Energy-Minimising Parallel Flows with Prescribed Vorticity Distribution

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Abstract

This note concerns a nonlinear differential equation problem in which both the nonlinearity in the equation and its solution are determined by prescribed data. The question under consideration arises from a study of two-dimensional steady parallel-flows of a perfect fluid governed by Euler's equations and a free-boundary condition, when the distribution of vorticity is arbitrary but prescribed.

1 Introduction

This is basically a study of the nonlinear ordinary differential equation $-u'' = \lambda(u)$ when both the nonlinearity λ and the solution u are unknowns, to be determined by additional constraints and boundary conditions. To begin, we outline the motivation in hydrodynamic wave theory, for otherwise the problem treated in Section 2 might appear arbitrary, if not a little bizarre. Section 2 is self-contained and possibly of independent interest.

1.1 Preamble: Euler's equations

Euler's equations [6, 13], which govern the velocity field $\vec{v}$ of an incompressible perfect fluid with pressure p and density ϱ in an external conservative force field $-\nabla\Phi$ are

$$\varrho(\vec{v}_t + \vec{v}\cdot\nabla\vec{v}) + \nabla\Phi + \nabla p = 0, \qquad \nabla\cdot\vec{v} = 0.$$

The first is Newton's law that the rate of change of momentum is the sum of external and internal forces and the second is the incompressibility condition. In two dimensions (when $\vec{v}$ is in the (x, y)-plane and $\vec{k}$ points in the z-direction) the vorticity, $\vec{\omega} = \nabla\times\vec{v} := \omega\vec{k}$ satisfies

$$\vec{\omega}\times\vec{v} = \vec{0} \text{ and the transport equation } \omega_t + \vec{v}\cdot\nabla\omega = 0.$$

If a domain $\Omega(0)$ at time 0 evolves under a smooth flow into $\Omega(t)$ at time t, it follows that

$$\int_{\Omega(t)} f(\omega(x,y,t))\,dxdy = \int_{\Omega(0)} f(\omega(x,y,0))\,dxdy$$

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for any reasonable function f. Thus the *vorticity distribution function* is conserved by Euler's equations. In different words, for smooth solutions of Euler's equations the vorticity at time t is a rearrangement of the vorticity at time zero, in the following sense.

Definition (Distribution functions and rearrangements). If Ω_1 and Ω_2 have the same finite measure, $\omega_1 : \Omega_1 \to \mathbb{R}$ and $\omega_2 : \Omega_2 \to \mathbb{R}$ are *rearrangements* of one another if

$$Z_1(a) := \text{meas}\,\{\omega_1 > a\} = \text{meas}\,\{\omega_2 > a\} =: Z_2(a) \quad \text{for all } a \in \mathbb{R}.$$

Equivalently the *distribution functions* Z_1 and Z_2 are equal. The set of rearrangements of a given function ω^* is denoted by $\mathcal{R}(\omega^*)$. □

Let $C(t)$ denote the position of a closed orientated curve evolving smoothly with the motion of its fluid particles governed by Euler's equations. Then the area within $C(t)$ is conserved, by incompressibility. Moreover,

$$\int_{C(t)} \vec{v} \cdot\, dS \text{ is independent of } t.$$

This, conservation of circulation, is known as Kelvin's Circulation Theorem.

In a simply connected domain in $\mathbb{R}^2$, the incompressibility equation, $\nabla \cdot \vec{v} = 0$, implies that there is a stream function Ψ with $\nabla^\perp \Psi = -\vec{v}$ where $\nabla^\perp = (-\partial_y, \partial_x)$. So Euler's equations can be re-written

$$\omega_t - (\nabla^\perp \Psi) \cdot \nabla \omega = 0, \qquad -\Delta \Psi = \omega. \tag{1}$$

1.2 Travelling waves with vorticity

A special case treated in [3, 5, 7, 8, 9, 14] is that of periodic waves of permanent form travelling with speed c on the surface of water in a channel above a horizontal bottom. The domain $\mathit{\Omega}(t)$ occupied by the fluid at time t is the region between the bottom and the surface $\mathscr{S}(t) := \{(x,y) : F(x-ct, y) = 0\}$, say, for some function $F : \mathbb{R}^2 \to \mathbb{R}$ which is periodic in x. Thus $\mathit{\Omega}(t) = \mathit{\Omega} + t(c,0)$, where $\mathit{\Omega} := \mathit{\Omega}(0)$. If $\Psi(\cdot,\cdot,t)$ and $\omega(\cdot,\cdot,t) : \mathit{\Omega}(t) \to \mathbb{R}$ are written

$$\Psi(x,y,t) = \psi(x-ct, y), \qquad \omega(x,y,t) = \zeta(x-ct,y),$$

where ψ, $\zeta : \mathit{\Omega} \to \mathbb{R}$, it is immediate from (1) that

$$\nabla^\perp \widehat{\psi} \cdot \nabla \zeta = 0 \text{ on } \mathit{\Omega} \text{ where } \widehat{\psi}(x,y) = \psi(x,y) - cy.$$

In particular, $-\Delta \psi = \zeta$ where ζ is constant on level sets of $\widehat{\psi}$; $\widehat{\psi}$ is called the relative stream function.

Moreover, if a point $(p(t), q(t))$ of the fluid moves on the surface $\mathscr{S}(t)$, then $F(p(t) - ct, q(t)) \equiv 0$ and, since $(\dot{p}(t), \dot{q}(t)) = -\nabla^\perp \psi(p(t) - ct, q(t))$,

$$\nabla F(p(t) - ct, q(t)) \cdot \nabla^\perp \widehat{\psi}(p(t) - ct, q(t)) \equiv 0.$$

Therefore, formally speaking, $\widehat{\psi}$ is constant on the fixed curve $\mathscr{S}(0)$ (henceforth denoted by $\mathscr{S}$) and the time-independent functions ψ, $\zeta : \varOmega \to \mathbb{R}$, and the wave speed c, satisfy

$$\left.\begin{array}{c} -\Delta\psi = \zeta \text{ in } \varOmega \\ \psi(x,0) = 0,\ x \in \mathbb{R}, \quad \psi(x,y) = C + cy,\ (x,y) \in \mathscr{S} \\ \zeta \text{ is constant on level sets of } \psi - cy \text{ in } \varOmega \end{array}\right\}, \tag{2}$$

where C is an unspecified constant. Constraints on the pressure in the flow at the surface $\mathscr{S}$ and on the flow vorticity distribution must also be satisfied.

Remark 1. The theory in [7, 8], and summarised in [14], focuses on cases in which the flow velocity relative to the moving frame is nowhere zero, equivalently $\psi_y - c$ does not change sign. Below this is called the monotone case because the relative stream function, $\psi(x,y) - cy$, is monotone in y. □

A variational approach

We begin by summarising a variational approach [5] which led to the case $c = 0$ in (2). For the analogous variational treatment in which non-zero c arises as a Lagrange multiplier due to a constraint on the momentum, see [3]. Both lead to the problem is Section 2 which, in Sections 3.1 and 3.2, is interpreted in terms of the travelling-wave problem. Throughout, let

$$\mu \in \mathbb{R} \text{ and } \zeta^* \in L^2(0,P) \times (0,Q) \text{ be given, where } P,\ Q > 0 \text{ are fixed.} \tag{3}$$

Now let $\mathscr{S}$ denote a P-periodic Jordan curve ($\mathscr{S} + (P,0) = \mathscr{S}$) in the open upper half plane, $\varOmega$ the region between $\mathscr{S}$ and the x-axis, Ω one period of $\varOmega$ and $\mathcal{S}$ one period of $\mathscr{S}$. More precisely, let $Q^\pm = \inf\{y > 0 : (\pm P/2, y) \in \mathscr{S}\}$, let $\mathcal{S}$ denote the component of $\mathscr{S}$ joining $(-P/2, Q^-)$ to $(P/2, Q^+)$ and let Ω be the domain with boundary segments $\mathcal{S}$, $\{\pm P/2\} \times [0, Q^\pm]$ and $[-P/2, P/2] \times \{0\}$.

Furthermore consider only those $\mathscr{S}$ for which the area of Ω is PQ.

