A REPARAMETRISATION OF THE SPHERE THROUGH A CONFORMAL MAPPING BETWEEN THE SPHERE AND A RIEMANN SURFACE

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ABSTRACT. Conformal transformations are of much interest to modelers of physical phenomena as they provide many attractive mathematical properties such as locally preserving the isotropy of scales, invariance of the structure of the operators such as Laplacian under the transformation. It is known to atmosphere and ocean modelers as to generate coordinate transformations on the sphere using the analytic functions belonging to the class of Mobius transformations which are linear and one-to-one in the complex plane. This work describe the method to use the analytic function that belongs to the class other than the Mobius transformations. Especially the complex power function is used to generate a reparametrisation of the sphere so as to provide variable resolution geomtry on the sphere. It is shown how the High resolution Tropical Belt Transformation is generated from this analytic function. While it is not possible to generate coordinate transformations on the sphere with this class of functions, it is indeed possible to achieve reparametrisation of the sphere. Construction of the Riemann surface is used to achieve this reparametrisation.

1. INTRODUCTION

Spherical harmonics based global spectral method under the triangular truncation provides an uniform resolution discretisation on the sphere. Variable resolution global spectral method is achieved by the spherical harmonics that have the spatially localised spectrum. The pioneering work

Key words and phrases. Riemann surface, spherical geometry, Schmidt transformation, High resolution Tropical Belt Transformation, conformal maps, reparametrisation maps.

by [Sch77] devised a conformal transformation of the spherical surface to itself. The standard spherical harmonics are transformed by the conformal transformation to generate a new spherical harmonic basis functions. This transformed spherical harmonic basis functions had spatially-localised spectrum so as to provide an non-uniform spectral discretisation on the sphere. A finer resolution spectral discretisation was achieved over one of the poles with a corresponding defocussing of resolution at the other pole. Specifically the conformal transformation used by [Sch77] belongs to the class of Mobius transformations of the complex plane [CG88]. In the context of global ocean modeling, [BEDJ99] generated a coordinate transformation based on the Mobius transformations. The class of transformations represented by the Mobius transformations provide a one-to-one correspondence for the points on the extended complex plane. These class of transformations are linear in nature. The search for other kinds of conformal transformations on the sphere which are different from the Mobius transformations is an ongoing quest. The work by [JNM12] came up with a variable resolution global spectral method using spherical harmonics. It used a reparametrisation map named 'High resolution Tropical Belt Transformation(HTBT)' to generate finer resolution spectral discretisation of the tropics on the sphere. This article describes the generation of HTBT from a complex power function. This is analytic function is of a different class than the Mobius transformation. By default, it is a many-to-one mapping in the complex plane. So the generation of a coordinate transformation on the sphere using this function is non-trivial. It requires the application of the concept of Riemann surface construction. Section 2 shows why it is not possible to generate a coordinate transformation on the sphere using a many-to-one functions like the complex power function. Section 3 describes the construction of a Riemann surface that induces a one-to-one correspondence with the points of the complex plane for the complex power function. Section 4 decribes a generation of an abstract 2-manifold from the Riemann surface. The generation of the Reparametrisation map using the conformal map between the sphere and the Riemann surface is described in section 5. The final section of this article provides the concluding remarks of the work.

2. TRANSFORMATION ON THE SPHERE USING THE COMPLEX POWER FUNCTION

Consider the unit sphere

$$S^2 := \left\{ (\cos\lambda\cos\phi, \sin\lambda\cos\phi, \sin\phi) \,|\, \lambda \in [0, 2\pi),\, \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}$$

A point P on S^2 is refered by the (λ, ϕ) coordinates where λ is the longitude and ϕ is the latitude. The north-pole (NP) and south-pole(SP) are refered by $\phi = \frac{\pi}{2}$ and $\phi = -\frac{\pi}{2}$ respectively.

Now we attempt to generate a coordinate transformation $\tau : S^2 \to S^2$ through the analytic function $f(z) = z^{\ell}, \ell > 1, \ell \in \mathrm{Z}_+$.

