Non-existence of Global Energy-Minimisers in Stokes Wave Problems

J. F. Toland*

Abstract

Recently it was shown that a wave profile which minimises total energy, elastic plus hydrodynamic, subject to the vorticity distribution being prescribed, gives rise to a steady hydroelastic wave. Using this formulation, the existence of non-trivial minimisers leading to such waves was established for certain non-zero values of the elastic constants. Here we show that when these constants are zero, global minimisers do not exist except in a unique set of circumstances.

Introduction

In two recent papers [2, 3] it was shown that minimisers of energy, expressed as a function of an unknown flow domain and a stream function with prescribed vorticity distribution, yield idealised, two-dimensional, steady, hydroelastic waves. This shape-optimisation formulation led to the existence of waves for which the surface is not flat for a range of positive surface-elasticity coefficients. However, classical water waves, with and without vorticity but with zero surface energy, were not accounted for by these methods, even though there are extensive alternative (but non-variational) global theories of existence. The purpose here is to show that, without surface energy, *global* minimisers do not exist, except for a single set of prescribed data for which the question is trivial.

To be precise, in the absence of surface elasticity, a minimiser of the energy in [3] exists if, and only if, the given data (ζ_Q, μ) (ζ_Q prescribes the vorticity distribution and μ the surface circulation per period) are $(0, 0)$, and a minimiser in [2] exists if, and only if, the given data (ζ_Q, μ, ν) (ν prescribes the horizontal momentum per period) are $(0, \mu, Q\mu)$. Thus, in the absence of surface energy effects, the steady water waves found by non-variational methods [1, 4, 5, 6, 7, 9, 10, 11] are not *global* minimisers of the hydrodynamic energy in [2, 3]. They are, presumably, critical points in some other sense, but their nature in a variational setting remains unexplored. In the irrotational case there is a rich theory of Morse indices [8] which, for the moment, seems inaccessible in the presence of vorticity. However, without surface energy there is no satisfactory global theory of the existence of water waves, with or without vorticity, by variational methods.

*St John's College, Cambridge, jft26@cam.ac.uk, 13th April 2013

Parallel flows do arise by variational methods because the arguments of [2, 3], when restricted to parallel-flow admissible functions with prescribed vorticity distribution, lead to the existence of mimimisers in that class. However the surface is flat and elasticity plays no role. A complete account of possible parallel-flow minimisers (which are not energy minimisers over general domains) is given in [12]. The following simple observation is the key to the non-existence of minimisers when general domains are admissible.

Estimates for a Boundary-value Problem

Let $\mathit{\Omega}$ be a P-periodic domain which is bounded below by the real axis and above by a Jordan curve $\mathscr{S}$ which is P-periodic in the x-direction $((P,0)+\mathscr{S}=\mathscr{S})$. Let $\mathcal{S}$, which is one period of $\mathscr{S}$, be the upper boundary of Ω, which is one period of $\mathit{\Omega}$. For given P-periodic $\zeta \in L^2_{\text{loc}}(\mathit{\Omega})$, and for $\mu \in \mathbb{R}$, consider the boundary-value problem

$$\left.\begin{array}{rr} & -\Delta\psi = \zeta \text{ on } \mathit{\Omega}, \\ \psi = 0 \text{ on } \mathbb{R}\times\{0\}, & \psi \text{ is } P\text{-periodic}, \\ \psi = C \text{ on } \mathscr{S}, & \displaystyle\int_{\mathcal{S}} \nabla\psi\cdot n\, dS = \mu, \end{array}\right\} \qquad (1) \quad \boxed{\texttt{BVP}}$$

for some constant C. This problem has a unique solution (ψ, C), which is the maximiser of

$$\min_{\psi\in\mathcal{A}(\Omega)} \left\{ \int_\Omega \left(-\frac{1}{2}|\nabla\psi|^2 + \zeta\psi \right) dxdy + \mu C(\psi) \right\}, \qquad (2) \quad \boxed{\texttt{minim}}$$

where

$$\mathcal{A}(\Omega) = \left\{ \begin{array}{c} \psi \in W^{1,2}_{\text{loc}}(\mathit{\Omega}) : \psi \text{ is } P\text{-periodic}, \\ \psi = 0 \text{ on } \mathbb{R}\times\{0\} \text{ and } \psi = C(\psi), \text{ a constant, on } \mathscr{S}. \end{array} \right\}$$