For any function $\zeta \in L^2_{\text{loc}}(\varOmega)$ which is P-periodic in x with $\zeta|_\Omega \in \mathcal{R}(\zeta^*)$, let $\psi = \psi(\Omega, \zeta) \in W^{1,2}(\Omega)$ be the weak solution of

$$\left.\begin{array}{c} -\Delta\psi = \zeta \text{ on } \varOmega, \\ \psi(x,0) = 0, \\ \psi \equiv C(\psi),\ \text{a constant, on } \mathscr{S}, \\ \psi \text{ is } P\text{-periodic in } x, \\ \int_{\mathcal{S}} \nabla\psi \cdot n dS = \mu, \end{array}\right\} \tag{4a}$$

in which the constant $C(\psi)$ is not prescribed. The classical water-wave problem is to find such a curve $\mathscr{S}$ and function ζ so that

$$\text{the vorticity } \zeta \text{ is constant on level sets of } \psi(\Omega, \zeta), \tag{4b}$$

and

$$-\frac{1}{2}|\nabla\psi(x,y)|^2 - g\,y \text{ is constant on } \mathcal{S}.$$

The first of these comes from (2) with $c = 0$ and second is the classical Bernoulli condition which says that for a steady flow under gravity, the pressure at the free surface is constant atmospheric pressure. In this formulation ψ yields the stream function and $\mathscr{S}$ the free boundary, and $\mu \in \mathbb{R}$ is the circulation on one period of the free boundary, as in the preceding section with $c = 0$.

More generally[1, 15], if an elastic membrane that nonlinearly resists stretching and bending is in contact with the surface, in its simplest form the Bernoulli condition becomes

$$\frac{1}{2}|\nabla\psi(x,y)|^2 + g\,y + E\Big(\sigma'' + \frac{1}{2}\sigma^3\Big) - \beta T(\ell(\mathcal{S}) - P)^{\beta-1}\sigma = \text{ constant}, \tag{4c}$$

where $'$ denotes differentiation with respect to arc length; $\ell(\mathcal{S})$ = length of $\mathcal{S}$; $E \geqslant 0$ is a coefficient of bending resistance; $T \geqslant 0$ and $\beta \geqslant 1$ measure nonlinear resistance to stretching and compression [5]. Note that (4c) includes the classical Bernoulli condition, $E = T = 0$, as a special case, and $\beta = 1$, $T > 0 = E$ corresponds to simple surface tension.

An energy functional, the sum of the kinetic and potential energies of the fluid and the elastic energy of the surface, is defined by

$$F(\mathscr{S}, \zeta) = \mathcal{E}(\mathcal{S}) + g\int_\Omega y\; dxdy + \frac{1}{2}\int_\Omega |\nabla\psi(\Omega,\zeta)|^2 dxdy, \tag{5}$$

where, with σ denoting the curvature of $\mathcal{S}$,

$$\mathcal{E} = E\int_{\mathcal{S}} |\sigma|^2\, dS + T(\ell(\mathcal{S}) - 2\pi)^\beta.$$

Then it is shown in [5] that minimizing the energy F over admissible curves $\mathscr{S}$ and $\zeta \in \mathcal{R}(\zeta^*)$ yields a solution of (4). Moreover, a necessary condition for a minimiser is that (4b) holds in the strong sense that $\zeta = \lambda(\psi)$ for some *non-increasing function* λ. In other words, an energy minimiser satisfies

$$-\Delta\psi = \lambda(\psi) = \zeta \in \mathcal{R}(\zeta^*) \text{ for some non-increasing function } \lambda. \tag{6}$$

The function λ is the infinite-dimensional Lagrange multiplier that arises from the constraint that the vorticity distribution coincides with that of ζ^*. It is *a priori* unknown as is ψ: both are components of solutions. In the absence of information about its regularity, the key to the analysis (Lemma 2) is that λ is non-increasing, which follows from the fact that it arises from minimisation in [3, 5], or from the analogous minimisation problem restricted to parallel flows. For suitable μ and positive values of the parameters, E, T it is shown in [3, 5] that minimisers, with prescribed vorticity distribution function over arbitrary periodic domains, exist. However, in the absence of surface energy effects, when $E = T = 0$, these infima of the energies in [3, 5] are not attained except in trivial circumstances [16]. (Minimizers in the narrow class of periodic flows, when surface energy plays no role, do exist.)

Parallel flows

A flow is called *parallel* if the stream function ψ and the vorticity ζ are functions of y only. In [5] it is remarked that 'for a given vorticity distribution it is easy to construct parallel-flow solutions by solving ordinary differential equations'. Consequently, the emphasis there was on finding *non-parallel-flow* solutions and on estimates to ensure that the minimisers found were not parallel flows. The purpose here is to examine the 'easy' task of finding which parallel flows, if any, could arise from minimizing energy, either over all domains, or over a strip domain on which admissible functions depend on y only.

For parallel flows $\mathscr{S} = \{(x, Q) : x \in \mathbb{R}\}$, $\Omega = (0, P) \times (0, Q)$, and ψ and ζ are functions of y only. Thus the P-periodicity in x of ψ and the generalised Bernoulli condition (4c) are satisfied automatically. Hence the problem reduces to $-u'' = \lambda(u)$, where λ is non-increasing but otherwise unknown, except that $\lambda(u)$ is a rearrangement of a known function on $[0, Q]$. We will return to this is Section 3.1 where the existence and behaviour of parallel-flow minimisers are shown to be determined by the parameters μ/P and $\int_\Omega \zeta^*$ alone. The minimization problem [3], in which horizontal momentum is prescribed on a slightly different admissible set, is considered in Section 3.2.

In [12] there is an exhaustive account of the parallel solutions of (6) when a *given* function λ prescribes the functional dependence of vorticity on the stream function. Because prescription of λ has no role in the initial-value problem, we prefer the present approach where all the constraints (vorticity distribution function, surface circulation per period, cross-sectional area or horizontal momentum) are invariants for smooth solutions of Euler's equations with free-boundary conditions.

Stability

Conserved quantities often have significance for questions of stability [2, 3, 4]. However, linearised criteria, such as Rayleigh's instability criterion for parallel flows [13, p.122], presume that the total energy is that of a flowing liquid in a conservative force field. By contrast, when $E, T > 0$ the energy (5) is shared between the fluid and the deformed elastic membrane once the surface is not flat. So Rayleigh's criterion is not relevant to the possible parallel-flow solutions of the hydroelastic wave free-boundary problem under discussion here. If, however, these parallel flows are, as they may be, considered as solutions in a domain which is a fixed strip of constant width, Rayleigh's condition may be relevant and checked using the properties of the explicit solutions that have been found.

Conclusions

When the prescribed vorticity distribution ζ^* is essentially one-signed, for all values of the circulation μ there is a unique parallel flow which satisfies the necessary conditions for an energy minimiser in [5]. If μ and $\mu + \int_\Omega \zeta^*$ have opposite signs (which is equivalent to the condition in Theorem 8), then that solution is non-monotone, otherwise it is monotone (Remark 1). However, when ζ^* changes sign all solutions are monotone, but there are values for μ for which no parallel flow satisfies the necessary condition for a minimiser of the energy in [5].

For the energy in [3] there are parallel flows that satisfy the necessary condition for minimisers if and only if $\kappa \in [k_0, k^0]$, where κ in (23) depends only on the prescribed momentum and circulation, and k_0, k^0 in (24) depend only on the given vorticity distribution

ζ^*. These parallel flows are monotone if and only if $\kappa \in \{k_0, k^0\}$, and there are infinitely many c with the same ψ corresponding to monotone solutions (Theorem 13).

If ζ^* changes sign parallel flows that satisfy the necessary conditions for minimizers in [3], if any, must be monotone. If ζ^* does not change sign, for every $\kappa \in (k_0, k^0)$ there is a unique solution and it is non-monotone (Theorem 15).

From given data, we calculate the stream function, the wave speed and the explicit dependence of vorticity on the stream function, for parallel-flow minimizers (indeed also for minimisers among the restricted class of parallel follows). For energy maximizers in the class of parallel flows, (6) holds with λ increasing. However, the analogue of Lemma 2 is not so straightforward because the regularity of the function λ is unknown. Note also, from Figure 1, that the stream function of non-monotone parallel-flow solutions may not be C^3, even when the prescribed vorticity distribution is real-analytic.

2 An ODE with Rearrangement Constraints

All the observations in this section are elementary. Let $\zeta \in L^2(0,Q)$ and suppose that

$$-u''(y) = \zeta(y) \text{ where } \zeta(y) = \lambda(u(y)), \quad y \in (0,Q), \quad u(0) = 0, \tag{7}$$

and $\lambda : \mathbb{R} \to \mathbb{R}$ is non-increasing. Since $u \in W^{2,2}(0,Q)$, u' is Hölder continuous on $[0,Q]$.

Lemma 2. *(a) Either (i) $u' \neq 0$ on $[0,Q]$ or (ii) there is a closed interval $[\alpha,\beta] \subset [0,Q]$ (possibly a singleton) such that $u' = 0$ on $[\alpha,\beta]$ and $u' \neq 0$ on $[0,Q] \setminus [\alpha,\beta]$.*

(b) Suppose (a)(ii) holds.

