IMMORTALITY OF PLATONIC SOLIDS

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1. Introduction

The existence of symmetry in nature and human creation is one of the most puzzling and exciting phenomena.The first manifestation of Platonic solids in nature is in the shape of viruses [4]. It is possible to say that Platonic solids appeared at the birth of life.

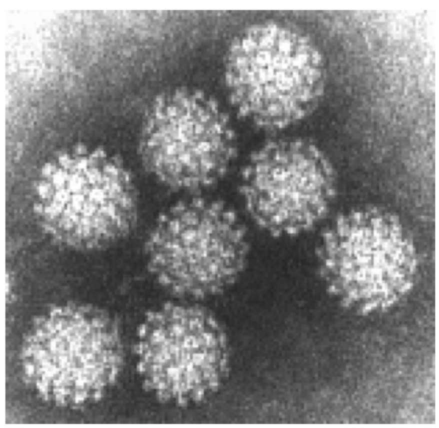
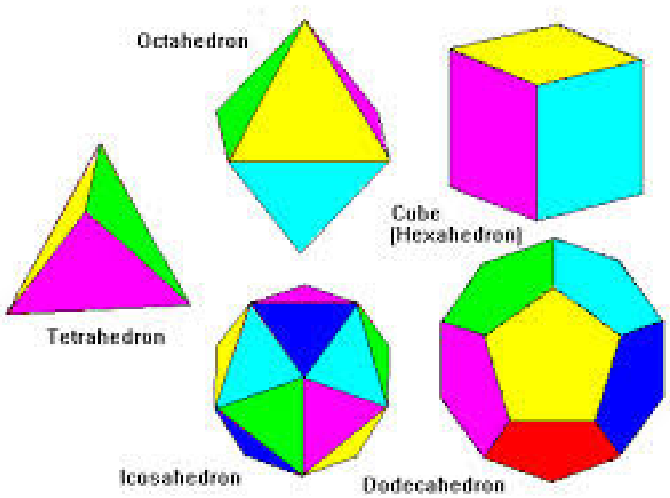


Figure 1. Icosahedron Papilloma Virus

The study of symmetry during the whole history of civilization led to the creation of the deepest theories in Science and became a source of inspiration in Art, Architecture and so on. We choose from this inexaustible theme one small fragment devoted to regular solids.



2. Some History

The most simplest examples of regular solids like, the tetrahedron, cube, and octahedron are found in crystals. More sophisticated forms like, the dodecahedron and icosaehedron do not appear as perfect crystals.[1]. This result was proved only in the nineteenth century. It is worth noting that a form close to the almost dodecahedron exists in nature. This is the mineral pyrite which was known from neolithic times. The Icosahedron does not exist as a natural crystal but in modern time does become a fundamental ingredient in the construction of quasicrystals. We discuss this very interesting result later.

The classification of platonic solids, i.e. the proof that there exist only 5 regular convex polyhedrons, appeared as a mathematical theorem in the XIII book of Euclid's elements, but was known before This was the very first classification theorem in the history of mathematics. Euclid's proof was based on the analysis of the numbers of angles and sides. It took almost two thousand years before there appeared the famous Euler formula $V - E + F = 2$ was appeared, in modern terms the Euler characteristic of Sphere S^2. Here V is the number of vertices of the polyhedron, E is the number of edges, and F is the number of faces. From this formula, the nice and elementary proof follows almost immediately.

This work of Euler, along with his other work about Königsberg's bridges became the base of modern topology. Let us jump hundred years ahead and open an beautiful book of Felix Klein [2] In this book, he relates the problem of unsolvability of algebric equations of the fifth order with the simplicity of the group of symmetry of the Icosahedron- the so called group of even permutations of 5 elements- the group A_5. This is one of the first examples of simple groups. The classification of simple groups is one of the greatest achievements of the XXI century. Now we

[1]The first dodecahedron may have long proceded tale.The North-Europeans 2500 B.C. seem to have molded docahedron from clay, perhaps by packing together 13 soft clay balls, arranged in a pleasing pattern,then pulling them appart to reveal the central, now-faceted form: a regular dodecahedron. In nature there exists a species Radiolaria, including Circogonia icosahedra, whose skeleton is shaped like a regular icosahedron.

consider some examples of modern research which show how unpredictably, but repeatedly Platonic solids appear in modern research.

