

Cauchy - Riemann differential equations of the spherical geometry

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Abstract

Schmidt's problem of constrained spherical coordinate transformation for numerical solution of partial differential equations in spherical geometry, involves solving a system of coupled first-order non-linear partial differential equations. It is established that this system is related to the canonical Cauchy - Riemann differential equations of complex variable theory. As a consequence of discovering this connection, many new non-trivial coordinate transformations on S^2 are generated through complex analytic functions.

Keywords: Cauchy - Riemann equations , spherical geometry, coordinate transformation, reparametrization map, variable resolution spectral method.

1. Introduction

The problem of finding a special class of spherical coordinate transformations which leave the differential operators $\nabla(\bullet) \cdot \nabla(\star)$, $\nabla(\bullet) \times \nabla(\star)$ and $\nabla^2(\bullet)$ invariant¹ was posed by F. Schmidt [1]. We refer to this problem as Schmidt's problem of spherical coordinate transformation . This problem arose in the context of variable-resolution spectral numerical discretization of the global atmospheric flows. The above -mentioned class of transformations were employed to generate localized spherical harmonic basis functions out of the standard spherical harmonics. However, only two types of non-trivial transformations have been discovered so far. They are the Schmidt transformation reported in Refs. [1, 2] and the other one is the High-resolution Tropical Belt Transformation reported Ref. [3]. The question remains open as to the existence of other transformations belonging to this class.

We show that it is possible to generate many more non-trivial coordinate transformations and reparameterization maps of S^2 . This is done by establish-

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¹ \bullet and \star are used to denote distinct scalar operands of the underlying gradient operators.

ing the connection between the system of coupled non-linear partial differential equations associated with Schmidt's problem and the canonical form of Cauchy - Riemann differential equations of analytic function theory. By virtue of this connection, it is demonstrated that complex analytic functions can be used to generate many non-trivial coordinate transformations on S^2 belonging to this special class.

2. Cauchy - Riemann differential equations of S^2

Let S^2 be the unit sphere and let (λ, ϕ) be the standard longitude - latitude parametrization such that $\lambda \in [0, 2\pi)$ and $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For a given (λ, ϕ) , an unique point P on S^2 is identified through the coordinate function

$$\sigma(\lambda, \phi) = (\cos \lambda \cos \phi, \sin \lambda \cos \phi, \sin \phi). \quad (2.1)$$

Consider a transformation of spherical coordinates τ given by

$$\tau(\lambda, \phi) = (\lambda', \phi').$$

This transformation can further be denoted as

$$\tau(\lambda, \phi) = (\tau_1(\lambda, \phi), \tau_2(\lambda, \phi)),$$

where

$$\begin{aligned} \lambda' &= \tau_1(\lambda, \phi), \\ \phi' &= \tau_2(\lambda, \phi). \end{aligned}$$

The inverse transformation is denoted as η and can be described as

$$\eta(\lambda', \phi') = (\eta_1(\lambda', \phi'), \eta_2(\lambda', \phi')) = (\lambda, \phi).$$

Let a be any sufficiently smooth scalar function on S^2 . The horizontal gradient operator is given by

$$\nabla a = \begin{pmatrix} \frac{1}{\cos \phi} \frac{\partial a}{\partial \lambda} \\ \frac{\partial a}{\partial \phi} \end{pmatrix}$$

The scalar function a defined on the sphere, transforms as

$$a(\lambda, \phi) = a'(\tau(\lambda, \phi)) = a'(\lambda', \phi'),$$

where prime denotes the dependence of a given quantity on the transformed coordinates (λ', ϕ') .

Then the gradient operator in the transformed coordinates is expressed as

$$\nabla' a' = \begin{pmatrix} \frac{1}{\cos \phi'} \frac{\partial a'}{\partial \lambda'} \\ \frac{\partial a'}{\partial \phi'} \end{pmatrix}$$

The relation between the operators ∇ and ∇' is given by the matrix-vector equation,

$$\nabla a = M \cdot \nabla' a'$$

where

$$M = \begin{pmatrix} \frac{\cos \tau_2}{\cos \phi} \frac{\partial \tau_1}{\partial \lambda} & \frac{1}{\cos \phi} \frac{\partial \tau_2}{\partial \lambda} \\ \cos \tau_2 \frac{\partial \tau_1}{\partial \phi} & \frac{\partial \tau_2}{\partial \phi} \end{pmatrix} \quad (2.2)$$

Note that M is the Jacobian matrix corresponding to the transformation of the gradient operator under τ .

