L. The loop l_{-1} is the analog of the loop $l_J \subset M^2$. Define the orientation reversed loop, which is analog of the loop $l_T \subset M^2$. Denote the corresponding cycle of l_{-1} by S_1, denote the corresponding cycle of l_T by S_2.

Lemma 8. *The form $2(M^2, \eta_M, \Xi_M)$ is equivalent to the form (L^2, η_L, Ξ_L) (probably, up to an element of the order 2 in W).*

Describe a regular cobordism of $2(L^2, \eta_L, \Xi_L)$ into a form (K^2, η_K, Ξ_K), where K^2 is the Klein bottle, as in the case of M^2 and L^2. Denote the orientation preserved cycle $R_2 \subset C \subset K^2$, which is the analog of the cycle $S_2 \subset C_{-1} \subset L^2$ and which is represented into the trivial element in $\aleph$, by η_K.

In this formula C in a thin cylinder, which is the analog of the cylinder C_{-1}. Denote the orientation reversed cycle $l_T \subset \mu_1 \subset K^2$ by R_1.

Recall that an immersion $f : K^2 \looparrowright \mathbb{R}^{10}$ with the prescribed isomorphism $\Xi_K : \nu_K \cong \eta_K^*(A) \oplus \eta_{\alpha;K}^*(\gamma) \oplus 3\eta_{\beta;K}^*(\gamma)$ of the normal bundle, where A is the universal 4-bundle over the subspace $B\mathbb{Z}/2(-1) \times B\mathbb{Z}/2(T) \subset B\aleph$, γ is the universal line bundle, is well-defined. The element $\eta_K(R_1)$ is the trivial element in $\aleph$. Moreover, the mapping $\eta_K(R_1)$ has the target a point in $B\aleph$.

The curve $f(R_1)$ is a framed curve in $\mathbb{R}^{10}$ and the stable Hopf invariant $h(R_1) \in \mathbb{Z}/2 = \{0, 1\}$ is well-defined.

Lemma 9. *The form $2(L^2, \eta_L, \Xi_L)$ is equivalent, probably, up to an element of the order 2 in W, to a form (K^2, η_K, Ξ_K), where the oriented framed loop R_1 has the Hopf invariant $h(R_1) \neq 0, h(R_1) \in \mathbb{Z}/2$.*

Proof of Lemma 8 and Lemma 9

Proofs of Lemmas are analogous. Let us prove Lemma 9. The characteristic mapping η_L takes the image in the subgroup $\mathbb{Z}/2(-1) \times \mathbb{Z}/2(T) \subset \aleph$, where the generators of the factors are $\{-1, T\}$.

Define the normal bundle structure Ξ_L as following. The normal bundle for (L^2, η_L, Ξ_L) is represented by a Whitney sum of 4-bundle, 3-bundle and the trivial line bundle $A \oplus B \oplus \varepsilon$.

The bundle B is splitted into the Whitney sum of 3 isomorphic line bundles: $B = B_1 \oplus B_2 \oplus B_3$. Each factor B_j, $j = 1, 2, 3$ is the line bundle, which is skew along the cycle R_2 by means of the element T, and is constant along the cycle R_1. The factors correspond to $\eta_{\beta;L}^*(\gamma)$.

The bundle A is splitted into the Whitney sum of 2 isomorphic copies of plane-bundles: $A = A_1 \oplus A_2$. The plane bundle A_1 (and A_1) should be looked as a line complex bundle. The each line complex bundle is equipped with the Hermitian conjugation long the cycle R_2 by means of the point symmetry, given by multiplication on -1 along the cycle R_1. The factors A_1, A_2 are inside the 4-dimensional block $\eta_L^*(A)$ of ν_{L_2} with generators $\{\pm 1, T\}$.

The factor ε corresponds to $\eta_{\alpha;L}^*(\gamma)$.

Denote two copies of (L^2, η_L, Ξ_L) by $(L_1^2, \eta_{L_1}, \Xi_{L_1})$, $(L_2^2, \eta_{L_2}, \Xi_{L_2})$. Define the following form $(L_2^2, \eta_{L_2}^{op}, \Xi_{L_3}^{op})$, which represents an element in W. The characteristic classes η_{L_2}, $\eta_{L_2}^{op}$ coincide, the normal bundle structure $\Xi_{L_2}^{op}$ is derived from Ξ_{L_2} by the reversing of the orientation of the each factors $B = B_1 \oplus B_2 \oplus B_3$ and by the complex conjugation on the factors A_1, A_2. In particular, the local orientations on the surfaces (L_2^2, Ξ_{L_2}) and $(L_2^2, \Xi_{L_2}^{op})$ with a prescribed normal bundle structure are opposite.

Let us prove that the forms $(L_2^2, \eta_{L_2}, \Xi_{L_2})$, $(L_2^2, \eta_{L_2}, \Xi_{L_2}^{op})$ are equivalent in W. Take a self-homotopy of η_{L_2} into itself such that the trace of a point $pt \in$

L_2 by this homotopy represents the generator $T \in \mathbb{Z}/2(-1)\times\mathbb{Z}/2(T) \subset \aleph$. By this homotopy the framing Ξ_{L_2} is transformed into a framing $\Xi^{op}_{L_2}$, where $\Xi^{op}_{L_2}$ is the composition of Ξ_{L_2} with the reflection in the factors B_1, B_2, B_3, A_1, A_2 as described above. The forms are equivalent.

Let us prove that the form $(L^2_1, \eta_{L_1}, \Xi_{L_1}) \cup (L^2_2, \eta_{L_2}, \Xi^{op}_{L_2})$ is regular cobordant to the form (K^2, η_K, Ξ_K), probably, up to a form (P^2, η_P, Ξ_P) with the characteristic class η_P takes the image in the central subgroup $\mathbb{Z}/2(-1) \subset \aleph$.

Take the restriction of Ξ_1 and of Ξ^{op}_2 over the cycle $R_1 \subset L^2_1$ and the cycle $-R'_1 \subset L^2_3$ correspondingly (in this formula $-R'_1$ is the cycle on L^2_2 which corresponds to the cycle R_1 with the opposite orientation, using the diffeomorphism $L^2_1 \cong L^2_2$). The restrictions $\Xi_1|_{R_1}$, $\Xi^{tw}_2|_{-R_1}$ are 4-dimensional $(-1,T)$-framings, which are stabilized in the codimension 4 by corresponding framings on $B \oplus \varepsilon$ ($B_i|_{R_1}, i = 1, 2, 3$ is the trivial bundle, the trivialization Ξ_2 over $B|_{R_1}$ is opposite to the trivialization $\Xi^{tw}_2|_{R_1}$, the trivialization Ξ_2 over $A|_{R_1}$ is conjugate to the trivialization $\Xi^{tw}_2|_{R_1}$).

Denote a $(-1,T)$-framing Ξ^{tw}_2 over (L_2, η_{L_2}) as following. Denote the line subbundles $\lambda_1 \subset A_1$, $\lambda_2 \subset A_2$, which correspond to the imaginary axis of the complex line bundles. The line bundle λ over L^2_2 is well-defined, and this bundle is skew over the cycle $R_2 \subset L^2_2$, which corresponds to the element T. Take the rotation trough the angle π inside the 4-bundle $B \oplus \lambda$ over L_2. As the result we get the new $(-1,T)$-framing over L^2_2, denoted by Ξ^{tw}_2. The framing Ξ^{tw}_2 coincides to the framing Ξ_1 everywhere, except the line bundle $\lambda_2 \subset \nu_{L_1}$, on this factor the framing Ξ^{tw}_2 is given by the reflection of Ξ_1. The framings Ξ^{tw}_2 is equivalent to the framing Ξ^{op}_2, and is equivalent to the framing Ξ_1.

Assume without loss of a generality that the restriction of the framing $\Xi_1|_{R_1}$ to the subbundle $B \oplus \varepsilon$ over the cycle R_1 is parallel to the coordinate axis e_7, e_8, e_9, e_{10}. Assume the framing $\Xi_1|_{R_1}$ on the factors λ_1, λ_2 is parallel to the vectors e_4, e_6 correspondingly. Assume the framing $\Xi_1|_{R_1}$ on the factors A_1, A_2 is parallel to the vectors $(e_3, e_4), (e_5, e_6)$ correspondingly. Then the skew-framing $\Xi^{tw}_2|_{R_1}$ coincides to the Ξ_1 along each directions, but the direction of the coordinate vector e_6, where Ξ_1, Ξ^{tw}_2 are opposite.

The -1-structure of skew framings $\Xi_1|_{R_1}$, $\Xi^{tw}_2|_{-R'_1}$ are distinguished only inside the factor A_2 of the normal bundle of $L^2_1 \cong L^2_2$, by a full rotation trough the angle 2π.

Take the regular cobordism transformation of the form $(L^2_1, \eta_{L_1}, \Xi_{L_1}) \cup (L^2_2, \eta_{L_2}, \Xi^{tw}_{L_2})$ by a surgery, with a support in small neighborhoods of a corresponding pair of points on R_1, $-R'_1$. As the result we get the form $(L^2_4, \eta_{L_4}, \Xi_{L_4})$. The image of η_{L_4} is in the space $B\mathbb{Z}/2(-1)\times B\mathbb{Z}/2(T)$. The cycle $R_1 \cup -R'_1$ is transformed into a cycle $R_3 \subset L^2_3$. The image of the characteristic class $\eta_{L_3}(R_3)$ is null-homotopic in the target space $B\mathbb{Z}/2(-1)\times B\mathbb{Z}/2(T)$.

The stable Hopf invariant $h(R_3)$ of the framed curve R_3 is non-trivial.

The form $(L_3^2, \eta_{L_3}, \Xi_{L_3}) \cup (K^2, \eta_K, \Xi_K)$ is cobordant to a form $(L_4^2, \eta_{L_4}, \Xi_{L_4})$, where the image of the mapping η_{L_4} is inside the space $B\mathbb{Z}/2(-1)$. The form $(L_4^2, \eta_{L_4}, \Xi_{L_4})$ is trivial, or, is of the order 2 in W. Theorem 8 is proved.