The aim is to construct a sequence of domains $\mathit{\Omega}_k$ of the kind described above such that each Ω_k has the same specified area, and a sequence of P-periodic functions $\zeta_k \in L^2_{\text{loc}}(\mathit{\Omega})$, each with the same prescribed norm $\|\zeta_k\|_{L^2(\Omega_k)}$, such that the corresponding solutions of (1) have $|C(\psi_k)| + \|\nabla\psi_k\|_{L^2(\Omega)} \to 0$ as $k\to\infty$.

Domains

Let P, $Q>0$ be fixed, let $n\in\mathbb{N}$, $\sigma\in(0,Q)$, $\delta\in(0,1)$, and let h denote the (P/n)-periodic extension of the lower semicontinuous function defined on $[-P/2n,\ P/2n]$ by

$$h(x) = \left\{ \begin{array}{ll} q := \dfrac{Q-\sigma\delta}{1-\delta} > Q, & x \in (-(1-\delta)P/2n, (1-\delta)P/2n) \\ \sigma, & x \in [-P/2n, -(1-\delta)P/2n] \cup [(1-\delta)P/2n, P/2n] \end{array} \right\}.$$

Let $\mathit{\Omega} = \{(x,y) : y \in (0,h(x)),\ x\in\mathbb{R}\}$. Then $\mathit{\Omega}$ is an open, P-periodic domain with Lipschitz boundary. For convenience let

$$\Omega = \mathit{\Omega} \cap \big((-P/2, P/2)\times(0,\infty)\big), \quad \partial\Omega_B = (-P/2,P/2)\times\{0\},$$

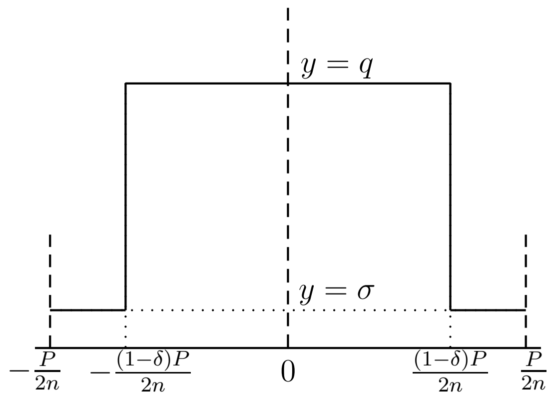

Figure 1. The domain ω.

$$\partial\Omega_T = \big(\partial\Omega \setminus \mathbb{R}\times\{0\}\big) \cap \big((-P/2, P/2)\times(0,\infty)\big)$$

and

$$\omega = \Omega \cap \big((-P/2n, P/2n)\times(0,\infty)\big), \quad \partial\omega_B = (-P/2n, P/2n)\times\{0\},$$
$$\partial\omega_T = \partial\Omega_T \cap \big((-P/2n, P/2n)\times(0,\infty)\big).$$

By construction meas $\Omega = n(\operatorname{meas}\omega) = PQ$.

