

DISPERSIVE BEHAVIOUR OF HIGH ORDER FINITE ELEMENT SCHEMES FOR THE ONE-WAY WAVE EQUATION

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ABSTRACT. We study the ability of high order numerical methods to propagate discrete waves at the same speed as the physical waves in the case of the one-way wave equation. A detailed analysis of the finite element method is presented including an explicit form for the discrete dispersion relation and a complete characterisation of the numerical Bloch waves admitted by the scheme. A comparison is made with the spectral element method and the discontinuous Galerkin method with centred fluxes. It is shown that all schemes admit a spurious mode. The spectral element method is always inferior to the finite element and discontinuous Galerkin schemes; a somewhat surprising result in view of the fact that, in the case of the *second order wave equation*, the spectral element method propagates waves with an accuracy *superior* to that of the finite element scheme. The comparative behaviour of the finite element and discontinuous Galerkin scheme is also somewhat surprising: the accuracy of the finite element method is superior to that of the discontinuous Galerkin method in the case of elements of odd order by two orders of accuracy, but worse, again by two orders of accuracy, in the case of elements of even order.

1. INTRODUCTION AND SUMMARY OF MAIN RESULTS

Consider the one-way wave equation for a given wave-speed $c > 0$,

$$\partial_t u + c\partial_x u = 0 \quad x \in \mathbb{R}, t > 0 \quad (1)$$

with suitable initial data. A key feature of the equation is the existence of non-trivial, spatially propagating solutions for each given temporal frequency ω ,

$$u(x, t) = e^{i\omega t} U(x) \quad (2)$$

where $U(x) = e^{-ikx}$, $k = \omega/c$. The relation between the wavenumber and the temporal frequency is known as the *dispersion relation* for the continuous problem. The function U satisfies a *Bloch wave condition*

$$U(x + h) = \lambda U(x), \quad x \in \mathbb{R}, h \in \mathbb{R} \quad (3)$$

where $\lambda = e^{-ikh}$ is the Floquet multiplier.

Let $X_{h,N}$ denote the space of continuous, piecewise polynomials of degree N on the grid $h\mathbb{Z}$,

$$X_{h,N} = \{v \in C(\mathbb{R}) : v|_{(x_m, x_{m+1})} \in \mathbb{P}_N, \quad m \in \mathbb{Z}\} \quad (4)$$

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where $x_m = mh$. A semi-discrete approximation of the one-way wave equation may be defined by seeking $u_{h,N} \in X_{h,N}$ such that

$$\int_{\mathbb{R}} (\partial_t u_{h,N} + c \partial_x u_{h,N}) \bar{v} dx = 0, \quad v \in X_{h,N} \quad (5)$$

along with appropriate initial conditions. A key issue [4] when assessing any spatial discretisation scheme for the one-way wave equation is the existence of non-trivial Bloch wave solutions of the discrete problem (5). These solutions again take the form (2),

$$u_{h,N}(x, t) = e^{i\omega t} U_{h,N}(x) \quad (6)$$

with the essential difference that the function $U_{h,N}$ must belong to the discrete space $X_{h,N}$, and satisfy a discrete Bloch wave condition

$$U_{h,N}(x + h) = \lambda_{h,N} U_{h,N}(x), \quad x \in \mathbb{R} \quad (7)$$

with the discrete Floquet multiplier $\lambda_{h,N}$ depending on the mesh-size h and the polynomial degree N .

The ability of the numerical scheme to propagate waves in space faithful to the true propagating waves depends on the accuracy with which the discrete Floquet multiplier approximates the true Floquet multiplier. The relative accuracy $R_{h,N}$ of the approximation is defined by

$$R_{h,N} = \frac{\lambda - \lambda_{h,N}}{\lambda}. \quad (8)$$

and our aim is to study the behaviour of this ratio as $\omega h/c \rightarrow 0$, for any polynomial order N . Some authors prefer to introduce a *discrete wavenumber*, $k_{h,N}$, satisfying

$$e^{-ihk_{h,N}} = \lambda_{h,N} \quad (9)$$

and to study the relative accuracy of the approximation $k \approx k_{h,N}$, where $k = \omega/c$ is the true wavenumber, given by

$$E_{h,N} = \frac{k - k_{h,N}}{k}. \quad (10)$$

These measures are related in the case where $k - k_{h,N}$ is small as follows

$$R_{h,N} = \frac{e^{-ikh} - e^{-ik_{h,N}h}}{e^{ikh}} = 1 - e^{-i(k_{h,N}-k)h} \approx -i(k - k_{h,N}) = -ikE_{h,N}. \quad (11)$$

As such, the choice of whether to study $R_{h,N}$ or $E_{h,N}$ is purely a matter of taste in the case where $k - k_{h,N}$ is small. However, it should be borne in mind that the condition (9) does not define a unique value of $k_{h,N}$. Care must be taken in selecting the value of $k_{h,N}$ satisfying (9) appropriately in order to avoid drawing incorrect conclusions. Moreover, if $k - k_{h,N}$ is *not* small, then there is no simple relation between $R_{h,N}$ and $E_{h,N}$. For these reasons, our preference is to study the relative accuracy of the discrete Floquet multiplier directly, since it is this quantity that appears in the Bloch wave condition and is uniquely defined.