Note that the function f $:\bar{\mathrm{C}} \longmapsto \bar{\mathrm{C}}$ is analytic everywhere except at $\mathrm{z} = \infty$. Here $\bar{\mathrm{C}} = \mathrm{C} \cup \infty$ is the extended complex plane. Since $f'(z) \neq 0$ everywhere in $\bar{\mathrm{C}}$ except at $\mathrm{z} = 0$, and $\mathrm{z} = \infty$, it is conformal in $\bar{\mathrm{C}}$ except at the points $\mathrm{z} = 0$, and $\mathrm{z} = \infty$.

Definition 1. The stereographic projection ρ can be expressed in terms of the (λ, ϕ) coordinates as

$$\rho : \mathrm{S}^2 \longmapsto \bar{\mathrm{C}} \text{ such that } \mathrm{z} = \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\exp(\iota\lambda)$$

Definition 2. The inverse of stereographic projection

$\rho^{-1} : \bar{\mathrm{C}} \longmapsto \mathrm{S}^2$ is given by $(\lambda, \phi) = (\arg(\mathrm{z}), 2\arctan(|\mathrm{z}|) - \frac{\pi}{2})$.

Remark 3. The stereographic projection ρ defines an homeomorphism between S^2 and $\bar{\mathrm{C}}$ [Ahl78].

Let w be the image of z under the mapping $\mathrm{f(z)} = \mathrm{z}^{\ell}$, $\ell > 1$.

Since $\mathrm{w} \in \bar{\mathrm{C}}$, it corresponds to a point P' on S^2. We denote P' by the coordinate pair (λ', ϕ'),

where $\lambda' = \mathrm{Arg(w)}, \phi' = 2\arctan(|w|) - \frac{\pi}{2}$.

To get the relationship between $P(\lambda, \phi)$ and $P'(\lambda', \phi')$, we make use of the functional relation $\mathrm{w} = \mathrm{z}^{\ell}$.

i.e. $\tan\left(\frac{\pi}{4} + \frac{\phi'}{2}\right)\exp(\imath\lambda') = \tan^{\ell}\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\exp(\imath\lambda)$.

Comparing the modulus and argument of the above complex valued expression, we get

$$\tan\left(\frac{\pi}{4} + \frac{\phi'}{2}\right) = \tan^{\ell}\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \tag{2.1}$$

$$\lambda' = \ell\lambda + 2k\pi, k = 0, \pm 1, \pm 2, \ldots \tag{2.2}$$

2.1. **Solution of eqn 2.2.** $\lambda' = \ell\lambda + 2\mathrm{k}\pi$, k is an integer.

Since $\lambda \in [0, 2\pi)$ and $\lambda' \in [0, 2\pi)$.

$$0 \leq \ell\lambda + 2\mathrm{k}\pi < 2\pi$$

$$\frac{2(\mathrm{k}-1)}{\ell}\pi \leq \lambda < \frac{2\mathrm{k}}{\ell}\pi$$

for $0 \leq \lambda < 2\pi$.

Let

$$\mathrm{I_k} := \left[\frac{2(\mathrm{k}-1)}{\ell}\pi, \frac{2\mathrm{k}}{\ell}\pi\right)$$

$$\mathrm{B_k} := \{(\lambda,\phi)|\lambda \in \mathrm{I_k}\}$$

If $\lambda \in \mathrm{I_k}$, $\lambda' = \ell\lambda + \sigma 2\pi$ for some integer σ and $\lambda' \in [0,2\pi)$, then $\sigma = -(\mathrm{k}-1)$.

i.e.

$$\lambda' = \ell\lambda - 2\pi(\mathrm{k}-1) \text{ if } \lambda \in \mathrm{I_k}, \mathrm{k} = 1,2,\ldots,\ell$$

2.2. **Solution of eqn 2.1.** $\tan\left(\frac{\pi}{4}+\frac{\phi'}{2}\right) = \tan^{\ell}\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$, $\ell > 1$.