r1 *Remark* 1. Let $\zeta \in L^2_{\text{loc}}(\Omega)$ be P/n-periodic and $\mu \in \mathbb{R}$. Then (1) has a solution which, by uniqueness, coincides on ω with the unique solution of the boundary-value problem

$$\left.\begin{aligned} -\Delta\psi &= \zeta \text{ on } \omega,\\ \psi = 0 \text{ on } \partial\omega_B, \quad \psi &\text{ is } P/n\text{-periodic},\\ \psi = \text{ is constant on } \partial\omega_T, \quad \int_{\partial\omega_T} \nabla\psi\cdot n\, dS &= \frac{\mu}{n}, \end{aligned}\right\} \tag{3}$$ bvp

which maximises of the analogue of (2) on ω. □

LC **Lemma 2.** *For given $\mu\in\mathbb{R}$, $n\in\mathbb{N}$ and a P/n-periodic function ζ, the solution of (1) satisfies*

$$|C| \leqslant \frac{2\sigma\mu}{P\delta} + \|\zeta\|_{L^2(\omega)}\sqrt{\frac{2n\sigma}{P\delta}}\sqrt{2\left(\frac{P(1-\delta)}{2n}\right)^2 + 4q(1-\delta)\frac{\sigma}{\delta} + \sigma^2}, \tag{4}$$ C

$$\|\nabla\psi\|^2_{L^2(\Omega)} \leqslant 2\|\zeta\|^2_{L^2(\Omega)}\left\{2\left(\frac{P(1-\delta)}{2n}\right)^2 + 4q(1-\delta)\frac{\sigma}{\delta} + \sigma^2\right\} + \frac{4\sigma\mu^2}{P\delta}. \tag{5}$$ D

Proof. Let $K = \|\nabla\psi\|_{L^2(\omega)}$ and note that

$$\frac{P\delta|C|}{2n} = \left|\int_{\frac{(1-\delta)P}{2n}}^{\frac{P}{2n}} \int_0^\sigma \psi_y(x,y)\; dxdy\right| \leqslant K\sqrt{\frac{P\delta\sigma}{2n}}.$$

Hence

$$|C| \leqslant K\sqrt{\frac{2n\sigma}{P\delta}}. \tag{6}$$ CC

Next note that for $x \in (-(1-\delta)P/2n, (1-\delta)P/2n)$, $y \in (\sigma, q)$,

$$|\psi(x,y)| = \left| \frac{1}{2} \int_{\frac{-(1-\delta)P}{2n}}^{x} \psi_x(t,y)\,dt - \frac{1}{2} \int_{x}^{\frac{(1-\delta)P}{2n}} \psi_x(t,y)\,dt + C \right| \leqslant \frac{1}{2} \int_{\frac{-(1-\delta)P}{2n}}^{\frac{(1-\delta)P}{2n}} |\psi_x(x,y)|\,dx + |C|.$$

Since $(a+b)^2 \leqslant 2(a^2+b^2)$, for $x \in (-(1-\delta)P/2n, (1-\delta)P/2n)$,

$$\begin{aligned} \int_\sigma^q |\psi(x,y)|^2\,dy &\leqslant \frac{1}{2} \int_\sigma^q \left(\int_{\frac{-(1-\delta)P}{2n}}^{\frac{(1-\delta)P}{2n}} |\psi_x(x,y)|\,dx \right)^2 dy + 2q|C|^2 \\ &\leqslant \frac{(1-\delta)P}{2n} \int_\sigma^q \int_{\frac{-(1-\delta)P}{2n}}^{\frac{(1-\delta)P}{2n}} |\psi_x(x,y)|^2\,dxdy + 2q|C|^2 \\ &\leqslant \frac{(1-\delta)P}{2n} K^2 + 2q|C|^2. \end{aligned}$$

Therefore

$$\int_{\frac{-(1-\delta)P}{2n}}^{\frac{(1-\delta)P}{2n}} \int_\sigma^q |\psi(x,y)|^2\,dxdy \leqslant 2\left(\frac{P(1-\delta)K}{2n}\right)^2 + \frac{2Pq(1-\delta)|C|^2}{n}. \qquad (7) \quad \boxed{\texttt{est1}}$$