Our first result establishes an algebraic condition on the discrete Floquet multiplier in terms of the order N , the mesh-size h and the wavenumber ω/c under which a non-trivial discrete Bloch wave may exist.

Theorem 1. *There exists a non-trivial Bloch wave solution of problem (5) of the form*

$$u_{h,N}(x, t) = e^{i\omega t} \sum_{m \in \mathbb{Z}} \lambda_{h,N}^m \phi(x - mh) \quad (12)$$

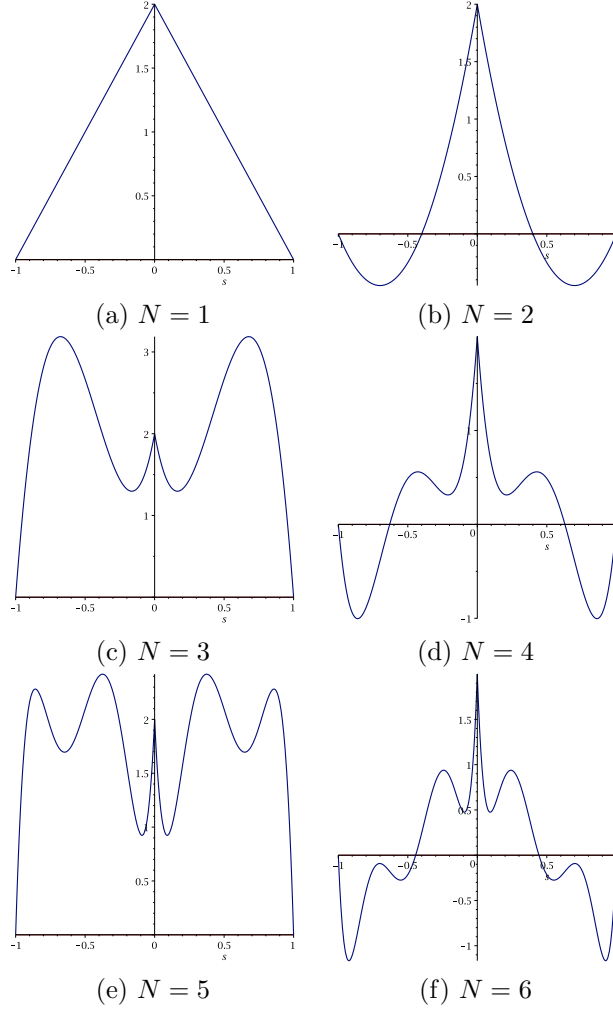


FIGURE 1. Plots of the basic function ϕ appearing in the Bloch wave expansion (12) in the case where $\Omega = 2$ for polynomial degree N from 1 to 6.

where $\phi \in X_{h,N}$, if and only if $\lambda_{h,N}$ is a solution of the algebraic equation

$$\overline{v_N(\omega h/c)} (\lambda - q_N(\omega h/c)) + (-1)^N v_N(\omega h/c) \left(\frac{1}{\lambda} - \overline{q_N(\omega h/c)} \right) = 0, \quad (13)$$

where $q_N(\Omega) = w_N(\Omega)/\overline{v_N(\Omega)}$, $v_N(\Omega)$ and $w_N(\Omega)$ are defined in Theorem 4.

The function ϕ appearing in the Bloch wave expansion (12) is a piecewise polynomial supported on $(-h, h)$ that depends on the polynomial degree N and $\Omega = \omega h/c$. The function is constructed as part of the proof of Theorem 1 given in Section 3. Figure 1 shows the function in the case $h = 1$ and $\omega = 2c$ for polynomial degree N from 1 to 6.

The next result, also proved in Section 3, addresses the accuracy of the Floquet multiplier (or if one prefers, thanks to (11), the accuracy of the discrete wavenumber):

Theorem 2. *The discrete dispersion relation (13) has two solutions $\lambda_{h,N}^P$ and $\lambda_{h,N}^S$, one of which corresponds to a physical mode $\lambda_{h,N}^P \approx e^{-i\omega h/c}$ and the other of which corresponds to a spurious mode. More precisely,*

$$\frac{e^{-i\omega h/c} - \lambda_{h,N}^P}{e^{-i\omega h/c}} = \frac{i}{2} \left[\frac{N!}{(2N+1)!} \right]^2 \begin{cases} -\frac{2N+1}{N+1} \left(\frac{\omega h}{c} \right)^{2N+1}, & N \text{ even,} \\ \frac{N+1}{2N+3} \left(\frac{\omega h}{c} \right)^{2N+3}, & N \text{ odd} \end{cases} \quad (14)$$

and

$$\lambda_{h,N}^S = (-1)^N \frac{v_N(\omega h/c)}{v_N(\omega h/c)\lambda_{h,N}^P} \approx (-1)^N \frac{v_N(\omega h/c)}{v_N(\omega h/c)} e^{i\omega h/c}. \quad (15)$$