6.2 Combinatorial invariants of links

6.2.1 Combinatorial asymptotic invariants

We shall give a combinatorial prove of the following statement. Analytical proof is much easy, see [A3].

Theorem 9. *For $m = 3$ there exists a combinatorial M of order 7 (see the formula (6.61) below), which is not a function of pairwise linking numbers of components. The combinatirial invariant M satisfies properties C1-C3 (C2 and C3 are reformulated below as (6.7), (6.8)) i.e. this is an asymptotic invariant.*

Let us reformulate conditions C2,C3 for combinatorial finite-type invariants in a more convenient form. Let $(\mathbf{L}, \xi)$ be an arbitrary m-component framed link. The framing $\xi = \cup_i \xi_i$ is determined a coordinate system on the boundary of a tubular neighborhood of the component $L_i \subset \mathbf{L}$, this coordinate system is given by a family of parallels and meridians.

For an arbitrary integer $r \in \mathbb{Z}$ let us define another framed m-component link $r(\mathbf{L}, \xi)$, the components of this link are defined by the replacement of the corresponding framed component (L_i, ξ_i) of the oriented framed link $(\mathbf{L}, \xi)$, $i = 1, \dots m$ to the component $r(L_i, \xi_i)$, which is the standard $(r, 1)$–time winding along L_i. The component $r(L_i, \xi_i)$ passes r times along the parallel and 1–time along the meridian on the boundary of the thin regular tubular neighborhood of the component L_i. In the case $r = 0$ we get the link with small non-knotted disjoin components.

The link $r(\mathbf{L}, \xi)$ is equipped with an induced framing $r\xi_i$. The framing $r\xi_i$ along an arbitrary component $r(L_i, \xi_i)$ of the link $r(\mathbf{L}, \xi)$ is defined by the (interior) normal vectors to the boundary of the regular neighborhood of each component of the link $\mathbf{L}$.

Let $(\mathbf{L}, \xi; L_0)$ be an arbitrary framed $(m-1)$–component link with a marked component $L_0 \subset \mathbf{L}$. Let us define m–component framed link $(\mathbf{L}, \xi; L_0)^{\uparrow}$. The last $(m-2)$ components of the link $(\mathbf{L}, \xi)$ are transformed by the identity. The marked framed component (L_0, ξ_0) of the link $\mathbf{L}$ is transformed to the pair of parallel framed components $(L_{0,1}^{\uparrow}, \xi_{0,1}^{\uparrow}; L_{0,2}^{\uparrow}, \xi_{0,2}^{\uparrow})$, the first component coincides with L_0, the second is defined by a small shift of the component L_0 along the frame ξ_0. The framed $\xi^{\uparrow}_{0,1}$, $\xi^{\uparrow}_{0,2}$ are defined as induced framed by ξ_0.

From a $(m-1)$–component framed link $(L,\xi;L_0)$ with one marked component and an integer r let us define two framed m-component links, denoted by $r((\mathbf{L},\xi;L_0)^{\uparrow})$, $(r(\mathbf{L},\xi;L_0))^{\uparrow}$. The link $r((\mathbf{L},\xi;L_0)^{\uparrow})$ is obtained by the composition of the operation of dubbing of the marked component and by the operation of r-time winding along each framed component (on the dubbing component the standard framings are defined). The link $(r(\mathbf{L},\xi;L_0))^{\uparrow}$ is obtained by the composition of the same operation with the opposite order. Namely, at the first stage each components, including the marked component, is transformed into r-time winding along the given framing. After the marked component is dubbed (the winding of the marked component is equipped with the prescribed framing.

Definition 5.

Let us say that a finite-type invariant I for m-component framed links (the invariant I depends not on its framing) is an asymptotic invariant of degree s (degree of the invariant I depends not from its order), if the following two equations are satisfied:

$$I(r(\mathbf{L},\xi)) = r^s I(\mathbf{L}) + o(r^s). \tag{6.7}$$

$$I(r((\mathbf{L},\xi;L_0)^{\uparrow})) = I((r(\mathbf{L},\xi;L_0))^{\uparrow}) + o(r^s), \tag{6.8}$$

where $o(r^s)$ is a polynomial of r of the degree less then s, coefficients of this polynomial depend only on the isotopy class of the framed link $(\mathbf{L},\xi)$.

Remark on the condition (6.8) **in Definition** 5

Assume that one-parameter family of magnetic fields $\mathbf{B}=\mathbf{B}(t)$, $t\in[0,1]$ is not frozen-in and at the initial moment $t=0$ has only closed magnetic lines inside the only magnetic tube; for $t\neq 0$ non-closed magnetic lines appear. Additionally, assume that in the moments $t_i=\frac{1}{i}$, $i\in\mathbb{N}$ all magnetic lines $\mathbf{B}$ are closed, and each magnetic line is characterized by the windings number along the central axis of the tube. For $t=t_i$ for each collection on m magnetic lines of the magnetic field $\mathbf{B}(t_i)$ the value I is well defined. The value $I(\mathbf{B}(t_i))$ is well defined as the mean value over each collections of m magnetic lines, normalized on lengths. Condition (6.8) means that $I(\mathbf{B}(t_i))\to I(\mathbf{B}(0))$, for $i\to+\infty$.

6.2.2 An asymptotic invariant: q-monomial linking coefficient, $q \in \mathbb{N}$

Let us start with the prove that q-monomial linking coefficient satisfies the properties (6.7), 6.8). $\mathbf{L} = L_1 \cup L_2$ be a 2-component link with linking coefficient k of components. Define q-monomial linking coefficient as k^q. In the case $q = 1$ q–monomial linking coefficient coincides with linking coefficient. The q-monomial linking coefficient is a finite-type invariant of the order q in the sense of V.A.Vassiliev. let us prove that the invariant k^q is an asymptotic invariant of the degree $s = 2q$. For this matter we shall prove the equalities (6.7), (6.8).

In the case $q = 1$ the linking number k satisfies the equations (6.7), (6.8). Namely, the linking number k is determined by the integral (3.1), after we replace the link $\mathbf{L}$ by the link $r\mathbf{L}$ the value of the integral is changed to the factor r^2. Therefore the value $I = k^q$ is changed to the factor r^{2q}, this proves the equation (6.7).

Let us check the formula (6.8) for $I = k^q$. Let us start with the case $q = 1$. Let $(\mathbf{L}, \xi)$ be a framed knot. Consider the 2-component link $(\mathbf{L}, \xi)^{\uparrow} = (\mathbf{L}^{\uparrow}, \xi^{\uparrow})$ and the 2-component link $r(\mathbf{L}^{\uparrow}, \xi^{\uparrow})$. The following formula is satisfied;

$$k(r((\mathbf{L}, \xi)^{\uparrow})) = r^2 k((\mathbf{L}, \xi)^{\uparrow}). \tag{6.9}$$

For a positive integer r let us consider a framed knot $r(\mathbf{L}, \xi)$ and the 2-component link $(r(\mathbf{L}, \xi))^{\uparrow}$. From geometrical point of view it is evident that $k((r(\mathbf{L}, \xi))^{\uparrow}) = kr^2 + r$. Therefore by the formula (6.9) we get

$$k((r(\mathbf{L}, \xi))^{\uparrow}) = k(r((\mathbf{L}, \xi)^{\uparrow})) + r. \tag{6.10}$$

The formula (6.8) in the case $q = 1$ is proved. For an arbitrary q the proof is evident: take the both sides of the equation (6.10) in the power q.

The q–monomial linking coefficient k^q is a combinatoric analog of the q–monomial helicity $\chi_{\mathbf{B}}^{[q]}$, which is constructed in Theorem (2).

To describe a finite-type invariant M°, which corresponds to the asymptotic invariant M, given by the formula (4.5), (see Conjecture the last line of the Appendix II) we need preliminaries. Let us recall simplest properties of finite-type invariants of multy-component links.

Let us consider a Conway polynomial for m–component link $\mathbf{L}$:

$$\nabla_{\mathbf{L}}(z) = z^{m-1}(c_0 + c_1 z^2 + \cdots + c_n z^{2n}). \tag{6.11}$$

(for definition and properties of Conway polynomial see [P-S], [Me], [Co]).

We shall consider invariants of links, which are expressed from the first two coefficients c_0, c_1 of this polynomial (the Conway polynomial is applied to the link $\mathbf{L}$ itself and to the all proper sublinks of $\mathbf{L}$).

The simplest invariant of 2-component links, which is not expressed from pairwise linking coefficients of components, is called the generalized Sato-Levine invariant, discovered by M.Polyak and O.Viro, see [[M-P], Section 5] for the definition. The simplest formula for this invariant was proposed in [K-L]:

$$\beta(\mathbf{L}) = c_1(\mathbf{L}) - c_0(\mathbf{L})(c_1(L_1) + c_1(L_2)). \tag{6.12}$$

In this formula $c_0(\mathbf{L})$ coincides with linking coefficient $lk(L_1, L_2)$, $c_1(L_1)$, $c_1(L_2)$ are called Casson's invariants of the corresponding knots, determined by the knotted components. For short we shall write k instead of $c_0(\mathbf{L})$ and $lk(L_1, L_2)$.

We shall define the simplest 2-component link, which is called k-Hopf link and is denoted by $\mathbf{L}^-_{Hopf}(k)$. The first component L_1 of the link $\mathbf{L}^-_{Hopf}(k)$ is the standard oriented circle in the plane, the second component L_2 is inside the tubular neighborhood, this component spends one time along the meridian in the opposite direction and $-k$ times along parallel, such that the linking coefficient of the component is equal to the prescribed integer k (see also [[M-P], Figure 7]).

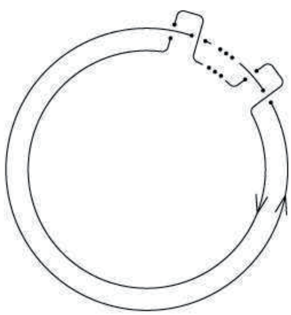

Figure 6.1: Link $\mathbf{L}^-_{Hopf}(k)$, $k < 0$.