Also, for $x \in (-P/2n, P/2n)$, $y \in (0,\sigma)$,

$$|\psi(x,y)|^2 = \left| \int_0^y \psi_y(x,s)ds \right|^2 \leqslant \sigma \int_0^\sigma |\psi_y(x,y)|^2 dy.$$

Hence, for $y \in (0,\sigma)$,

$$\int_{-P/2n}^{P/2n} |\psi(x,y)|^2 dx \leqslant \sigma \int_{-P/2n}^{P/2n} \int_0^\sigma |\psi_y(x,y)|^2 dxdy \leqslant \sigma K^2$$

and so

$$\int_0^\sigma \int_{-P/2n}^{P/2n} |\psi(x,y)|^2 dxdy \leqslant \sigma^2 K^2. \qquad (8) \quad \boxed{\texttt{est2}}$$

Therefore, by (6), (7) and (8),

$$\int_\omega |\psi(x,y)|^2\,dxdy \leqslant K^2 \left\{ 2\left(\frac{(1-\delta)P}{2n}\right)^2 + 4q(1-\delta)\frac{\sigma}{\delta} + \sigma^2 \right\}. \qquad (9) \quad \boxed{\texttt{est3}}$$

From (3), (6) and (9) it follows that

$$\begin{aligned} K^2 = \int_\omega |\nabla\psi|^2 dxdy &= \int_\omega \zeta\psi\,dxdy + \frac{\mu C}{n} \\ &\leqslant K\|\zeta\|_{L^2(\omega)} \sqrt{2\left(\frac{P(1-\delta)}{2n}\right)^2 + 4q(1-\delta)\frac{\sigma}{\delta} + \sigma^2} + K\mu\sqrt{\frac{2\sigma}{nP\delta}}. \end{aligned} \qquad (10) \quad \boxed{\texttt{sq}}$$

Cancelling K and substituting in (6) gives (4), and squaring (10) gives

$$K^2 \leqslant 2\|\zeta\|^2_{L^2(\omega)} \left\{ 2\left(\frac{P(1-\delta)}{2n}\right)^2 + 4q(1-\delta)\frac{\sigma}{\delta} + \sigma^2 \right\} + \frac{4\sigma\mu^2}{nP\delta}.$$

Since ζ is (P/n)-periodic, it follows by Remark 1 that

$$\int_\Omega |\nabla\psi|^2 dxdy = nK^2 \leqslant 2\|\zeta\|^2_{L^2(\Omega)} \left\{ 2\left(\frac{P(1-\delta)}{2n}\right)^2 + 4q(1-\delta)\frac{\sigma}{\delta} + \sigma^2 \right\} + \frac{4\sigma\mu^2}{P\delta},$$

which proves (5). □

Remark. An almost identical calculation leads to the same conclusion when the Lipschitz domains Ω are replaced by C^∞ or real-analytic domains. Smoothness is not the issue. □

Minimising Sequences

Denote the sets Ω, ω and the parameter q above by Ω_k, ω_k and q_k when $n = k$, $\delta = 1/k$, $\sigma = 1/k^2$, $k \geqslant 2$. See Figure 2 for Ω_k and note that $q_k \to Q$ as $k \to \infty$. Let $\mathcal{S}_k = \partial\Omega_{k_T}$.

Suppose ζ_Q is any prescribed P-periodic, locally square-integrable function on $\mathit{\Omega}_Q := \mathbb{R} \times (0, Q)$. Let $\Omega_Q = (-P/2, P/2) \times (0, Q)$ and, for Ω with $\operatorname{meas}\Omega = PQ$, let $\mathcal{R}_Q(\Omega)$ denote the rearrangements on Ω of $\zeta_Q\big|_{\Omega_Q}$.

Let $\zeta_k \in \mathcal{R}_Q(\Omega_k)$ be P/k-periodic and let (ψ_k, C_k) be the corresponding solution of (1) on Ω_k. By Remark 1, ψ_k is a solution of (3) on ω_k.