Let $\alpha = \frac{\pi}{4}+\frac{\phi'}{2}$, $\beta = \frac{\pi}{4}+\frac{\phi}{2}$. for $\alpha,\beta \in [0,\pi/2]$.

Then $\tan\alpha = \tan^{\ell}\beta$.

If we set $\tan\beta = \gamma$.

This implies $\alpha = \arctan[\gamma^{\ell}]$.

$\implies \alpha = \psi + m\pi, \qquad m = 0,\pm1,\pm2,\ldots$ and $0 \le \psi \le \frac{\pi}{2}$.

Since ϕ' is a latitudinal coordinate and by choosing $\phi' \in [-\frac{\pi}{2},\frac{\pi}{2}]$, we get $\alpha \in [0,\pi/2]$.

So $0 \le \psi + m\pi \le \frac{\pi}{2} \implies m = 0$.

i.e. $\phi' = 2\arctan\left(\tan^{\ell}\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right) - \frac{\pi}{2}$.

Claim 4. Given a λ' such that $(\lambda',\phi') \in (S^2)'$, there are exactly ℓ points that map to (λ',ϕ') , unless it is a pole.

Proof. Suppose $(\lambda,\phi) \longmapsto (\lambda',\phi')$ and $(\lambda',\phi') \in B_k$.

i.e. For $\lambda' = \ell\lambda - 2\pi(k-1)$ then $\rho^{-1}\circ f \circ \rho : (\lambda + \frac{2\pi n}{\ell},\phi) \longmapsto (\lambda',\phi')$. where $n = -(k-1), -(k-1)+1, -(k-1)+2, \ldots (\ell-k)$.

This transformation is conformal on the sphere excepting at the poles. But due to the multi-valued nature, it does not qualify as a coordinate transformation on the sphere. □

3. CONSTRUCTION OF THE RIEMANN SURFACE R_w

It was shown in the previous section that it was not possible to construct a coordinate transformation $\tau : S^2 \to S^2$ using the complex analytic function f :$\bar{\mathrm{C}} \longmapsto \bar{\mathrm{C}}$ defined by $\mathrm{f(z)} = \mathrm{z}^\ell$, $\ell > 1$. It was due to the fact that this function was not one-to-one in the extended complex plane $\bar{\mathrm{C}}$. Given a $z \neq 0$, and defining $\omega := e^{\frac{2\pi}{\ell}}$, the points $\{z\omega^k : k = 1, 2, \ldots.., \ell\}$all have the same image under this function.

We proceed to construct a new set R_w which will have the property $f : \bar{\mathrm{C}} \mapsto R_w$ is one-to-one. Here, R_w:= the Riemann surface for the ℓ-th root function, inducing a conformal and one-to-one map $F_\ell \circ f = I$ on R_w.

Definition 5. Let us define, for $z \neq 0$, Arg z to be the angle made by the vector z with respect to the positive real axis, measured counter-clockwise, and so normalized that $0 \leq$ Arg $z < 2\pi$.

Definition 6. Let $A_k := \left\{ z | \frac{2(k-1)}{\ell}\pi \leq \mathrm{Arg}\, z < \frac{2k}{\ell}\pi \right\} \cup \{0\}$ be a subset of C.

Definition 7. Now define the set $B_k := \left\{ w | w = z^\ell, z \in A_k \right\}$.

Fact. *By definition B_k is the entire complex plane, except that it holds the property $Arg\, w < 2\pi$.*

Fact.

$$\bigcup_{k=1}^{\ell} A_k = C.$$

Fact.

$$\bigcap_{k=1}^{\ell} A_k = \{z | z = 0\}$$

Definition 8. Let us define, for $z \neq 0$, arg z to be the angle made by the vector z with respect to the positive real axis, measured counter-clockwise. It has any one of an infinite number of real vaules differing by an integral multiples of 2π.

Fact.

$$w \in B_k \implies 2(k-1)\pi \leq \arg w < 2k\pi \;\; \textit{and} \;\; 0 \leq \operatorname{Arg} w < 2\pi$$

Remark. The function $f(z) = z^\ell$ defines an one-to-one correspondence between the sets A_k and B_k.