If $\alpha > 0$, then either $u' > 0$, u is concave and $\zeta > 0$ is non-increasing on $[0,\alpha)$, or $u' < 0$, u is convex and $\zeta < 0$ is non-decreasing on $[0,\alpha)$.

If $\beta < Q$, then either $u' > 0$, u is convex and $\zeta < 0$ is non-increasing on $(\beta,Q]$, or $u' < 0$, u is concave and $\zeta > 0$ is non-decreasing on $(\beta,Q]$.

(c) If $u(y_1) = u(y_2)$, where $0 \leqslant y_1 < y_2 \leqslant Q$, then u is even on $[y_1,y_2]$ about $\frac{1}{2}(y_1+y_2)$.

Proof. (a) Suppose (i) and (ii) are both false. Then there is a closed interval $[z_1,z_2] \subset [0,Q]$ such that $u'(z_1) = u'(z_2) = 0$ but $u'(z) \neq 0$ for all $z \in (z_1,z_2)$. If $u'(z) > 0$ for $z \in (z_1,z_2)$, then

$$0 < u'(z) = -\int_z^{z_2} u''(y)dy = \int_z^{z_2} \lambda(u(y))dy \text{ for all } z \in (z_1,z_2), \tag{8}$$

and it follows that $\limsup_{z \nearrow z_2} \lambda(u(z)) \geqslant 0$. Since u is increasing on (z_1,z_2) and λ is non-increasing, $\lambda(u(z)) \geqslant 0$ for all $z \in (z_1,z_2)$. However,

$$0 = u'(z_1) - u'(z_2) = \int_{z_1}^{z_2} \lambda(u(z))dz.$$

It follows that $\lambda(u(z)) = 0$ for all $z \in (z_1,z_2)$ which contradicts (8). A contradiction arises similarly when $u' < 0$ on (z_1,z_2). Hence (i) or (ii) must hold.

(b) If (a)(ii) holds, let $[\alpha,\beta]$ denote the maximal subset of $[0,Q]$ on which $u'=0$. Suppose that $\alpha>0$ and $u'>0$ on $[0,\alpha)$. Since, for $y\in(0,\alpha)$,

$$0<u'(y)=\int_y^\alpha \zeta(z)dz$$

it follows that $\limsup_{y\nearrow\alpha}\zeta(y)\geqslant 0$. Also, because $\zeta=\lambda\circ u$ is non-increasing on $(0,\alpha)$, it follows that $\zeta\geqslant 0$ on $(0,\alpha)$. Consequently if $\zeta(\hat\alpha)=0$, $\hat\alpha\in(0,\alpha)$, then $\zeta(y)=0$ for all $y\in(\hat\alpha,\alpha)$. This implies that $u'(\hat\alpha)=0$ which contradicts the maximality of $[\alpha,\beta]$. Hence $\zeta(y)>0$ and ζ is non-increasing when $u'>0$ on $(0,\alpha)$. If $u'<0$, it follows similarly that $\zeta<0$ is non-decreasing on $[0,\alpha)$.

When $\beta<Q$ let $u_1(y):=u(Q-y)$, $\zeta_1(y):=\zeta(Q-y)$, $y\in(0,\alpha_1)$ where $\alpha_1=Q-\beta$. Then u_1, ζ_1 satisfies the same equation as u, ζ and the required conclusion about u on (β,Q) is equivalent to the preceding observation about u_1 on $(0,\alpha_1)$.

(c) When $u(y_1)=u(y_2)$, let $v(y)=u(y_1+y_2-y)$, $y\in[y_1,y_2]$. Then

$$-u''(y)=\lambda(u(y)),\quad -v''(y)=\lambda(v(y)),\quad y\in[y_1,y_2],$$

$v(y_2)=u(y_1)=u(y_2)=v(y_1)$. Since λ is non-increasing, it follows that

$$0\leqslant\int_{y_1}^{y_2}(u'(y)-v'(y))^2dy=\int_{y_1}^{y_2}(\lambda(u(y))-\lambda(v(y))(u(y)-v(y))\,dy\leqslant 0.$$

Hence $u\equiv v$, which proves (c). □

Remark 3. If u is not monotone, the maximal interval $[\alpha,\beta]\subset(0,Q)$ with the property that $u'=0$ on $[\alpha,\beta]$ is well defined (see Lemma 2 (a)), u is either convex or concave, and there exists $\underline{\delta}$, $\overline{\delta}>0$ with $u(\alpha-\underline{\delta})=u(\beta+\overline{\delta})$. Therefore, from part (c), $u(y)=u(\alpha+\beta+\overline{\delta}-\underline{\delta}-y)$ for $y\in[\alpha-\underline{\delta},\beta+\overline{\delta}]$. It follows that

$$u(\alpha)=u(\beta+\overline{\delta}-\underline{\delta})\text{ and }u(\beta)=u(\alpha+\overline{\delta}-\underline{\delta}).$$

From this and the maximality of $[\alpha,\beta]$, it follows that $0\leqslant\overline{\delta}-\underline{\delta}\leqslant 0$. Hence $\underline{\delta}=\overline{\delta}$ and u is even about p on an interval $[p-\delta,p+\delta]$ for some $\delta>0$ where $p=\frac{1}{2}(\alpha+\beta)$. Let I_p denote the maximal such interval. Then $[0,Q]\setminus I_p$ is an interval $[0,2p-Q]$, an interval $(2p,Q]$, or $\emptyset$, according to whether $p>\frac{1}{2}Q$, $p<\frac{1}{2}Q$ or $p=\frac{1}{2}Q$. Note that $\zeta=\lambda(u)$ is even about p on I_p.

On intervals where u is monotone, $\zeta=\lambda(u)$ has the opposite monotonicity (not necessarily strict) to that of u, because λ is non-increasing.

We refer to (a)(ii) with $0<\alpha\leqslant\beta<Q$ as the *non-monotone* case. □

When ζ is a rearrangement of ζ_0

Now we examine what more can be said when ζ in (7) is a rearrangement of a known function $\zeta_0\in L^2[0,Q]$. Without loss of generality suppose that ζ_0 is non-increasing and denote by $\zeta^0(y):=\zeta_0(Q-y)$, $y\in(0,Q)$, the non-decreasing rearrangement of ζ_0. Because of Lemma 2 and Remark 3, two specific families of rearrangements of ζ_0, parametrized by $p\in(0,Q)$, are important.

I(a) When $p \in (0, \frac{1}{2}Q]$ and $I_p = [0, 2p]$, let $\hat{\zeta}_{(p)} = \zeta_0$ on $[2p, Q]$ and on I_p let $\hat{\zeta}_{(p)}$ be the rearrangement on I_p of $\zeta_0\big|_{I_p}$ which is even about p and non-decreasing on $[0, p]$. Thus

$$\hat{\zeta}_{(p)}(y) = \begin{cases} \zeta_0(2(p-y)), & y \in [0, p], \\ \zeta_0(2(y-p)), & y \in [p, 2p], \\ \zeta_0(y), & y \in [2p, Q]. \end{cases}$$

I(b) When $p \in [\frac{1}{2}Q, Q)$ and $I_p = [2p-Q, Q]$, let $\hat{\zeta}_{(p)} = \zeta^0$ on $[0, 2p-Q]$ and on I_p let $\hat{\zeta}_{(p)}$ be the rearrangement on I_p of $\zeta^0\big|_{I_p}$ which is even about p and non-decreasing on $[2p-Q, p]$. Thus

$$\hat{\zeta}_{(p)}(y) = \begin{cases} \zeta^0(y), & y \in [0, 2p-Q], \\ \zeta^0(Q-2p+2y), & y \in [2p-Q, p], \\ \zeta^0(2p+Q-2y), & y \in [p, Q]. \end{cases}$$

II(a) When $p \in (0, \frac{1}{2}Q]$ and $I_p = [0, 2p]$, let $\check{\zeta}_{(p)} = \zeta^0$ on $[2p, Q]$ and on I_p let $\check{\zeta}_{(p)}$ be the rearrangement on I_p of $\zeta^0\big|_{I_p}$ which is even about p and non-increasing on $[0, p]$. Thus

$$\check{\zeta}_{(p)}(y) = \begin{cases} \zeta^0(2(p-y)), & y \in [0, p], \\ \zeta^0(2(y-p)), & y \in [p, 2p], \\ \zeta^0(y), & y \in [2p, Q]. \end{cases}$$