Theorem 2 shows that the discrete dispersion relation always has a solution corresponding to the physical wave $e^{i\Omega x}$ but, in addition, there is always a second solution corresponding to a spurious propagating wave that is non-physical. Figure 2 shows the true physical wave plotted along with both propagating discrete Bloch waves in the case $h = 1$, $\omega = 2c$. Figure 2(a) corresponds to piecewise linear approximation which is completely unable to resolve the physical wave. However, Figures 2(b)-(e) show that the physical wave is rapidly resolved as the polynomial degree is increased on a fixed mesh of size $h = 1$.

Interestingly, the nature of the spurious discrete Bloch wave is revealed as a higher frequency mode characterised by sharp peaks. The presence of a spurious mode is less than ideal, but has not proved to be a serious problem in applications provided one adopts the common practice of employing a numerical filter [6,8] that removes the mode as a post-processing operation.

2. COMPARISON WITH SPECTRAL ELEMENT AND CENTRED DISCONTINUOUS GALERKIN SCHEMES

A number of alternative numerical schemes have been advocated for the approximation of problems of type (1). The spectral element method [5,9,11] is equivalent to the finite element method used in conjunction with a Gauss-Lobatto quadrature rule for the evaluation of the element matrices, and is attractive computationally because it leads to a diagonal mass matrix. The discontinuous Galerkin method [3,7] is widely used for computational wave propagation and is also computationally attractive because it leads to a block diagonal mass matrix. We confine our attention to the variant with centred fluxes since, like the finite and spectral element schemes, it is conservative and as such makes for a more natural comparison. All three schemes exhibit a single, undamped, spurious mode. The interested reader may consult [1] for the analysis of the dispersive properties of the discontinuous Galerkin method with non-centred fluxes.

Table 1 presents the leading term in the relative error in the approximation of the Floquet multiplier for each of the methods in the cases $N = 1, \dots, 5$. The results for the finite element scheme are special cases of the general result (14) proved in Theorem 2, whilst the results for the centred discontinuous Galerkin scheme are

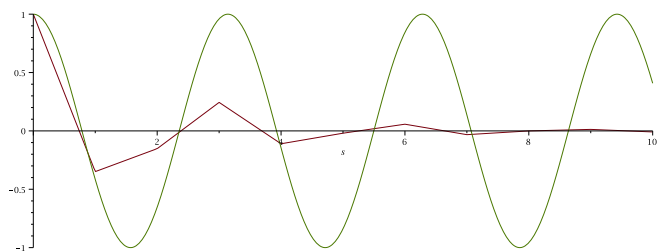
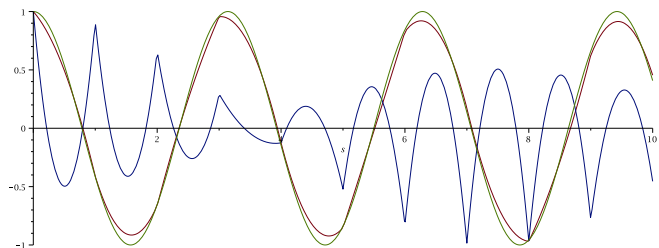
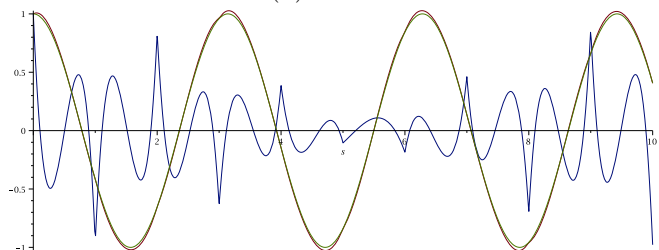
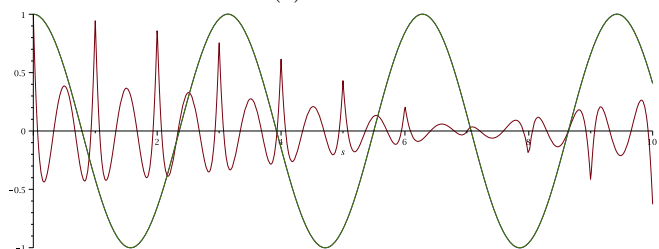
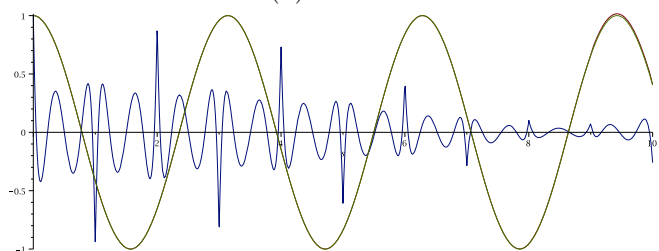
(a) $N = 1$ (b) $N = 2$ (c) $N = 3$ (d) $N = 4$ (e) $N = 5$