We started from the result which we extract from the paper [1]

3. Caustics with symmetries of platonic solids

Let us consider the following simple model: the envelope of the rays emanating from convex wave front invariant under the actions of the group of symmetry of Platonic solids–the polyhedral groups. How does reflect this on the topology of singularities in the model, the well known duality between platonic solids? We call such singularities as polyhedral caustics. Let us recall this classical duality (by exchange the number of verices with edge) : 1.The tetrahedron is self-dual. 2.The cube is dual to the octahedron. 3.The icosahedron is dual to the dodecahedron. The remarkable result of the paper [1] is that this duality leads to the duality in umbilics for all types of Platonic solids. To be precise, if the wave front is invariant under the symmetry group G, then the set of singularities, so called caustic surfaces, are invariant under the same group G but acting on a dual space.field H play the role of physical coordinates x, y, z. The equilibrium conditions define the energy E as a function of H and the level surface $E(H) = const$ is a compact surface W. Its caustics represent the fields for which the magnetization orientation undergoes a jump(swtching fields). For example, in the case of cubic symmetry, the caustics of switching fields present the topology of caustics for the octahedron.

4. Platonic solids and McKay correspondence

At the end of the 1970s, J. McKay found an unpredictable relation between representations of finite groups and Coxeter graphs of simple Lie algebras [3]. This result tied together the classical works of Klein (1872)for invariants of platonic solids,the classification of algebraic surfaces with isolated singularities, and the classification of Lie algebras and simple groups. Let G be a finite subgroup of $SU_2(\mathbf{C})$ - the binary group of invariance of Platonic solids. If $\mathbf{C}^2$ is a two-dimensional complex space, then $\mathbf{C}^2/G$ is a complex surfaces with an isolated singularity. It is possible to show that the resolution of this singularity for all Platonic solids relates to exceptional Lie algebras. On the other hand these singularities are determined by an algebra A invariant under a group G,generated by three variables X, Y, Z with one relation. For example, in the case of the symmetry group of Icosahedron, this relation is the follows:$X^5 + Y^3 + Z^2 = 0$ [9]. We leave this very active field of research by quotating McKay: "Would not the Greeks appreciate the result that the simple Lie algebras may be derived from the Platonic solids?".

5. Quasicrystals

Our next example is from the theory of quasicrystals. We quote from Klein's book : "The theory of Icosahedron has during the last years obtained a place of such importance for nearly all departments of modern analysis, that it seemed expedient to publish a systematic exposition of the same" [2]. Exactly one hundred years later, a group of physicists published the paper where the Icosahedron appeared in a very unpredictable situation in the theory of crystals[8]. As is well known from the work of Bravais and others, crystals with symmetry of 5th, 8th, 10th, and 12th orders do not exist, since such symmetries are inconsistent with translation-invariance of the crystal lattice. So this structure having local symmetry, but lacking spatial periodicity. Structures of this type were called quasicrystals. Such quasicrystals relate to the representation of the group of the Icosahedron I acting in R^6 and then projecting by an "irrational" angle to R^3 embedded in R^6.

Representation of the group I can be decomposed in two invariant subspaces R^3 and $R^{3\perp}$. We embed the space R^3 in R^6 by an "irrational" angle. It means that the intersection of a lattice Z^6 in R^6 consists of the origin alone. The projection of Z^6 to R^3 defines a tiling in R^6 which is our quasicrystal.

6. Duality in Spin systems with the symmetry of finite subgroups of $SO(3)$

In this section we describe the final example from our collection. It illustrates complex relations between mathematics and physics. We begin with some mathematics.

In the beginning of the 1930s, L. Pontryagin and E. R. van Kampen built the duality theory for abelian groups. The main result was the following theorem.

Let G be a locally compact Abelian group. Consider the group of characters of G, i.e. the set of mappings $\chi : G \to U(1)$ satisfying

$$\chi(g_1 \times g_2) = \chi(g_1) \times \chi(g_2)$$

This set of mappings make up a group $\widehat{G}$, called the character group (or the dual group) of G. Then $\widehat{\widehat{G}}$, the dual group of $\widehat{G}$, coincides with G.