II(b) When $p \in [\frac{1}{2}Q, Q)$ and $I_p = [2p-Q, Q]$, let $\check{\zeta}_{(p)} = \zeta_0$ on $[0, 2p-Q]$ and on I_p let $\check{\zeta}_{(p)}$ be the rearrangement on I_p of $\zeta_0\big|_{I_p}$ which is even about p and non-increasing on $[2p-Q, p]$. Thus

$$\check{\zeta}_{(p)}(y) = \begin{cases} \zeta_0(y), & y \in [0, 2p-Q], \\ \zeta_0(Q-2p+2y), & y \in [2p-Q, p], \\ \zeta_0(2p+Q-2y), & y \in [p, Q]. \end{cases}$$

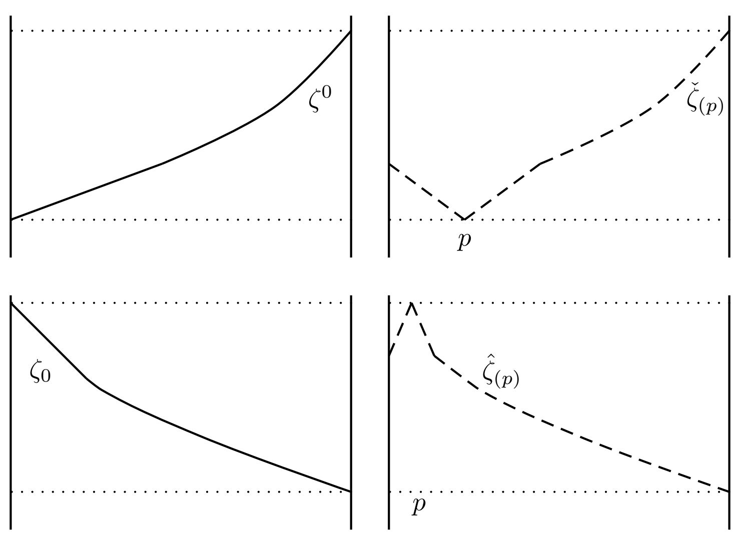

Figure 1: Special rearrangements

Remark 4. When $\zeta \in \mathcal{R}(\zeta_0)$, in the non-monotone case it follows from Remark 3 that ζ_0 must be essentially one-signed and ζ in (7) must be either $\check{\zeta}_{(p)}$ or $\hat{\zeta}_{(p)}$, where $p = \frac{1}{2}(\alpha+\beta) \in (0, Q)$. $\square$

These rearrangements have a certain symmetry: for $p \in (0, \frac{1}{2}Q]$,

$$\hat{\zeta}_{(p)}(y) = \hat{\zeta}_{(Q-p)}(Q-y) \text{ and } \check{\zeta}_{(p)}(y) = \check{\zeta}_{(Q-p)}(Q-y), \tag{9}$$

and if extended to $[0, Q]$, $\check{\zeta}_{(Q)} = \hat{\zeta}_{(0)} = \zeta_0$ and $\check{\zeta}_{(0)} = \hat{\zeta}_{(Q)} = \zeta^0$. Let

$$[\zeta] = \int_0^Q \zeta(y)\, dy, \tag{10}$$

$$k_0 := \int_0^Q y\zeta_0(y)dy, \quad k^0 := \int_0^Q y\zeta^0(y)dy, \quad K := [k_0, k^0]. \tag{11}$$

Since $\hat{\zeta}_{(p)}$ and $\check{\zeta}_{(p)}$ are a rearrangements of ζ_0, it follows that $[\hat{\zeta}_{(p)}] = [\check{\zeta}_{(p)}] = [\zeta_0] = [\zeta^0]$ and, for $y \in [0, Q]$,

$$\int_y^Q \hat{\zeta}_{(p)}\, dz + \int_{(Q-y)}^Q \hat{\zeta}_{(Q-p)}\, dz = [\zeta_0] = \int_y^Q \check{\zeta}_{(p)}\, dz + \int_{(Q-y)}^Q \check{\zeta}_{(Q-p)}\, dz. \tag{12}$$

Remark 5. Note that for distinct p the functions $\hat{\zeta}_{(p)}$ and $\check{\zeta}_{(p)}$ are not distinct if there exist $a_1 > 0$ with $\zeta_0(0) = \zeta_0(a_1) > \zeta_0(y)$ for $y \in (a_1, Q]$. In that case

$$\hat{\zeta}_{(p)} = \zeta_0, \quad p \in [0, \tfrac{1}{2}a_1], \quad \hat{\zeta}_{(p)} = \zeta^0, \quad p \in [Q - \tfrac{1}{2}a_1, Q].$$

Similarly if there exists $b_1 > 0$ with $\zeta_0(Q) = \zeta_0(Q - b_1) < \zeta_0(y)$ for $y \in [0, Q - b_1)$, then

$$\check{\zeta}_{(p)} = \zeta^0, \quad p \in [0, \tfrac{1}{2}b_1], \quad \check{\zeta}_{(p)} = \zeta_0, \quad p \in [Q - \tfrac{1}{2}b_1, Q].$$

□

Functionals. We introduce functionals, f and g, that play a role in the hydrodynamic application. First, let

$$\hat{f}(p) := \int_p^Q \hat{\zeta}_{(p)}\, dy, \quad \check{f}(p) := \int_p^Q \check{\zeta}_{(p)}\, dy, \tag{13}$$

and note that

$$\hat{f}(p) = \frac{1}{2}[\zeta_0] + \frac{1}{2}\int_{2p}^Q \zeta_0 dy, \; \check{f}(p) = \frac{1}{2}[\zeta_0] + \frac{1}{2}\int_{2p}^Q \zeta^0 dy, \; p \in (0, \tfrac{1}{2}Q],$$
$$\hat{f}(p) = \frac{1}{2}[\zeta_0] - \frac{1}{2}\int\limits_{2(Q-p)}^{Q} \zeta_0 dy, \; \check{f}(p) = \frac{1}{2}[\zeta_0] - \frac{1}{2}\int\limits_{2(Q-p)}^{Q} \zeta^0 dy, \; p \in [\tfrac{1}{2}Q, Q).$$

Thus $\check{f}$, $\hat{f}$ are continuous on $[0, Q]$,

$$\hat{f}(0) = \check{f}(0) = [\zeta_0], \quad \hat{f}(\tfrac{1}{2}Q) = \check{f}(\tfrac{1}{2}Q) = \frac{1}{2}[\zeta_0], \quad \hat{f}(Q) = \check{f}(Q) = 0,$$

$\check{f} - \frac{1}{2}[\zeta_0]$ and $\hat{f} - \frac{1}{2}[\zeta_0]$ are odd about $\frac{1}{2}Q$, and

$$\check{f}(p) + \hat{f}(\tfrac{1}{2}Q - p) = \tfrac{3}{2}[\zeta_0], \quad p \in [0, \tfrac{1}{2}Q],$$
$$\check{f}(p) + \hat{f}(\tfrac{3}{2}Q - p) = \tfrac{1}{2}[\zeta_0], \quad p \in [\tfrac{1}{2}Q, Q].$$

Moreover, from the monotonicity of ζ_0 and ζ^0, $\hat{f}$ is convex while $\check{f}$ is concave on $[0, \frac{1}{2}Q]$, and $\hat{f}$ is concave while $\check{f}$ is convex on $[\frac{1}{2}Q, Q]$. Therefore

$$\hat{f}(p) \leqslant \check{f}(p), \ p \in [0, \tfrac{1}{2}Q] \text{ and } \hat{f}(p) \geqslant \check{f}(p), \ p \in [\tfrac{1}{2}Q, Q]. \tag{14}$$

If ζ_0 is not a constant, these inequalities are strict on $(0, \frac{1}{2}Q) \cup (\frac{1}{2}Q, Q)$.