It is easy to check that after the renumbering of the components of the link $\mathbf{L}^-_{Hopf}(k)$ we have a new link in the same isotopy class. It is easy to check the equality:

$$r(L_0,\xi_0)^{\uparrow} = \mathbf{L}^{-}_{Hopf}(1), \tag{6.13}$$

where (L_0,ξ_0) is the standard circle in the plane, equipped with the trivial framing, which is parallel to the plane. Obviously, we have $c_0(\mathbf{L}^{-}_{Hopf}(1)) = 1$.

With the link $\mathbf{L}^{-}_{Hopf}(k)$ let us define the link $\mathbf{L}^{+}_{Hopf}(k)$, which is obtained from $\mathbf{L}^{-}_{Hopf}(-k)$ by the opposition of the orientation of one (an arbitrary) of the components. Linking coefficient of $\mathbf{L}^{+}_{Hopf}(k)$, obviously, is k. The link $\mathbf{L}^{-}_{Hopf}(k)$ is more simple then $\mathbf{L}^{+}_{Hopf}(k)$ from algebraic point of view, because for $\mathbf{L}^{-}_{Hopf}(k)$ all coefficients in the Conway polynomial, except the coefficient c_0, are trivial.

The link $\mathbf{L}^{+}_{Hopf}(k)$ is natural when we investigate asymptotic limits. The link $\mathbf{L}^{+}_{Hopf}(k)$ is modeled a pair of closed magnetic lines, linked with the coefficient k.

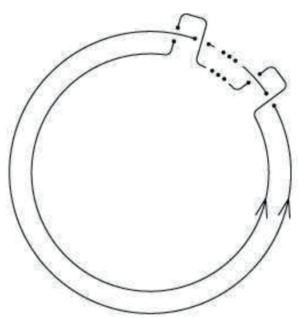

Figure 6.2: Link $\mathbf{L}^{+}_{Hopf}(k)$, $k > 0$.

The link $\mathbf{L}^{+}_{Hopf}(k)$ is equipped with the natural framing. This framing is defined such that the self-linking coefficient of each component of $\mathbf{L}^{+}_{Hopf}(k)$ is equal to k. The framing of the component L_2 coincides with the interior normal vector to the regular neighborhood of the component L_1 (recall that L_2 is on this surface). The framing along the component L_1 is defined by the collection of vectors, each vector is in the normal plane of the component L_1 at the corresponding point, the end of this vector coincides with the intersection point of the considered plane with component L_2. In particular, for $k = 0$ the standard framing is parallel to the standard plane.

Lemma 10. *The generalized Sato-Levine invariant, defined by the formula* (6.12), *satisfies the following equation:*

$$\beta(\mathbf{L}^{+}_{Hopf}(k)) = \frac{(k+1)k(k-1)}{6}. \tag{6.14}$$

Proof of Lemma 10

We shall present two different proof of the lemma.

The first proof. The standard calculation of the left side of the formula (6.14) by means of the formula (6.11) for the Conway polynomial proves that $\beta(\mathbf{L}^{+}_{Hopf}(k))$ is the polynomial of the degree 3 of the variable k. For $k = -1, 0$, or $+1$, we have $\beta(\mathbf{L}^{+}_{Hopf}(k)) = 0$, and $\beta(\mathbf{L}^{+}_{Hopf}(2)) = 1$. The equality (6.14) is the only possible.

Second proof. For the link $\mathbf{L}^{+}_{Hopf}(k)$ let us consider the link $\mathbf{L}^{-}_{Hopf}(-k)$. As we had mentioned above, we have $c_1(\mathbf{L}^{-}_{Hopf}(-k)) = 0$. Components of $\mathbf{L}^{-}_{Hopf}(-k)$ are non-knotted and by the formula (6.12) we get $\beta(\mathbf{L}^{-}_{Hopf}(-k)) = 0$. The formula (6.14) is a special case of the formula, when one component of the link changes the orientation. This theorem is proved in [N]. Lemma 10 is proved.

Remark

The definition of the generalized Sato-Levine Invariant, introduced in [A-R], is different with respect to (6.12). This definition is given by the formula:

$$\beta^{\circ}(\mathbf{L}) = c_1(\mathbf{L}) - k(c_1(L_1) + c_1(L_2)) - P(k), \tag{6.15}$$

where $P(k)$ is a polynomial of the degree 3 of k, given by the formula

$$P(k) = \frac{(k+1)k(k-1)}{6}. \tag{6.16}$$

In particular, for the invariant β° we have:

$$\beta^{\circ}(\mathbf{L}^{+}_{Hopf}(k)) = 0, \tag{6.17}$$

$$\beta^{\circ}(\mathbf{L}^{-}_{Hopf}(k)) = -P(k). \tag{6.18}$$

For the invariant β we have:

$$\beta(\mathbf{L}^{+}_{Hopf}(k)) = P(k), \tag{6.19}$$

$$\beta(\mathbf{L}^{-}_{Hopf}(k)) = 0. \tag{6.20}$$

The invariant β°, given by the formula (6.17), is called the normalized generalized Sato-Levine invariant. The normalization β° as well as the generalized Sato-Levine invariant β have the order 3.

In the case $k(\mathbf{L}) = 0$ the generalized Sato-Levine invariant is called the Sato-Levine invariant. In the paper [A-R2] an integral formula for the Sato-Levine invariant is proposed.

Lemma 11.

1. For an arbitrary 2-component link $(\mathbf{L}, \xi)$ the following equations are satisfied:

$$Q(r) + \beta^{\circ}(rL_1, L_2) = r^2\beta^{\circ}(\mathbf{L}), \tag{6.21}$$

$$Q(r) + \beta^{\circ}(L_1, rL_2) = r^2\beta^{\circ}(\mathbf{L}), \tag{6.22}$$

where $Q(r)$ is an r-polynomial, the coefficients of this polynomial depends only on the two parameters, namely, on the parameters $k(\mathbf{L})$, $k(L_1, \xi_1)$ in the case of the equation (6.21) and on the parameters $k(\mathbf{L})$, $k(L_2, \xi_1)$ in the case of the equation (6.22). In the case $k(\mathbf{L}) = 0$ we get $Q(r) = 0$.

2. For an arbitrary framed knot $(\mathbf{K}, \xi)$ in the case $k(K, \xi) \neq 0$ the following equation is satisfied:

$$\beta^{\circ}((r(\mathbf{K}, \xi))^{\uparrow}) = 2r^5 c_1(\mathbf{K}), \tag{6.23}$$

where $c_1(\mathbf{L})$ is the Casson invariant of the knot $\mathbf{K}$.

Corollary

The generalized (normalized) Sato-Levine invariant β° is not an asymptotic invariant of the degree ≤ 5 (Definition 5).

Remark

Of coarse, $\deg(\beta^{\circ}) = 3$, $\deg(\beta) = 3$, and by [B-M] β°, β are asymptotic invariants of the degree 6, but the asymptotic limits coincide with k^3.

Proof of Corollary

By Statement 2 the asymptotic denominator cannot be less then 5, hence it is exactly 5. In the case $k(\mathbf{L},\xi) \neq 0$ the equation $\beta^{\circ}(r(\mathbf{K},\xi)^{\uparrow}) = P(r)+O(r^4)$ is satisfied, where P is a polynomial, the coefficients of the polynomial depends on $k(\mathbf{K},\xi)$ only (the proof repeats arguments of Statement 1). Even if $\deg(P(r)) \leq 5$, this polynomial depends not of $C_2(\mathbf{K})$. We get, the formula (6.23) contradicts to the asymptotic axiom (6.8) $\beta^{\circ}(r(\mathbf{K},\xi)^{\uparrow}) = \beta^{\circ}(r(\mathbf{K},\xi)^{\uparrow}) + O(r^4)$.

The Δ-moves are more convenient with respect to crossing moves for investigation of invariants of low orders (see [Na] for detailed definition and references).

Accordingly to calculations, presented in [A-M-R], the generalized Sato-Levine invariant β° is totally defined by the condition (6.17) and by the following two formula (6.24),(6.25), which are described the jumps of the invariant by Δ–moves of the following two kinds:

$$\beta^{\circ}(\mathbf{L})|_{t=t_0+\varepsilon} - \beta^{\circ}(\mathbf{L})|_{t_0-\varepsilon} = O(x)O(y)(lk(L_x^+, L') - lk(L_x^-, L') - O(x)) \tag{6.24}$$

where the Δ-move involves two branches of a one component and one branch of another component.

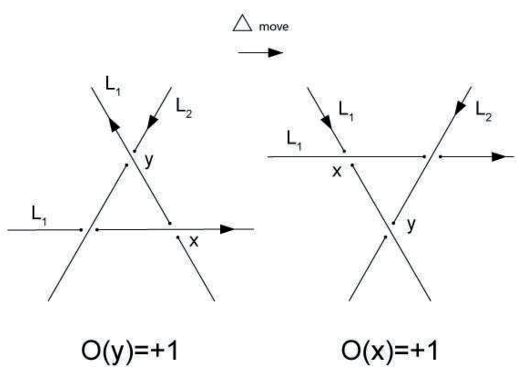

Figure 6.3: Δ-move for 2-component links.

$$\beta^\circ(\mathbf{L})|_{t=t_0+\varepsilon} - \beta^\circ(\mathbf{L})_{t_0-\varepsilon} = 0, \tag{6.25}$$

where is assumed that the Δ-move involves 3 branches of a one component.

In the formula (6.24) by $O(x)$ a vertex is denoted the algebraic value of the vertex x of the disappeared triangle, in which the projection of a component of $\mathbf{L}$, say the component L_1 (the case of the component L_2 is analogous); $O(y)$ is the algebraic value of an arbitrary last vertex of the disappeared triangle on the diagram; L_x^+ is a closed loop on the diagram with the vertex x, which contains two sides of the disappeared triangle (by the assumption this loop is on the projection of L_1), L_x^- is the last loop on the projection on the same component L_1, L' is the last component of the link (by assumption, $L' = L_2$).

