Version: today is 2 August 2013.

The action of the orthogonal group on planar vectors: invariants, covariants, and syzygies

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Abstract. The construction of invariant and covariant polynomials from the x, y components of n planar vectors under the SO(2) and O(2) orthogonal groups is addressed. Molien functions determined under the SO(2) symmetry group are used as a guide to propose integrity bases for the algebra of invariants and the modules of covariants. The Molien functions that describe the structure of the algebra of invariants and the free modules of (m)–covariants, $m \leq n-1$, are written as a ratio of a numerator in λ with positive coefficients over a $(1-\lambda^2)^{2n-1}$ denominator. This form of single rational function is standard in invariant theory and has a clear symbolic interpretation. However, its usefulness is lost for the non–free modules of (m)–covariants, $m \geq n$, due to negative coefficients in the numerator. We propose for these non–free modules a new representation of the Molien function as a sum of n rational functions with positive coefficients in the numerators and different numbers of terms in the denominators. This non–standard form is symbolically interpreted in term of a generalized integrity basis. Integrity bases are explicitly given for $n = 2, 3, 4$ planar vectors and m ranging from 0 to 5. The integrity bases obtained under the SO(2) symmetry group are subsequently extended to the O(2) group.

PACS numbers: 02.20.-a, 02.20.Hj, 31.15.xh, 31.30.jy, 33.20.Vq

AMS classification scheme numbers: 13A50, 15A72, 16W22

Submitted to: *J. Phys. A: Math. Gen.*

1. Introduction

1.1. Invariant theory

Group theory is the natural mathematical framework to describe the consequences of symmetry. It was introduced in physics by Weyl [1] and Wigner [2] to efficiently treat problems with symmetry emanating from quantum mechanics. One typical use of group theory is the development of phenomenological descriptions. Theoretical assumptions or experimental data often lead to consider a symmetry group G. The issue is then to build a quantity transforming according to an irreducible representation Γ_f of group G from basic elements that span a possibly reducible representation Γ_i.

In molecular physics, the symmetry of a quasi–rigid linear polyatomic molecule in its equilibrium configuration is either the $C_{\infty v}$ or $D_{\infty h}$ point group [3, 4]. These two groups are respectively isomorphic to the orthogonal O(2) and O(2) $\times Z_2$ abstract groups. They both admit the special orthogonal group SO(2) as a proper subgroup. A quasi–rigid linear molecule with n atoms has $n-2$ doubly degenerate vibrational bending modes [5]. The displacement vectors of these modes are contained in planes perpendicular to the symmetry axis of the molecule and enter as basic elements in the construction of invariants such as the effective Hamiltonian or covariants such as the electric dipole moment. The carbon dioxide CO_2 [6] and acetylene C_2H_2 [7] are two examples of linear molecules where the theory of effective operators [8, 9, 10] was successfully employed to analyze high–resolution rotation–vibration spectra. Similar considerations with respect to the O(2) orthogonal group occur in a transversely isotropic material [11]. A privilegied axis exists and the properties are the same in all directions of an isotropy plane normal to this axis. Such a material is axially symmetric and the group SO(2) is a subgroup of its symmetry group.

The invariant or covariant functions are typically generated through a lengthy step by step approach that constructs all possible terms of degree n compatible with the final representation of the symmetry group from simpler terms of lower degree [12, 13]. Invariant theory [14] is a branch of mathematics based on group theory and algebra which gives an alternative way to construct these objects. Its main concern is the global description of the structure of the ring $R = \mathbb{C}\left[x_1, \ldots, x_n\right]^{\Gamma_0}$ of G–invariant polynomials in the variables x_i. The symbol Γ_0 represents the totally symmetric irreducible representation of group G. The extension of invariant theory to covariant polynomials considers the modules of covariants. Invariant theory has been applied successfully in numerous fields of physics: molecular physics [15], crystallography [16, 17], liquid crystals [18], continuum mechanics [19, 20, 21], high–energy physics [22, 23], theory of qubits [24, 25], and qualitative analysis of physical systems [26, 27].

1.2. Representation theory and Molien function

The Taylor expansion of the Molien function $M^G\left(\Gamma_f; \Gamma_i; \lambda\right)$ at $\lambda = 0$ gives information about the number c_k of linearly independent objects of degree k transforming as the irreducible representation Γ_f of group G that can be built up from elementary objects that span the n–dimensional Γ_i representation [28, 29]:

$$M^G\left(\Gamma_f;\Gamma_i;\lambda\right)=\sum_{k=0}^{\infty}c_k\lambda^k.$$

Counting the number of linearly independent objects of degree k generated at the end of the step by step construction evocated in section 1.1 returns the c_k value. Molien however found that the generating function could be directly determined without requiring any construction of the final objects [30]. Equation (1) states that the Molien function of a finite group G of order $|G|$ depends on the character $\chi\left(\Gamma_f;g\right)$ of the final irreducible representation and on the $n\times n$ matrices $M\left(\Gamma_i;g\right)$ of the initial reducible linear representation:

$$M^G\left(\Gamma_f;\Gamma_i;\lambda\right)=\frac{1}{|G|}\sum_{g\in G}\frac{\bar{\chi}\left(\Gamma_f;g\right)}{\det\left(1_{n\times n}-\lambda M\left(\Gamma_i;g\right)\right)}. \qquad (1)$$

The bar over the character χ refers to the complex conjugation, $1_{n\times n}$ denotes the $n\times n$ unit matrix and the sum runs over all the elements of the group G. The determinant in the denominator assures that the Molien function is independent of the basis chosen to write the matrix representation of Γ_i. For continuous groups, the discrete sum over the group elements g is replaced by an integration over the continuous parameters of the group [2]. Two different symbolic interpretations of the right–hand side of (1) have arisen along the development of invariant theory.

1.3. Method of syzygies

The first point of view deals with a set of polynomial generators $\gamma_1,\ldots,\gamma_s$ of the ring of invariants [31]. Any invariant polynomial of the ring of invariants decomposes as a (possibly non–unique) polynomial in these generators. The γ_i are algebraically independent in the special case of groups generated by reflections. The Molien function for the ring of invariants is then simply given by (2), where each generator of degree d_i contributes to a factor $\left(1-\lambda^{d_i}\right)$ in the denominator of the rational function M^G.

$$M^G\left(\Gamma_0;\Gamma_i;\lambda\right)=\frac{1}{\prod_{i=1}^{s}\left(1-\lambda^{d_i}\right)}. \qquad (2)$$

In the general case, however, the generators are often algebraically dependent, *i.e.* there exist first–order syzygies $\sigma_i\left(\gamma_1,\ldots,\gamma_s\right)=0$ where σ_i is a polynomial of degree f_i. The decomposition of any invariant as a polynomial in the generators is then not unique. One invariant may be counted several times in the Taylor expansion of (2). Taking into account the first–order syzygies of degree f_i among the generators implies to correct the numerator of the Molien function (2) as $\left(1-\sum\lambda^{f_i}\right)$ to eliminate the overcounting. The syzygies σ_i themselves may not be algebraically independent and second–order syzygies occur between them. Too many polynomials have then been removed and the numerator of the Molien function should now be corrected as $\left(1-\sum\lambda^{f_i}+\sum\lambda^{g_i}\right)$. The construction is continued as necessary with even higher–order syzygies. The Hilbert syzygy theorem [32, 33] assures that the procedure finishes after a finite number of steps and implies that the numerator is a finite polynomial in λ. The method of syzygies results in a ratio of a polynomial with alternating coefficients in λ over a product of $\left(1-\lambda^{d_i}\right)$ terms:

$$M^G\left(\Gamma_0;\Gamma_i;\lambda\right)=\frac{1-\sum\lambda^{f_i}+\sum\lambda^{g_i}-\cdots}{\prod_{i=1}^{s}\left(1-\lambda^{d_i}\right)}.$$

1.4. Integrity basis

A second point of view interprets the Molien functions in term of integrity bases [34], also called homogeneous systems of parameters [31]. Such bases are often associated with Cohen–Macaulay rings of invariants [31]. Such a ring presents the remarkable structure of a free module $M\left(\Gamma_0\right)$ over a subring $R_1\subset R$ of invariants, where $\left(M\left(\Gamma_0\right),+\right)$ is an additive group of polynomial invariants. This decomposition is known in the mathematical literature as an Hironaka decomposition [35]. The corresponding integrity basis contains D algebraically independent denominator invariant polynomials θ_k that generate the subring D and N linearly independent numerator invariant polynomials φ_k that span the vector space $\left(M_{\text{invar}},+\right)$. The denominator and numerator polynomials are also named primary and secondary invariants by other authors [35]. Any invariant polynomial in $x_1,\ldots,x_n$ is uniquely decomposed as a polynomial in the numerator and denominator polynomials [28]:

$$\sum_{k=1}^{N}\varphi_k\left(x_1,\ldots,x_n\right)\times p_k\left(\theta_1\left(x_1,\ldots,x_n\right),\ldots,\theta_D\left(x_1,\ldots,x_n\right)\right), \tag{3}$$

where the p_k are polynomials in D variables. The two different roles of the denominator and numerator polynomials clearly appears in (3). The denominator polynomials can be exponentiated to any non–negative integer while the numerator polynomials only occur linearly.

It is straightforward to write a Molien function that admits a symbolic interpretation when the algebraic structure of the ring of invariants or the module of covariants is known. The reverse problem guesses the set of denominator and numerator polynomials from the expression of the Molien function. This question is more important for physical applications. An integrity basis may be suggested when the Molien function is appropriately expressed as a ratio of a numerator polynomial with non–negative coefficients in λ over a denominator written as a product of $\left(1-\lambda^{\delta_k}\right)$ terms:

$$M^G\left(\Gamma_f;\Gamma_i;\lambda\right)=\frac{\sum_{k=1}^{k=N}\lambda^{\nu_k}}{\prod_{k=1}^{k=D}\left(1-\lambda^{\delta_k}\right)},\ \ \nu_k\in\mathbb{N},\ \ \delta_k\in\mathbb{N}_0=\mathbb{N}\backslash\left\{0\right\}, \tag{4}$$

where the integers δ_k and ν_k are respectively the degree of the polynomials θ_k and φ_k. The number D of denominator polynomials in the right–hand side of equation (4) is generally different from the number s of generators considered in the method of syzygies.

The above description of a Cohen–Macaulay ring of invariants is valid too for a free module of covariants transforming according the irreducible representation Γ_f. The algebraic structure of a free module of Γ_f–covariants is a module $M(\Gamma_f)$ over a subring $R_1 \subset R$ of invariants where $(M(\Gamma_f), +)$ is an additive group of Γ_f–covariant polynomials. The denominator polynomials of the Molien function (4) are still invariant polynomials but the numerator polynomials are now Γ_f–covariants. Not all the modules of covariants are free however. The present paper shows in the case of the action of the orthogonal group on n planar vectors that a generalized integrity basis can still be proposed when the module of covariants is non–free.

1.5. Example of integrity bases with one planar vector

We give an example of integrity bases by presenting the action of the SO(2) group on the ring $\mathcal{P}_1 = \mathbb{C}[x, y]$ of polynomials that depend on the x, y components of one planar vector. A point M of coordinates (x, y) is rotated under the action of the SO(2) group by an angle φ in the (Ox, Oy) plane. The action of the element g_φ of SO(2) on the (x, y) coordinates is simply related to a rotation matrix: $g_\varphi \diamond (x, y) = (x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi)$. The action of g_φ on any polynomial $p \in \mathcal{P}_1$ is then defined by:

$$(g_\varphi \bullet p)(x, y) = p\left(g_\varphi^{-1} \diamond (x, y)\right) = p\left(x\cos\varphi + y\sin\varphi, -x\sin\varphi + y\cos\varphi\right). \quad (5)$$

The irreducible representations of the SO(2) group are all unidimensional and are labelled as (m) with $m \in \mathbb{Z}$ a relative integer. Equation (5) shows that the action of the SO(2) group on the polynomials of degree one $\pi(x, y) = x - iy$ and $\mu(x, y) = x + iy$ is diagonal (i is the imaginary number defined by $i^2 = -1$):

$$(g_\varphi \bullet \pi)(x, y) = e^{i\varphi}\pi(x, y), \quad (6a)$$

$$(g_\varphi \bullet \mu)(x, y) = e^{-i\varphi}\mu(x, y). \quad (6b)$$

The π and μ functions transform respectively as the $(+1)$ and (-1) irreducible representations and the pair of functions (π, μ) span a two–dimensional reducible representation $\Gamma_1 = (-1) \oplus (+1)$ of the SO(2) group.