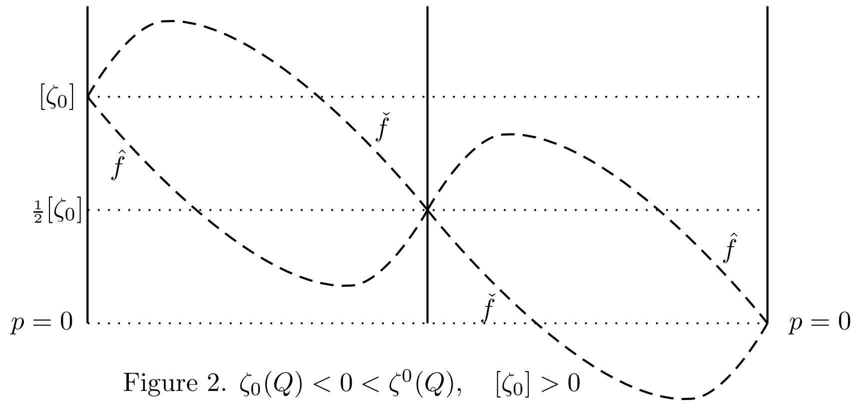

Figure 2. $\zeta_0(Q) < 0 < \zeta^0(Q), \quad [\zeta_0] > 0$

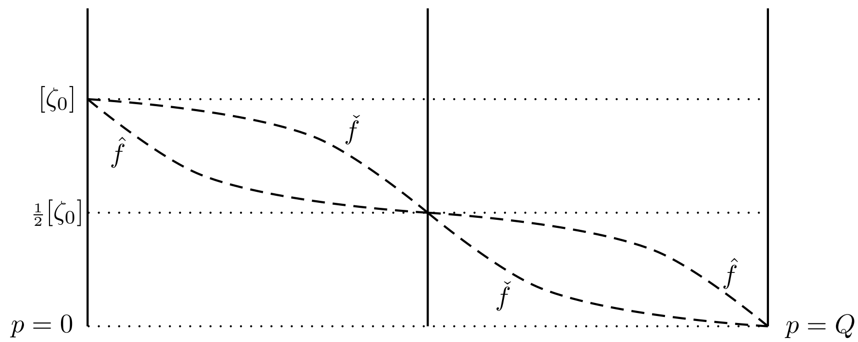

Figure 3. $\zeta_0 \geq 0$

Figures 1 and 2 illustrate some of these features in two cases: in the first ζ_0 has positive mean value while changing sign from positive to negative; in the second it is essentially positive.

Remark 6. (i) Suppose ζ_0 is zero on $[a, b] \subset [0, Q]$, and non-zero elsewhere. Then $\hat{f}'$ is zero on $[\frac{1}{2}a, \frac{1}{2}b] \cup [Q - \frac{1}{2}b, Q - \frac{1}{2}a]$ and non-zero elsewhere, and $\check{f}'$ is zero on $[\frac{1}{2}Q - \frac{1}{2}b, \frac{1}{2}Q - \frac{1}{2}a] \cup [\frac{1}{2}Q + \frac{1}{2}a, \frac{1}{2}Q + \frac{1}{2}b]$, and non-zero elsewhere.

(ii) When $\zeta_0 = 0$ on $[a, Q]$ and positive elsewhere, $\check{f}$ is bijective from $[\frac{1}{2}Q - \frac{1}{2}a, \frac{1}{2}Q]$ onto $\big[\frac{1}{2}[\zeta_0]], [\zeta_0]\big]$ and $\check{f}$ is a bijection from $(\frac{1}{2}Q - \frac{1}{2}a, \frac{1}{2}Q + \frac{1}{2}a)$ onto the interval $\big(0, [\zeta_0]\big)$.

(iii) When $\zeta_0 = 0$ on $[0, b]$ and negative elsewhere, $\hat{f}$ is bijective from $[\frac{1}{2}b, \frac{1}{2}Q]$ onto $\big[[\zeta_0], \frac{1}{2}[\zeta_0]\big]$ and $\hat{f}$ is a bijection from $(\frac{1}{2}b, Q - \frac{1}{2}b)$ onto $([[\zeta_0], 0)$. □

The second functional g is defined as follows:

$$\hat{g}(p) = \int_0^Q y\hat{\zeta}_{(p)}(y)dy, \quad \check{g}(p) = \int_0^Q y\check{\zeta}_{(p)}(y)dy, \quad p \in [0, Q]. \tag{15}$$

Since $\hat{\zeta}_{(p)}$ and $\check{\zeta}_{(p)}$are even, and $y - p$ is odd, about p on I_p,

$$\int_{I_p} y\hat{\zeta}_{(p)}(y)dy = \int_{I_p} (y-p)\hat{\zeta}_{(p)}(y)dy + p\int_{I_p} \zeta_{(p)}(y)dy = p\int_{I_p} \hat{\zeta}_{(p)}(y)dy,$$

and similarly for $\check{\zeta}_{(p)}$. Therefore, from the definitions of $\hat{\zeta}_{(p)}, \check{\zeta}_{(p)}$, and the monotonicity of ζ_0 and ζ^0,

$$\begin{aligned}
\text{on } [0, \tfrac{1}{2}Q]: \quad \hat{g}(p) &= \int_{2p}^{Q} y\zeta_0(y)dy + p\int_0^{2p} \zeta_0(y)dy,\\
\hat{g}'(p) &= \int_0^{2p} \zeta_0(y)dy - 2p\zeta_0(2p) \geqslant 0,\\
\text{on } [\tfrac{1}{2}Q, Q]: \quad \hat{g}(p) &= \int_0^{2p-Q} y\zeta^0(y)dy + p\int_{2p-Q}^{Q} \zeta^0(y)dy,\\
\hat{g}'(p) &= \int_{2p-Q}^{Q} \zeta^0(y)dy - 2(Q-p)\zeta^0(2p-Q) \geqslant 0,\\
\text{on } [0, \tfrac{1}{2}Q]: \quad \check{g}(p) &= \int_{2p}^{Q} y\zeta^0(y)dy + p\int_0^{2p} \zeta^0(y)dy,\\
\check{g}'(p) &= \int_0^{2p} \zeta^0(y)dy - 2p\zeta^0(2p) \leqslant 0,\\
\text{on } [\tfrac{1}{2}Q, Q]: \quad \check{g}(p) &= \int_0^{2p-Q} y\zeta_0(y)dy + p\int_{2p-Q}^{Q} \zeta_0(y)dy,\\
\check{g}'(p) &= \int_{2p-Q}^{Q} \zeta_0(y)dy - 2(Q-p)\zeta_0(2p-Q) \leqslant 0.
\end{aligned}$$

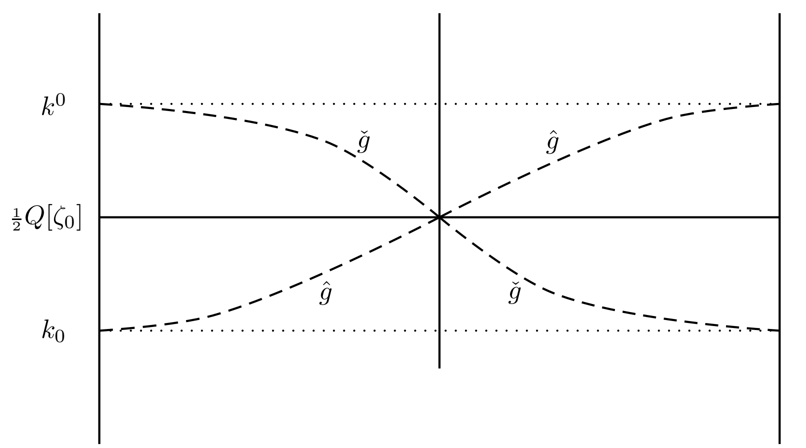

Figure 4. Note $k_0 + k^0 = Q[\zeta_0]$

Note that $\check{g}$ is non-increasing and $\hat{g}$ is non-decreasing on $[0, Q]$, $\hat{g}$ is convex and $\check{g}$ is concave on $[0, \frac{1}{2}Q]$, while $\hat{g}$ is concave and $\check{g}$ is convex on $[\frac{1}{2}Q, Q]$. The range of both is the closed interval K in (11) and

$$\begin{gathered}
\hat{g}(0) = \check{g}(Q) = k_0, \quad \hat{g}(Q) = \check{g}(0) = k^0, \quad \hat{g}(\tfrac{1}{2}Q) = \check{g}(\tfrac{1}{2}Q) = \tfrac{1}{2}Q[\zeta_0],\\
\hat{g}(p) + \hat{g}(Q-p) = Q[\zeta_0], \qquad \check{g}(p) + \check{g}(Q-p) = Q[\zeta_0].
\end{gathered}$$

Remark 7. As in Remark 5, suppose that a_1, $b_1 \in [0, Q]$ with $\zeta_0(0) = \zeta_0(a_1) > \zeta_0(y)$ for $y \in (a_1, Q]$, and $\zeta_0(Q) = \zeta_0(Q - b_1) < \zeta_0(y)$ for $y \in [0, Q - b_1)$. Then $\hat{g} \equiv k_0$ on $[0, \frac{1}{2}a_1]$ and $\hat{g} \equiv k^0$ on $[Q - \frac{1}{2}a_1, Q]$, while $\check{g} \equiv k^0$ on $[0, \frac{1}{2}b_1]$ and $\check{g} \equiv k_0$ on $[Q - \frac{1}{2}b_1, Q]$. On $(\frac{1}{2}a_1, Q - \frac{1}{2}a_1)$ the function $\hat{g}$ is strictly increasing, on $(\frac{1}{2}b_1, Q - \frac{1}{2}b_2)$ the function $\check{g}$ is strictly decreasing, and the range of both on these sets is (k_0, k^0). $\square$

3 Parallel-Flows Minimizers

Now we study parallel-flows that satisfy necessary conditions for minimizers of the energies in [3, 5]. There is no loss of generality in supposing henceforth that $\zeta^*(x,y) := \zeta_0(y)$ in (3), where $\zeta_0 \in L^2(0,Q)$ is non-increasing.