Assume that the link $\mathbf{L}$ for $t = t_0$ is transformed by means of Δ–move with the formula (6.24). Then the link $r\mathbf{L}$ for $t = t_0$ is transformed by means of the associated family of r^2 Δ–moves. Let us prove the following formula:

$$r^4\beta^\circ(\mathbf{L})|_{t=t_0+\varepsilon} - r^4\beta^\circ(\mathbf{L})|_{t_0-\varepsilon} = \beta^\circ(r\mathbf{L})|_{t=t_0+\varepsilon} - \beta^\circ(r\mathbf{L})_{t_0-\varepsilon}, \tag{6.26}$$

In this formula and below we write $r\mathbf{L}$ instead of $r(\mathbf{L},\xi)$ for short.

On the diagram of the link $r\mathbf{L}$ let us consider r^2 self-intersection points x_i, $i = 1,\dots,r^2$, which are in a neighborhood x of the disappeared triangle on the diagram $\mathbf{L}$. In each point x_i on the diagram of $r\mathbf{L}$ let us consider the loop $(rL)_x^+$, which starts and ends the same side that the loop L_x^+ on the corresponding diagram of $\mathbf{L}$. Denote the linking number $lk(L_1, L_2)$ by k, the linking numbers $lk(L_x^+, L')$, $lk(L_x^-, L')$ by $\lambda_+(x)$, $\lambda_-(x)$ correspondingly. Evidently, we get $\lambda_+(x) + \lambda_-(x) = k$.

Let us calculate in each point x_i the coefficients $\lambda_+(x_i) = lk((rL)_{x_i}^+, rL')$, $\lambda_-(x_i) = lk((rL)_{x_i}^+, rL')$ and then calculate the sum of the coefficients over all the self-intersection points. By the straightforward calculation we get:

$$\sum_{i=1}^{r^2} \lambda_+(x_i) = r^2(\lambda_+(x)+(\lambda_+(x)+k)+\dots+(\lambda_+(x)+ki)+\dots+(\lambda_+(x)+k(r-1))).$$

$$\sum_{i=1}^{r^2} \lambda_-(x_i) = r^2(\lambda_-(x)+k(r-1))+(\lambda_-(x)+k(r-2))+\dots+(\lambda_-(x)+ki)+\dots+\lambda_-(x).$$

Therefore for each family of Δ–moves, which consists of r^2 elementary Δ–moves, determined by the moves of one of the r copies of the branches of the component rL' in a neighborhood of the point x, the sum of jumps of β° is equal to

$$\sum_{i=1}^{r^2} \lambda_+(x_i) - \lambda_-(x_i) = r^3(\lambda_+(x) - \lambda_-(x)).$$

the sum of jumps of β° for branches of the component rL' are equal. The equation (6.26) is proved.

To prove the equations (6.21), (6.22) for an arbitrary $(\mathbf{L}, \xi)$ it is sufficient to check the each equation for the link $\mathbf{L}^+_{Hopf}(k)$ with arbitrary framings of the components. This calculation is evident.

Let us check the equation (6.23). By a result of [A-M-R] the generalized Sato-Levine invariant satisfies the formula:

$$\beta^\circ((\mathbf{K}, \xi)^\uparrow) = 2c_1(\mathbf{K})k((\mathbf{K}, \xi)^\uparrow), \tag{6.27}$$

where the link $(\mathbf{K}{\cdot}\xi)^\uparrow$ is defined by the dubbing of the framed knot $(\mathbf{K}, \xi)$, $c_1(\mathbf{K})$ is the Casson invariant of the knot $\mathbf{K}$, $k((\mathbf{K}, \xi)^\uparrow)$ is the linking coefficient of the components.

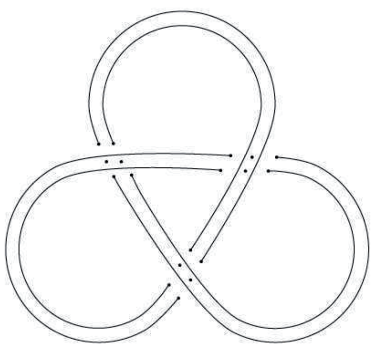

Figure 6.4: An illustration of the formula (6.27).

The following formula is satisfied:

$$c_1(r\mathbf{K}) = r^3 c_1(\mathbf{K}). \tag{6.28}$$

Indeed, by a result of [A-M-R] the Casson invariant $c_1(\mathbf{K})$ is jumped by a Δ–move to the sign of the disappeared triangle. The sides of the disappeared triangle are formed by the 3 segments of branches on the diagram of the knot in the neighborhood of the critical point of the Δ–move. Therefore the equation (6.28) is evident, and from the equations (6.27), (6.10), (6.28) we have the equality (6.23).

The equation (6.23) follows from the formula (6.27) and from (6.17). Lemma 11 is proved.

Let us start by the definition of an axillary invariant $\tilde{M}$ and its normalization $\tilde{M}^{\circ}$.

Let $\mathbf{L} = L_1 \cup L_2 \cup L_3$ be a 3-component link. Let us consider the invariant $\gamma(\mathbf{L})$, which is defined by S.A.Melikhov in the paper [Me]. This invariant is a function of the coefficients c_1 and $k = c_0$ of the Conway polynomial of various sublinks of the link $\mathbf{L}$, this invariant is defined by the following formula:

$$\gamma(\mathbf{L}) = c_1(\mathbf{L}) - \tag{6.29}$$

$$((1,2)(2,3) + (2,3)(3,1) + (3,1)(1,2))(c_1(L_1) + c_1(L_2) + c_1(L_3))$$
$$-((3,1) + (2,3))(c_1(L_1 \cup L_2) - (1,2)(c_1(L_1) + c_1(L_2)))$$
$$-((1,2) + (3,1))(c_1(L_2 \cup L_3) - (2,3)(c_1(L_2) + c_1(L_3)))$$
$$-((2,3) + (1,2))(c_1(L_3 \cup L_1) - (3,1)(c_1(L_3) + c_1(L_1))),$$

where by (i,j) the linking number $k(L_i \cup L_j)$ of the pair of components L_i, L_j, $i,j = 1,2,3$, $i \neq j$, of the link $\mathbf{L}$ is defined.

Define a 3-component link $\mathbf{L}^-_{Hopf}((1,2);(2,3);(3,1))$, this link depends on 3 integer parameters $(1,2),(2,3),(3,1) \in \mathbb{Z}$. Let us consider three 2-components links $\mathbf{L}^-_{Hopf}((2,3))$, $\mathbf{L}^-_{Hopf}((3,1))$, $\mathbf{L}^-_{Hopf}((1,2))$, which are located in small neighborhoods of the standard triangle ABC on the plane. Define a 3-component link $\mathbf{L}^-_{Hopf}((1,2);(2,3);(3,1))$, the components of this link are defined by the connected sum of the first component of the link $\mathbf{L}^-_{Hopf}((2,3))$ with the second component of the link $\mathbf{L}^-_{Hopf}((3,1))$, the first component of the link $\mathbf{L}^-_{Hopf}((3,1))$ with the second component of the link $\mathbf{L}^-_{Hopf}((1,2))$, and the first component of the link $\mathbf{L}^-_{Hopf}((1,2))$ with the second component of the link $\mathbf{L}^-_{Hopf}((2,3))$.

The summation of the corresponding components of the corresponding 2-component links in vertexes of the triangle is defined along corresponding sides of the triangle without twist along the plan of the triangle ABC. The components of the link $\mathbf{L}^-_{Hopf}((1,2);(2,3);(3,1))$ correspond to the sides of the triangle, denote this components by L_1, L_2, L_3. The denotations are taken such that $k(L_1,L_2) = (1,2)$, $k(L_2,L_3) = (2,3)$, $k(L_3,L_1) = (3,1)$.

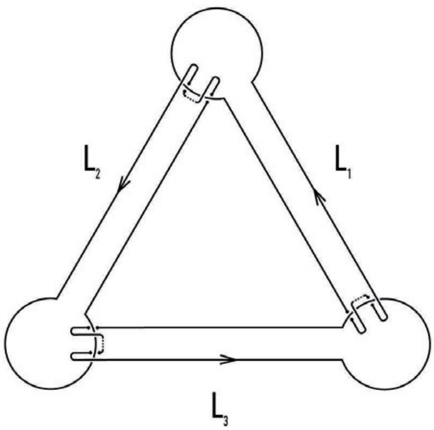

Figure 6.5: Link $Hopf^{-}((1,2)(2,3)(3,1))$.

Let us define also the link $\mathbf{L}^{+}_{Hopf}((1,2),(2,3),(3,1))$ with the prescribed linking coefficients. The differences between $\mathbf{L}^{-}_{Hopf}((1,2),(2,3),(3,1))$ and $\mathbf{L}^{+}_{Hopf}((1,2),(2,3),(3,1))$ is the following: the summation of the components is defined by means of the collection of the links $\mathbf{L}^{+}_{Hopf}((2,3))$, $\mathbf{L}^{+}_{Hopf}((3,1))$, $\mathbf{L}^{-}_{Hopf}((1,2))$, instead of the collection of the links $\mathbf{L}^{-}_{Hopf}((2,3))$, $\mathbf{L}^{-}_{Hopf}((3,1))$, $\mathbf{L}^{-}_{Hopf}((1,2))$.