A monomial $\pi^{n_1}\mu^{n_2}$ is invariant under the SO(2) group action if $n_1 = n_2$. As a consequence, we can construct one invariant of degree two, $\pi\mu$, one invariant of degree four, $\pi^2\mu^2$,.... The polynomial 1 is an invariant of degree zero. The corresponding Molien function is:

$$M^{\mathrm{SO}(2)}\left((0); \Gamma_1; \lambda\right) = 1 + \lambda^2 + \lambda^4 + \lambda^6 + \cdots = \frac{1}{1 - \lambda^2}. \quad (7)$$

The integrity basis for the invariants can now be read from (7): it contains one denominator polynomial of degree 2 and one numerator polynomial of degree 0. They are respectively chosen as $\pi\mu$ and 1.

A monomial $\pi^{n_1}\mu^{n_2}$ is (1)–covariant if $n_1 = n_2 + 1$. The (1)–covariant of lowest degree is π_1, the next one is $\pi^2\mu$ of degree three. The Molien function (8) suggests one denominator polynomial of degree 2 and one numerator polynomial of degree 1, respectively chosen as $\pi\mu$ and π.

$$M^{\mathrm{SO}(2)}\left((1)\,;\Gamma_1;\lambda\right) = \lambda + \lambda^3 + \lambda^5 + \cdots = \frac{\lambda}{1-\lambda^2}. \tag{8}$$

This last result can be generalized to the set of (m)–covariants, $m \geq 1$. The Molien function (9) suggests one denominator polynomial of degree 2, which is chosen as $\pi\mu$, and one numerator polynomial of degree m, which is chosen as π^m.

$$M^{\mathrm{SO}(2)}\left((m)\,;\Gamma_1;\lambda\right) = \lambda^m + \lambda^{m+2} + \lambda^{m+4} + \cdots = \frac{\lambda^m}{1-\lambda^2},\ m \geq 1. \tag{9}$$

1.6. Outline of the paper

Section 1.5 demonstrates that the Molien function for the ring of polynomial invariants and for the modules of polynomial (m)–covariants built from the pair of components (x, y) of one planar vector can be written as a single rational function which admits a symbolic interpretation in term of integrity basis. This paper considers the ring of polynomial invariants and modules of polynomial (m)–covariants constructed from the components of n planar vectors under the SO(2) and O(2) group actions. The Molien function $M^{\mathrm{SO}(2)}\left((m)\,;\Gamma_i;\lambda\right)$ is introduced in section 2, where two different forms of the Molien function, $M_\alpha^{\mathrm{SO}(2)}$ and $M_\beta^{\mathrm{SO}(2)}$, are presented and discussed. Their explicit expressions for two, three and four vectors and $0 \leq m \leq 5$ are given. The extension to the O(2) group is presented in section 3. The ring of invariants and modules of (m)–covariants decomposes under SO(2) as free modules when $0 \leq m \leq n-1$. The rational fraction $M_\alpha^{\mathrm{SO}(2)}$ admits a standard symbolic interpretation in term of integrity basis for these irreducible representations but negative coefficients appear in the numerator for $m \geq n$. This paper shows the remarkable result that a symbolic interpretation of the Molien function can still be given for $m \geq n$ if the $M_\beta^{\mathrm{SO}(2)}$ form is used. It is an expansion as a sum of several rational functions with different numbers of terms in the denominator. This remarkable decomposition has not yet been discussed in the literature. The module of the corresponding (m)–covariants is non–free and we show that representations with lattices help to better understand the difference between the free and non–free situations. Section 4 presents integrity bases for the polynomials built from two vectors at low m. This work is extended to three vectors in section 5 and to four vectors in section 6. Finally, section 7 generalizes the results to the O(2) orthogonal group.

2. Molien functions for n planar vectors under SO(2)

2.1. Computation of the Molien functions via an integral

Let us consider n planar vectors of components $(x_i, y_i)_{1\leq i\leq n}$, and the ring $\mathcal{P}_n = \mathbb{C}\left[x_1, y_1, x_2, y_2, \cdots, x_n, y_n\right]$ of polynomials in these $2n$ components. The aim of this paper is to propose integrity bases for the ring of invariants $\mathcal{P}_n^{(0)} \subset \mathcal{P}_n$ and the modules of

(m)–covariants $\mathcal{P}_n^{(m)} \subset \mathcal{P}_n$ under the SO(2) symmetry. The disjoint union $\cup_{m\in\mathbb{Z}}\mathcal{P}_n^{(m)}$ of all the subsets $\mathcal{P}_n^{(m)}, m \in \mathbb{Z}$ simply equals the $\mathcal{P}_n$ set of all polynomials in $2n$ variables. As discussed in section 1.5, each pair of functions $\pi_j = \pi(x_j, y_j) = x_j - iy_j$, $\mu_j = \mu(x_j, y_j) = x_j + iy_j$ spans a two–dimensional reducible representation $\Gamma_1 = (-1) \oplus (+1)$ of the SO(2) group. The n two–dimensional vectors span a reducible representation which is the direct sum of the representations of each π_j, μ_j:

$$\Gamma_n = \underbrace{\Gamma_1 \oplus \cdots \oplus \Gamma_1}_{\text{n times}}.$$

The initial representation Γ_n contains n times the $(+1)$ representation and n times the (-1) representation. The Molien functions $M^G(m; \Gamma_n; \lambda)$ and $M^G(-m; \Gamma_n; \lambda)$ are identical and the integrity basis associated with the second Molien function is the complex conjugate of the integrity basis associated with the first Molien function. The integer m is considered from now on to be non–negative: $m = 0, 1, 2, \cdots$. We deduce the Molien generating function for the action of the SO(2) group on the space $\mathcal{P}_n$ from the general formula (1) adapted to the continuous groups and the characters of the SO(2) group:

$$M^{\mathrm{SO}(2)}((m); \Gamma_n; \lambda) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-im\varphi}}{(1-\lambda e^{i\varphi})^n (1-\lambda e^{-i\varphi})^n} d\varphi. \tag{10}$$

Evaluating the integral in equation (10) with the theorem of residues (the detailed steps of the derivation are given in supplementary data XXX) gives an explicit expression for the Molien function:

$$\begin{aligned} &M^{\mathrm{SO}(2)}((m); \Gamma_n; \lambda) \\ &= \frac{\lambda^{2(n-1)+m}}{(1-\lambda^2)^{2n-1}} \sum_{k=0}^{n-1} \begin{pmatrix} n-1+m \\ k \end{pmatrix} \begin{pmatrix} 2(n-1)-k \\ n-1 \end{pmatrix} \left(\frac{1-\lambda^2}{\lambda^2}\right)^k. \end{aligned} \tag{11}$$

It is noteworthy that the arguments of both binomial coefficients are all non–negative for $n \geq 1$ and $m \geq 0$. Furthermore, the numerator of each binomial coefficient is greater or equal than the corresponding denominator. This is the standard domain of definition of the binomial coefficient $\begin{pmatrix} n \\ k \end{pmatrix}$ with $0 \leq k \leq n$. Equation (11) is thus well defined for all values of n and m.

The Taylor expansion of the Molien function (11) gives the number of invariant or (m)–covariant polynomials of a given degree. No direct symbolic interpretation can however been given to expression (11). Two other equivalent expressions of the Molien function are proposed in section 2.2 and section 2.3. They have the same Taylor expansion as (11) and, furthermore, a symbolic interpretation in term of integrity bases is possible for $M_\alpha^{\mathrm{SO}(2)}$ on the range $0 \leq m \leq n-1$ and $M_\beta^{\mathrm{SO}(2)}$ when $m \geq n$.

2.2. Molien function as one rational function

The Molien function (11) can be rewritten as a ratio $M_\alpha^{\mathrm{SO}(2)}((m); \Gamma_n; \lambda) = N_\alpha(n, m; \lambda) / (1-\lambda^2)^{2n-1}$, between two polynomials in λ, with the polynomial N_α

defined as:

$$N_\alpha\left(n,m;\lambda\right)=\lambda^m\sum_{k=0}^{n-1}\left(\begin{array}{c}n-1+m\\k+m\end{array}\right)\left(\begin{array}{c}n-1-m\\k\end{array}\right)\lambda^{2k}.$$

The function $M_\alpha^{\mathrm{SO}(2)}$ is exactly the same function as (11) provided the generalized definition of binomial coefficients to negative integer arguments defined in [36, 37, 38] is adopted. This extended definition is consistent with the one employed by the Maple computer algebra system [39]. The numerator N_α is a polynomial in λ of degree $2\left(n-1\right)-m$ for $0\leq m\leq n-1$ and of degree $2\left(n-1\right)+m$ for $m\geq n$.

The form $M_\alpha^{\mathrm{SO}(2)}$ is appealing when $0\leq m\leq n-1$. The four arguments in the binomial coefficients of N_α are non–negative in such a case. Furthermore, a binomial coefficient $\left(\begin{array}{c}n\\k\end{array}\right)$ vanishes for $0\leq n<k$, and the sum over k needs only to range from 0 to $n-1-m$. The coefficients of the λ^i terms in the numerator N_α are all non–negative. The form of the Molien function $M_\alpha^{\mathrm{SO}(2)}$ is identical to the standard form (4) and it can be used to suggest integrity bases in the range $0\leq m\leq n-1$.

The sum over k of all the binomial coefficients entering the numerator N_α is a central binomial coefficient:

$$\sum_{k=0}^{n-1}\left(\begin{array}{c}n-1+m\\k+m\end{array}\right)\left(\begin{array}{c}n-1-m\\k\end{array}\right)=\left(\begin{array}{c}2\left(n-1\right)\\n-1\end{array}\right).\qquad(12)$$

While the left hand side of (12) depends on n and m, the right hand side depends only on n and is independent of the label m of the final irreducible representation. The binomial coefficient in the right–hand side of (12) is the number of multivariate monomials of degree $n-1$ in n variables.

2.3. Molien function as a sum of rational functions

The Molien function (11) can be written as a sum of n rational functions, $M_\beta^{\mathrm{SO}(2)}\left(\left(m\right);\Gamma_n;\lambda\right)=\sum_{k=0}^{n-1}N_\beta\left(n,m;k;\lambda\right)/\left(1-\lambda^2\right)^{2n-1-k}$, where the numerators N_β are polynomials of degree m:

$$N_\beta\left(n,m;k;\lambda\right)=\left(\begin{array}{c}2\left(n-1\right)-k\\n-1\end{array}\right)\left(\begin{array}{c}m-n+k\\k\end{array}\right)\lambda^m.$$

The powers in the denominators of $M_\beta^{\mathrm{SO}(2)}$ range from n to $2n-1$.

The form $M_\beta^{\mathrm{SO}(2)}$ is appealing for $m\geq n$: the two binomial coefficients appearing in $N_\beta\left(n,m;k;\lambda\right)$ are always positive, whereas they can be non–positive for $0\leq m\leq n-1$. For $m\geq n$, the form $M_\beta^{\mathrm{SO}(2)}$ is a sum of n rational functions where all the numerators have non–negative coefficients. Each term in the sum is a rational function that gives a partial contribution to the generalized integrity basis construction. It is noteworthy that the number of terms in the denominators of these n rational functions ranges from

n to $2n-1$. Such form of a Molien function was not yet suggested in the mathematical or physics literature as a form suitable for symbolic interpretation.

The total number of numerator polynomials is given by the sum of all the binomial coefficients that appear in $N_\beta\left(n,m;k;\lambda\right)$:

$$\sum_{k=0}^{n-1}\left(\begin{array}{c} 2\left(n-1\right)-k \\ n-1 \end{array}\right)\left(\begin{array}{c} m-n+k \\ k \end{array}\right)=\left(\begin{array}{c} n-1+m \\ m \end{array}\right). \tag{13}$$

The right–hand side corresponds to the number of multivariate monomials of degree m in n variables.