Now we construct a new set R_w as follows

$$R_w := B_1 \times \{1\} \cup B_2 \times \{2\} \cup \cdots \cup B_\ell \times \{\ell\}$$

An arbitrary point P belonging to R_w be denoted by $w;k$. The notation $w;k$ refers to a complex number w belonging to the set B_k.

i.e. $w;k \in R_w \Rightarrow w \in B_k$, for some $k \in \{1,2,\ldots,\ell\}$.

Following properties are imposed on the construction of R_w.

(1) For $m,n \in \{1,2,\ldots,\ell\}$,

$$m \neq n \Rightarrow w_1;m \neq w_2;n \quad \text{even if} \quad w_1 = w_2.$$

(2) For given $w;k \in R_w$,

$$\lim_{\operatorname{Arg} w \longrightarrow 2\pi} w;k \longrightarrow w';(k+1) \bmod \ell$$

where $|w'| = |w| \quad \& \quad \operatorname{Arg} w' = 0$.

(3) $0;1 = 0;2 = 0;3 = \cdots = 0;\ell$. We denote $0;k$ as 0. Then $\cap_{k=1}^{\ell} B_k = 0$.

This construction of the set R_w is the Riemann surface on which the function $z = w^{\frac{1}{\ell}}$ is well-defined.

4. CONSTRUCTION OF AN ABSTRACT 2-MANIFOLD M

We proceed to construct an abstract 2-manifold M by applying the inverse stereographic projection ρ^{-1} on the Riemann surface R_w.

For $w;k \in R_w$ (i.e. $w \in B_k$, $k \in \{1,2,....,\ell\}$),

$$\rho^{-1}(w;k) = (\lambda_k, \phi_k)$$

where $\lambda_k =$ Arg(w) and $\phi_k =$ 2arctan(|w|)-$\frac{\pi}{2}$.

Let

$$S_k := \left\{ (\cos\lambda_k \cos\phi_k, \sin\lambda_k \cos\phi_k, \sin\phi_k) \,|\, \lambda_k \in [0,2\pi),\, \phi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \right\}$$

We can define a new map $\hat{\rho}$:$B_k \mapsto S_k$ given by

$\hat{\rho}(w;k) = (\cos\lambda_k \cos\phi_k, \sin\lambda_k \cos\phi_k, \sin\phi_k)$

Fact. *$\hat{\rho}$ defines an homeomorphism between B_k and S_k.*

Now we construct a new set M as follows

$$M := S_1 \times \{1\} \cup S_2 \times \{2\} \cup \cdots \cup S_\ell \times \{\ell\}$$

By virtue of the continuity of $\hat{\rho}$ and the construction of R_w, M inherits the following properties.

(1) For $m, n \in \{1,2,...,\ell\}$, $m \neq n \implies$

$$(\cos\lambda_m \cos\phi_m, \sin\lambda_m \cos\phi_m, \sin\phi_m) \quad \neq \quad (\cos\lambda_n \cos\phi_n, \sin\lambda_n \cos\phi_n, \sin\phi_n)$$

,even if $(\lambda_m, \phi_m) = (\lambda_n, \phi_n)$.

(2) For given $(\cos\lambda_k \cos\phi_k, \sin\lambda_k \cos\phi_k, \sin\phi_k) \in M$,

$$\lim_{\lambda_k \longrightarrow 2\pi} (\cos\lambda_k \cos\phi_k, \sin\lambda_k \cos\phi_k, \sin\phi_k) \quad \longrightarrow \quad (\cos\lambda_n \cos\phi_n, \sin\lambda_n \cos\phi_n, \sin\phi_n)$$

where $n = (k+1)$ mod ℓ and $\lambda_n = 0,\ \phi_n = \phi_k$.

(3) $\bigcap_{k=1}^{\ell} S_k = \{(0,0,1),(0,0,-1)\}$.