From the formula (6.29) the following equalities follow:

$$\gamma(\mathbf{L}^{-}_{Hopf}((1,2),(2,3),(3,1))) = 0, \tag{6.30}$$

$$\gamma(\mathbf{L}^{+}_{Hopf}((1,2),(2,3),(3,1))) = R((1,2),(2,3),(3,1)), \tag{6.31}$$

where the polynomial $R((1,2),(2,3),(3,1))$ of the pairwise linking numbers of components is defined by the formula:

$$\begin{gathered} R((1,2),(2,3),(3,1)) = \\ \tfrac{(1,2)^2(2,3)(3,1)+(2,3)^2(3,1)(1,2)+(3,1)^2(1,2)(2,3)}{2} + \tfrac{3(1,2)(2,3)(3,1)}{2}. \end{gathered} \tag{6.32}$$

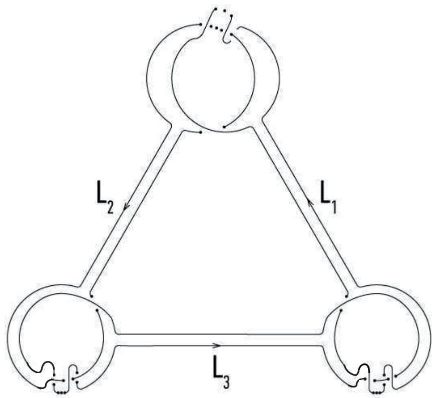

Figure 6.6: Link $Hopf^{+}((1,2)(2,3)(3,1))$.

Define the normalized invariant $\gamma^{\circ}(\mathbf{L})$ by the formula:

$$\gamma^{\circ}(\mathbf{L}) = \gamma(\mathbf{L}) - R((1,2),(2,3),(3,1)). \tag{6.33}$$

It is easy to check the equations:

$$\gamma^{\circ}(\mathbf{L}^{+}_{Hopf}((1,2),(2,3),(3,1))) = 0, \tag{6.34}$$

$$\gamma^{\circ}(\mathbf{L}^{-}_{Hopf}((1,2),(2,3),(3,1))) = -R((1,2),(2,3),(3,1)). \tag{6.35}$$

The normalized invariant γ° and the invariant γ have the order 4.

In the following lemma we write-down the formula by S.A.Melikhov of jumps of the invariant γ (the formula of jumps for the invariant γ° are the same) for homotopy of links with self-intersection but without intersections between different components. Let $\mathbf{L}_{sing;3}$ be a singular 3-component link, the components L_1, L_2 are regular, the component $L_{sing,3}$ has the only self-intersection point. Denote by $\mathbf{L}_{+;3}$, $\mathbf{L}_{-;3}$ two 3-component links, the first 2 components of this links coincides with L_1, L_2, the third component $L_{3;+}$ of the link $\mathbf{L}_{3;+}$ is defined by means of the prescribed resolution of the self-intersection of the component $L_{sing,3}$, the third component $L_{3;-}$ of the link $\mathbf{L}_{-;3}$ is defined by means of the opposite resolution of the self-intersection of the component of the link $L_{sing,3}$.

Let us define 4-component link $\mathbf{L}_{s;3}$ with the components $(L_1, L_2, L_{3+}, L_{3-})$. The first two components of the links $\mathbf{L}_{s;3}$, $\mathbf{L}_{sing;3}$ coincide, the components L_{3+}, L_{3-} of the link $\mathbf{L}_{s;3}$ are obtained by the orientation-preserved smoothing of the singular component $L_{sing,3}$ of the link $\mathbf{L}_{sing;3}$.

Denote by $(1,3^+)$ $(1,3^-)$ the linking coefficients of the component L_1 with the components L_{3+}, L_{3-} correspondingly. Define by $(2,3^+)$ $(2,3^-)$ the linking coefficients of the component L_2 with the components L_{3+}, L_{3-} correspondingly.

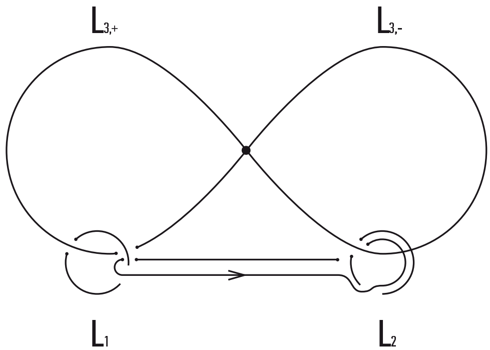

Figure 6.7: A skein relation for γ: $(1,3^+) = +1$, $(2,3^+) = 0$, $(1,3^-) = 0$, $(2,3^-) = +2$, $(1,2) = +1$; $\gamma(\mathbf{L}_+) - \gamma(\mathbf{L}_-) = 2$.

Analogical denotations are well-defined after the replacement $3 \to 1$, $3 \to 2$ of the numbers of the components.

Lemma 12. *The invariant $\gamma(\mathbf{L})$ of 3-component links satisfies the following equations:*

$$\gamma(\mathbf{L}_{+;3}) - \gamma(\mathbf{L}_{-;3}) = (1,2)((1,3^+)(2,3^-) + (1,3^-)(2,3^+)), \tag{6.36}$$

$$\gamma(\mathbf{L}_{+;1}) - \gamma(\mathbf{L}_{-;1}) = (2,3)((2,1^+)(3,1^-) + (2,1^-)(3,1^+)), \tag{6.37}$$

$$\gamma(\mathbf{L}_{+;2}) - \gamma(\mathbf{L}_{-;2}) = (3,1)((3,2^+)(1,2^-) + (3,2^-)(1,2^+)), \tag{6.38}$$

The normalized invariant $\gamma^\circ(\mathbf{L})$ satisfies the same equations.

Proof of Lemma 12

The formula (6.36),(6.37),(6.38) of the jump of the invariant γ are proved in [Me], p. 11. The formula for γ° are, evidently, one-to-one. Lemma 12 is proved.

Let us study the formula for jumps of the invariant γ° with respect to Δ–moves of components. Let us consider the case when a Δ–move involves all 3 components of the link $\mathbf{L}$. Let us introduce the following denotations.

Let us consider the links $\mathbf{L}_- = (L_{1,-} \cup L_{2,-} \cup L_{3,-})$, $\mathbf{L}_+ = (L_{1,+} \cup L_{2,+} \cup L_{3,+})$, the disappeared triangle ABC_-, formed by 3 branches of the projection of $\mathbf{L}_-$, which are on different components of the link, and the appeared triangle ABC_+, which is formed by the corresponding branches of the link L_+. Let us take the order of the vertexes of the triangle, such that the vertex A of the appeared and disappeared triangles is on the intersection of projections of components L_2 and L_3, the vertex B of the appeared and disappeared triangles is on the intersection of projections of components L_1 and L_2,the vertex C of the appeared and disappeared triangles is on the intersection of projections of components L_1 and L_2.

Denote by $(L_{1,\pm} \odot L_{2,\pm}, L_{3,\pm})$ the 2–component link, which is defined by the standard smoothing of the components $L_{1,\pm}$, $L_{2,\pm}$ at the vertex $C_\pm$. The sign $\pm$ means one of the two possible position on $\mathbf{L}_+$, or, on $\mathbf{L}_-$.

Analogously denotations $(L_{2,\pm} \odot L_{3,\pm}, L_{1,\pm})$, $(L_{3,\pm} \odot L_{1,\pm}, L_{2,\pm})$ are introduced.

Lemma 13. *The invariant γ satisfies the following equations with respect to Δ–moves, which involve all 3 components of the link:*

$$\gamma(\mathbf{L}_+) - \gamma(\mathbf{L}_-) = \beta(L_{1,+} \odot L_{2,+}, L_{3,+}) - \beta(L_{1,-} \odot L_{2,-}, L_{3,-}) = \tag{6.39}$$

$$\beta(L_{2,+} \odot L_{3,+}, L_{1,+}) - \beta(L_{2,-} \odot L_{3,-}, L_{1,-}) =$$

$$\beta(L_{3,+} \odot L_{1,+}, L_{2,+}) - \beta(L_{3,-} \odot L_{1,-}, L_{2,-}).$$

For normalized invariants γ°, β° the analogous equations are satisfied.

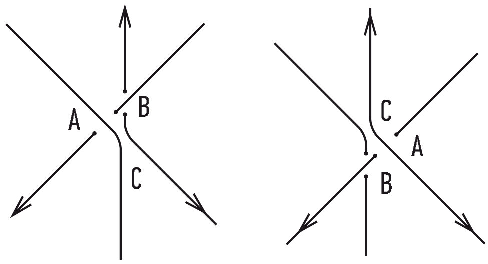

Figure 6.8: $\mathbf{L}_{-} = L_{1,-} \odot L_{2,-}, L_{3,-}$ $\mathbf{L}_{+} = L_{1,+} \odot L_{2,+}, L_{3,+}$
A Δ-move for γ; all components of links are involved.

Proof of Lemma 13

let us apply formula (6.29), (6.30). The following equation is satisfied:

$$c_1(\mathbf{L}_+) - c_1(\mathbf{L}_-) = \gamma(\mathbf{L}_+) - \gamma(\mathbf{L}_-). \tag{6.40}$$

Indeed, for a Δ–move, which involves all 3 components of the link $\mathbf{L}$, only the first term in the formula (6.29) is jumped. Let us recall that an arbitrary Δ–move is presented as the composition of the homotopy with intersection of the prescribed pair of branches components, the isotopy for which the projection of the last component L_3 is moved thought the vertex A of the disappeared triangle, and the homotopy this the opposite crossing of the same pair of branches of the components. There are 3 different decompositions of the given Δ–homotopy of this kind.

Let us apply the formula for transformations of the coefficients in the Conway polynomial for each of this three decompositions. We have the following relation:

$$c_1(\mathbf{L}_+) - c_1(\mathbf{L}_-) = c_1(L_{1,+} \odot L_{2,+}, L_{3,+}) - c_1(L_{1,-} \odot L_{2,-}, L_{3,-}). \tag{6.41}$$

Let us express the coefficient c_1 in the formula (6.41) by means of the invariant β°, using the formula (6.15). Evidently, the following formula is satisfied:

$$c_1(L_{1,+} \odot L_{2,+}, L_{3,+}) - c_1(L_{1,-} \odot L_{2,-}, L_{3,-}) = \tag{6.42}$$

$$\beta(L_{1,+} \odot L_{2,+}, L_{3,+}) - \beta(L_{1,-} \odot L_{2,-}, L_{3,-}).$$

Take the expression (6.42) in the right side of the formula (6.41) and use this formula in the formula (6.40). This gives one of the three required equities in (6.39). The last two equations is proved analogously. Lemma 13 is proved.