3.1 Prescribed Circulation μ

We begin with the energy in [5], discussed in Section 1.2, which leads to a solution of (2) with $c = 0$, and (6). If a parallel flow minimizer exists, then $\psi(x,y) = u(y)$, $x \in \mathbb{R}$, $y \in [0,Q]$, where u is solution of (7) in which ζ is a rearrangement of ζ_0, λ is non-increasing and $Pu'(Q) = \mu$. Hence

$$u'(y) = \frac{\mu}{P} + \int_y^Q \zeta(z)dz, \quad \zeta \in \mathcal{R}(\zeta_0). \tag{16}$$

From Lemma 2 and Remark 4, there are only two possibilities:

$$\begin{aligned}
&u \text{ is non-monotone, } \zeta_0 \text{ is essentially one-signed,} \\
&\zeta \in \{\check{\zeta}_{(p)},\, \hat{\zeta}_{(p)}\} \text{ where } u'(p) = 0,\ p \in (0,Q);
\end{aligned} \tag{17a}$$

$$u \text{ is monotone and } \zeta \in \big\{\zeta_0,\, \zeta^0\big\}. \tag{17b}$$

It is obvious from (16) that if μ/P and $\mu/P + [\zeta_0]$ are non-zero with opposite signs, no solution is monotone. If ζ_0 does not change sign and μ/P and $\mu/P + [\zeta_0]$ have the same sign, then all solutions are monotone. When ζ_0 changes sign it is more complicated (Remark 12).

Theorem 8. *A necessary and sufficient condition for the existence of a non-monotone solution is that ζ_0 is essentially one-signed with μ/P and $[\zeta_0] + \mu/P$ non-zero of opposite signs. For such ζ_0, P and μ, the solution of* (7) *with $u'(Q) = \mu/P$, and the function λ, are uniquely determined.*

Proof. Suppose u is a non-monotone solution. Then ζ does not change sign, by Lemma 2, and hence $\mu = 0$ is impossible. Moreover, u' is monotone and $u'(Q) = \mu/P$ and $u'(0) = \mu/P + [\zeta_0]$ have opposite signs.

For the converse, suppose ζ_0 is essentially one-signed and μ/P and $\mu/P + [\zeta_0]$ have opposite signs. By (16), the only possibilities are $u = \check{u}_{(\check{p})}$ where, in the notation of (13),

$$\check{u}_{(\check{p})}(y) := \frac{\mu y}{P} + \int_0^y \int_s^Q \check{\zeta}_{(\check{p})}(z)dz\,ds,\ \frac{\mu}{P} + \check{f}(\check{p}) = 0,$$

or $u = \hat{u}_{(\hat{p})}$ where

$$\hat{u}_{(\hat{p})}(y) := \frac{\mu y}{P} + \int_0^y \int_s^Q \hat{\zeta}_{(\hat{p})}(z)dz\,ds,\ \frac{\mu}{P} + \hat{f}(\hat{p}) = 0.$$

Since ζ is required to be a non-increasing function of u, the (not-necessarily-strict) monotonicity of $\hat{\zeta}_{(\hat{p})}$ and $\hat{u}_{(\hat{p})}$, which change at $\hat{p}$, are opposite. Similarly the monotonicities of $\check{\zeta}_{(\check{p})}$ and $\check{u}_{(\check{p})}$, which change at $\check{p}$, are opposite. Hence, under our hypothesis, if $\mu/P < 0 < \mu/P + [\zeta_0]$, we must choose $\check{u}_{(\check{p})}$, and if $\mu/P + [\zeta_0] < 0 < \mu/P$, we must choose $\hat{u}_{(\hat{p})}$, if suitable $\hat{p}$ and $\check{p}$ exist.

However, from Remark 4 and the convexity/concavity of $\check{f}$, $\hat{f}$, note that there exists a *unique* $\check{p} \in (0, Q)$ with $\mu/P + \check{f}(\check{p}) = 0$ if $-[\zeta_0] < \mu/P < 0$, and there is a *unique* $\hat{p} \in (0, Q)$ with $\rho + \hat{f}(\hat{p}) = 0$ if $0 < \mu/P < -[\zeta_0]$.

Both $\check{u}_{(\check{p})}$ and $\hat{u}_{(\hat{p})}$ are Hölder continuous and therefore $\check{\lambda}_{(\check{p})}$, $\hat{\lambda}_{(\hat{p})}$ defined by

$$\check{\lambda}_{(\check{p})}(\check{u}_{(\check{p})}(y)) = \check{\zeta}_{(\check{p})}(y) \text{ or } \hat{\lambda}_{(\hat{p})}(\hat{u}_{(\hat{p})}(y)) = \hat{\zeta}_{(\hat{p})}(y)$$

are non-increasing, by construction. Thus if $\mu > 0$, $\hat{u}_{(\hat{p})}$ is a solution, and if $\mu < 0$, $\check{u}_{(\check{p})}$ is a solution. This completes the proof. □

Remark 9. Note that for such values of P, μ and $[\zeta_0]$ another solution is uniquely determined, but with non-decreasing λ. □

To relate the monotone and non-monotone cases, it is useful to recast the hypotheses of Theorem 8 using the parameter

$$\rho := \frac{1}{2}[\zeta_0] + \frac{\mu}{P}. \tag{18}$$

Lemma 10. $|\rho| < \frac{1}{2}\big|[\zeta_0]\big|$ *if and only if* μ/P *and* $\mu/P + [\zeta_0]$ *are non-zero and have opposite signs.*

Proof. Suppose $|\rho| < \frac{1}{2}\big|[\zeta_0]\big|$. Then, from (18), $\mu \neq 0$, $[\zeta_0] \neq 0$, and μ and $[\zeta_0]$ must have opposite signs. When $\mu > 0$ and $[\zeta_0] < 0$ it follows that $\mu/P + [\zeta_0] < 0$ (because either $\mu/P + \frac{1}{2}[\zeta_0] = |\rho| < \frac{1}{2}\big|[\zeta_0]\big| = -\frac{1}{2}[\zeta_0]$, or $\mu/P + [\zeta_0] < \mu/P + \frac{1}{2}[\zeta_0] < 0$). On the other hand, when $\mu < 0$ and $[\zeta_0] > 0$ it follows that $\mu/P + [\zeta_0] > 0$ (because either $-\mu/P - \frac{1}{2}[\zeta_0] = |\rho| < \frac{1}{2}\big|[\zeta_0]\big| = \frac{1}{2}[\zeta_0]$, or $\mu/P + [\zeta_0] > \mu/P + \frac{1}{2}[\zeta_0] > 0$).

Conversely if μ/P and $\mu/P + [\zeta_0]$ are non-zero and have opposite signs, then either $\mu/P > 0$ and $\frac{1}{2}[\zeta_0] < \mu/P + \frac{1}{2}[\zeta_0] < -\frac{1}{2}[\zeta_0]$, or $\mu/P < 0$ and $-\frac{1}{2}[\zeta_0] < \mu/P + \frac{1}{2}[\zeta_0] < \frac{1}{2}[\zeta_0]$. In both cases $|\rho| < \big|[\zeta_0]\big|$, and the proof is complete. □

Turning now to monotone solutions, note that ζ_0 may change sign but that $r(\zeta_0) \geqslant 0$ defined below is zero when it does not (see Figures 2 and 3).

When $[\zeta_0] \geqslant 0$, let $\max_{[0,\frac{1}{2}Q]} \check{f} = [\zeta_0] + \frac{1}{2}r(\zeta_0)$ and $\min_{[0,\frac{1}{2}Q]} \hat{f} = \frac{1}{2}[\zeta_0] - \frac{1}{2}r(\zeta_0)$.
When $[\zeta_0] \leqslant 0$, let $\min_{[0,\frac{1}{2}Q]} \hat{f} = [\zeta_0] - \frac{1}{2}r(\zeta_0))$ and $\max_{[0,\frac{1}{2}Q]} \check{f} = \frac{1}{2}[\zeta_0] + \frac{1}{2}r(\zeta_0)$.
Let $\hat{m}(\zeta_0) = 2\min_{[0,\frac{1}{2}Q]} \hat{f}$ and $\check{M}(\zeta_0) = 2\max_{[0,\frac{1}{2}Q]} \check{f}$.

Theorem 11. *Monotone parallel-flow solutions exist if and only if*

$$|\rho| \geqslant \tfrac{1}{2}\big|[\zeta_0]\big| + r(\zeta_0). \tag{19}$$

For such values of ρ, there is only one monotone solution.