FIGURE 2. Plots of the Bloch wave expansion (12) in the case where $\Omega = 2$ for polynomial degree N from 1 to 5. The true physical mode $e^{i\Omega}$ is shown along with the Bloch waves corresponding to both the physical zero $\lambda_{h,N}^P$ and the spurious zero $\lambda_{h,N}^S$.

Degree	Centred DG	Finite Element	Spectral Element
1	$\frac{i\Omega^3}{48}$	$\frac{i\Omega^5}{180}$	$\frac{i\Omega^3}{6}$
2	$-\frac{i\Omega^7}{16800}$	$-\frac{i\Omega^5}{4320}$	$\frac{i\Omega^5}{1080}$
3	$\frac{i\Omega^7}{806400}$	$\frac{i\Omega^9}{3175200}$	$\frac{i\Omega^7}{75600}$
4	$-\frac{i\Omega^{11}}{1005903360}$	$-\frac{i\Omega^9}{254016000}$	$\frac{i\Omega^9}{31752000}$
5	$\frac{i\Omega^{11}}{120708403200}$	$\frac{i\Omega^{13}}{479480601600}$	$\frac{i\Omega^{11}}{838252800}$

TABLE 1. Leading terms for the relative error in the approximation of the Floquet multiplier for Centred Discontinuous Galerkin, Finite Element and Spectral Element schemes applied to (1)

special cases of the general result proved in Theorem 2 of [1]. It seems that a general result is not available concerning the the spectral element scheme [9]; we have obtained the entries in Table 1 by direct computation.

Table 1 shows that the accuracy of the spectral element method is always inferior to the finite element and centred discontinuous Galerkin methods both in terms of the order of convergence and the magnitude of the coefficient of the leading term in the error. It would be easy to dismiss the inferior behaviour of the spectral element scheme as an inevitable by-product arising from the use of reduced order integration were it not for the somewhat surprising fact [2] that, in the case of the *second order wave equation*, the spectral element method propagates waves with an accuracy *superior* to that of the finite element scheme.

The comparison of the finite element method and centred discontinuous Galerkin schemes is less clear-cut with both methods exhibiting *superconvergence*. That is to say, the order of convergence for the centred discontinuous Galerkin scheme in the case of *even* order elements is two orders larger than one might expect. Conversely, the finite element scheme exhibits superconvergence in the case of *odd* order elements. The relative behaviour of the finite element and discontinuous Galerkin schemes becomes even more remarkable if one compares the expression for the leading term in the relative error (14) for the finite element scheme with that of the centred discontinuous Galerkin scheme presented in Theorem 2 of [1]:

$$\frac{e^{-i\omega h/c} - \lambda_{h,N}^{DG}}{e^{-i\omega h/c}} = \frac{i}{2} \left[\frac{N!}{(2N+1)!} \right]^2 \begin{cases} \frac{2N+1}{N+1} \left(\frac{\omega h}{c} \right)^{2N+1}, & N \text{ odd} \\ -\frac{N+1}{2N+3} \left(\frac{\omega h}{c} \right)^{2N+3}, & N \text{ even.} \end{cases} \quad (16)$$

The reader will observe that the expression is virtually identical to the corresponding expression (14) for the finite element scheme (the only difference being that the

'odd' and 'even' cases are interchanged) despite the quite different nature of the schemes.

3. PROOFS

Let ${}_1F_1$ denote the confluent hypergeometric function defined by the series

$${}_1F_1(a, b, z) = 1 + \frac{a}{b}z + \frac{a}{b} \frac{a+1}{b+1} \frac{z^2}{2!} + \frac{a}{b} \frac{a+1}{b+1} \frac{a+2}{b+2} \frac{z^3}{3!} + \dots \quad (17)$$

so that, in particular,

$${}_1F_1(-n; -2n; z) = \sum_{\ell=0}^n \frac{(2n-\ell)!}{(2n)!} \frac{n!}{(n-\ell)!} \frac{z^\ell}{\ell!}. \quad (18)$$