Schmidt [1] proved that whenever M satisfies the matrix equation

$$M^T M = \det(M) I, \quad (2.3)$$

the transformation τ ensures the invariance of the differential operators as

$$\begin{aligned} \nabla(\bullet) \cdot \nabla(\star) &\equiv f \cdot (\nabla'(\bullet) \cdot \nabla'(\star)) \\ \nabla(\bullet) \times \nabla(\star) &\equiv f \cdot (\nabla'(\bullet) \times \nabla'(\star)) \\ \Delta(\bullet) &\equiv f \cdot \Delta'(\bullet) \end{aligned} \quad (2.4)$$

where $f = \det(M)$ is the determinant of the matrix M and I is the identity matrix. The symbols \bullet and \star correspond to two distinct scalar operands of the gradient operator. Here $\nabla(\bullet) \cdot \nabla(\star)$ denotes the scalar product of the gradients, $\nabla(\bullet) \times \nabla(\star)$ is the vector product of the gradients and Δ is the Laplacian operator on S^2 .

Lemma. *A 2×2 real matrix M satisfies the equation (2.3), if and only if it is of the form*

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where $a, b \in \mathbf{R}$ and $a^2 + b^2 \neq 0$.

Proof. The converse part of the statement is trivial. So we prove only the forward part.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the 2×2 matrix such that $a, b, c, d \in \mathbf{R}$.

Then $\det(M) = ad - bc$.

$$\text{Eqn. (2.3)} \implies \begin{aligned} a^2 + c^2 &= ad - bc \\ b^2 + d^2 &= ad - bc \\ ab + cd &= 0 \end{aligned} \implies \begin{aligned} (a-d)^2 + (b+d)^2 &= 0 \\ ab + cd &= 0 \end{aligned} \implies \begin{aligned} a &= d \\ b &= -c \end{aligned}$$

□

Since the matrix (2.2) satisfies equation (2.3), the above lemma lead to

$$\begin{aligned} \frac{\cos \tau_2}{\cos \phi} \frac{\partial \tau_1}{\partial \lambda} &= \frac{\partial \tau_2}{\partial \phi}, \\ \cos \tau_2 \frac{\partial \tau_1}{\partial \phi} &= -\frac{1}{\cos \phi} \frac{\partial \tau_2}{\partial \lambda}. \end{aligned} \quad (2.5)$$

Any coordinate transformation τ satisfying the equation (2.5) will have the special property given by (2.4).

A non-trivial solution to eqn. (2.5) was given in Ref. [1] and it is of the form

$$\begin{aligned} \tau_1(\lambda, \phi) &= \lambda, \\ \tau_2(\lambda, \phi) &= \arcsin \left[\frac{(1 - c^2) + (1 + c^2) \sin \phi}{(1 + c^2) + (1 - c^2) \sin \phi} \right], \quad c > 1. \end{aligned} \quad (2.6)$$

Eqn.(2.6) is known in the meteorological literature as ‘Schmidt’s transformation’. Schmidt’s transformation is a pole-symmetric dilatation on the sphere. Subsequent works such as [4, 2] explored on the question of finding other non-trivial transformations and arrived at the conclusion that eqn. (2.6) is the only non-trivial one of the type $\tau : [0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow [0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

[3] showed by relaxing the assumption that $\lambda' \in [0, 2\pi)$, the system (2.5) can admit other solutions.

A reparametrization map named ‘High-resolution Tropical Belt Transformation(HTBT)’ was found as yet another non-trivial solution to eqn. (2.5) and it is of the form

$$\begin{aligned}\tau_1(\lambda, \phi) &= \ell \lambda, \\ \tau_2(\lambda, \phi) &= \arcsin \left[\frac{(1 + \sin \phi)^\ell - (1 - \sin \phi)^\ell}{(1 + \sin \phi)^\ell + (1 - \sin \phi)^\ell} \right].\end{aligned}\quad (2.7)$$

, where $\ell > 1$.

It is clear that HTBT is bijective map of the type $\tau : [0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow [0, 2\pi\ell) \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Specifically $\lambda' \in [0, 2\pi\ell)$.