Proof. Suppose a monotone solution u exists. Then $\zeta \in \{\zeta_0, \zeta^0\}$ and, since λ is non-increasing, by (16), either

$$u'(y) = \frac{\mu}{P} + \int_y^Q \zeta_0(z)dz \geqslant 0 \text{ for all } y \in [0, Q] \tag{20a}$$

or

$$u'(y) = \frac{\mu}{P} + \int_y^Q \zeta^0(z)dz \leqslant 0 \text{ for all } y \in [0, Q]. \tag{20b}$$

It follows from the identities after (13) that this is equivalent to either

$$\frac{\mu}{P} + 2\hat{f}\left(\frac{y}{2}\right) \geqslant [\zeta_0], \text{ or } \frac{\mu}{P} + 2\check{f}\left(\frac{y}{2}\right) \leqslant [\zeta_0], \text{ for all } y \in [0, Q].$$

Hence there is a monotone solution in the form

$$\hat{u}_{(0)}(y) := \frac{\mu y}{P} + \int_0^y \int_s^Q \zeta_0(z)dz\, ds \text{ only if } \frac{\mu}{P} + \hat{m}(\zeta_0) \geqslant [\zeta_0] \tag{21a}$$

and in the form

$$\check{u}_{(0)}(y) := \frac{\mu y}{P} + \int_0^y \int_s^Q \zeta^0(z)dz\, ds \text{ only if } \frac{\mu}{P} + \check{M}(\zeta_0) \leqslant [\zeta_0]. \tag{21b}$$

From the definition of $r(\zeta_0)$ it follows that (19) is a necessary condition for the existence of a monotone solution.

Conversely, if (19) holds the definition of $r(\zeta_0)$, in the cases $\mu/P + \frac{1}{2}[\zeta_0] \geqslant 0$ and $\mu/P + \frac{1}{2}[\zeta_0] \leqslant 0$, leads to (21). The existence of solutions in the form (20) follows. This completes the proof. $\square$

Remark 12. Since $r(\zeta_0) = 0$ when ζ_0 does not change sign, there is a unique solution, which is monotone, if $\left|\mu/P + \frac{1}{2}[\zeta_0]\right| \geqslant \frac{1}{2}\left|[\zeta_0]\right|$, and a unique solution, which is non-monotone, otherwise. However, when ζ_0 changes sign and

$$\tfrac{1}{2}\left|[\zeta_0]\right| + r(\zeta_0) > \left|\mu/P + \tfrac{1}{2}[\zeta_0]\right| \geqslant \tfrac{1}{2}\left|[\zeta_0]\right|,$$

no parallel-flow solutions of any kind exist. $\square$

3.2 Prescribed Circulation μ and Momentum ν

Paper [3] considers the problem of minimizing an energy similar to (5) subject to an additional requirement that the horizontal momentum has the prescribed value ν:

$$\int_\Omega \partial_y \psi(x, y)\, dxdy = \nu.$$

It is shown there that a minimizer satisfies the third condition in (2) in the strong sense that $\zeta = \lambda(\psi - cy)$ where λ is non-increasing and c is the Lagrange multiplier corresponding to the momentum constraint. Thus when a minimizer is a parallel flow, $\psi(x,y) = u(y)$ and, for some constant c,

$$-u''(y) = \zeta(y), \quad \zeta \in \mathcal{R}(\zeta_0), \tag{22a}$$

$$u(0) = 0, \qquad u(Q) = \frac{\nu}{P}, \qquad u'(Q) = \frac{\mu}{P}, \tag{22b}$$

$$\zeta(y) = \lambda(u(y) - cy) \text{ for some non-increasing function } \lambda. \tag{22c}$$

By a solution of (22) is meant a pair (u, c), and a solution is called monotone if $u(y) - cy$ is monotone on $[0, Q]$. In Remark 16 we will see how there can exist non-monotone solutions for which ζ is monotone. Let

$$\kappa = \frac{\nu - \mu Q}{P}. \tag{23}$$

Suppose (22a), (22b) have a solution u. Then

$$\frac{\nu}{P} = \int_0^Q u'(y)\, dy = \int_0^Q \Big(u'(Q) - \int_y^Q u''(z)\, dz \Big)\, dy = \frac{Q\mu}{P} + \int_0^Q y\zeta(y)\, dy.$$

Hence a necessary condition for the existence of a solution of (22a), (22b) is that

$$\kappa = \int_0^Q y\zeta(y) dy \in \Big[\int_0^Q y\zeta_0(y)\, dy, \int_0^Q y\zeta^0(y)\, dy \Big] =: [k_0, k^0]. \tag{24a}$$

If (22c) is also satisfied and g is defined in (15), then

$$\kappa = \check{g}(p) \text{ if } \zeta = \check{\zeta}_{(p)}, \quad \kappa = \hat{g}(p) \text{ if } \zeta = \hat{\zeta}_{(p)}. \tag{24b}$$

Theorem 13. *(a) There exists a monotone solution* (u, c) *of* (22) *if and only if* $\kappa \in \{k_0, k^0\}$.
(b) If ζ_0 *is a constant,* $(\kappa, \zeta) = (k_0, \zeta_0) = (k^0, \zeta^0)$ *and* (u, c) *is a monotone solution of* (22) *if and only if* $u = u_0$ *where*

$$u_0(y) = \frac{\mu y}{P} - \frac{\zeta_0}{2}\big((Q-y)^2 - Q^2\big).$$

and c *does not lie strictly between* μ/P *and* $\mu/P - \zeta_0 Q$.
(c)If $\kappa = k_0$ *and* ζ_0 *is not a constant, all monotone solutions* (u, c) *are of the form* $(\hat{u}_{(0)}, c)$ *where* $\hat{u}_{(0)}$ *is given in* (21a) *and* $c \leqslant \mu/P + \hat{m}(\zeta_0) - [\zeta_0]$ *so that* $\hat{u}_{(0)} - cy$ *is non-decreasing.*
(d) If $\kappa = k^0$ *and* ζ_0 *is not a constant, all monotone solutions* (u, c) *are of the form* $(\check{u}_{(0)}, c)$ *where* $\check{u}_{(0)}$ *is given by* (21b) *and* $c \geqslant \mu/P + \check{M}(\zeta_0) - [\zeta_0]$ *so that* $\check{u}_{(0)} - cy$ *is non-increasing.*

Proof. (a) If (22) has a monotone solution (u, c), then $\zeta(y) = \lambda(u(y) - cy)$ is monotone, whence $\zeta \in \{\zeta_0, \zeta^0\}$ and the calculation for (24) yields that $\kappa \in \{k_0, k^0\}$. For the converse, suppose that $\kappa = k_0$. Then for $\zeta \in \mathcal{R}(\zeta_0)$,

$$\int_0^Q y\zeta(y)\, dy = k_0 = \int_0^Q y\zeta_0(y)\, dy = \min_{\zeta \in \mathcal{R}(\zeta_0)} \Big\{ \int_0^Q y\zeta(y)\, dy \Big\},$$

by the classical rearrangement inequality ([10, Ch. 10.2, Thm. 378] or [11, Thm. 3.4]). Hence $\zeta = \zeta_0$ and $(\hat{u}_{(0)}, c)$, where $\hat{u}_{(0)}$ is given by (21a), is a monotone solution for all

$c<0$ sufficiently large. Similarly, if $\kappa = k^0$, $(\check u_{(0)}, c)$, where $\check u_{(0)}$ is given by (21b), is a monotone solution for all $c>0$ sufficiently large.

(b) If ζ_0 is constant, $k_0 = k^0 = \zeta_0 Q^2/2$, and the conclusion follows by direct calculation.

(c) Suppose ζ_0 is not constant and that (u,c) is any monotone solution, so that $\zeta \in \{\zeta_0, \zeta^0\}$. Suppose $\zeta = \zeta_0$. If $u - cy$ is non-increasing, then $\zeta_0 = \lambda(u-cy)$ must be non-decreasing. Since ζ_0 is non-increasing and not a constant, this is impossible. Hence $u-cy$ is non-decreasing. From (21a), $u = \hat u_{(0)}$ and $\hat u_{(0)}(y) - cy$ is non-decreasing if and only if $\mu/P - c + \hat m(\zeta_0) \geqslant [\zeta_0]$.