2.4. Expressions of the Molien functions $M_\alpha^{\mathrm{SO}(2)}$ and $M_\beta^{\mathrm{SO}(2)}$ for $n=2,3,4$ and low m

The explicit expressions of the $M_\alpha^{\mathrm{SO}(2)}$ and $M_\beta^{\mathrm{SO}(2)}$ Molien functions for $n=2,3,4$ and $0\leq m\leq 5$ are given in table 1. It is easy to check the property expressed by equation (12). The sum of the coefficients of the numerator polynomial of M_α depends on n but is independent of m.

3. Molien functions for n planar vectors under $\mathrm{O}(2)$

The irreducible representations of the $\mathrm{O}(2)$ group are the two one–dimensional representations A_1 and A_2 and an infinite number of two–dimensional representations E_m. The reflection in a line $\Delta\left(\varphi\right)$ that makes an angle φ with the Ox axis acts on the coordinates as: $g_{\Delta(\varphi)}\diamond\left(x,y\right)=\left(x\cos 2\varphi+y\sin 2\varphi, x\sin 2\varphi-y\cos 2\varphi\right)$, and the action of $g_{\Delta(\varphi)}$ on any polynomial $p\in\mathcal{P}_1$ is:

$$\left(g_{\Delta(\varphi)}\bullet p\right)\left(x,y\right)=p\left(g_{\Delta(\varphi)}^{-1}\diamond\left(x,y\right)\right)=p\left(x\cos 2\varphi+y\sin 2\varphi, x\sin 2\varphi-y\cos 2\varphi\right).$$

Each pair of functions $\pi_j=\pi\left(x_j,y_j\right)=x_j-iy_j$, $\mu_j=\mu\left(x_j,y_j\right)=x_j+iy_j$ transforms as

$$\begin{aligned}\left(g_{\Delta(\varphi)}\bullet\pi\right)\left(x,y\right)&=e^{-2i\varphi}\mu\left(x,y\right),\\ \left(g_{\Delta(\varphi)}\bullet\mu\right)\left(x,y\right)&=e^{2i\varphi}\pi\left(x,y\right),\end{aligned}$$

and spans a two–dimensional irreducible representation $\Gamma_1=E_1$ of the $\mathrm{O}(2)$ group and the initial representation Γ_n is again the direct sum of the Γ_1. The expressions of the Molien functions for the $\mathrm{O}(2)$ point group can be constructed from the Molien functions of the $\mathrm{SO}(2)$ group:

$$\begin{aligned}M^{\mathrm{O}(2)}\left(A_1;\Gamma_n;\lambda\right)&=\frac{1}{2}\left(M^{\mathrm{SO}(2)}\left(\left(0\right);\Gamma_n;\lambda\right)+\frac{1}{\left(1-\lambda^2\right)^n}\right),\\ M^{\mathrm{O}(2)}\left(A_2;\Gamma_n;\lambda\right)&=\frac{1}{2}\left(M^{\mathrm{SO}(2)}\left(\left(0\right);\Gamma_n;\lambda\right)-\frac{1}{\left(1-\lambda^2\right)^n}\right),\\ M^{\mathrm{O}(2)}\left(E_m;\Gamma_n,\lambda\right)&=M^{\mathrm{SO}(2)}\left(\left(m\right);\Gamma_n;\lambda\right).\end{aligned}$$

The sum $M^{\mathrm{O}(2)}\left(A_1;\Gamma_n;\lambda\right)+M^{\mathrm{O}(2)}\left(A_2;\Gamma_n;\lambda\right)$ of the Molien functions for the final A_1 and A_2 representations of the $\mathrm{O}(2)$ group gives back the Molien function for the (0)

Table 1. Expressions of $M_{\alpha}^{\mathrm{SO}(2)}\left(\Gamma_f;\Gamma_n;\lambda\right)$ and $M_{\beta}^{\mathrm{SO}(2)}\left(\Gamma_f;\Gamma_n;\lambda\right)$ Molien functions for $n=2,3,4$ planar vectors and final $\Gamma_f=(m)$, $0\leq m\leq 5$ irreducible representations of SO(2). The distinct contributions of the A_1 and A_2 invariants of O(2) are separated in the numerator of the rational function for SO(2)–invariants.

n	Γ_f	$M_{\alpha}^{\mathrm{SO}(2)}\left(\Gamma_f;\Gamma_n;\lambda\right)$	$M_{\beta}^{\mathrm{SO}(2)}\left(\Gamma_f;\Gamma_n;\lambda\right)$
2	(0)	$\frac{(1)_{A_1}+\left(\lambda^2\right)_{A_2}}{(1-\lambda^2)^3}$	
2	(1)	$\frac{2\lambda}{(1-\lambda^2)^3}$	
2	(2)		$\frac{2\lambda^2}{(1-\lambda^2)^3}+\frac{\lambda^2}{(1-\lambda^2)^2}$
2	(3)		$\frac{2\lambda^3}{(1-\lambda^2)^3}+\frac{2\lambda^3}{(1-\lambda^2)^2}$
2	(4)		$\frac{2\lambda^4}{(1-\lambda^2)^3}+\frac{3\lambda^4}{(1-\lambda^2)^2}$
2	(5)		$\frac{2\lambda^5}{(1-\lambda^2)^3}+\frac{4\lambda^5}{(1-\lambda^2)^2}$
3	(0)	$\frac{\left(1+\lambda^2+\lambda^4\right)_{A_1}+\left(3\lambda^2\right)_{A_2}}{(1-\lambda^2)^5}$	
3	(1)	$\frac{3\lambda+3\lambda^3}{(1-\lambda^2)^5}$	
3	(2)	$\frac{6\lambda^2}{(1-\lambda^2)^5}$	
3	(3)		$\frac{6\lambda^3}{(1-\lambda^2)^5}+\frac{3\lambda^3}{(1-\lambda^2)^4}+\frac{\lambda^3}{(1-\lambda^2)^3}$
3	(4)		$\frac{6\lambda^4}{(1-\lambda^2)^5}+\frac{6\lambda^4}{(1-\lambda^2)^4}+\frac{3\lambda^4}{(1-\lambda^2)^3}$
3	(5)		$\frac{6\lambda^5}{(1-\lambda^2)^5}+\frac{9\lambda^5}{(1-\lambda^2)^4}+\frac{6\lambda^5}{(1-\lambda^2)^3}$
4	(0)	$\frac{\left(1+3\lambda^2+6\lambda^4\right)_{A_1}+\left(6\lambda^2+3\lambda^4+\lambda^6\right)_{A_2}}{(1-\lambda^2)^7}$	
4	(1)	$\frac{4\lambda+12\lambda^3+4\lambda^5}{(1-\lambda^2)^7}$	
4	(2)	$\frac{10\lambda^2+10\lambda^4}{(1-\lambda^2)^7}$	
4	(3)	$\frac{20\lambda^3}{(1-\lambda^2)^7}$	
4	(4)		$\frac{20\lambda^4}{(1-\lambda^2)^7}+\frac{10\lambda^4}{(1-\lambda^2)^6}+\frac{4\lambda^4}{(1-\lambda^2)^5}+\frac{\lambda^4}{(1-\lambda^2)^4}$
4	(5)		$\frac{20\lambda^5}{(1-\lambda^2)^7}+\frac{20\lambda^5}{(1-\lambda^2)^6}+\frac{12\lambda^5}{(1-\lambda^2)^5}+\frac{4\lambda^5}{(1-\lambda^2)^4}$

representation of the SO(2) group. The expressions of these functions are given in table 1 for $n=2,3,4$. Terms in the numerator of $M^{\mathrm{O}(2)}\left(A_1;\Gamma_n;\lambda\right)$ are labelled as $(x)_{A_1}$ and a similar notation is followed for A_2. The Molien function for the E_m representation of O(2) is identical to the Molien function for the (m) representation of SO(2). We can associate to each (m)–covariant its complex conjugate (which is a $(-m)$–covariant) and the pair transforms according to representation E_m.

4. Integrity bases for two planar vectors under SO(2)

4.1. Invariants

The Molien function $M^{\mathrm{SO}(2)}\left((0)\,;\Gamma_2;\lambda\right)$ for the construction of SO(2)–invariants from the Γ_2 representation suggests an integrity basis composed of three denominator polynomials and two numerator polynomials. The polynomials $\pi_i\mu_j, 1\leq i,j\leq 2$ are manifestly SO(2)–invariant. They are the generators of the ring of invariants $\mathcal{P}_2^{(0)}$ [14]. The four generators may be alternatively chosen as scalar products $r_1 = \mathbf{r}_1 \cdot \mathbf{r}_1$, $r_2 = \mathbf{r}_2 \cdot \mathbf{r}_2$, $s_{1,2} = \mathbf{r}_1 \cdot \mathbf{r}_2$ and z–components $t_{1,2}$ of vector products between two planar vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ [14, 28]:

$$\begin{aligned} r_1 &= \pi_1\mu_1 = x_1^2 + y_1^2, \\ r_2 &= \pi_2\mu_2 = x_2^2 + y_2^2, \\ s_{1,2} &= (\pi_1\mu_2 + \pi_2\mu_1)\,/2 = x_1x_2 + y_1y_2, \\ t_{1,2} &= (\pi_1\mu_2 - \pi_2\mu_1)\,/(2i) = x_1y_2 - x_2y_1. \end{aligned} \tag{16}$$

The first numerator polynomial is the polynomial 1 of degree zero while the second one is a quadratic polynomial of symmetry A_2 in O(2). It is natural to choose it as $t_{1,2}$. The three algebraically independent quadratic denominator polynomials are selected as r_1, r_2 and $s_{1,2}$. Furthermore, the Molien function $M^{\mathrm{SO}(2)}\left((0)\,;\Gamma_2;\lambda\right)$ can be written as the rational function $\left(1-\lambda^4\right)/\left(1-\lambda^2\right)^4$. This expression admits a symbolic interpretation in the spirit of the method of syzygies of section 1.3. The four terms in the denominator correspond to the four generators. The λ^4 term in the numerator indicates that the quartic polynomial in the coordinates of the planar vectors vanishes [14].

4.2. Free module of (1)*–covariants*

The three denominator invariants are chosen to be the same r_1, r_2 and $s_{1,2}$ polynomials as in the analysis of invariants. The two linearly independent (1)–covariants of degree one suggested by the numerator of $M_\alpha^{\mathrm{SO}(2)}\left((1)\,;\Gamma_2;\lambda\right)$ are selected as π_1 and π_2. To describe the structure of the free module of (1)–covariants, we remark that any (1)–covariant decomposes as a linear combination of an infinite set of (1)–covariants:

$$\sum_{(n_1,n_2,n_3)\in\mathbb{N}^3} c^{(1)}_{n_1,n_2,n_3} r_1^{n_1} r_2^{n_2} s_{1,2}^{n_3}\pi_1 + \sum_{(n_1,n_2,n_3)\in\mathbb{N}^3} c^{(2)}_{n_1,n_2,n_3} r_1^{n_1} r_2^{n_2} s_{1,2}^{n_3}\pi_2. \tag{17}$$

The coefficients $c^{(1,2)}_{n_1,n_2,n_3}$ in the decomposition are complex numbers. The monomials $r_1^{n_1} r_2^{n_2} s_{1,2}^{n_3}\pi_{1,2}$ constitute the $\mathbb{C}$–basis of the module of (1)–covariants. Any (1)–covariant decomposes as a $\mathbb{C}$–linear combination of elements in the corresponding $\mathbb{C}$–basis. The $\mathbb{C}$–basis is different from the concept of basis of a free module. The module structure described by (17) consists of a ring of invariant polynomials and two (1)–covariant polynomials, the rank of the module is the cardinality of the basis and is here equal to two.

A geometric picture of the $\mathbb{C}$–basis is given by lattices of points. Two three–dimensional sets of points $\mathbb{N}^3$ are considered, one for π_1 and one for π_2. To the $r_1^{n_1} r_2^{n_2} s_{1,2}^{n_3}\pi_i$ polynomial is associated a point of coordinates (n_1, n_2, n_3). For example, the point in figure 1a represents the polynomial $r_1^2 r_2^2 s_{1,2}\pi_1$, while the point in figure 1b represents the polynomial $r_1 r_2^2 s_{1,2}^2\pi_2$. The $\mathbb{C}$–basis of the (1)–covariants is then viewed as the lattice of points of figure 2.