HTBT provides a reparametrization of S^2 . i.e. given an unique pair of (λ', ϕ') , an unique point on S^2 is identified through the coordinate function

$$\sigma(\eta(\lambda', \phi')) = (\cos \eta_1(\lambda', \phi') \cos \eta_2(\lambda', \phi'), \sin \eta_1(\lambda', \phi') \cos \eta_2(\lambda', \phi'), \sin \eta_2(\lambda', \phi')).\quad (2.8)$$

Here η is the inverse of HTBT given by

$$\begin{aligned}\eta_1(\lambda', \phi') &= \frac{\lambda'}{\ell}, \\ \eta_2(\lambda', \phi') &= \arcsin \left[\frac{(1 + \sin \phi) \frac{1}{\ell} - (1 - \sin \phi) \frac{1}{\ell}}{(1 + \sin \phi) \frac{1}{\ell} + (1 - \sin \phi) \frac{1}{\ell}} \right].\end{aligned}\quad (2.9)$$

[3].

Thus by relaxing the assumption that the solutions need not be of the type $\tau : [0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow [0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$, one more solution to eqn. (2.5) was generated.

The Schmidt transformation referred by eqn. (2.6) and HTBT referred by eqn. (2.7) are the only known non-trivial solutions of eqn. (2.5) so far.

Consider a change of variables into the system of eqn. (2.5) with following substitution

$$\begin{aligned}(u, v) &= \left(\frac{1}{2} \ln \left[\frac{1 + \sin \tau_2}{1 - \sin \tau_2} \right], \tau_1 \right), \\ (x, y) &= \left(\frac{1}{2} \ln \left[\frac{1 + \sin \phi}{1 - \sin \phi} \right], \lambda \right).\end{aligned}\quad (2.10)$$

As a result of this substitution, the system of eqns. (2.5) takes the form

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}\quad (2.11)$$

which is the canonical form of the Cauchy - Riemann differential equations of complex function theory. The change of variables (eqn. (2.10)) established the relation between the canonical form of Cauchy - Riemann equations and the system of coupled first-order PDEs governing the special class of spherical coordinate transformations (eqn. (2.5)). Henceforth the above mentioned system of coupled first-order PDEs will be called as ‘Cauchy - Riemann differential equations of spherical geometry’.

3. Nontrivial Solutions of Cauchy-Riemann equations of S^2

We now seek to find further reparametrization maps of the type $\tau : [0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \Omega \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\Omega \subset \mathbf{R}$.

It is well known that the complex analytic functions of the form $u(x, y) + i v(x, y) = f(x + iy)$ are the solutions of the standard Cauchy-Riemann equations [5, 6]. Any solution of eqn.(2.11) induces a solution of eqn. (2.5) by virtue of the relationship given by eqn.(2.10) . i.e. the solutions of Cauchy - Riemann equations of S^2 are given by

$$\frac{1}{2} \ln \left[\frac{1 + \sin \tau_2}{1 - \sin \tau_2} \right] + i \tau_1 = f \left(\frac{1}{2} \ln \left[\frac{1 + \sin \phi}{1 - \sin \phi} \right] + i \lambda \right) \quad (3.1)$$

For a given analytic function f , by comparing the real and imaginary parts in the eqn. (3.1) , we obtain the transformation $\tau(\lambda, \phi) = (\tau_1(\lambda, \phi), \tau_2(\lambda, \phi))$. This fact leads us to the following result.

Theorem. *The general solution of the system of eqns. (2.5) is of the form*

$$\begin{aligned} \tau_1(\lambda, \phi) &= \text{Im} \left[f \left(\frac{1}{2} \ln \left[\frac{1 + \sin \phi}{1 - \sin \phi} \right] + i \lambda \right) \right] \\ \frac{1}{2} \ln \left[\frac{1 + \sin \tau_2(\lambda, \phi)}{1 - \sin \tau_2(\lambda, \phi)} \right] &= \text{Re} \left[f \left(\frac{1}{2} \ln \left[\frac{1 + \sin \phi}{1 - \sin \phi} \right] + i \lambda \right) \right], \end{aligned} \quad (3.2)$$

where f is an arbitrary complex analytic function .

Remark 1. Among the non-trivial solutions $\tau = (\tau_1, \tau_2)$, only those transformations $\tau : (\lambda, \phi) \rightarrow (\lambda', \phi')$ that are bijective can qualify either as a coordinate transformation or as a reparametrization map of S^2 .

These coordinate transformations and reparameterization maps can be used to generate variable resolution grid point configuration on the sphere as follows.

Remark 2. Let τ be a given transformation. From its inverse transformation η , and an equi-spaced distribution of points in the space of (λ', ϕ') , a variable resolution mesh point configuration is generated on the sphere through $\sigma(\eta(\lambda', \phi'))$ as given by eqn. (2.8).