Let \mathcal{L} denote the operator defined by

$$\mathcal{L} v = v' + \frac{1}{2}i\Omega v. \quad (19)$$

Theorem 3. For $N \in \mathbb{N}$, define

$$\Phi_N = \sum_{\ell=0}^N (-i\Omega)^\ell \frac{(2N+2-\ell)!}{(2N+2)!} P_\ell^{(N-\ell+1, N-\ell+1)} \quad (20)$$

where $P_\ell^{(\alpha, \alpha)}$ denotes the Jacobi polynomial of degree ℓ , orthogonal with respect to the weight function $(1-x^2)^\alpha$. Then, Φ_N is a polynomial of degree N satisfying

(i)

$$\Phi_N(\pm 1) = {}_1F_1(-N-1; -2N-2; \mp i\Omega) - \frac{(N+1)!}{(2N+2)!} (\mp i\Omega)^{N+1}, \quad (21)$$

so that, in particular, $\Phi_N(-1) = \overline{\Phi_N(1)}$,

(ii)

$$\mathcal{L} \Phi_N = -\frac{1}{2} \frac{(N+2)!}{(2N+2)!} (-i\Omega)^{N+1} P_N^{(1,1)} \quad (22)$$

(iii) for all $v \in \mathbb{P}_{N+1} \cap H_0^1(-1, 1)$ there holds

$$\int_{-1}^1 \mathcal{L} \Phi_N(x) v(x) dx = 0, \quad (23)$$

(iv)

$$\int_{-1}^1 \mathcal{L} \Phi_N(x) dx = -2 \frac{(N+1)!}{(2N+2)!} (-i\Omega)^{N+1} \begin{cases} 1, & N \text{ even} \\ 0, & N \text{ odd} \end{cases} \quad (24)$$

and

$$\int_{-1}^1 \mathcal{L} \Phi_N(x) x dx = -2 \frac{(N+1)!}{(2N+2)!} (-i\Omega)^{N+1} \begin{cases} 0, & N \text{ even} \\ 1, & N \text{ odd.} \end{cases} \quad (25)$$

Proof. (i) Inserting the identity

$$P_\ell^{(N-\ell+1, N-\ell+1)}(\pm 1) = (\pm 1)^\ell \frac{(N+1)!}{\ell!(N-\ell+1)!}$$

into the expression for $\Phi_N(\pm 1)$ gives

$$\Phi_N(\pm 1) = \sum_{\ell=0}^{N+1} \frac{(2N-\ell+2)!}{(2N+2)!} \frac{(N+1)!}{(N-\ell+1)!} \frac{(\mp i\Omega)^\ell}{\ell!} - \frac{(N+1)!}{(2N+2)!} \frac{(\mp i\Omega)^{N+1}}{(N+1)!},$$

and then comparing with (18) gives (ii).

(ii) A simple computation using the identity

$$\frac{d}{dx} P_\ell^{(N-\ell+1, N-\ell+1)} = \frac{1}{2}(2N-\ell+3)P_{\ell-1}^{(N-\ell+2, N-\ell+2)}, \quad \ell = 1, 2, \dots$$

along with the fact that $P_0^{(N+1, N+1)}$ is constant gives the first result.

(iii) Let $v \in \mathbb{P}_{N+1} \cap H_0^1(-1, 1)$ so that $v = (1-x^2)w$ for some $w \in \mathbb{P}_{N-1}$. The result then follows at once from (ii) in view of the orthogonality of $P_N^{(1,1)}$ to \mathbb{P}_{N-1} with respect to the inner product with weight $1-x^2$.

(iv) The first identity follows by integrating (22) and using the result

$$P_N^{(1,1)}(x) = \frac{2}{N+2} \frac{d}{dx} P_{N+1}^{(0,0)}. \quad (26)$$

The second identity follows by taking the product of (22) with x , and integrating by parts again making use of (26), and the orthogonality properties of $P_{N+1}^{(0,0)}$. \square

The polynomials defined in Theorem 3 are used to construct a pair of new polynomials $\Phi_E, \Phi_O \in \mathbb{P}_N$ as follows.

Theorem 4. *For given $N \in \mathbb{N}$, there exists a pair of polynomials Φ_E, Φ_O of degree N satisfying $\Phi_E(\pm 1) = \Delta_N(\Omega)$, $\Phi_O(\pm 1) = \pm \Delta_N(\Omega)$, where*

$$\Delta_N(\Omega) = \Phi_{N-1}(1)\overline{\Phi_N(1)} - \Phi_N(1)\overline{\Phi_{N-1}(1)},$$

and in addition,

$$\int_{-1}^1 \mathcal{L} \Phi_E(x)v(x) dx = \int_{-1}^1 \mathcal{L} \Phi_O(x)v(x) dx = 0 \quad (27)$$

for all $v \in \mathbb{P}_N \cap H_0^1(-1, 1)$. Moreover,

$$\int_{-1}^1 (\mathcal{L} \Phi_E(x) + x \mathcal{L} \Phi_O(x)) dx = -\frac{2N!}{(2N)!} (-i\Omega)^N \left((-1)^N w_N(\Omega) + \overline{w_N(\Omega)} \right) \quad (28)$$

where

$$w_N(\Omega) = {}_1F_1(-N-1; -2N-2; -i\Omega) - \frac{i\Omega}{2(2N+1)} {}_1F_1(-N; -2N; -i\Omega). \quad (29)$$