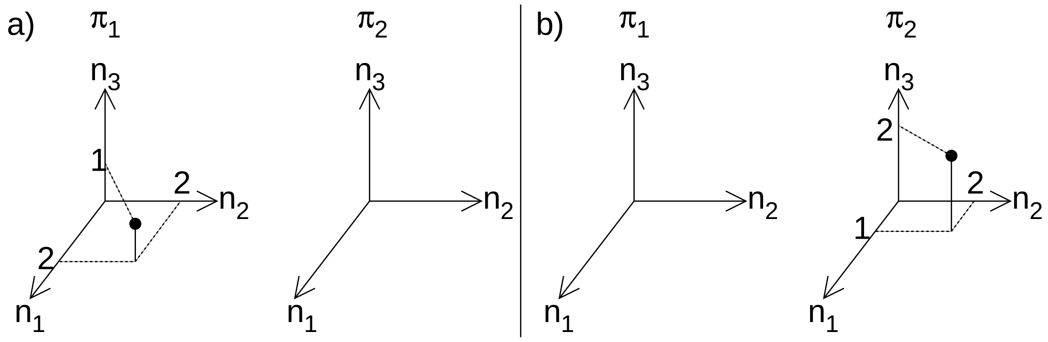

Figure 1. a) Polynomial $r_1^2 r_2^2 s_{1,2} \pi_1$. b) Polynomial $r_1 r_2^2 s_{1,2}^2 \pi_2$.

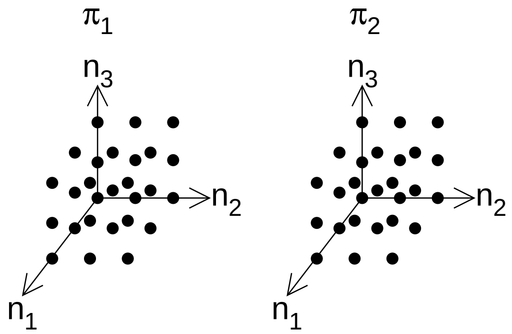

Figure 2. Lattices for the $\mathbb{C}$–basis of (1)–covariants built from two planar vectors.

4.3. Non–free module of (m)*–covariants,* $m \geq 2$

The structure of the (m)–covariants, $m \geq 2$, is more involved. The α form of the Molien function, $M_\alpha^{\mathrm{SO}(2)}\left((m)\,;\Gamma_2;\lambda\right) = \left[(m+1)\,\lambda^m - (m-1)\,\lambda^{m+2}\right] / \left(1-\lambda^2\right)^3$, suggests three denominator polynomials, chosen as r_1, r_2 and $s_{1,2}$. The first term in the numerator, $(m+1)\,\lambda^m$ corresponds to the $(m+1)$ linearly independent (m)–covariants of degree m. The second term in the numerator, $\left[-(m-1)\,\lambda^{m+2}\right]$, refers to $(m-1)$ relations of degree $m+2$ between the r_1, r_2, $s_{1,2}$ denominator invariants and the $m+1$ numerator (m)–covariants of degree m. We detail in the next subsections the algebraic structure of the non–free modules of (m)–covariants, $m \geq 2$.

4.3.1. Non–free module of (2)*–covariants* The expression of the β form of the Molien function $M_\beta^{\mathrm{SO}(2)}\left((2)\,;\Gamma_2;\lambda\right)$ is given by $2\lambda^2/\left(1-\lambda^2\right)^3 + \lambda^2/\left(1-\lambda^2\right)^2$. The set $\mathcal{N}^{2,2} = \{\pi_1^2, \pi_1\pi_2, \pi_2^2\}$ contains three linearly independent (2)–covariants of lowest degree. They are all three chosen as numerator polynomials in order to span the three–dimensional vector space of (2)–covariants of degree two. However, they cannot be multiplied by arbitrary polynomials of three invariants like in (18) because this leads to the Molien function $3\lambda^2/\left(1-\lambda^2\right)^3$. which counts some (2)–covariants several times due to relation $2s_{1,2}\pi_1\pi_2 - r_1\pi_2^2 - r_2\pi_1^2 = 0$ among r_1, r_2, $s_{1,2}$, π_1^2, $\pi_1\pi_2$, and π_2^2.

$$p_1\left(r_1, r_2, s_{1,2}\right)\pi_1^2 + p_2\left(r_1, r_2, s_{1,2}\right)\pi_1\pi_2 + p_3\left(r_1, r_2, s_{1,2}\right)\pi_2^2, \tag{18}$$

The overcounting problem is eliminated by substituting every $r_1^n\pi_2^2$ term that appear in expansion (18) with its equivalent expression $2r_1^{n-1}s_{1,2}\pi_1\pi_2 - r_1^{n-1}r_2\pi_1^2$. Any (2)–covariant is then *uniquely* decomposed according to equation (19):

$$p_{2,0}\left(r_1,r_2,s_{1,2}\right)\pi_1^2 + p_{1,1}\left(r_1,r_2,s_{1,2}\right)\pi_1\pi_2 + p_{0,2}\left(r_2,s_{1,2}\right)\pi_2^2. \tag{19}$$

The (2)–covariants π_1^2 and $\pi_1\pi_2$ are multiplied by polynomial functions in the three denominator polynomials while the (2)–covariant π_2^2 is multiplied by a polynomial function of r_2 and $s_{1,2}$ only. The first two terms in the sum (19) are a product of a polynomial in three variables with a (2)–covariant: they are associated with the $2\lambda^2/\left(1-\lambda^2\right)^3$ part of the Molien function $M_\beta^{\mathrm{SO}(2)}\left((2);\Gamma_2;\lambda\right)$. The last term is a product of a polynomial in two variables with a (2)–covariant and is related to the $\lambda^2/\left(1-\lambda^2\right)^2$ part of the Molien function. The associated generalized integrity basis of $M_\beta^{\mathrm{SO}(2)}\left((2);\Gamma_2;\lambda\right)$ would contain the denominators $r_1, r_2, s_{1,2}$ and numerators $\pi_1^2, \pi_1\pi_2$ for the $2\lambda^2/\left(1-\lambda^2\right)^3$ contribution and the denominators $r_2, s_{1,2}$ and numerator π_2^2 for the $\lambda^2/\left(1-\lambda^2\right)^2$ contribution. These results are summed up in table 2. The corresponding $\mathbb{C}$–basis contains the elements $r_1^{n_1}r_2^{n_2}s_{1,2}^{n_3}\pi_1^2$, $r_1^{n_1}r_2^{n_2}s_{1,2}^{n_3}\pi_1\pi_2$, and $r_2^{n_2}s_{1,2}^{n_3}\pi_2^2$. Figure 3 gives a picture of the corresponding lattice of points.

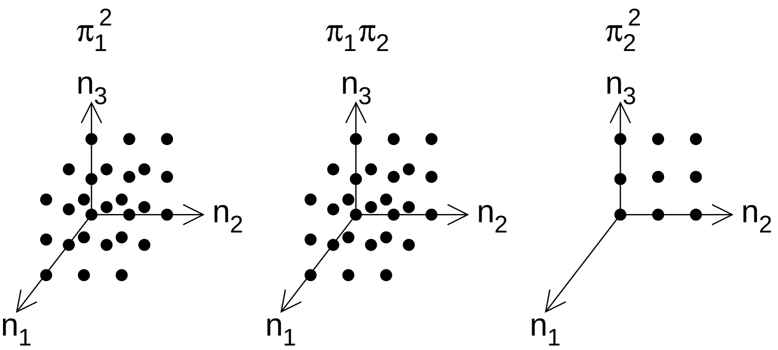

Figure 3. Lattices for the $\mathbb{C}$–basis of (2)–covariants built from two planar vectors and decomposition (19).

The proposed generalized integrity basis is nevertheless not unique. We could have chosen to remove all the $s_{1,2}^n\pi_1\pi_2$ terms in (18) instead of the $r_1^n\pi_2^2$. Substituting every occurrence of $s_{1,2}^n\pi_1\pi_2$ by $\left(r_1s_{1,2}^{n-1}\pi_2^2 + r_2s_{1,2}^{n-1}\pi_1^2\right)/2$, any (2)–covariant is expressed as the combination (20):

$$p'_{2,0}\left(r_1,r_2,s_{1,2}\right)\pi_1^2 + p'_{1,1}\left(r_1,r_2\right)\pi_1\pi_2 + p'_{0,2}\left(r_1,r_2,s_{1,2}\right)\pi_2^2, \tag{20}$$

which suggests to relate the denominators $r_1, r_2, s_{1,2}$ and numerators π_1^2, π_2^2 to the $2\lambda^2/\left(1-\lambda^2\right)^3$ rational function and the denominators r_1, r_2 and numerator $\pi_1\pi_2$ to the $\lambda^2/\left(1-\lambda^2\right)^2$ rational function. Figure 4 illustrates this decomposition of the non–free module of (2)–covariants.

One yet another possibility substitutes every $r_2^n\pi_1^2$ term in (18) by $2s_{1,2}r_2^{n-1}\pi_1\pi_2 - r_1r_2^{n-1}\pi_2^2$. Any (2)–covariant is then expressed as the combination (21),

$$p''_{2,0}\left(r_1,s_{1,2}\right)\pi_1^2 + p''_{1,1}\left(r_1,r_2,s_{1,2}\right)\pi_1\pi_2 + p''_{0,2}\left(r_1,r_2,s_{1,2}\right)\pi_2^2, \tag{21}$$

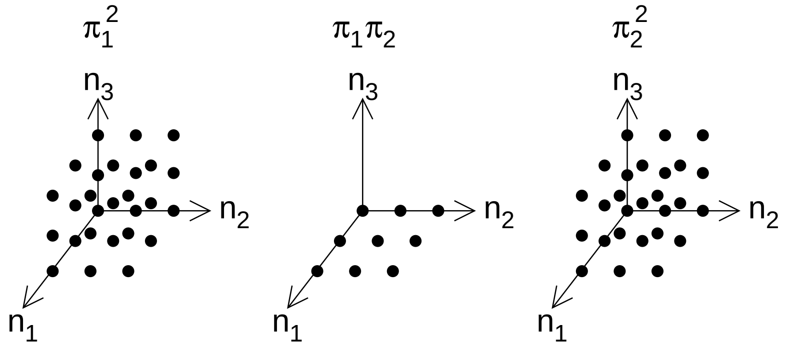

Figure 4. Lattices for the $\mathbb{C}$–basis of (2)–covariants built from two planar vectors and decomposition (20).

which suggests to associate the denominators $r_1, r_2, s_{1,2}$ and numerators $\pi_2^2, \pi_1\pi_2$ to the $2\lambda^2/\left(1-\lambda^2\right)^3$ rational function and the denominators $r_1, s_{1,2}$ and the numerator π_1^2 to the $\lambda^2/\left(1-\lambda^2\right)^2$ rational function. This decomposition is pictured in Figure 5.

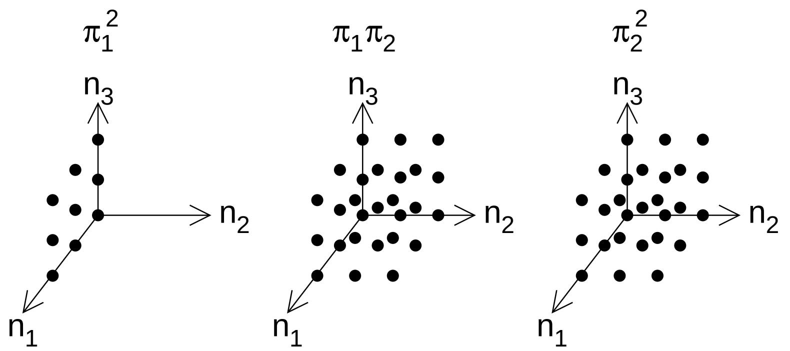

Figure 5. Lattices for the $\mathbb{C}$–basis of (2)–covariants built from two planar vectors and decomposition (21).

4.3.2. Non–free module of (3)*–covariants* The β form of the Molien function for the (3)–covariants is a sum of two rational functions, $2\lambda^3/\left(1-\lambda^2\right)^3 + 2\lambda^3/\left(1-\lambda^2\right)^2$. The π_1^3, $\pi_1^2\pi_2$, $\pi_1\pi_2^2$ and π_2^3 polynomials are the four linearly independent (3)–covariants of lowest degree (degree three) and are chosen as numerator invariants.