Let us define an axillary invariant $\tilde{M}$ and its normalization $\tilde{M}^\circ$.

Definition 6. *Let* $\mathbf{L} = (L_1 \cup L_2 \cup L_3)$ *be an arbitrary 3-component link. Define the invariant* $\tilde{M}(\mathbf{L})$ *by the following formula:*

$$\tilde{M}(\mathbf{L}) = (1,2)(2,3)(3,1)\gamma(\mathbf{L}) - \tag{6.43}$$

$$((1,2)^2(1,3)^2\beta(L_2 \cup L_3) + (2,3)^2(2,1)^2\beta(L_3 \cup L_1) + (2,3)^2(2,1)^2\beta(L_3 \cup L_1)).$$

Define the normalized invariant $\tilde{M}^\circ(\mathbf{L})$ *by the following formula:*

$$\tilde{M}^\circ(\mathbf{L}) = (1,2)(2,3)(3,1)\gamma^\circ(\mathbf{L}) - \tag{6.44}$$

$$((1,2)^2(1,3)^2\beta^\circ(L_2 \cup L_3) + (2,3)^2(2,1)^2\beta^\circ(L_3 \cup L_1) + (2,3)^2(2,1)^2\beta^\circ(L_3 \cup L_1)).$$

Theorem 10.

1. The invariant $\tilde{M}$ *satisfies the following equations for homotopies of links with one self-intersection point on the corresponding component.*

$$\tilde{M}(\mathbf{L}_{+;3}) - \tilde{M}(\mathbf{L}_{-;3}) = (1,2)^2(2,3)(3,1)((1,3^+)(2,3^-) + (1,3^-)(2,3^+)) \tag{6.45}$$

$$-(1,2)^2(3,1)^2(2,3^+)(2,3^-) - (2,3)^2(2,1)^2(1,3^+)(1,3^-),$$

$$\tilde{M}(\mathbf{L}_{+;1}) - \tilde{M}(\mathbf{L}_{-;1}) = (1,2)(2,3)^2(3,1)((2,1^+)(3,1^-) + (2,1^-)(3,1^+)) \tag{6.46}$$

$$-(2,3)^2(1,2)^2(3,1^+)(3,1^-) - (3,1)^2(3,2)^2(2,1^+)(2,1^-),$$

$$\tilde{M}(\mathbf{L}_{+;2}) - \tilde{M}(\mathbf{L}_{-;2}) = (1,2)(2,3)(3,1)^2((3,2^+)(1,2^-) + (3,2^-)(1,2^+)) \tag{6.47}$$

$$-(3,1)^2(2,3)^2(1,2^+)(1,2^-) - (1,2)^2(1,3)^2(3,2^+)(3,2^-).$$

For the normalized invariant $\tilde{M}^\circ$ *the same equations are satisfied.*

The invariant $\tilde{M}$ *satisfies the following equations for a* Δ*–move, which involves all 3 components of the link:*

$$\tilde{M}(\mathbf{L}_+) - \tilde{M}(\mathbf{L}_-) = \tag{6.48}$$

$$(1,2)(2,3)(3,1)(\beta^\circ(L_{1,+} \odot L_{2,+}, L_{3,+}) - \beta^\circ(L_{1,+} \odot L_{2,+}, L_{3,+})) =$$

$$(1,2)(2,3)(3,1)(\beta^\circ(L_{2,+} \odot L_{3,+}, L_{1,+}) - \beta^\circ(L_{2,+} \odot L_{3,+}, L_{1,+})) =$$
$$(1,2)(2,3)(3,1)(\beta^\circ(L_{3,+} \odot L_{1,+}, L_{2,+}) - \beta^\circ(L_{3,+} \odot L_{1,+}, L_{2,+})).$$

For normalized invariants $\tilde{M}^\circ$, β° the same equations are satisfied.

The invariant $\tilde{M}$ satisfies the following equation:

$$\tilde{M}(\mathbf{L}^{-}_{Hopf}((2,3),(1,2),(3,1))) = 0. \tag{6.49}$$

The normalized invariant $\tilde{M}^\circ$ satisfies the following equation:

$$\tilde{M}^\circ(\mathbf{L}^{+}_{Hopf}((2,3),(1,2),(3,1))) = 0. \tag{6.50}$$

2. The invariants $\tilde{M}$ and $\tilde{M}^\circ$ are uniquely well-defined by means of the equations (6.45)-(6.50)*. The invariants $\tilde{M}$ and $\tilde{M}^\circ$ are of order* 7 *in the sense of V.A. Vassiliev.*

3. In the particular case $(1,2) = (2,3) = (3,1) = k$ *the formulas for the invariants $\tilde{M}$ and $\tilde{M}^\circ$ are simplified:*

$$\tilde{M} = k^3 c_1(\mathbf{L}) - 3k^4[c_1(L_1,L_2) + c_1(L_2,L_3) + c_1(L_3,L_1)] \tag{6.51}$$
$$-k^5[c_1(L_1) + c_2(L_2) + c_3(L_3)].$$

$$\tilde{M}^\circ = k^3 c_1(\mathbf{L}) - 3k^4[c_1(L_1,L_2) + c_1(L_2,L_3) + c_1(L_3,L_1)] \tag{6.52}$$
$$-k^5[c_1(L_1) + c_2(L_2) + c_3(L_3)] - 2k^7 - \frac{3k^6}{2} + \frac{k^5}{2}.$$

Proof of Theorem 10**.**

The formulas (6.45),(6.46),(6.47),(6.48) are followed from the formulas (6.36), (6.37), (6.38) for jumps of the invariant γ° and from analogous well-known formulas for jumps of the invariant β° for elementary homotopies of links, see [A-R],[Me],[Ni]. The formula (6.49) follows from the formulas (6.29), (6.49). The second and the third parts of the theorem are evident. Theorem 10 is proved.

To construct the asymptotic invariant M and to prove Theorem 9 we shall need two lemmas. In the lemmas properties of the invariant $\tilde{M}$ are studied. The first lemma could be interesting by itself.

Let $(\mathbf{L},\xi) = ((L_1,\xi_1),(L_2,\xi_2),(L_3,\xi_3))$ be a 3-component framed link. Let us consider the link $(r(L_1,\xi_1),L_2,L_3)$, $r \in \mathbb{N}$. Denote this link by ${}_1r\mathbf{L}$ for short. This freedom of the denotation is possible, because of the formula (6.54) below. Analogously define ${}_2r\mathbf{L} = (L_1, r(L_2,\xi_2), L_3)$, ${}_3r\mathbf{L} = (L_1,L_2,r(L_3,\xi_3))$.

Lemma 14. *Let the link ${}_1r\bar{\mathbf{L}}$ is obtain from the link ${}_1r\mathbf{L}$ after the first component of $r(L_1,\xi_1)$ is replaced to another component by the gluing of an arbitrary colored braid with r strings along a short segment on the component L_1 of the link $\mathbf{L}$. The following equation*

$$\tilde{M}({}_1r\bar{\mathbf{L}}) = \tilde{M}({}_1r\mathbf{L}) \tag{6.53}$$

is satisfied. In particular, as an evident corollary, if we take another framing ξ_1' of the component L_1 and obtain another link $({}_1r\mathbf{L})' = (r(L_1,\xi_1'),L_2,L_3)$ for which the following formula is satisfied:

$$\tilde{M}(({}_1r\mathbf{L})') = \tilde{M}({}_1r\mathbf{L}). \tag{6.54}$$

The formulas, analogously to the formulas (6.53), (6.54)*, are satisfied for an arbitrary component of* $\mathbf{L}$.

The same formulas are satisfied for the normalized invariant $\tilde{M}^\circ$.

Proof of Lemma 14

It is sufficiently to proof the lemma for the case when the braid is presented by the elementary full twist of the two strings. Such a transformation is determined by an elementary homotopy with the only self-intersection point of the pair of parallel strings of the component $r(L_1,\xi_1)$.

Denote by $(1,2),(2,3),(3,1)$ the linking coefficients of the corresponding components of the link ${}_1r\mathbf{L}$. Let r_1, r_2 be positive integers, which are equal to the numbers of the wings of the components rL_1^+, rL_1^- around the component L_1 and satisfied the relation $r_1+r_2=r$. Let us apply the formula (6.46) to calculate the jumps of the invariant $\tilde{M}$, when the link ${}_1r\mathbf{L}$ is transformed into the link $\bar{\mathbf{L}}$. It is not hard to see that the first term is changed by the value $2(1,2)^2(2,3)^2(3,1)^2r_1r_2$, the second and the third terms are jumped by the value $-(1,2)^2(2,3)^2(3,1)^2r_1r_2$. Therefore the value $\tilde{M}$ is fixed. Lemma 14 is proved.

Lemma 15. *1. The following property is satisfied:*

$$\tilde{M}({}_1r\mathbf{L}) = r^4\tilde{M}(\mathbf{L}) +_1 Q(r), \tag{6.55}$$

where the polynomial ${}_1Q(r)$ is defined by the formula:

$${}_1Q(r) = \tilde{M}({}_1r\mathbf{L}^-_{Hopf}((1,2),(2,3),(3,1)) - \tilde{M}(\mathbf{L}^-_{Hopf}(r(1,2),(2,3),r(3,1))),$$

and the coefficients of the polynomial ${}_1Q(r)$ depends only on the pairwise linking numbers $(1,2),(2,3),(3,1)$ of components of the link $\mathbf{L}$. The same relations are satisfied for the links ${}_2r\mathbf{L}$, ${}_3r\mathbf{L}$.

2. The following equation is satisfied:

$$\tilde{M}(r\mathbf{L}) = r^{12}\tilde{M}(\mathbf{L}) + Q(r), \tag{6.56}$$

where the polynomial $Q(r)$ is depended by the formula:

$$Q(r) = \tilde{M}(r\mathbf{L}^-_{Hopf}((1,2),(2,3),(3,1)) - \tilde{M}(\mathbf{L}^-_{Hopf}(r^2(1,2),r^2(2,3),r^2(3,1))),$$

the coefficients of the polynomial $Q(r)$ depends only of pairwise linking numbers $(1,2),(2,3),(3,1)$ of the link $\mathbf{L}$.