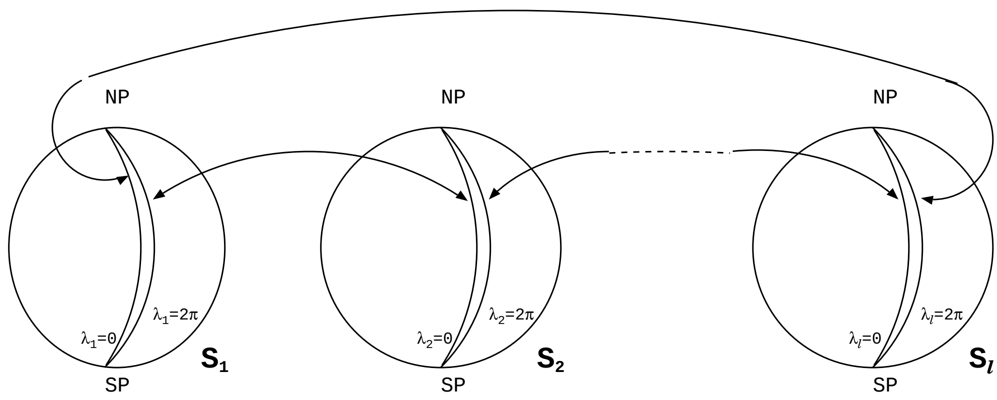

FIGURE 4.1. Illustration of the abstract 2-manifold M

Remark 9. (λ_k, ϕ_k) are the local coordinates of M , with k indiactes the points of the set $S_k \subset M$.

Remark 10. A global coordinate system (λ', ϕ') can be defined for the points of M.

Let $(\lambda', \phi') = (2(k-1)\pi + \lambda_k, \phi_k)$ for $k \in \{1, 2, ..., \ell\}$.

Clearly, $\lambda' \in \{0, 2\pi\ell)$ and $\phi' \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

For a given point P' refered by the global coordinates (λ', ϕ') , the local coordinate (λ_k, ϕ_k) is given by the following relations.

$k = \lambda' | 2\pi$, $\lambda_k = \lambda' \mod 2\pi$ and $\phi_k = \phi'$.

Fig. 4.1 illustrates the abstract 2-manifold M . It is a manifold (Riemann surface) with ℓ sheets . Each sheet S_k is a sphere with local coordinates (λ_k, ϕ_k), $1 \leq k \leq \ell$. Each of the spherical sheet S_k has an infinitesimal cut along the longitude $\lambda_k = 2\pi$. The longitude $\lambda_k = 2\pi$ of the sheet S_k is conneted to the sheet S_{k+1} at the longitude $\lambda_{k+1} = 0$, $1 \leq k < \ell$. The longitude $\lambda_\ell = 2\pi$is connected to $\lambda_1 = 0$ of the sheet S_1.

5. A NEW REPARAMETRIZATION OF THE SPHERE

Notice that there is an one-to-one relation between the points on the sphere S^2 and the points of the 2-manifold M .

The commutative diagram given below illustrates this correspondence.

$$\begin{tikzcd} S^2 \arrow[r, "\rho"] \arrow[d, dashed, "\tau"'] & \bar{C} \arrow[d, "f"] \\ M & R_w \arrow[l, "\hat{\rho}"] \end{tikzcd}$$

The correspondence is given by

$$\tau \equiv \hat{\rho} \circ f \circ \rho$$

The map can be described by $\tau : S^2 \longmapsto M$ with

$$\tau(\lambda,\phi) = (\lambda',\phi') = \left(\ell\lambda, 2\arctan\left[\tan^{\ell}\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right] - \frac{\pi}{2}\right) \tag{5.1}$$

where (λ',ϕ') are the global coordinates of M.

The inverse map can be described by $\eta : M \longmapsto S^2$ with

$$\eta(\lambda',\phi') = (\lambda,\phi) = \left(\lambda'/\ell, 2\arctan\left[\tan^{\frac{1}{\ell}}\left(\frac{\pi}{4}+\frac{\phi'}{2}\right)\right] - \frac{\pi}{2}\right) \tag{5.2}$$

Notice that the one-to-one correspondence between the coordinates of S^2 and those of M induces a new reparametrisation for S^2.