As in section 4.3.1, we select r_1, r_2, and $s_{1,2}$ as denominator invariants of the first rational function and we start from the redundant decomposition of a (3)–covariant as:

$$\begin{aligned}&p_{3,0}\left(r_1, r_2, s_{1,2}\right)\pi_1^3 + p_{2,1}\left(r_1, r_2, s_{1,2}\right)\pi_1^2\pi_2 + p_{1,2}\left(r_1, r_2, s_{1,2}\right)\pi_1\pi_2^2 \\ &+p_{0,3}\left(r_1, r_2, s_{1,2}\right)\pi_2^3,\end{aligned} \tag{22}$$

corresponding to a Molien function equal to $4\lambda^3/\left(1-\lambda^2\right)^3$. Redundancies are eliminated by first using the quintic relation $2s_{1,2}\pi_1\pi_2^2 - r_1\pi_2^3 - r_2\pi_1^2\pi_2 = 0$ to eliminate the $r_1^n\pi_2^3$ terms of (22). The quintic relation $2s_{1,2}\pi_1^2\pi_2 - r_1\pi_1\pi_2^2 - r_2\pi_1^3 = 0$ is used in a second step to eliminate the $r_1^n \pi_1 \pi_2^2$ terms. Any (3)–covariant then uniquely decomposes as:

$$p''_{3,0}(r_1, r_2, s_{1,2})\,\pi_1^3 + p''_{2,1}(r_1, r_2, s_{1,2})\,\pi_1^2\pi_2 + p''_{1,2}(r_2, s_{1,2})\,\pi_1\pi_2^2 \\ +p''_{0,3}(r_2, s_{1,2})\,\pi_2^3. \qquad (23)$$

Decomposition (23) corresponds to the symbolic interpretation of the β form of the Molien function. The first rational function $2\lambda^3/(1-\lambda^2)^3$ is associated with r_1, r_2, and $s_{1,2}$ as denominator polynomials and π_1^3, $\pi_1^2\pi_2$ as numerator polynomials. The second contribution $2\lambda^3/(1-\lambda^2)^2$ is associated with r_2 and $s_{1,2}$ as denominator polynomials and $\pi_1\pi_2^2$, π_2^3 as numerator polynomials. Figure 6 gives the geometric interpretation of the $\mathbb{C}$–basis of (3)–covariants in term of two $\mathbb{N}^3$ spaces associated with π_1^3, $\pi_1^2\pi_2$ and two $\mathbb{N}^2$ spaces associated with $\pi_1\pi_2^2$, π_2^3.

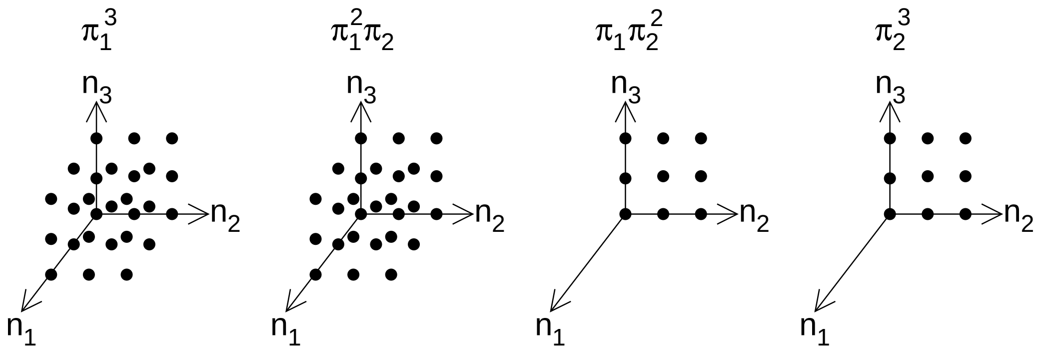

Figure 6. Lattices for the $\mathbb{C}$–basis of (3)–covariants built from two planar vectors and decomposition (23).

4.3.3. Non–free module of (4)*–covariants and* (5)*–covariants* The same principles used in the description of the non–free modules of (2)–covariants and (3)–covariants are still valid for the modules of (4)–covariants and (5)–covariants. The successive application of the three sextic relations $2s_{1,2}\pi_1\pi_2^3 - r_1\pi_2^4 - r_2\pi_1^2\pi_2^2 = 0$, $2s_{1,2}\pi_1^2\pi_2^2 - r_1\pi_1\pi_2^3 - r_2\pi_1^3\pi_2 = 0$, and $2s_{1,2}\pi_1^3\pi_2 - r_1\pi_1^2\pi_2^2 - r_2\pi_1^4 = 0$, allows to uniquely decompose any (4)–covariant as:

$$p_{4,0}(r_1, r_2, s_{1,2})\,\pi_1^4 + p_{3,1}(r_1, r_2, s_{1,2})\,\pi_1^3\pi_2 \\ +p_{2,2}(r_2, s_{1,2})\,\pi_1^2\pi_2^2 + p_{1,3}(r_2, s_{1,2})\,\pi_1\pi_2^3 + p_{0,4}(r_2, s_{1,2})\,\pi_2^4. \qquad (24)$$

The generalized integrity basis related to the Molien function $M_\beta^{\mathrm{SO}(2)}((4);\Gamma_2;\lambda)$ consists in r_1, r_2, $s_{1,2}$ as denominator polynomials and π_1^4, $\pi_1^3\pi_2$ as numerator polynomials for the $2\lambda^4/(1-\lambda^2)^3$ contribution and in r_2, $s_{1,2}$ as denominator polynomials and $\pi_1^2\pi_2^2$, $\pi_1\pi_2^3$, and π_2^4 as numerator polynomial for the $3\lambda^4/(1-\lambda^2)^2$ contribution. Figure 7 gives a geometric view of the $\mathbb{C}$–basis of (4)–covariants in term of two $\mathbb{N}^3$ spaces associated with π_1^4, $\pi_1^3\pi_2$ and three $\mathbb{N}^2$ spaces associated with $\pi_1^2\pi_2^2$, $\pi_1\pi_2^3$, and π_2^4.

Finally, the Molien function $M_\beta^{\mathrm{SO}(2)}((5);\Gamma_2;\lambda)$ and the four relations of degree seven $2s_{1,2}\pi_1\pi_2^4 - r_1\pi_2^5 - r_2\pi_1^2\pi_2^3 = 0$, $2s_{1,2}\pi_1^2\pi_2^3 - r_1\pi_1\pi_2^4 - r_2\pi_1^3\pi_2^2 = 0$, $2s_{1,2}\pi_1^3\pi_2^2 - r_1\pi_1^2\pi_2^3 - r_2\pi_1^4\pi_2 = 0$, $2s_{1,2}\pi_1^4\pi_2 - r_1\pi_1^3\pi_2^2 - r_2\pi_1^5 = 0$ between the quadratic invariants and

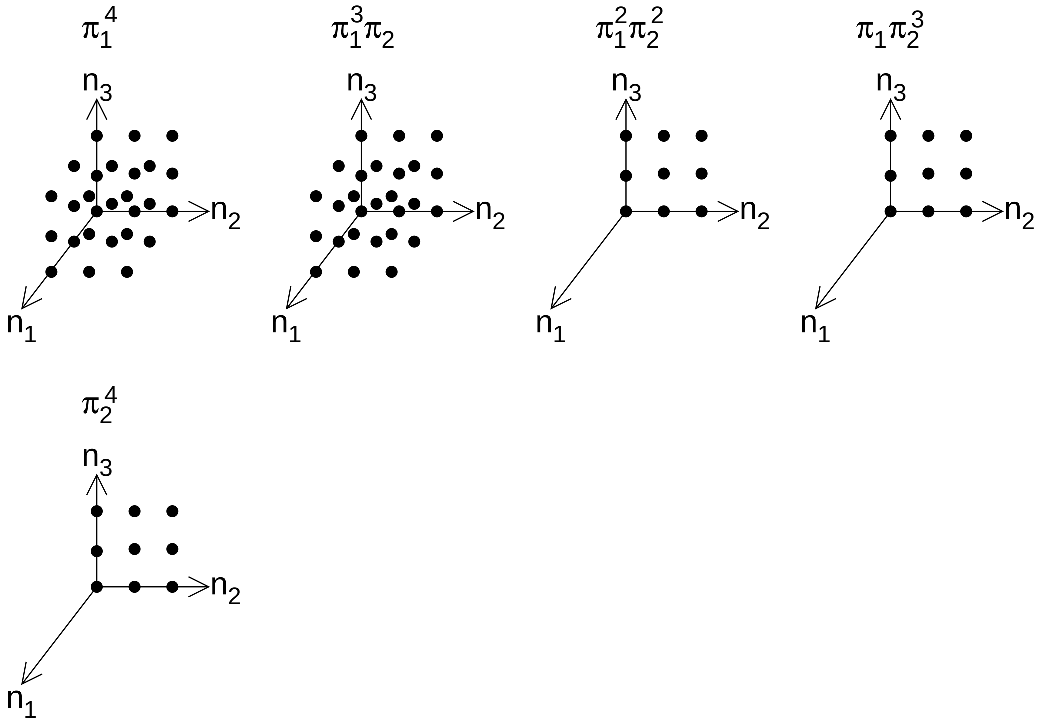

Figure 7. Lattices for the ℂ–basis of (4)–covariants built from two planar vectors and decomposition (24).

the set of (5)–covariants of lowest degree imply that any (5)–covariant can be uniquely written as:

$$p_{5,0}\left(r_1,r_2,s_{1,2}\right)\pi_1^5+p_{4,1}\left(r_1,r_2,s_{1,2}\right)\pi_1^4\pi_2+p_{3,2}\left(r_2,s_{1,2}\right)\pi_1^3\pi_2^2$$
$$+p_{2,3}\left(r_2,s_{1,2}\right)\pi_1^2\pi_2^3+p_{1,4}\left(r_2,s_{1,2}\right)\pi_1\pi_2^4+p_{0,5}\left(r_2,s_{1,2}\right)\pi_2^5, \qquad (25)$$

and suggest a generalized integrity basis composed of r_1, r_2, $s_{1,2}$ as denominator polynomials and π_1^5, $\pi_1^4\pi_2$ as numerator polynomials for the $2\lambda^5/\left(1-\lambda^2\right)^3$ contribution and of r_2, $s_{1,2}$ as denominator polynomials and $\pi_1^3\pi_2^2$, $\pi_1^2\pi_2^3$, $\pi_1\pi_2^4$, and π_2^5 as numerator polynomials for the $4\lambda^5/\left(1-\lambda^2\right)^2$ contribution.

5. Integrity bases for three planar vectors under SO(2)

The integrity bases for the invariants and low (m)–covariants are given in table 3.