It is natural for mathematicians to try to generalize this theorem to a non-Abelian case. One of the main examples of non-Abelian groups provide the finite subgroups of $SO(3)$, the group of rotation of our friends the groups of symmetries of Platonic solids and the Dihedral group D_{2n}. We consider this topic in broader context.

This is a nontrivial task, since in the non-commutative case, the product of irreducible representations is not irreducible and so the set $\widehat{G}$ is not a group. Nevertheless, this problem was solved in some sence by Japanese mathematician T.Tannaka in 1938. Idependently, it was solved by the Soviet mathematician M. Krein in 1941, who didn't know about the work of Tannaka. The paper of Tannaka attracted the attention of J. von Neumann who noted the above-mentioned difficulty and indicated several important general properties of $\widehat{G}$. A dual to a non-commutative group is not a group but a commutative space, endowed with a multiplicative operation. The papers of Tannaka and Krein were practically forgotten for almost thirty years, until the first papers appeared on non-commutative integration and ring groups. But the real value of these works has been appreciated only later, in the 1980s, when the theory of quantum groups was created. Quantum groups are closely related to integrable quantum systems. These systems appeared earlier in physics.

Now we turn to physics.

In statistical physics, within the theory of phase transitions, for a long time a number of models have been proposed and analysed, describing lattice aproximation for various kinds of physical matter. One of the first such models was the one-dimensional Ising model (1925). E. Ising (1900-1998), a student of W. Lenz, wrote a paper were he found an analytical formula for the free energy of the model.

A generalization to the two-dimensional case led to serious difficulties. The two-dimensional Ising model was solved only in 1944 by L. Onsager and till now is a rare example of an exactly solvable model in Statistical physics.

Consequently, physicists tried to find approximate methods to identify points of phase transitions. In 1941, two Dutch physicists H. Kramers and G. Wannier have found a very nice method of calculating the point of phase transition in the two-dimentional Ising model. They constructed a transformation between the low-temperature and the high-temperature phases. It was latter called the Kramers–Wannier duality. From the mathematical point of view, it is a very interesting object: an infinite-dimensional bundle with the structure group $G = Z_2$. The Kramers–Wannier duality consists in passing to the dual lattice (homological duality à la Poincaré) and to the dual group $\widehat{G}$, in this case coinciding with the same Z_2. Later, physicists generalized the KW-duality to systems with a Z_n-symmetry (so-called Potts models).

At the end of the 1970s, in connection with problems in quantum field theory (quark confinement),physicists became interested in a generalization of the KW-duality to non-Abelian groups. Just at this time several papers appeared related to the KW-duality for some non-Abelian groups; (see [5]). One of the interesting results of this paper was the construction of the dual object to the symmetric groups S_n. For instance, the dual object to S_3 is the octahedron. At that time, the authors of this paper had no idea about the results of Tannaka–Krein.

Then, 25 years later, V. Buchstaber and one of the authors of this paper(M.M) constructed the KW-duality for non-commutative finite groups based on the ideas of quantum groups [6, 7] Only later we learned about the works of Tannaka and Krein.

Although our results were not covered by Tannaka–Krein duality it is evident that the earlier knowledge of their ideas would have allowed us to complete our work much earlier. It is an additional example to the famous collection of F. Dyson [10]. In the course of our work we have found interesting relations with an old (and almost forgotten) paper by Frobenius [11]. Ferdinand Gotfried Frobenius (1849-1917) was one of the founders of the theory of group representations (mainly for finite groups). His famous theorems about irreducible representations of groups are presented in all textbooks on the theory of groups representations. However, one of his papers, full of interesting ideas, was shelved for decades. For instance, Frobenius introduced for non-commutative groups the concept of generalized characters. He posed the question of whether generalized characters determine a group in the same way as in the commutative case. It is well known that there exist non-isomorphic groups with the same table of characters, viz., the group of unit quaternions Q and the dihedral group D_2, both of order 8.

7. Conclusion

For lack of space we omitted some other interesting examples of the appearence of Platonic (regular) solids in science, e.g. minimal surfaces with Platonic symmetries. But we hope that the examples we mentioned, show the surprising and unique role of Platonic solids in nature, science, and culture. There is no doubt that Platonic solids will continue surprise and encourage us in the future.

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