3. For an arbitrary 2-component framed link $\mathbf{L}$ with the marked first component the following equation is satisfied:

$$\tilde{M}(\mathbf{L}^{\uparrow},\xi) = S(k(L_1,\xi_1),k(\mathbf{L})), \tag{6.57}$$

where the polynomial $S(k(L_1,\xi_1),k(\mathbf{L}))$ is equal to $\tilde{M}((\mathbf{L}^-)^{\uparrow}_{Hopf}(k),\xi')$ and is depended only of the self-linking coefficient of the framed component (L_1,ξ_1) and of the linking coefficient $k(\mathbf{L})$ of components of the link $\mathbf{L}$.

Analogous formula are satisfied for the normalized invariant $\tilde{M}^{\circ}$.

Proof of Lemma 15

Let us proof Claim 1. let us consider the following list 1-4 of elementary transformations of the link $\mathbf{L}$.

–1. A Δ–move, which involves only the first component L_1.

–2. A homotopy with a self-intersection point on the component L_2.

–3. A homotopy with a self-intersection point on the component L_3.

–4. A Δ–move, which involves all the 3 components of the link $\mathbf{L}$.

Evidently, for an arbitrary link $\mathbf{L}$ there exists a sequence Ξ of transformations 1-4, which transforms the link $\mathbf{L}$ into the link $\mathbf{L}^-_{Hopf}((2,3),(1,2),(3,1))$. Denote the components of the link $\mathbf{L}^-_{Hopf}((2,3),(1,2),(3,1))$ by

$(L^-_{1;Hopf}, L^-_{2;Hopf}, L^-_{3;Hopf})$. The sequence Ξ detects the sequence $r\Xi$ of elementary transformations from the list 1-4 of the link ${}_1r\mathbf{L}$ into the link ${}_1r\mathbf{L}^-_{Hopf}$.

Let us consider an arbitrary elementary transformation ξ_i of the type 1-4. Denote by $r\xi_i$ an elementary transformation of the type 2,3,4 or a sequence of elementary transformations of the type 1, which corresponds to the elementary transformation ξ_i of the link ${}_1r\mathbf{L}$. Denote by $\mathbf{L}_-$, $\mathbf{L}_+$ the links, which are related by the transformation ξ_i. Denote by ${}_1r\mathbf{L}_-$, ${}_1r\mathbf{L}_+$ the links, constructed from the links $\mathbf{L}_-$, $\mathbf{L}_+$ by means of the r-time winding of the first component. The links ${}_1r\mathbf{L}_-$, ${}_1r\mathbf{L}_+$ are related by the corresponding sequence of elementary transformations of the types 2,3,4, or by a transformation of the type 1. For an arbitrary of elementary transformation ξ_i of the link $\mathbf{L}$, listed above, and for the corresponding transformation $r\xi_i$ of the link $r\mathbf{L}$ the following equality is satisfied:

$$r^4(\tilde{M}(\mathbf{L}_+) - \tilde{M}(\mathbf{L}_-)) = \tilde{M}({}_1r\mathbf{L}_+) - \tilde{M}({}_1r\mathbf{L}_-). \qquad (6.58)$$

This equality is followed from the formulas (6.45) - (6.48). It is sufficiently to check the equality (6.55) for the link $\mathbf{L}^-_{Hopf}((2,3),(1,2),(3,1)))$. This calculation is evident, using Lemma 14. For the links ${}_2r\mathbf{L}$, ${}_3r\mathbf{L}$ the proves are analogous. Clam 1 is proved.

Let us prove Clam 2. The equality (6.56) follows from the equality (6.55) and from the analogous equalities for the last two components. Claim 2 is proved.

Let us prove Clam 3. Consider the following list 1-2 of elementary transformations of 2-component links.

–1. Δ–move, which involves only the component L_1.

–2. The elementary homotopy with self-intersection point of the component L_2.

It is easy to check, that for an arbitrary 2-component framed link $(\mathbf{L}, \xi)$ with $c_0(\mathbf{L}) = k$ there exists a sequence Ξ of elementary transformations 1-2, which transforms the framed link $(\mathbf{L}, \xi)$ into the standard framed link $(\mathbf{L}_{Hopf}(k), \xi')$. The sequence Ξ induces the corresponding sequence $\Xi^{\uparrow}$ of transformations of the 3-component link $(\mathbf{L}^{\uparrow})$ into the 2-component link $(\mathbf{L}_{Hopf}(k)^{\uparrow})$. For an arbitrary framing ξ_1 of the component $L_{Hopf;1}$ of the link $\mathbf{L}_{Hopf}(k)$ the following equalities are satisfied: $\tilde{M}(\mathbf{L}_{Hopf}(k)^{\uparrow}) = \tilde{M}({}_3k\mathbf{L}_{Hopf}(1)^{\uparrow}) = k^4\tilde{M}(\mathbf{L}_{Hopf}(1)^{\uparrow}) = k^4\tilde{M}(\mathbf{L}_{Hopf}(1,1,1)) = 0$. Let us check that for each elementary transformation, listed above, the value $\tilde{M}(\mathbf{L}^{\uparrow})$ is fixed.

Let us start with the elementary transformation of type 1. In the formula (6.49) only the following terms are changed: $(1,2)(2,3)(3,1)\gamma^{\circ}$ and

$(2,3)(1,3)\beta^\circ(L_{1,1},L_{1,2})$, where components $L_{1,1},L_{1,2}$ are defined as the dubbing of the component L_1 by means of the given framing ξ. Each therm is changed by the value

$$2\sigma k(L_{1,1},L_{1,2})k(L_{1,1},L_3)k(L_{1,2},L_3),$$

where σ is the sign of the disappeared triangle in the singular point of the Δ-move. The jumps of the two summands are opposite and the total jump is trivial. It is proved that the value $\tilde{M}(\mathbf{L}^\uparrow)$ is not changed by the transformation of the type 1.

Let us consider an elementary transformation of type 2. In the formula (6.58) only the terms $(1,2)(2,3)(3,1)\gamma^\circ(\mathbf{L}^\uparrow)$, $(1,2)^2(2,3)^2\beta^\circ(L_{1,1},L_3)$, $(1,2)^2(1,3)^2\beta^\circ(L_{1,2},L_3)$ are changed. The jump of the first term is compensated by the sum of jumps of the second and the third terms. It is proved that the value $\tilde{M}(\mathbf{L}^\uparrow)$ is not changed by the transformation of the type 2. Claim 3 is proved. Lemma 15 is proved.

Let us define an asymptotic finite-type invariant, which we denote by M°. Let us consider the link $\mathbf{L}^+_{Hopf}((2,3),(3,1),(1,2))$, the component of this link denote by $L_{Hopf,i}$, $i=1,2,3$, the framings of the components could be arbitrary. Denote by $\mathbf{L}^{norm}_{Hopf}((2,3),(3,1),(1,2))$ the link with components $((2,3)L_{Hopf,1},(3,1)L_{Hopf,2},(1,2)L_{Hopf,3})$. Let us define function $Q((1,2),(2,3),(3,1))$, which gives a normalization of the invariant (below we shall prove that the function Q is a polynomial). The function Q depends on the 3 variables $(1,2)(2,3)(3,1)\neq 0$ by the formula:

$$Q((1,2),(2,3),(3,1))=\frac{\tilde{M}^\circ(\mathbf{L}^{norm}_{Hopf}((1,2),(2,3),(3,1)))}{(1,2)^4(2,3)^4(3,1)^4}, \tag{6.59}$$

and denote $Q=0$ in the case $(1,2)(2,3)(3,1)=0$. The required invariant is given by the following formula:

$$M^\circ(\mathbf{L})=\tilde{M}^\circ(\mathbf{L})-Q((1,2),(2,3),(3,1)). \tag{6.60}$$

For an arbitrary framed 3-component link $(\mathbf{L},\xi) = ((L_1,\xi_1),(L_2,\xi_2),(L_3,\xi_3))$, let us re-denote the framed link $((2,3)(L_1,\xi_1),(3,1)(L_2,\xi_2),(1,2)(L_3,\xi_3))$ by $(\mathbf{L}^{norm},\xi^{norm})$. Below the framings would be omitted, because the invariant is not depended on this framings (see Lemma 15). Obviously all the linking numbers of components of the link $\mathbf{L}^{norm}$ coincide and equal to the product $(1,2)(2,3)(3,1)$ of the pairwise linking numbers of the link $\mathbf{L}$.

Lemma 16.

1. Assuming $(1,2)(2,3)(3,1) \neq 0$*, the invariant* M°*, which is defined by the formula* (6.60)*, is not depended of a framing* ξ^{norm} *and is given by the following formula:*

$$M^\circ(\mathbf{L}) = \frac{\tilde{M}^\circ(\mathbf{L}^{norm}, \xi^{norm})}{(1,2)^4(2,3)^4(3,1)^4} =, \tag{6.61}$$

$$\frac{\gamma^\circ(\mathbf{L}^{norm})}{(1,2)(2,3)(3,1)} - \beta^\circ(L_1^{norm}, L_2^{norm}) - \beta^\circ(L_2^{norm}, L_3^{norm}) - \beta^\circ(L_3^{norm}, L_1^{norm}).$$

2. Assuming $(1,2)(2,3)(3,1) = 0$*, more precisely, assuming* $(1,2) = 0$*, the invariant* M° *is given by the following formula (comp. with* (4.6)*):*

$$M^\circ(\mathbf{L}) = (2,3)^2(3,1)^2\beta^\circ(L_1, L_2). \tag{6.62}$$

3. The function $Q((1,2),(2,3),(3,1))$*, which is defined by the formula* (6.59)*, is a polynomial.*

Proof of Lemma 16

Let us prove the statement 1. Using the formula (6.55) for each component, let us prove that the sides of the formula (6.61 coincide, if they coincide for an arbitrary link $\mathbf{L}$ with the given linking coefficients. By the normalized condition we may take $\mathbf{L} = L_{Hopf}((2,3),(3,1),(2,3))$. The statement is proved.