5.1. Generators of invariants and syzygies between them

The expression of the Molien function for SO(2)–invariants built up from three vectors is $M_\alpha^{\mathrm{SO}(2)}\left((0)\,;\Gamma_3;\lambda\right)=\left[\left(1+\lambda^2+\lambda^4\right)_{A_1}+\left(3\lambda^2\right)_{A_2}\right]/\left(1-\lambda^2\right)^5$. Its numerator separates in two contributions from A_1 and A_2 final representation under the O(2) group. This rational function suggests an integrity basis containing five quadratic denominator invariants, a numerator constant, four quadratic numerator invariants (one

Table 2. Integrity bases for invariants and covariants of the SO(2) group built up from two planar vectors. The underscored polynomial transform as the A_2 irreducible representation of O(2) group.

m	Term	Polynomial
Denominators		$d_1 = r_1$, $d_2 = r_2$, $d_3 = s_{1,2}$
≥ 0	$\left(1-\lambda^2\right)^3$	d_1, d_2, d_3
≥ 2	$\left(1-\lambda^2\right)^2$	d_2, d_3
Numerators		
0	$1+\lambda^2$	1, $\underline{t_{1,2}}$
1	2λ	π_1, π_2
2	$2\lambda^2$	π_1^2, $\pi_1\pi_2$
	λ^2	π_2^2
3	$2\lambda^3$	π_1^3, $\pi_1^2\pi_2$
	$2\lambda^3$	$\pi_1\pi_2^2$, π_2^3
4	$2\lambda^4$	π_1^4, $\pi_1^3\pi_2$
	$3\lambda^4$	$\pi_1^2\pi_2^2$, $\pi_1\pi_2^3$, π_2^4
5	$2\lambda^5$	π_1^5, $\pi_1^4\pi_2$
	$4\lambda^5$	$\pi_1^3\pi_2^2$, $\pi_1^2\pi_2^3$, $\pi_1\pi_2^4$, π_2^5

has symmetry A_1 and three have symmetry A_2 in O(2)), and one quartic numerator invariant of symmetry A_1 in O(2). The nine invariants $\pi_i\mu_j$, $1 \leq i,j \leq 3$ or equivalently the $r_i = x_i^2 + y_i^2$, $s_{i,j} = x_i x_j + y_i y_j$, and $t_{i,j} = x_i y_j - x_j y_i$ constitute a set of generators for all the SO(2) invariant polynomials built up from Γ_3 [14]. The three quadratic $t_{i,j}$ polynomials change sign upon any reflection in a line. They are identified with the three numerator polynomials of symmetry A_2. Syzygies exist among the nine generators. The expression (26) is much suitable for a symbolic interpretation of the Molien function under the method of syzygies.

$$M^{\mathrm{SO}(2)}\left((0)\,;\Gamma_3;\lambda\right) = \frac{1-9\lambda^4+16\lambda^6-9\lambda^8+\lambda^{12}}{\left(1-\lambda^2\right)^9}. \tag{26}$$

The nine syzygies $\sigma_i = 0$ of degree four suggested by (26) are given by Weyl [14]:

$$\begin{aligned}
\sigma_1 &= t_{1,2}^2 - r_1 r_2 + s_{1,2}^2, & \sigma_2 &= t_{1,3}^2 - r_1 r_3 + s_{1,3}^2,\\
\sigma_3 &= t_{2,3}^2 - r_2 r_3 + s_{2,3}^2, & \sigma_4 &= t_{1,2}t_{1,3} - r_1 s_{2,3} + s_{1,2}s_{1,3},\\
\sigma_5 &= t_{1,2}t_{2,3} - s_{1,2}s_{2,3} + r_2 s_{1,3}, & \sigma_6 &= t_{1,3}t_{2,3} - r_3 s_{1,2} + s_{1,3}s_{2,3},\\
\sigma_7 &= s_{1,3}t_{1,2} + r_1 t_{2,3} - s_{1,2}t_{1,3}, & \sigma_8 &= s_{2,3}t_{1,2} + s_{1,2}t_{2,3} - r_2 t_{1,3},\\
\sigma_9 &= r_3 t_{1,2} + s_{1,3}t_{2,3} - s_{2,3}t_{1,3}.
\end{aligned}$$

The nine first–order syzygies are not independent and the numerator of (26) predicts sixteen second–order syzygies. The syzygy investigated in section 4.1 between the generators of two planar vectors was simple enough that a construction of an integrity basis for invariants was done by hand. The syzygies between the generators of three planar vectors are much more intricate and integrity bases have to be determined with a different approach that relies on a computer algebra system.

5.2. Integrity basis of invariants with an algorithmic approach

The algorithmic approach for bringing up an integrity basis bypasses the description of the syzygies among the generators. It rather uses the Molien function as a guide to propose an educated guess for the integrity basis. The symbolic interpretation of the Molien function $M_{\alpha}^{\mathrm{SO}(2)}\left((0)\,;\Gamma_3;\lambda\right)$ first requires to pick up a set of denominator polynomials θ_k. The number D and degree of these polynomials are directly read from the denominator of the Molien function. Gröbner bases are used to check the algebraically independency of the denominator polynomials [35]. In a second step, the numerator of the Molien function gives an indication on the number and degree of the linearly independent numerator polynomials. A vector space $\mathcal{V}_k$ is spanned by the polynomials of degree k built from the tentative integrity basis, keeping in mind that powers of denominator polynomials may be used but numerator polynomials appear only linearly. The integrity basis is acceptable if the rank of the vector space $\mathcal{V}_k$ is equal for any value of k to the coefficient c_k in the Taylor expansion of the Molien function. In practice, this is checked for the lowest values of k. After some trial and error, we found that the $d_1 = r_1$, $d_2 = r_2$, $d_3 = r_3$, $d_4 = s_{1,2} + s_{1,3}$, $d_5 = s_{1,2} + s_{2,3}$ denominator polynomials and 1 , $s_{2,3}$, $t_{1,2}$, $t_{1,3}$, $t_{2,3}$, $s_{2,3}^2$ numerator polynomials defines an integrity basis for the invariants built up from Γ_3. The ring of invariants is Cohen–Macaulay and any SO(2)–invariant decomposes as:

$$\begin{aligned}&p_1\left(d_1,\ldots,d_5\right)+p_2\left(d_1,\ldots,d_5\right)s_{2,3}+p_3\left(d_1,\ldots,d_5\right)t_{1,2}\\&+p_4\left(d_1,\ldots,d_5\right)t_{1,3}+p_5\left(d_1,\ldots,d_5\right)t_{2,3}+p_6\left(d_1,\ldots,d_5\right)s_{2,3}^2,\end{aligned}$$

where the notation $d_i,\ldots,d_j$ stands for the set of $(j-1+1)$ variables $d_i, d_{i+1},\ldots,d_{j-1},d_j$.

5.3. Free module of (1)–covariants

The Molien function for the (1)–covariants asks for three (1)–covariants of degree one and three (1)–covariants of degree three. The denominator polynomials of the integrity basis of the invariants are chosen as denominator polynomials for the (1)–covariants. The algorithmic procedure described in section 5.2 is adapted by using linearly independent (1)–covariants as numerator polynomials. It shows that π_1 , π_2 , π_3 , $\pi_1 s_{2,3}$, $\pi_2 s_{1,3}$, $\pi_3 s_{2,3}$ can be chosen as numerator invariants. The module of (1)–covariants is free and any (1)–covariant decomposes as:

$$\begin{aligned}&p_1\left(d_1,\ldots,d_5\right)\pi_1+p_2\left(d_1,\ldots,d_5\right)\pi_2+p_3\left(d_1,\ldots,d_5\right)\pi_3\\&+p_4\left(d_1,\ldots,d_5\right)\pi_1 s_{2,3}+p_5\left(d_1,\ldots,d_5\right)\pi_2 s_{1,3}+p_6\left(d_1,\ldots,d_5\right)\pi_3 s_{2,3}.\end{aligned}$$

5.4. Free module of (2)–covariants

The denominator polynomials of $M_{\alpha}^{\mathrm{SO}(2)}\left((2)\,;\Gamma_3;\lambda\right) = 6\lambda^2/\left(1-\lambda^2\right)^5$ are again chosen to be identical to the denominator polynomials of the invariants. The six linearly independent (2)–covariants of degree two are the monomials $\mathcal{N}^{3,2} =$

$\{\pi_1^2, \pi_1\pi_2, \pi_1\pi_3, \pi_2^2, \pi_2\pi_3, \pi_3^2\}$ in the expansion of $(\pi_1 + \pi_2 + \pi_3)^3$. They are chosen as numerator invariants. Any (2)–covariant decomposes as:

$$\begin{aligned} & p_1\,(d_1, \ldots, d_5)\,\pi_1^2 + p_2\,(d_1, \ldots, d_5)\,\pi_1\pi_2 + p_3\,(d_1, \ldots, d_5)\,\pi_1\pi_3 \\ & + p_4\,(d_1, \ldots, d_5)\,\pi_2^2 + p_5\,(d_1, \ldots, d_5)\,\pi_2\pi_3 + p_6\,(d_1, \ldots, d_5)\,\pi_3^2. \end{aligned}$$

The module of (2)–covariants is free. Each of the six (2)–covariant polynomials is associated with a five–dimensional lattice $\mathbb{N}^5$.

5.5. Non–free modules of (m)*–covariants,* $m \geq 3$

The coefficient in front of λ^{m+2} in the α form of the Molien function $M_\alpha^{\mathrm{SO}(2)}\left((m)\,;\Gamma_3;\lambda\right) = N_{3,m}\,(\lambda)\,/\,(1-\lambda^2)^5$, with

$$N_{3,m}\,(\lambda) = (m+2)\,(m+1)\,\lambda^m/2 - (m+2)\,(m-2)\,\lambda^{m+2} + (m-1)\,(m-2)\,\lambda^{m+4}/2,$$

becomes negative for $m \geq 3$. As in the two vector case, relations between denominator and numerator polynomials are expected and the corresponding modules of (m)–covariants are not free.

5.5.1. Non–free module of (3)*–covariants* There are exactly ten linearly independent (3)–covariants of degree three, and the negative coefficient in $N_{3,3}\,(\lambda)$ suggests five relations between the denominator and numerator polynomials, in the same spirit as section 4.3.1 for two vectors. The β form $M_\beta^{\mathrm{SO}(2)}\left((3)\,;\Gamma_3;\lambda\right) = 6\lambda^3/\,(1-\lambda^2)^5 + 3\lambda^3/\,(1-\lambda^2)^4 + \lambda^3/\,(1-\lambda^2)^3$ can be used to propose a generalized integrity basis. The expression suggests a total of ten (3)–covariants of degree three to be partitioned among the three numerators. The denominator of the first rational function contains five terms, the second denominator four terms and the third one only three terms. The denominator polynomials associated to the first rational fraction are chosen to be the five denominator polynomials of the invariants. The four denominator polynomials corresponding to the second contribution are selected by removing the r_1 invariant from this set, and the three denominator polynomials of the third contribution are obtained by removing r_1 and r_2 together from this set.

We can methodically define the numerator polynomials for each of the three rational function of $M_\beta^{\mathrm{SO}(2)}\left((3)\,;\Gamma_3;\lambda\right)$ by looking back to the two vector case. Equation (13) states that the total number of numerator polynomials in $M_\beta^{\mathrm{SO}(2)}\left((m)\,;\Gamma_2;\lambda\right)$ is simply $m+1$. Note that this number is the cardinal of the set of monomials $\mathcal{N}^{2,m} = \{\pi_1^{m_1}\pi_2^{m_2}, (m_1, m_2) \in \mathbb{N}^2, m_1 + m_2 = m\}$. The set $\mathcal{N}^{2,m}$ is partioned into two disjoint subsets $\mathcal{N}^{2,m,1}$ and $\mathcal{N}^{2,m,2}$ respectively associated with the first and the second rational function of $M_\beta^{\mathrm{SO}(2)}\left((m)\,;\Gamma_2;\lambda\right)$. Following table 2, $\mathcal{N}^{2,m,1}$ contains the monomials π_1^m and $\pi_1^{m-1}\pi_2$. The monomials of $\mathcal{N}^{2,m}$ not in $\mathcal{N}^{2,m,1}$ belong to the subset $\mathcal{N}^{2,m,2}$. The partitioning in two parts of the set $\mathcal{N}^{2,m}$ of numerator polynomials is certainly not unique as shown by the complete study in section 4.3.

We define the set $\mathcal{N}^{n,m,i}$ of numerator polynomials attached to the i^{th} rational fraction of $M_\beta^{\mathrm{SO}(2)}\left((m)\,;\Gamma_n;\lambda\right)$ as the (m)–monomials that appear in the multivariate polynomial (27).

$$\left(\pi_i + \pi_{i+1} + \cdots + \pi_n\right)^{n-1} \times \left(\pi_1 + \pi_2 + \cdots + \pi_i\right)^{m-n} \times \pi_i, m \geq n \geq 2. \quad (27)$$

Such a definition coincides with the sets given in table 2 for $m \geq 2$. The expression of the $\mathcal{N}^{n,m,i}$ numerator polynomials is explicitly given in Appendix A for $2 \leq n \leq 4$ and $n \leq m \leq 5$. The number of monomials in polynomials $\left(\pi_i + \pi_{i+1} + \cdots + \pi_n\right)^{n-1}$ and $\left(\pi_1 + \pi_2 + \cdots + \pi_i\right)^{m-n}$ are respectively equal to the first and second binomial coefficients of the left–hand side of (13). The polynomial $\mathcal{N}_j^{n,m,i}$ is the j^{th} element of the set $\mathcal{N}^{n,m,i}$.