Furthermore,

$$\int_{-1}^1 (1+x) (\mathcal{L} \Phi_E(x) - \mathcal{L} \Phi_O(x)) dx = 2 \frac{2N!}{(2N)!} (-i\Omega)^N v_N(\Omega) \quad (30)$$

and

$$\int_{-1}^1 (1-x) (\mathcal{L} \Phi_E(x) + \mathcal{L} \Phi_O(x)) dx = 2 \frac{2N!}{(2N)!} (i\Omega)^N \overline{v_N(\Omega)} \quad (31)$$

where

$$v_N(\Omega) = {}_1F_1(-N-1; -2N-2; -i\Omega) + \frac{i\Omega}{2(2N+1)} {}_1F_1(-N; -2N; -i\Omega). \quad (32)$$

Proof. Existence of Φ_E (and likewise Φ_O) is by construction. We seek Φ_E in the form

$$\Phi_E = \alpha_{N-1}\Phi_{N-1} + \alpha_N\Phi_N$$

where α_{N-1} and α_N are coefficients, and observe that $\mathcal{L}\Phi_E$ satisfies condition (27) automatically thanks to Theorem 3(iii). The coefficients are chosen so that

$$\alpha_{N-1} = -\Phi_N(1) + \overline{\Phi_N(1)}; \quad \alpha_N = \Phi_{N-1}(1) - \overline{\Phi_{N-1}(1)}$$

and

$$\beta_{N-1} = \Phi_N(1) + \overline{\Phi_N(1)}; \quad \beta_N = -\Phi_{N-1}(1) - \overline{\Phi_{N-1}(1)}.$$

Straightforward computation using (24) and (25) gives

$$\begin{aligned} \int_{-1}^1 (\mathcal{L}\Phi_E(x) + x\mathcal{L}\Phi_O(x)) dx = \\ -2 \frac{N!}{(2N)!} (-i\Omega)^N \begin{cases} \alpha_{N-1} - \frac{i\Omega}{2(2N+1)}\beta_N, & N \text{ odd} \\ \beta_{N-1} - \frac{i\Omega}{2(2N+1)}\alpha_N, & N \text{ even.} \end{cases} \end{aligned}$$

Substituting for the coefficients and simplifying gives (28) with the difference that $w_N(\Omega)$ is replaced by $w'_N(\Omega)$ given by

$$w'_N(\Omega) = \Phi_N(1) - \frac{i\Omega}{2(2N+1)}\Phi_{N-1}(1).$$

Substituting for $\Phi_N(1)$ and $\Phi_{N-1}(1)$ using (21) and simplifying gives

$$w'_N(\Omega) = w_N(\Omega) - \frac{N!}{(2N+1)!} (-i\Omega)^{N+1}$$

which, in turn, gives

$$(-1)^N w'_N(\Omega) + \overline{w'_N(\Omega)} = (-1)^N w_N(\Omega) + \overline{w_N(\Omega)}$$

and (28) follows.

The proofs of (30) and (31) proceed in a similar fashion using (24) and (25) and substituting for the coefficients to obtain (30) and (31) with the difference that $v_N(\Omega)$ is replaced by $v'_N(\Omega)$ given by

$$v'_N(\Omega) = \Phi_N(1) + \frac{i\Omega}{2(2N+1)}\Phi_{N-1}(1).$$

Substituting for $\Phi_N(1)$ and $\Phi_{N-1}(1)$ as before and simplifying reveals that $v'_N(\Omega) = v_N(\Omega)$ and the results follow as claimed. \square

Lemma 1. Let $\mathcal{E}_N(\Omega)$ be defined by

$$\mathcal{E}_N(\Omega) = \frac{e^{i\Omega} - \overline{q_N(\Omega)}}{e^{i\Omega}}. \quad (33)$$

Then,

$$\mathcal{E}_N(\Omega) = - \left[\frac{N!}{(2N+1)!} \right]^2 \frac{\Omega^{2N+2}}{2} \left\{ 1 - \frac{2N+2}{(2N+1)(2N+3)} i\Omega + \mathcal{O}(\Omega^2) \right\}. \quad (34)$$

Proof. Define $\mathcal{D}_N(\Omega)$ by the condition

$$e^{i\Omega}\mathcal{D}_N(\Omega) = e^{i\Omega} - \frac{{}_1F_1(-N; -2N; i\Omega)}{{}_1F_1(-N; -2N; -i\Omega)}, \quad (35)$$

so that $\mathcal{D}_N(\Omega)$ is the relative error in the diagonal Padé approximation of the exponential [10]. It may be shown [12] that

$$\mathcal{D}_N(\Omega) = i \left[\frac{N!}{(2N)!} \right]^2 \frac{\Omega^{2N+1}}{2N+1} \{1 + \mathcal{O}(\Omega^2)\}. \quad (36)$$