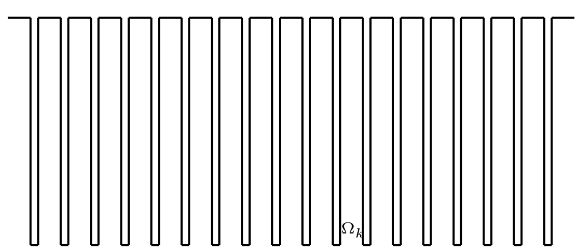

Figure 2. $\mathit{\Omega}_k$ is P/k periodic; meas $(\Omega_k) = PQ$; troughs at $y = 1/k^2$; trough width $= 1/k^2$.

cor **Corollary 3.** *With $\zeta = \zeta_k$ and $\Omega = \Omega_k$ in Lemma 2,*

$$|C_k| \to 0 \text{ and } \int_{\Omega_k} |\nabla\psi_k|^2 \to 0 \text{ as } n \to \infty.$$

Proof. This is immediate since $\|\zeta_k\|_{L^2(\Omega_k)} = \|\zeta_Q\|_{L^2(\Omega_Q)}$ for all k, $q_k \to Q$ as $k \to \infty$, and the right sides of (4) and (5) tend to zero when $n = k$, $\sigma = 1/k^2$ and $\delta = 1/k$. □

Without surface energy in [3]

In the notation of [3], $P = 2\pi$ and the infimum of the energy is

$$M := \inf_{\Omega,\zeta}\left\{\max_{\psi\in\mathcal{A}(\Omega)}\left\{\int_\Omega\left(-\frac{1}{2}|\nabla\psi|^2+\zeta\psi\right)dxdy+\mu C(\psi)\right\}+g\int_\Omega y\,dxdy\right\},$$

over all admissible domains Ω with area $2\pi Q$ and all $\zeta\in\mathcal{R}_Q(\Omega)$. We have observed that

$$M \leqslant \inf_{n\geqslant 2}\int_{\Omega_k}\left(\frac{1}{2}|\nabla\psi_k|^2+gy\right)dxdy \text{ where } \psi_k \text{ satisfies (1) with } \zeta=\zeta_k \text{ on } \Omega_k.$$

Since it is obvious that

$$\int_{\Omega_k} y\,dxdy \to \int_{-\pi}^{\pi}\int_0^Q y\,dxdy = \pi Q^2,$$

it follows from Corollary 3 that

$$\int_{\Omega_k}\left(\frac{1}{2}|\nabla\psi_k|^2+gy\right)dxdy \to \pi g Q^2$$

and hence that $M \leqslant \pi g Q^2$. On the other hand,

$$M \geqslant g\int_\Omega y\,dxdy \geqslant g\int_{\Omega_Q} y\,dxdy = g\pi Q^2.$$

by [3], Lemma 4.6. Hence $M = \pi g Q^2$. If it is attained by (Ω,ψ,ζ), then $\nabla\psi = 0$ almost everywhere and $\Omega = \Omega_Q$. Hence ψ does not satisfy (1) if $\Omega \neq \Omega_Q$ and $(\zeta_Q,\mu)\neq(0,0)$.

The conclusion is that M is not attained when $(\zeta_Q,\mu)\neq(0,0)$. Obviously if $(\zeta_Q,\mu) = (0,0)$, then $\psi\equiv 0$ attains the infimum M.