(d) The proof when $\zeta = \zeta^0$ is similar. This completes the proof. □

Remark 14. In the preceding theorem $u \in \{\hat u_{(0)}, \check u_{(0)}\}$ is independent of c and since, by the identities following (13),

$$\hat u'_{(0)}(y) = \frac{\mu}{P} + \int_y^Q \zeta_0(z)dz = \frac{\mu}{P} + 2\hat f\left(\frac{y}{2}\right) - [\zeta_0], \tag{25}$$

$$\check u'_{(0)}(y) = \frac{\mu}{P} + \int_y^Q \zeta^0(z)dz = \frac{\mu}{P} + 2\check f\left(\frac{y}{2}\right) - [\zeta_0], \tag{26}$$

the number of critical points of u is determines by the number of solutions of $2\hat f(p) + \mu/P - [\zeta_0] = 0$ or $2\check f(p) + \mu/P - [\zeta_0] = 0$ in $[0, Q/2]$. If ζ_0 changes sign, from Figure 2 this is zero, one or two. □

Theorem 13 leaves open the possibility that there are non-monotone solutions when $\kappa \in \{k_0, k^0\}$. The next result characterizes all the non-monotone solutions. Recall ζ_0 does not change sign and $[\zeta_0] \neq 0$ are both necessary for existence of non-monotone solutions.

Theorem 15. *Suppose* $[\zeta_0] \neq 0$ *and* ζ_0 *does not change sign.*

(a) If $\kappa \in (k_0, k^0)$ *there exists a unique solution of* (22). *It is non-monotone and given by* $(\check u_{(\check p)}, \check c)$ *when* $[\zeta_0] > 0$ *where*

$$\check u_{(\check p)}(y) := \frac{\mu y}{P} + \int_0^y \int_s^Q \check\zeta_{(\check p)}(z)dz\,ds, \ \frac{\mu}{P} + \check f(\check p) = \check c, \ \kappa = \check g(\check p), \tag{27a}$$

or $(u,c) = (\hat u_{(\hat p)}, \hat c)$ *when* $[\zeta_0] < 0$ *where*

$$\hat u_{(\hat p)}(y) := \frac{\mu y}{P} + \int_0^y \int_s^Q \hat\zeta_{(\hat p)}(z)dz\,ds, \ \frac{\mu}{P} + \hat f(\hat p) = \hat c, \ \kappa = \hat g(\hat p). \tag{27b}$$

(b) When ζ_0 *is strictly monotone in neighbourhoods of* 0 *and* Q *(equivalently,* $a_1 = b_1 = 0$ *in Remark 7),* (22) *has a non-monotone solution only if* $\kappa \in (k_0, k^0)$.

(c) When $\kappa = k_0$ *and* $a_1 + b_1 > 0$ *in Remark 7, there are infinitely many non-monotone solutions, of the form* $(\check u_{(\check p)}, \check c)$ *in* (27a) *if* $[\zeta_0] > 0$, *and* $(\hat u_{(\hat p)}, \hat c)$ *in* (27b) *if* $[\zeta_0] < 0$, *where* $\check p \in (0, \frac12 a_1)$ *and* $\hat p \in (Q - \frac12 b_1, Q)$ *satisfy* (28).

(d) When $\kappa = k^0$ *and* $a_1 + b_1 > 0$ *in Remark 7, there are infinitely many non-monotone solutions, of the form* $(\check u_{(\check p)}, \check c)$ *in* (27a) *if* $[\zeta_0] > 0$, *and* $(\hat u_{(\hat p)}, \hat c)$ *in* (27b) *if* $[\zeta_0] < 0$, *where* $\check p \in (0, \frac12 b_1)$ *and* $\hat p \in (Q - \frac12 a_1, Q)$ *satisfy* (28).

Proof. (a) When $\kappa \in (k_0, k^0)$, no solution is monotone. Therefore, from (17a) with $u'(p)$ replaced by $u'(p) - c$, and (24b), the only possibilities are that (u, c) is $(\check{u}_{(\check{p})}, \check{c})$ or $(\hat{u}_{(\hat{p})}, \hat{c})$ where

$$\kappa = \hat{g}(\hat{p}) \text{ or } \kappa = \check{g}(\check{p}) \tag{28}$$

and

$$\frac{\mu}{P} + \check{f}(\check{p}) = \check{c} \text{ or } \frac{\mu}{P} + \hat{f}(\hat{p}) = \hat{c}. \tag{29}$$

Because ζ and $u - cy$ in (22) have opposite monotonicities, we must have $(\check{u}_{(\check{p})}, \check{c})$ if $[\zeta_0] > 0$ and $(\hat{u}_{(\hat{p})}, \hat{c})$ if $[\zeta_0] < 0$, with $\check{c}$, $\hat{c}$ determined by (29), if (28) has a solution.

However, when $\kappa \in (k_0, k^0)$ and the sign of $[\zeta_0]$ is given, the relevant equation from (28) has unique solutions in $(0, Q)$ (in fact in $(a_1/2, Q - b_1/2)$ where a_1 and b_1 are defined in Remark 7). The uniqueness is immediate from this construction.

(b) Suppose that (22) has a non-monotone solution (u, c) and $a_1 = b_1 = 0$ in Remark 7. It is then immediate that (u, c) is either $(\check{u}_{(\check{p})}, \check{c})$ or $(\hat{u}_{(\hat{p})}, \hat{c})$, for some $\check{p}, \hat{p} \in (0, Q)$. Since the hypothesis implies that ζ_0, $\zeta^0 \notin \{\check{\zeta}_{(p)}, \hat{\zeta}_{(p)}\}$, it follows that $\kappa \neq k_0$, k^0, and $\kappa \in (k_0, k^0)$ follows from (24).

An elementary adaptation of the proof of (a), in the light of Remark 7 yields (c) and (d), and completes the proof. $\square$

Remark 16. The solutions given by parts (c) and (d) involve $\{\check{\zeta}_{(\check{p})}, \hat{\zeta}_{(\hat{p})}\}$ which, by Remark 7, coincides with $\{\zeta_0, \zeta^0\}$ in these cases. Note that although $\hat{g}$ or $\check{g}$ are independent of p in these intervals, $\hat{f}$ and $\check{f}$, and consequently c, are not. Thus there are intervals of wave speeds for which the vorticity profiles are the same, as in part (a). Note also that parts (c) and (d) give non-monotone solutions for which the vorticity is monotone. $\square$

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References

[1] P. Baldi and J.F. Toland, Steady periodic water waves under nonlinear elastic membranes, *J. Reine Angew. Math.* **652** (2011), 67-112. DOI: 10.1515/crelle.2011.015.

[2] T. B. Benjamin, The alliance of practical and analytical insights into the nonlinear problems of fluid mechanics. In *Applications of Methods of Functional Analysis to Problems in Mechanics.* (Joint Sympos., IUTAM/IMU, Marseille, 1975), pp. 8–29. Lecture Notes in Math., 503. Springer, Berlin, 1976.

[3] B. Buffoni and G. R. Burton, On the stability of travelling waves with vorticity obtained by minimisation. To appear in *Nonlinear Differential Equations Appl.* http://arxiv.org/abs/1207.7198

[4] G. R. Burton, Global nonlinear stability for steady ideal fluid flow in bounded planar domains. *Arch. Ration. Mech. Anal.* **176** (2005), 149-163.

[5] G. R. Burton and J. F. Toland, Surface waves on steady perfect-fluid flows with vorticity *Comm. Pure Appl. Math.* **LXIV** (2011), 975-1007.

[6] A. J. Chorin and J. E. Marsden, *An Introduction to Mathematical Fluid Mechanics.* Springer, New York, 1993.

[7] A. Constantin and W. Strauss, Exact steady periodic water waves with vorticity, *Comm. Pure Appl. Math.*, Vol. **LVII** (2004), 481–527.

[8] A. Constantin and W. Strauss, Stability properties of steady water waves with vorticity, *Comm. Pure Appl. Math.*, Vol. **LX** (2007), 911–950.

[9] M.-L. Dubreil-Jacotin, Sur la détermination rigoureuse des ondes permanentes périodiques d'ampleur finie, *J. Math. Pures Appl.* **13** (1934), 217–291.

[10] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities.* 2nd. Edition. Cambridge University Press, Cambridge, 1954.

[11] E. H. Lieb and M. Loss, *Analysis.* Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, R. I. 1997.

[12] V. Kozlov and N. Kuznetsov, Steady free-surface vortical flows parallel to the horizontal bottom. *Quart. J. Mech. Appl. Math.* **64** (2011), 371-399.

[13] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Non-viscous Fluids.* Applied Mathematical Sciences 96, Springer-Verlag, New York, 1994.

[14] W. A. Strauss, Steady water waves, *Bull. Am. Math. Soc.*, New Ser. 47, No. 4, 671-694 (2010).

[15] J. F. Toland, Steady periodic hydroelastic waves, *Arch. Rational Mech. Anal.* **189** (2) (2008), 325–362. (DOI 10.1007/s00205-007-0104-2)

[16] J. F. Toland, Non-existence of global minimisers of energy in Stokes-wave problems. *To appear.*