The β form of the Molien function is a sum of rational functions with different number of terms in the denominators and the algorithmic procedure for one rational function exposed in section 5.2 is modified to take care of the new representation. The numerators in $\mathcal{N}^{3,m,i}$ can only be multiplied by a polynomial in the denominator polynomials attached to the i^{th} rational fraction. The computation confirms that the choice of $\mathcal{N}^{3,3,1}$ as numerators of the first rational function, $\mathcal{N}^{3,3,2}$ as numerators of the second rational function, and $\mathcal{N}^{3,3,3}$ as numerator of the third rational function is a generalized integrity basis for the module of (3)–covariants.

The module of (3)–covariants is not free. The expansion

$$\begin{aligned}&\sum_{i=1}^{6} p_{1,i}\left(d_1, \ldots, d_5\right) \mathcal{N}_i^{3,3,1} + \sum_{i=1}^{3} p_{2,i}\left(d_1, \ldots, d_5\right) \mathcal{N}_i^{3,3,2}\\ &+ p_{3,1}\left(d_1, \ldots, d_5\right) \mathcal{N}_1^{3,3,3},\end{aligned} \quad (28)$$

can be reduced by taking into account the five relations (2.1a)–(2.1e) of Appendix B between the numerator and denominator polynomials. Once the terms $d_1^n \mathcal{N}_1^{3,3,2}$, $d_1^n \mathcal{N}_2^{3,3,2}$, $d_1^n \mathcal{N}_3^{3,3,2}$, $d_1^n \mathcal{N}_1^{3,3,3}$ and $d_2^n \mathcal{N}_1^{3,3,3}$ are removed from expansion (28) using relations (2.1a)–(2.1e), the decomposition of any (3)–covariant finally reads as:

$$\begin{aligned}&\sum_{i=1}^{6} p_{1,i}\left(d_1, \ldots, d_5\right) \mathcal{N}_i^{3,3,1} + \sum_{i=1}^{3} p_{2,i}\left(d_2, \ldots, d_5\right) \mathcal{N}_i^{3,3,2}\\ &+ p_{3,1}\left(d_3, \ldots, d_5\right) \mathcal{N}_1^{3,3,3}.\end{aligned} \quad (29)$$

The six covariants $\mathcal{N}_i^{3,3,1}$ are associated to a lattice in $\mathbb{N}^5$, the three covariants $\mathcal{N}_i^{3,3,2}$ are associated to a lattice in $\mathbb{N}^4$, and the covariant $\mathcal{N}_1^{3,3,3}$ is associated to a lattice in $\mathbb{N}^3$. Figure 8 gives a geometrical point of view of this result.

5.5.2. Non–free module of (4)*–covariants* The β form of the Molien function is $M_\beta^{\mathrm{SO}(2)}\left((4)\,;\Gamma_3;\lambda\right) = 6\lambda^3/\left(1-\lambda^2\right)^5 + 6\lambda^3/\left(1-\lambda^2\right)^4 + 3\lambda^3/\left(1-\lambda^2\right)^3$. The module of (4)–covariants is not free, the relations (2.2a)–(2.2l) of Appendix B between denominator polynomials d_i and numerator polynomials $\mathcal{N}_j^{3,4,i}$ hold. In a first step, relations (2.2d), (2.2e), (2.2f), (2.2h), (2.2i) are used to remove the terms $d_1^n \mathcal{N}_4^{3,4,2}$, $d_1^n \mathcal{N}_5^{3,4,2}$, $d_1^n \mathcal{N}_6^{3,4,2}$, $d_1^n \mathcal{N}_2^{3,4,3}$, $d_1^n \mathcal{N}_3^{3,4,3}$. The right hand side of these relations contains $\mathcal{N}_1^{3,4,2}$, $\mathcal{N}_2^{3,4,2}$, $\mathcal{N}_3^{3,4,2}$,

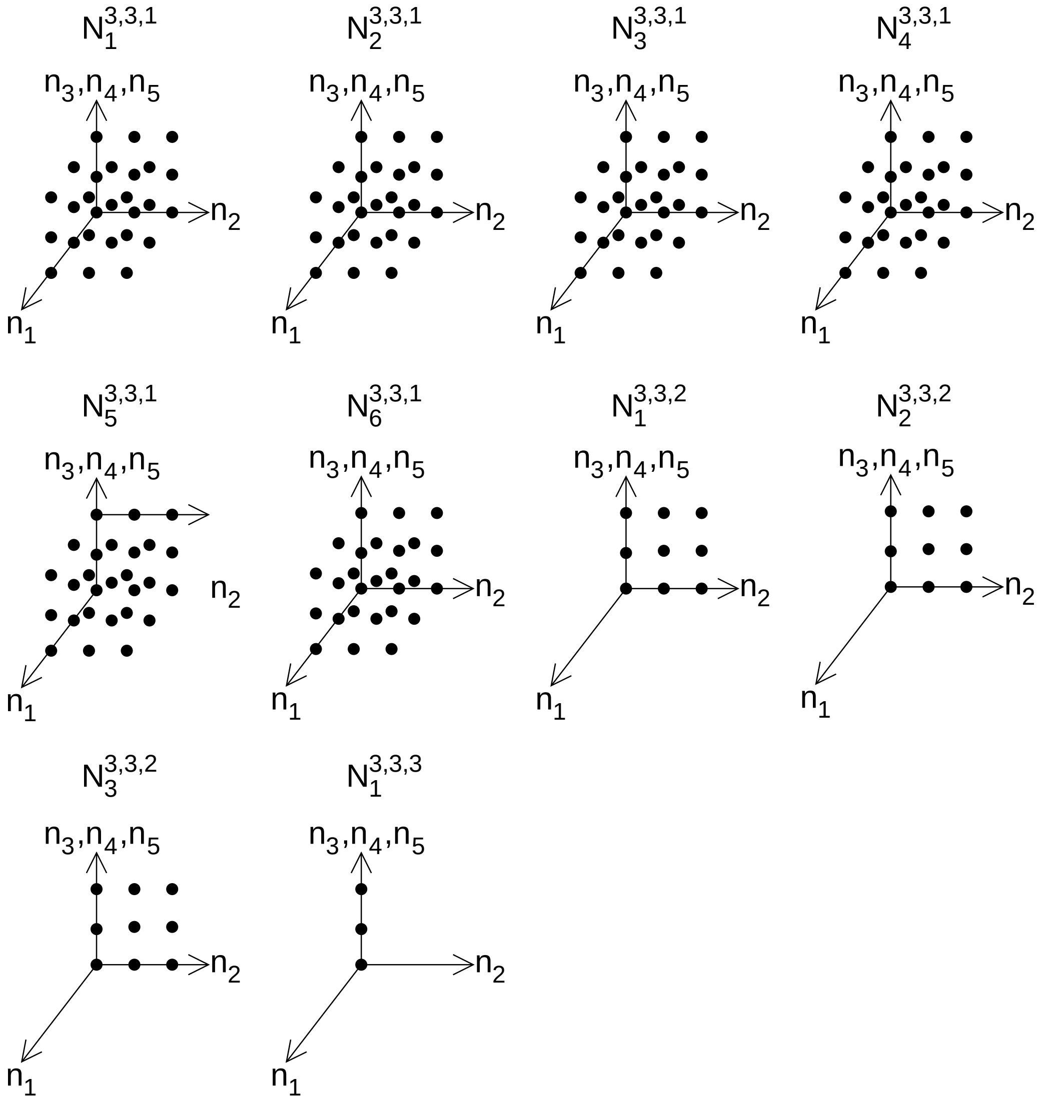

Figure 8. Lattices for the $\mathbb{C}$–basis of (3)–covariants built from three planar vectors and decomposition (29). The arrow labelled n_3, n_4, n_5 represents a three–dimensional space $\mathbb{N}^3$.

$\mathcal{N}_1^{3,4,3}$, but the terms $d_1^n\mathcal{N}_1^{3,4,2}$, $d_1^n\mathcal{N}_2^{3,4,2}$, $d_1^n\mathcal{N}_3^{3,4,2}$, $d_1^n\mathcal{N}_1^{3,4,3}$ can be removed in a second step using relations (2.2a), (2.2b), (2.2c), (2.2g). In a third step, the $d_2^n\mathcal{N}_3^{3,4,3}$ terms are removed using (2.2l). The right–hand side of this relation contains a $\mathcal{N}_2^{3,4,3}$ term, but $d_2^n\mathcal{N}_1^{3,4,3}$ and $d_2^n\mathcal{N}_2^{3,4,3}$ terms can be removed using (2.2j) and (2.2k). Any (4)–covariant finally decomposes as:

$$\sum_{i=1}^{6} p_{1,i}\left(d_1,\ldots,d_5\right)\mathcal{N}_i^{3,4,1} + \sum_{i=1}^{6} p_{2,i}\left(d_2,\ldots,d_5\right)\mathcal{N}_i^{3,4,2}$$
$$+\sum_{i=1}^{3} p_{3,i}\left(d_3,\ldots,d_5\right)\mathcal{N}_i^{3,4,3}.$$

In terms of lattices of points, the six covariants $\mathcal{N}_i^{3,4,1}$ are associated with a $\mathbb{N}^5$ lattice, the six covariants $\mathcal{N}_i^{3,4,2}$ are associated with a $\mathbb{N}^4$ lattice, and the three covariants $\mathcal{N}_i^{3,4,3}$ are associated with a $\mathbb{N}^3$ lattice.

5.5.3. Non–free module of (5)*–covariants* The β form of the Molien function is $M_\beta^{\mathrm{SO}(2)}\left((4);\Gamma_3;\lambda\right) = 6\lambda^3/\left(1-\lambda^2\right)^5 + 9\lambda^3/\left(1-\lambda^2\right)^4 + 6\lambda^3/\left(1-\lambda^2\right)^3$. As in the treatment of the non–free modules of (3)–covariants and (4)–covariants, relations (2.3a)–(2.3u) of Appendix B indicate that the module of (5)–covariants is not free. In a first step, the terms $d_1^n\mathcal{N}_7^{3,5,2}$, $d_1^n\mathcal{N}_8^{3,5,2}$, $d_1^n\mathcal{N}_9^{3,5,2}$, $d_1^n\mathcal{N}_4^{3,5,3}$ $d_1^n\mathcal{N}_5^{3,5,3}$ $d_1^n\mathcal{N}_6^{3,5,3}$ are eliminated by using the relations (2.3g), (2.3h), (2.3i), (2.3m), (2.3n), (2.3o). However the right–hand side of these relations produce the terms $d_1^n\mathcal{N}_1^{3,5,2}$, $d_1^n\mathcal{N}_2^{3,5,2}$, $d_1^n\mathcal{N}_3^{3,5,2}$, $d_1^n\mathcal{N}_4^{3,5,2}$, $d_1^n\mathcal{N}_5^{3,5,2}$, $d_1^n\mathcal{N}_6^{3,5,2}$, $d_1^n\mathcal{N}_1^{3,5,3}$, $d_1^n\mathcal{N}_2^{3,5,3}$, $d_1^n\mathcal{N}_3^{3,5,3}$. In a second step, the terms $d_1^n\mathcal{N}_4^{3,5,2}$, $d_1^n\mathcal{N}_5^{3,5,2}$, $d_1^n\mathcal{N}_6^{3,5,2}$, $d_1^n\mathcal{N}_2^{3,5,3}$, $d_1^n\mathcal{N}_3^{3,5,3}$, are eliminated by the use of relations (2.3d), (2.3e), (2.3f), (2.3k), (2.3l). The products $d_1^n\mathcal{N}_1^{3,5,2}$, $d_1^n\mathcal{N}_2^{3,5,2}$, $d_1^n\mathcal{N}_3^{3,5,2}$, $d_1^n\mathcal{N}_1^{3,5,3}$ are generated in the right–hand side. In a third step, the terms $d_1^n\mathcal{N}_1^{3,5,2}$, $d_1^n\mathcal{N}_2^{3,5,2}$, $d_1^n\mathcal{N}_3^{3,5,2}$ $d_1^n\mathcal{N}_1^{3,5,3}$ are eliminated by relations (2.3a), (2.3b), (2.3c), (2.3j). At this point, any (5)–covariant decomposes as:

$$\sum_{i=1}^{6} p_{1,i}\left(d_1,\ldots,d_5\right)\mathcal{N}_i^{3,5,1} + \sum_{i=1}^{9} p_{2,i}\left(d_2,\ldots,d_5\right)\mathcal{N}_i^{3,5,2}$$
$$+\sum_{i=1}^{6} p_{3,i}\left(d_2,\ldots,d_5\right)\mathcal{N}_i^{3,5,3}.$$