If we denote the standard longitude-latitude parametrisation on the sphere as

$$\sigma = (\cos\lambda\cos\phi, \sin\lambda\cos s\phi, \sin\phi)$$

and if we denote $\eta(\lambda',\phi') = (\eta_1,\eta_2)$.

Then $\sigma \circ \eta$ provides a new parametrisation of S^2 with respect to (λ',ϕ') coordinates.

i.e. the coordinate function

$$(\cos\eta_1(\lambda')\cos\eta_2(\phi'), \sin\eta_1(\lambda')\cos\eta_2(\phi'), \sin\eta_2(\phi'))$$

relate a given pair (λ', ϕ') to an unique point on the sphere S^2 (the north and south pole will be refered by $\phi' = -\frac{\pi}{2}$ and $\phi' = \frac{\pi}{2}$ respectively).

Refer to textbooks such as [Pre01] for a detailed description of Reparametrisation maps.

The eqns. 5.1 and 5.2 both refer to the 'High resolution Tropical Belt Transformation(HTBT)' described in [JNM12].

6. CONCLUDING REMARKS

We have constructed an abstract 2-manifold M from the Riemann surface R_w . The one-to-one correspondence between the points on the sphere S^2 and the manifold M is achieved by this construction. R_w being a Riemann surface, is itself a 2-dimensional C^∞ manifold [Mir95]. M being diffeomorphic to R_w offers some useful mathematical properties. They are

(1) M is a smooth manifold diffeomorphic to the sphere S^2 .
(2) τ is a C^∞ diffeomorphism between S^2 and M.
(3) τ is not conformal at the poles of the sphere.

From the previous works ([CG88, BEDJ99]), it is known that it is possible to generate a coordinate transformation on the sphere $\tau : S^2 \to S^2$ using the class of Mobius transformations which are linear and one-to-one in the complex plane.

In this work, it is shown that it is not possible to generate a coordinate transformation $\tau : S^2 \to S^2$ using a multi-valued complex function such as $f(z) = z^\ell$, $\ell > 1$. But it is possible to create diffeomorphism to a manifold M such that $\tau : S^2 \to M$. This is achieved through the construction of a Riemann surface. Noteworthy is the fact that the coordinates (λ', ϕ') of M through the coordinate function $\sigma \circ \eta$, become a new coordinate system for the sphere as well. This aspect has some important implications for generating variable resolution methods on the sphere. To generate variable resolution methods, one could create numerical discretisation with the uniform distribution of points on M. The computed results upon mapping back to the sphere will be of variable resolution. Equivalently, to achieve variable resolution one could do computations on the sphere S^2 with the uniform distribution of points in (λ', ϕ') through the coordinate function $\sigma \circ \eta$. The latter approach was in fact demonstrated in the work of [JNM12].

The future work will be to identify other class of complex analytic functions using which multiple regions of the sphere can be studied with fine resolution.

ACKNOWLEDGEMENTS

I gratefully acknowledge the Isaac Newton Institute for Mathematical Sciences and the programme on 'Multiscale Numerics of the Atmosphere and Ocean(AMM)' for hosting my participation and visit. I thank Dr. Gautam Bharali, Department of Mathematics, Indian Institute of Scince for his valuable inputs in refining the proofs of the section 1 and also verifying the Riemann surface construction presented in section 2. It is my pleasure to acknowledge the useful discussions with Dr. Jason Cantarella, (Department of Mathematics, University of Atlanta,GA) a participant of the Newton Institute programme on 'Topological dynamics in the Physical and Biological Scinces(TOD)'. I fondly remember the stimulating discussions with Rob Kusner(Department of mathematics and statistics,University of Massachusettes,Amherst,MA) of the TOD programme. My sincere thanks are to Dr. Hilary Weller and other organisers of the AMM programme for the encouragement and support.

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