Without surface energy in [2]

With $\Omega = \Omega_k$ in Lemma 2, let $\widehat{\psi}_k$ denote ψ when $\zeta = 0$ and $\mu = 1$, $\overline{\psi}_k$ denote ψ when $\zeta = \zeta_k$ and $\mu = 0$, and let $\widetilde{\psi}_k(x,y) = y$. Note that

$$\int_{\Omega_k}\frac{\partial\widehat{\psi}_k}{\partial y} = \widehat{C}_k P \text{ where } \hat{\psi}_k = \widehat{C}_k \text{ on } \mathcal{S}_k,$$
$$\int_{\Omega_k}\frac{\partial\overline{\psi}_k}{\partial y} = \overline{C}_k P \text{ where } \overline{\psi}_k = \overline{C}_k \text{ on } \mathcal{S}_k,$$
$$\int_{\Omega_k}\frac{\partial\widetilde{\psi}_k}{\partial y} = PQ \text{ and } \int_{\mathcal{S}_k}\nabla\widetilde{\psi}_k\cdot n\,dS = P.$$

Let $\psi_k = a_k\widehat{\psi}_k+\overline{\psi}_k+b_k\widetilde{\psi}_k$ and, in the notation of [2], let $\xi_k = \psi_k\big|_{\mathcal{S}_k}$. Then for given real numbers μ and ν, the triple (Ω_k,ξ_k,ζ_k) is admissible for the variational problem in [2] if

$$a_k + Pb_k = \mu; \qquad \widehat{C}_k P a_k + PQb_k = \nu - \overline{C}_k P,$$

and hence

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \frac{1}{P(Q-\widehat{C}_kP)} \begin{pmatrix} PQ & -P \\ -\widehat{C}_kP & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \nu-\overline{C}_kP \end{pmatrix} \tag{11}$$ ab

if $Q \neq \widehat{C}_kP$. However $\widehat{C}_k$ depends only on Ω_k and, by Lemma 2, $\widehat{C}_k \to 0$ as $n \to \infty$. Hence (a_k, b_k) is uniquely determined when k is sufficiently large. Similarly from Lemma 2, $\overline{C}_k \to 0$ as $k \to \infty$. Moreover $a_k\widehat{\psi}_k + \overline{\psi}_k$ coincides with ψ in Lemma 2 when $\mu = a_k$ and $\zeta = \zeta_k$. Hence

$$\int_{\Omega_k} |\nabla(a_k\widehat{\psi}_k + \overline{\psi}_k)|^2 dxdy \to 0 \text{ and } \int_{\Omega_k} |b_k\nabla\widetilde{\psi}_k|^2 dxdy \to \frac{\nu^2}{PQ} \text{ as } k \to \infty.$$

Hence, in the notion of[2], the infimum of the energy in the absence of surface elasticity, is

$$\begin{aligned} m &:= \inf\{\mathcal{L}(\Omega,\xi,\zeta) : \Omega \in \mathfrak{O},\ \xi \in H^{1/2}_{\text{loc}}(\mathcal{S}),\ \zeta \in \mathcal{R}_Q(\Omega),\ C=\mu,\ \ I=\nu\} \\ &\leqslant \inf_{k\geqslant 2}\left\{\frac{1}{2}\int_{\Omega_k} |\nabla\psi_k|^2 dxdy + g\int_{\Omega_k} ydxdy\right\} \leqslant \frac{1}{2}\left(\frac{\nu^2}{PQ} + gPQ^2\right). \end{aligned} \tag{12}$$

However, by the Cauchy-Schwarz inequality, for $\Omega \in \mathfrak{O}$ and any admissible ψ,

$$\nu = \int_\Omega \frac{\partial\psi}{\partial y} dxdy \text{ implies that } \frac{1}{2}\int_\Omega |\nabla\psi|^2 dxdy \geqslant \frac{1}{2}\left(\frac{\nu^2}{PQ}\right).$$

Since, as before,

$$g\int_{\Omega_k} ydxdy \geqslant \frac{gPQ^2}{2}, \text{ it follows that } m \geqslant \frac{1}{2}\left(\frac{\nu^2}{PQ} + gPQ^2\right).$$

This shows that

$$m = \frac{1}{2}\left(\frac{\nu^2}{PQ} + gPQ^2\right)$$

and this value is attained if and only if $\psi(x,y) = \nu y/PQ$ and $\Omega = \Omega_Q$ and $\zeta_Q = 0$.

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