Identity (35) allows one to write ${}_1F_1(-N; -2N; i\Omega)$ and ${}_1F_1(-N-1; -2N-2; i\Omega)$ in terms of $\mathcal{D}_N(\Omega)$ and $\mathcal{D}_{N+1}(\Omega)$ respectively. Inserting these expressions into the formula for $q_N(\Omega)$ gives, after some computation, the following expression for $\mathcal{E}_N(\Omega)$:

$$\frac{{}_1F_1(-N-1; -2N-2; -i\Omega)\mathcal{D}_{N+1}(\Omega) + \frac{i\Omega}{2(2N+1)}{}_1F_1(-N; -2N; -i\Omega)\mathcal{D}_N(\Omega)}{{}_1F_1(-N-1; -2N-2; -i\Omega) + \frac{i\Omega}{2(2N+1)}{}_1F_1(-N; -2N; -i\Omega)}.$$

Expanding the denominator as a series in Ω gives

$$1 - \frac{N}{2N+1}i\Omega + \mathcal{O}(\Omega^2),$$

whilst, along with the aid of identity (36), the numerator is given by

$$\left\{1 - \frac{1}{2}i\Omega + \mathcal{O}(\Omega^2)\right\} \left\{\mathcal{D}_{N+1}(\Omega) + \frac{i\Omega}{2(2N+1)}\mathcal{D}_N(\Omega)\right\}.$$

Combining these expressions gives the claimed result. \square

3.1. Proof of Theorem 1.

Proof. A simple change of variable in the summation gives

$$u_{h,N}(x+rh, t) = e^{i\omega t} \sum_{\ell \in \mathbb{Z}} \lambda_{h,N}^{\ell+r} \phi(x-\ell h) = \lambda_{h,N}^r u_{h,N}(x, t)$$

for $x \in \mathbb{R}$, $r \in \mathbb{Z}$, showing that $u_{h,N}$ is indeed a discrete Bloch wave with Floquet multiplier $\lambda_{h,N}$. The function $\phi \in X_{h,N}$ is chosen to be supported on $(-h, h)$ and given by

$$\phi(x) = \begin{cases} \frac{1}{2}(\Phi_E + \Phi_O)(2x/h + 1), & x \in (-h, 0) \\ \frac{1}{2}(\Phi_E - \Phi_O)(2x/h - 1), & x \in (0, h) \end{cases}$$

where Φ_E and Φ_O are defined in Theorem 4. The first assertion in Theorem 4 shows that ϕ is a continuous piecewise polynomial of degree N .

The first step is to show that condition (5) holds for $u_{h,N}$ of the form (12). Let $v \in \mathbb{P}_N \cap H_0^1(0, h)$ and observe that, with $\Omega = \omega h/c$, applying the change of variable $s = 2x/h - 1$ and letting $V(s) = v(x)$, there holds

$$\int_{\mathbb{R}} (i\omega\phi(x) + c\phi'(x)) \overline{v(x)} dx = \frac{c}{2} \int_{-1}^1 (\mathcal{L}\Phi_E - \mathcal{L}\Phi_O) \overline{V(s)} ds = 0$$

and

$$\int_{\mathbb{R}} (i\omega\phi(x-h) + c\phi'(x-h)) \overline{v(x)} dx = \frac{c}{2} \int_{-1}^1 (\mathcal{L}\Phi_E + \mathcal{L}\Phi_O) \overline{V(s)} ds = 0$$

thanks to $V \in \mathbb{P}_N \cap H_0^1(-1, 1)$ and (27). Consequently, since $\phi(\cdot - mh)$ is supported on $(mh - h, mh + h)$, the only non-zero terms arising from substituting $u_{h,N}$ into condition (5) arise when $m = 0$ and $m = 1$. Consequently, (5) holds for all $v \in \mathbb{P}_N \cap H_0^1(0, h)$. In turn, the Bloch wave property means that (5) holds for all $v \in \mathbb{P}_N \cap H_0^1(mh, mh + h)$ for all $m \in \mathbb{Z}$.