The statement 2 is evident.

The statement 3 follows from the formula (6.61), which is applied to the link $\mathbf{L}^{norm}_{Hopf}((2,3),(3,1),(1,2))$. The right side of the formula (6.61) is transformed by means of the formula (6.52). By the formula (6.52) it is sufficient to prove that the coefficient $c_1(\mathbf{L}^{norm}_{Hopf}, \xi^{norm}_{Hopf})$in the Conway polynomial is zero if at least one of the linking coefficients $(1,2)$, $(2,3)$, or $(3,1)$ is zero. In this case the link $(\mathbf{L}^{norm}, \xi^{norm})$ contains a small non-linked component and by the basic property of the Conway polynomial this condition is satisfied (see [P-S]). Lemma 16 is proved.

Proof of Theorem 9

Let us prove that the invariant M°, defined by the formula (6.61), or, by (6.60), satisfies the equation (6.7) for $s = 12$ (in this formula the invariant I has to be replaced by M°).

Let $(\mathbf{L}, \xi)$ be an arbitrary framed link, $(\mathbf{L}^{norm}, \xi^{norm})$ be its normalization, $r(\mathbf{L}, \xi)$, $r(\mathbf{L}^{norm}, \xi^{norm})$ be the links, which are obtained using r-time windings of the links $(\mathbf{L}, \xi)$, $(\mathbf{L}^{norm}, \xi^{norm})$ correspondingly. Let us consider the move Ξ of the link $(\mathbf{L}, \xi)$ into the link $(\mathbf{L}^{+}_{Hopf}((1,2),(2,3),(3,1))$, which consists of a sequence of Δ–moves, the operation of a changing of framings, and homotopies (self-intersections of different components is forbidden). Consider the corresponding move $r\Xi$ of the link $r(\mathbf{L}, \xi)$ into the link $r\mathbf{L}^{+}_{Hopf}((1,2),(2,3),(3,1))$, which is induced by Ξ. Using the formula (6.61) from Lemma 15 we may prove that the jumps $\delta M^\circ(\Xi)$, $\delta M^\circ(r\Xi)$ of the invariant M° by means of Ξ and by means of $r\Xi$ are related by the formula:

$$r^{12}\delta M^\circ(\Xi) = \delta M^\circ(r\Xi).$$

Let us re-denote $\mathbf{L}^{+}_{Hopf}((1,2)(2,3)(3,1))$ by $\mathbf{L}_{Hopf}$, and $(\mathbf{L}^{+}_{Hopf}((1,2)(2,3)(3,1)))^{norm}$ and $\mathbf{L}^{norm}_{Hopf}$ for short. Let us prove the following equation:

$$r^{12}M^\circ(\mathbf{L}_{Hopf}) + o(r^{12}) = \qquad (6.63)$$

$$M^\circ(r\mathbf{L}_{Hopf}).$$

After evident transformations the first term in the left side of the formula (6.63) is the following:

$$r^{12}((1,2)(2,3)(3,1))^{-4}\tilde{M}^\circ((\mathbf{L}_{Hopf})^{norm}).$$

The right side of the formula (6.63) is the following:

$$(r^6(2,2)(2,3)(3,1))^{-4})\tilde{M}^\circ(r^3(\mathbf{L}_{Hopf})^{norm}).$$

After the rescaling $r^3 \mapsto r$ the equation (6.63) is transformed in the following:

$$r^{12}\tilde{M}^\circ((\mathbf{L}_{Hopf})^{norm}) + o(r^{12}) = \qquad (6.64)$$

$$\tilde{M}^\circ(r(\mathbf{L}_{Hopf})^{norm}).$$

From Lemma 15 using the short denotations we get that the equation (6.64) is equivalent to the equation:

$$r^{12}\tilde{M}^\circ(\mathbf{L}_{Hopf}(k,k,k)) + o(r^{12}) = \tilde{M}^\circ(r(\mathbf{L}_{Hopf}(k,k,k))), \qquad (6.65)$$

where $k = (1,2)(2,3)(3,1)$. Using the equation (6.50) the previous equation is transformed as following:

$$\tilde{M}^{\circ}(r(\mathbf{L}^{+}_{Hopf}(k,k,k))) = o(r^{12}). \tag{6.66}$$

Let us prove the equation (6.66). The link $\mathbf{L}^{+}_{Hopf}(k,k,k)$ is represented by 3 parallel components of the type $(1,k)$ on the standard embedding torus. The link $r\mathbf{L}^{+}_{Hopf}(k,k,k)$ is defined using r-time winding of the each component of the link $\mathbf{L}^{+}_{Hopf}(k,k,k)$ along the given framing. Take the link $r\mathbf{L}^{+}_{Hopf}(k,k,k)$ in its isotopy class such that the 3-d component is in a neighborhood of the axis (OZ), and the components 1 and 2 are in the neighborhood of the standard circle S in the plane (x,y), this circle is the central line of the origin solid torus. With the links $\mathbf{L}^{+}_{Hopf}(k,k,k)$, $r\mathbf{L}^{+}_{Hopf}(k,k,k)$ let us consider also 2-component links $\mathbf{L}^{+}_{Hopf}(k,k)$, $r\mathbf{L}^{+}_{Hopf}(k,k)$, each such link is obtained by elimination of the 3-d component of the corresponding link. Denote the first link by (L_1, L_2), and the second link by (rL_1, rL_2) for short.

Let us transform the link $r(\mathbf{L}^{+}_{Hopf}(k,k,k))$ into the link $\mathbf{L}^{+}_{Hopf}(r^2k, r^2k, r^2k)$ by means of the following composition of homotopies $\Xi_3 \circ \Xi_2 \circ \Xi_1$.

Let us consider the central line S with the framing ξ, of the self-linking coefficient k. Denote the thin solid-torus U, its boundary by ∂U. On the boundary the coordinate system, which is related with ξ is well defined. Without loss of a generality we may assume that the component rL_1 coincides with r-time winding of S of the type $(1,r)$. The homotopy Ξ_1 is fixed on rL_1. Therefore we have $rL_1 = L'_1$.

Without loss of the generality we may assume that the component rL_1 is on the surface ∂V of the solid torus V, the thin of this torus is much more less then the thin of the solid torus U the solid torus V is closed to the surface $\partial U'$ of a bigger solid torus U', which is concentric of the solid torus U. The homotopy Ξ_1 is defined as a result of a shift of the solid torus V thought a short segment on the component rL_1 from $\partial U'$ to the central line S. The homotopy Ξ_1 has r intersections between a the short segment on rL_1 with the component rL_2. Let us denote by (L'_1, L'_2) the result of the homotopy.

Define the homotopy Ξ_2, which is a simplest homotopy in a neighborhood of the two closed parallel short segments on L'_1, L'_2, and which has exactly r intersection points between the components. This homotopy change the linking coefficient of the components L'_1, L'_2, from $kr^2 + r$ to kr^2.

As the result the homotopy the link $\Xi_1(r\mathbf{L}^{+}_{Hopf}(k,k,k))$ is transformed into a link $\Xi_2 \circ \Xi_1(r\mathbf{L}^{+}_{Hopf}(k,k,k))$, denoted by $(L"_1, L"_2, L"_3)$. Denote the homotopy $\Xi_2 \circ \Xi_1$ by Ψ for short. The homotopy Ψ is decomposed as the r elementary homotopies Ψ_i, $i = 1,\dots r$, each elementary homotopy has two intersection points of the components with different signs and keeps the linking coefficient:

$$\Psi = \Psi_r \circ \dots \Psi_1.$$

Define the homotopy Ξ_3, which transforms the link $(L"_1, L"_2, L"_3)$ into the link $\mathbf{L}^{+}_{Hopf}(r^2k, r^2k, r^2k)$. This homotopy is fixed on the component $L"_1$, on the components $L"_2$, $L"_3$, this homotopy, generally speaking, has self-intersection points. By the considered homotopy the component $L"_2$ (correspondingly $L"_3$) is transformed into an interior (correspondingly exterior) winding of the torus ∂U, which is parallel to the winding $L"_1$. An analogous homotopy is considered in Lemma 15. The homotopy Ξ_3 keeps the value M°.

Let us denote by δM° the jump of the invariant M° by the homotopy Ψ. Let us re-denote the first two components of the link by L_1, L_2. Denote by $\delta c_1(L_1, L_2)$ the jump of the coefficient c_1, when the homotopy Ψ is restricted on the sublink of the 1-th and the 2-d components, denote by $\delta c_1(L_1, L_2, L_3)$ the jump of the coefficient c_1 of the homotopy of the 3-component link. Using the equation (6.52) it is sufficient to prove the following equations:

$$\delta c_1(L_1, L_2) = O(r^3), \tag{6.67}$$

$$\delta c_1(L_1, L_2, L_3) = O(r^5). \tag{6.68}$$

Investigate the jump of the coefficient $c_1(L_1, L_2)$ by the elementary homotopy Ψ_i. By the homotopy Ψ_i the jump $\delta(c_1)_i$ of the coefficient c_1 has the order ir, the sum of the jumps $\sum_{i=1}^{r} \delta(c_1)_i$ has the order r^3. The formula (6.67) is proved.

Investigate the jump of the coefficient $c_1(L_1, L_2, L_3)$ by the elementary homotopy Ψ_i. By the homotopy Ψ_i the jump $\delta(c_1)_i$ of the coefficient c_1 has the order ir^3, the sum of the jumps $\sum_{i=1}^{r} \delta(c_1)_i$ has the order r^5. The formula (6.68) is proved. The formula (6.7) for $I = M^{\circ}$ is proved.

The formula (6.8) is proved by analogously calculations. The asymptotic invariant M° is well-defined and Theorem 9 is proved.

6.2.3 Conjecture

The invariant M° for 3-component links given by (6.60), or, by (6.61), coincides with the integral invariant M, given by (4.5).

This Conjecture is proved in [A3] up to polynomial, which depends on pairwise linking numbers of the components.

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