We illustrate the construction of few coordinate transformations and reparametrization maps of S^2 through some elementary analytic functions.

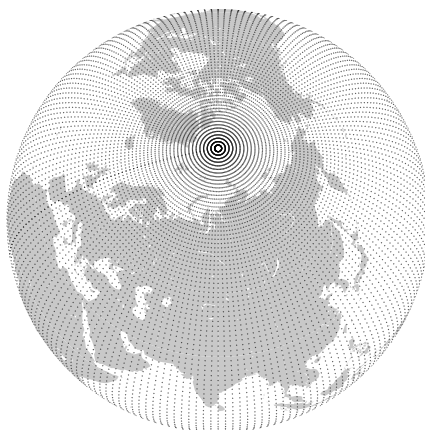


Figure 3.1: Variable resolution mesh configuration generated from the transformation corresponding to the analytic function $f(z) = z + \frac{1}{2} \ln c^2$, $c = 2$. This variable resolution mesh configuration has high resolution grid centered at the north pole. The density of the grid-point distribution falls off smoothly away from the north-pole and it become coarsest at the south pole.

3.1. $f(z) = z + k$, where k is a real constant

Eqn. (3.1) \implies

$$\frac{1}{2} \ln \left[\frac{1 + \sin \tau_2}{1 - \sin \tau_2} \right] + i \tau_1 = \frac{1}{2} \ln \left[\frac{1 + \sin \phi}{1 - \sin \phi} \right] + i \lambda + k$$

For the special case of $k = \frac{1}{2} \ln c^2$ for some $c > 1$, we get the Schmidt transformation (eqn. (2.6)).

Fig. 3.1 illustrate the variable resolution mesh point configuration corresponding to Schmidt transformation with $c = 2$. The grid-points are clustered at the north pole and the density of the grid-points decrease smoothly away from the northern polar region. Similarly Fig. 3.2 illustrates the variable resolution mesh-point configuration corresponding to a case where $0 < c < 1$ and specifically $c = \frac{1}{2}$. In this case, the mesh-points are clustered around the south-pole with the density of the mesh-point distribution falling off smoothly away from this region.

3.2. $f(z) = \ell z$, where $\ell > 1$ is a constant

Eqn. (3.1) \implies

$$\frac{1}{2} \ln \left[\frac{1 + \sin \tau_2}{1 - \sin \tau_2} \right] + i \tau_1 = \ell \left(\frac{1}{2} \ln \left[\frac{1 + \sin \phi}{1 - \sin \phi} \right] + i \lambda \right)$$

We get the High resolution Tropical Belt Transformation (HTBT) (referred by eqn. (2.7)). Fig. 3.3 illustrates the variable mesh-point configuration corresponding to HTBT. The mesh-point are clustered in the equatorial belt region with their density decreasing smoothly as we away from the tropical belt.

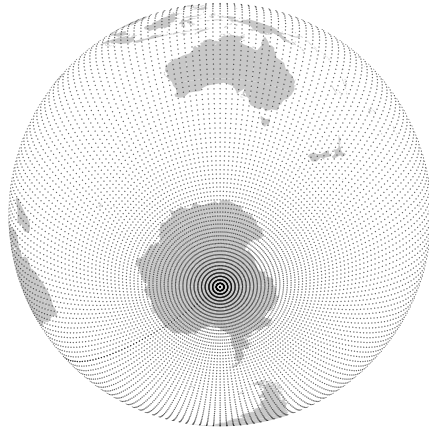


Figure 3.2: Variable resolution mesh configuration generated from the transformation corresponding to the analytic function $f(z) = z + \frac{1}{2} \ln c^2$, $c = \frac{1}{2}$. This variable resolution mesh configuration has high- resolution grid center at the south-pole. The density of grid point distribution decreases smoothly away from the south-pole and it become coarsest at the north-pole.



Figure 3.3: Variable resolution mesh configuration generated from the transformation corresponding to the analytic function $f(z) = \ell z$, $\ell = 3$. This variable resolution mesh configuration has an high-resolution zone over Tropical Belt centred at the equator. The density of mesh-point distribution decreases smoothly from the equatorial belt and become coarsest at the polar regions.

3.3. $f(z) = az + b$, where a and b are some constants

Let us take $a = \ell$ and $b = \frac{1}{2} \ln c^2$.