It therefore remains only to show that (5) holds when v is continuous piecewise, linear on the grid $h\mathbb{Z}$. Moreover, thanks again to the Bloch wave property, it suffices to consider the case where v is supported on $(-h, h)$ and satisfies $v(\pm h) = 0$, $v(0) = 1$. The only non-zero terms arising from substituting $u_{h,N}$ into (5) for this choice of v occur when $m = 0$ and $m = \pm 1$. Direct computation and simple changes of variable reveal that

$$\begin{aligned} T^0 &= \int_{\mathbb{R}} (i\omega\phi(x) + c\phi'(x)) \overline{v(x)} dx \\ &= \frac{c}{4} \int_{-1}^1 (\mathcal{L}\Phi_E + \mathcal{L}\Phi_O)(1+s) ds + \frac{c}{4} \int_{-1}^1 (\mathcal{L}\Phi_E - \mathcal{L}\Phi_O)(1-s) ds \\ &= \frac{c}{2} \int_{-1}^1 (\mathcal{L}\Phi_E + s\mathcal{L}\Phi_O) ds. \end{aligned}$$

Likewise

$$T^- = \int_{\mathbb{R}} (i\omega\phi(x-h) + c\phi'(x-h)) \overline{v(x)} dx = \frac{c}{4} \int_{-1}^1 (\mathcal{L}\Phi_E + \mathcal{L}\Phi_O)(1-s) ds$$

and

$$T^+ = \int_{\mathbb{R}} (i\omega\phi(x+h) + c\phi'(x+h)) \overline{v(x)} dx = \frac{c}{4} \int_{-1}^1 (\mathcal{L}\Phi_E - \mathcal{L}\Phi_O)(1+s) ds.$$

Consequently, condition (5) holds if and only if $\lambda_{h,N} = \lambda$ where

$$\frac{T^+}{\lambda} + T^0 + T^- \lambda = 0.$$

The coefficients in this condition are evaluated explicitly in terms of Ω and N in identities (28)-(31). Inserting these identities and simplifying shows that the condition on λ is equivalent to (13). \square

3.2. Proof of Theorem 2.

Proof. Let $\Omega = \omega h/c$ and use (33) to write $\overline{q_N(\Omega)} = e^{i\Omega}(1 - \mathcal{E}_N(\Omega))$ and $q_N(\Omega) = e^{-i\Omega}(1 - \mathcal{E}_N(\Omega))$ where $\mathcal{E}_N(\Omega)$ is defined in (34). Substituting in the discrete dispersion relation (13) and rearranging gives

$$\frac{e^{-i\Omega} - \lambda}{\lambda} = \frac{e^{-i\Omega} \overline{v_N(\Omega)} \overline{\mathcal{E}_N(\Omega)} + (-1)^N e^{i\Omega} v_N(\Omega) \mathcal{E}_N(\Omega)}{\lambda v_N(\Omega) + (-1)^{N+1} e^{i\Omega} v_N(\Omega)}.$$

Inspection of (34) shows that $\mathcal{E}_N(\Omega) \rightarrow 0$ as $\Omega \rightarrow 0$. Consequently, $\lambda \rightarrow e^{-i\Omega}$ or $\lambda \rightarrow (-1)^N v_N(\Omega) e^{i\Omega} / \overline{v_N(\Omega)}$ as $\Omega \rightarrow 0$.

Letting $\Omega \rightarrow 0$ and passing along the branch on which $\lambda \rightarrow e^{-i\Omega}$ gives

$$\frac{e^{-i\Omega} - \lambda}{e^{-i\Omega}} = \frac{e^{-i\Omega} \overline{v_N(\Omega)} \overline{\mathcal{E}_N(\Omega)} + (-1)^N e^{i\Omega} v_N(\Omega) \mathcal{E}_N(\Omega)}{e^{-i\Omega} v_N(\Omega) + (-1)^{N+1} e^{i\Omega} v_N(\Omega)} + \mathcal{O}(|\mathcal{E}_N(\Omega)|^2).$$

The remainder of the proof consists of simply expanding the numerator and denominator as series in Ω . In particular, the denominator is found to be

$$e^{-i\Omega\overline{v_N(\Omega)}} + (-1)^{N+1}e^{i\Omega}v_N(\Omega) = \begin{cases} 2 + \mathcal{O}(\Omega^2), & N \text{ odd} \\ -\frac{2(N+1)}{2N+1}i\Omega, & N \text{ even.} \end{cases}$$

Likewise, using the expression (34) for $\mathcal{E}_N(\Omega)$ gives the following expression for the numerator

$$e^{-i\Omega\overline{v_N(\Omega)\mathcal{E}_N(\Omega)}} + (-1)^N e^{i\Omega}v_N(\Omega)\mathcal{E}_N(\Omega) = \left[\frac{N!}{(2N+1)!} \right]^2 \Omega^{2N+2} \begin{cases} \frac{N+1}{2N+3}i\Omega + \mathcal{O}(\Omega^2), & N \text{ odd} \\ -1 + \mathcal{O}(\Omega^2), & N \text{ even.} \end{cases}$$

Taking the ratio and expanding as a series in Ω gives the result claimed for the error in $\lambda_{h,N}^P$.

The expression for the spurious solution follows simply from the fact that the product of the zeros of the (associated quadratic version of the) discrete dispersion relation (13) is given by $(-1)^N v_N(\Omega)/\overline{v_N(\Omega)}$. \square

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