Eqn. (3.1) \implies

$$\frac{1}{2} \ln \left[\frac{1 + \sin \tau_2}{1 - \sin \tau_2} \right] + i \tau_1 = \ell \left(\frac{1}{2} \ln \left[\frac{1 + \sin \phi}{1 - \sin \phi} \right] + i \lambda \right) + \frac{1}{2} \ln c^2$$

,We get a new transformation

$$\begin{aligned} \tau_1(\lambda, \phi) &= \ell \lambda \\ \tau_2(\lambda, \phi) &= \arcsin \left[\frac{c^2(1 + \sin \phi)^\ell - (1 - \sin \phi)^\ell}{c^2(1 + \sin \phi)^\ell + (1 - \sin \phi)^\ell} \right] \end{aligned} \quad (3.3)$$

Notice that the above transformation reduces to the schmidt transformation (eqn. (2.6)) for the case $c \neq 1$ and $\ell = 1$, whereas it reduces to HTBT(referred by eqn. (2.7)) for $c = 1$ and $\ell \neq 1$. Thus , a new reparametrization map which is more general than the earlier one can be constructed.

Fig. 3.4 and Fig. 3.5 illustrate the two different variable resolution mesh-point configuration corresponding to the transformation τ referred by eqn. 3.3. Fig. 3.4 shows an variable resolution mesh-point configuration with clustering points around a middle latitude region of the northern hemisphere, while Fig. 3.5 shows a variable resolution mesh-point configuration which clusters the points around a middle latitude of the southern hemisphere. In both the cases, the density of mesh points decrease smoothly as move away from the middle latitude. The centre of higher resolution depends on the choice of c . When $c > 1$, the centre of high resolution occurs on the northern hemisphere midlatitude wheareas for $0 < c < 1$, the centre of high resolution occurs on the southern middle latitude of the sphere.

3.4. $f(z) = z^2$

Eqn. (3.1) \implies

$$\frac{1}{2} \ln \left[\frac{1 + \sin \tau_2}{1 - \sin \tau_2} \right] + i \tau_1 = \left[\frac{1}{2} \ln \left[\frac{1 + \sin \phi}{1 - \sin \phi} \right] + i \lambda \right]^2$$

The above equation gives the transformation

$$\begin{aligned} \tau_1(\lambda, \phi) &= \ln \left(\frac{1 + \sin \phi}{1 - \sin \phi} \right)^\lambda \\ \frac{1}{2} \ln \left[\frac{1 + \sin \tau_2(\lambda, \phi)}{1 - \sin \tau_2(\lambda, \phi)} \right] &= \frac{1}{4} \ln^2 \left(\frac{1 + \sin \phi}{1 - \sin \phi} \right) - \lambda^2 \end{aligned} \quad (3.4)$$



Figure 3.4: Variable resolution mesh configuration from the transformation corresponding to the analytical function $f(z) = az + b$, $a = 2$, $b = \frac{1}{2}ln2^2$. This variable resolution mesh configuration has high-resolution zone over a middle latitude of northern hemisphere. The density of mesh-point distribution decreases smoothly as away from this centre. The resolution becomes coarser at both the poles.



Figure 3.5: Variable resolution mesh configuration from the transformation corresponding to the analytical function $f(z) = az + b$, $a = \frac{1}{2}$, $b = \frac{1}{2}ln2^2$. This variable resolution mesh configuration has a high-resolution zone over a middle latitude of southern hemisphere. The density of mesh-point distribution decreases smoothly as away from this centre. The resolution becomes coarser at both the poles.

Note here that both of the coordinate functions τ_1 and τ_2 are dependent on both λ and ϕ . This is unlike the previous cases where τ_1 was dependent on only λ and τ_2 was dependent only ϕ . At present, it is not obvious as how to generate variable resolution mesh-point configuration from an equi-spaced distribution of points in the space of (λ', ϕ') for this transformation.

4. Conclusions

Schmidt's problem of finding a class of spherical coordinate transformations which leave the differential operators $\nabla(\bullet) \cdot \nabla(\star)$, $\nabla(\bullet) \times \nabla(\star)$ and $\nabla^2(\bullet)$ invariant was addressed in this work. i.e. the class of coordinate transformations and reparameterization maps of S^2 satisfying the condition as given by eqn. (2.4). These are the solutions of the equations (2.5). We established that the system of eqns. (2.5) is connected to the canonical Cauchy - Riemann equations of the complex function theory. By virtue of this connection, the said system of equations are named as Cauchy-Riemann differential equations of S^2 . i.e. given a complex analytic function f , a solution to the Schmidt's problem can be generated through system of eqns. (3.2). We have achieved the general solution to the Schmidt's problem of spherical coordinate transformation as a consequence.

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