In a fourth step, the term $d_2^n\mathcal{N}_6^{3,5,3}$ is eliminated by (2.3u), but products $d_2^n\mathcal{N}_2^{3,5,3}$, $d_2^n\mathcal{N}_4^{3,5,3}$, $d_2^n\mathcal{N}_5^{3,5,3}$ appear in the right–hand side. In a fifth step, the terms $d_2^n\mathcal{N}_3^{3,5,3}$ and $d_2^n\mathcal{N}_5^{3,5,3}$ are eliminated by relations (2.3r), (2.3t). The relations generate the products $d_2^n\mathcal{N}_2^{3,5,3}$ and $d_2^n\mathcal{N}_4^{3,5,3}$. In a sixth step, the terms $d_2^n\mathcal{N}_1^{3,5,3}$, $d_2^n\mathcal{N}_2^{3,5,3}$, $d_2^n\mathcal{N}_4^{3,5,3}$ are eliminated by relations (2.3p), (2.3q), (2.3s). After the six steps of rewriting, any (5)–covariant decomposes as:

$$\sum_{i=1}^{6} p_{1,i}\left(d_1,\ldots,d_5\right)\mathcal{N}_i^{3,5,1} + \sum_{i=1}^{9} p_{2,i}\left(d_2,\ldots,d_5\right)\mathcal{N}_i^{3,5,2}$$
$$+\sum_{i=1}^{6} p_{3,i}\left(d_3,\ldots,d_5\right)\mathcal{N}_i^{3,5,3}.$$

The six covariants $\mathcal{N}_i^{3,5,1}$ are associated with a $\mathbb{N}^5$ lattice, the nine covariants $\mathcal{N}_i^{3,5,2}$ are associated with a $\mathbb{N}^4$ lattice, and the six covariants $\mathcal{N}_i^{3,5,3}$ are associated with a $\mathbb{N}^3$ lattice.

Table 3. Integrity bases for invariants and covariants of the SO(2) group built up from three planar vectors. The underscored polynomials transform as the A_2 irreducible representation of O(2) group. The other numerator polynomials of the (0) irreducible representation transform as the totally symmetric representation A_1 of the O(2) group. See Appendix A for the explicit expression of the sets of numerator polynomials $\mathcal{N}^{3,m,i}$.

m	Term	Polynomial
Denominators		$d_1 = r_1,\ d_2 = r_2,\ d_3 = r_3,\ d_4 = s_{1,2} + s_{1,3},\ d_5 = s_{1,2} + s_{2,3}$
≥ 0	$\left(1-\lambda^2\right)^5$	$d_1,\ d_2,\ d_3,\ d_4,\ d_5$
≥ 3	$\left(1-\lambda^2\right)^4$	$d_2,\ d_3,\ d_4,\ d_5$
≥ 3	$\left(1-\lambda^2\right)^3$	$d_3,\ d_4,\ d_5$
Numerators		
0	$1+4\lambda^2+\lambda^4$	$1,\ s_{2,3},\ \underline{t_{1,2}},\ \underline{t_{1,3}},\ \underline{t_{2,3}},\ s_{2,3}^2$
1	$3\lambda+3\lambda^3$	$\pi_1,\ \pi_2,\ \pi_3,\ \pi_1 s_{2,3},\ \pi_2 s_{1,3},\ \pi_3 s_{2,3}$
2	$6\lambda^2$	$\pi_1^2,\ \pi_1\pi_2,\ \pi_1\pi_3,\ \pi_2^2,\ \pi_2\pi_3,\ \pi_3^2$
3	$6\lambda^3$	$\mathcal{N}^{3,3,1}$
	$3\lambda^3$	$\mathcal{N}^{3,3,2}$
	λ^3	$\mathcal{N}^{3,3,3}$
4	$6\lambda^4$	$\mathcal{N}^{3,4,1}$
	$6\lambda^4$	$\mathcal{N}^{3,4,2}$
	$3\lambda^4$	$\mathcal{N}^{3,4,3}$
5	$6\lambda^5$	$\mathcal{N}^{3,5,1}$
	$9\lambda^5$	$\mathcal{N}^{3,5,2}$
	$6\lambda^5$	$\mathcal{N}^{3,5,3}$

6. Integrity bases for four planar vectors under SO(2)

6.1. Invariants and free modules of (1)–, (2)–, *and* (3)*–covariants*

The Molien function for SO(2)–invariants built up from four vectors, $M_\alpha^{\mathrm{SO}(2)}\left((0)\,;\Gamma_4;\lambda\right) = \left[(1+3\lambda^2+6\lambda^4)_{A_1} + (6\lambda^2+3\lambda^4+\lambda^6)_{A_2}\right]/\left(1-\lambda^2\right)^7$, suggests to find seven quadratic denominator invariants, one numerator constant, nine quadratic numerator invariants (of which three have A_1 symmetry and six have A_2 symmetry), nine quartic numerator invariants (of which six have A_1 symmetry and three have A_2 symmetry), and one sextic numerator polynomial of symmetry A_2. The set of generators $r_i,\ s_{i,j},\ t_{i,j}$, $1 \leq i < j \leq 4$ contains sixteen linearly independent quadratic polynomials. The Molien function is rewritten in the form $M^{\mathrm{SO}(2)}\left((0)\,;\Gamma_4;\lambda\right) = N_{4,0}\left(\lambda\right)/\left(1-\lambda^2\right)^{16}$ compatible with the method of syzygies, where the numerator reads as:

$$N_{4,0}\left(\lambda\right) = 1 - 36\lambda^4 + 160\lambda^6 - 315\lambda^8 + 288\lambda^{10} - 288\lambda^{14} + 315\lambda^{16} - 160\lambda^{18} + 36\lambda^{20} - \lambda^{24}.$$

Thirty–eight first order syzygies of degree four among the generators are given in Appendix C, but they span a vector space of only thirty–six linearly independent relations as confirmed by the $36\lambda^4$ term in the numerator $N_{4,0}\left(\lambda\right)$. As in the case with three planar vectors, the intricate structure of the syzygies calls for an algorithmic approach. The results are given in table 4. The twenty numerator polynomials $\mathcal{N}^{4,3}$

for the free module of (3)–covariants are the monomials that belong to the expansion of $(\pi_1 + \pi_2 + \pi_3 + \pi_4)^3$.

Table 4. Integrity bases for invariants and covariants of the SO(2) group built up from four planar vectors. The underscored polynomials transform as the A_2 irreducible representation of O(2) group. The other numerator polynomials of the (0) irreducible representation transform as the totally symmetric representation A_1 of the O(2) group. See Appendix A for the explicit expression of the sets of numerator polynomials $\mathcal{N}^{4,m,i}$.

m	Term	Polynomial
Denominators		$d_1 = r_1$, $d_2 = r_2$, $d_3 = r_3$, $d_4 = r_4$, $d_5 = s_{1,2} + s_{1,3} + s_{3,4}$, $d_6 = s_{1,3} + s_{1,4} + s_{2,4}$, $d_7 = s_{1,4} + s_{2,3} + s_{3,4}$
≥ 0	$\left(1 - \lambda^2\right)^7$	$d_1, d_2, d_3, d_4, d_5, d_6, d_7$
≥ 4	$\left(1 - \lambda^2\right)^6$	$d_2, d_3, d_4, d_5, d_6, d_7$
≥ 4	$\left(1 - \lambda^2\right)^5$	d_3, d_4, d_5, d_6, d_7
≥ 4	$\left(1 - \lambda^2\right)^4$	d_4, d_5, d_6, d_7
Numerators		
0	$1 + 9\lambda^2 + 9\lambda^4 + \lambda^6$	1, $s_{2,3}$, $s_{2,4}$, $s_{3,4}$, $\underline{t_{1,2}}$, $\underline{t_{1,3}}$, $\underline{t_{1,4}}$, $\underline{t_{2,3}}$, $\underline{t_{2,4}}$, $\underline{t_{3,4}}$, $s_{2,3}^2$, $s_{2,4}^2$, $s_{3,4}^2$, $s_{2,3}s_{2,4}$, $s_{2,3}s_{3,4}$, $s_{2,4}s_{3,4}$, $\underline{t_{2,3}s_{2,3}}$, $\underline{t_{2,4}s_{2,4}}$, $\underline{t_{3,4}s_{3,4}}$, $\underline{s_{3,4}^2 t_{3,4}}$
1	$4\lambda + 12\lambda^3 + 4\lambda^5$	π_1, π_2, π_3, π_4, $\pi_1 s_{2,3}$, $\pi_1 s_{2,4}$, $\pi_1 s_{3,4}$, $\pi_2 s_{1,3}$, $\pi_2 s_{1,4}$, $\pi_2 s_{2,3}$, $\pi_3 s_{1,2}$, $\pi_3 s_{1,3}$, $\pi_3 s_{2,4}$, $\pi_4 s_{1,2}$, $\pi_4 s_{1,3}$, $\pi_4 s_{2,3}$, $\pi_1 s_{2,3}^2$, $\pi_2 s_{1,3}^2$, $\pi_3 s_{1,2}^2$, $\pi_4 s_{1,2}^2$
2	$10\lambda^2 + 10\lambda^4$	π_1^2, π_2^2, π_3^2, π_4^2, $\pi_1\pi_2$, $\pi_1\pi_3$, $\pi_1\pi_4$, $\pi_2\pi_3$, $\pi_2\pi_4$, $\pi_3\pi_4$, $\pi_1^2 s_{1,2}$, $\pi_2^2 s_{1,2}$, $\pi_3^2 s_{1,2}$, $\pi_4^2 s_{1,2}$, $\pi_1\pi_2 s_{3,4}$, $\pi_1\pi_3 s_{2,4}$, $\pi_1\pi_4 s_{2,3}$, $\pi_1\pi_4 s_{1,3}$, $\pi_2\pi_3 s_{2,4}$, $\pi_3\pi_4 s_{1,4}$
3	$20\lambda^3$	π_1^3, $\pi_1^2\pi_2$, $\pi_1^2\pi_3$, $\pi_1^2\pi_4$, $\pi_1\pi_2^2$, $\pi_1\pi_2\pi_3$, $\pi_1\pi_2\pi_4$, $\pi_1\pi_3^2$, $\pi_1\pi_3\pi_4$, $\pi_1\pi_4^2$, π_2^3, $\pi_2^2\pi_3$, $\pi_2^2\pi_4$, $\pi_2\pi_3^2$, $\pi_2\pi_3\pi_4$, $\pi_2\pi_4^2$, π_3^3, $\pi_3^2\pi_4$, $\pi_3\pi_4^2$, π_4^3
4	$20\lambda^4$	$\mathcal{N}^{4,4,1}$
	$10\lambda^4$	$\mathcal{N}^{4,4,2}$
	$4\lambda^4$	$\mathcal{N}^{4,4,3}$
	λ^4	$\mathcal{N}^{4,4,4}$
5	$20\lambda^5$	$\mathcal{N}^{4,5,1}$
	$20\lambda^5$	$\mathcal{N}^{4,5,2}$
	$12\lambda^5$	$\mathcal{N}^{4,5,3}$
	$4\lambda^5$	$\mathcal{N}^{4,5,4}$

6.2. Non–free modules of (4) *and* (5)*–covariants*

Negative coefficients appear for $m \geq 4$ in the numerator of the α form $M_\alpha^{\mathrm{SO}(2)}\left((m)\,;\Gamma_4;\lambda\right) = N_{4,m}\left(\lambda\right) / \left(1 - \lambda^2\right)^7$, with

$$N_{4,m}\left(\lambda\right) = \left(m+3\right)\left(m+2\right)\left(m+1\right)\lambda^m/6 - \left(m-3\right)\left(m+3\right)\left(m+2\right)\lambda^{m+2}/2$$
$$+ \left(m-3\right)\left(m-2\right)\left(m+3\right)\lambda^{m+4}/2 - \left(m-3\right)\left(m-2\right)\left(m-1\right)\lambda^{m+6}/6.$$

Generalized integrity basis for the modules of (4) and (5)–covariants are constructed following the same lines seen in section 4.3 and section 5.5 for the non–free modules with two or three vectors. Their elements are given in table 4. As a consequence, any (4)–covariant uniquely decomposes as:

$$\sum_{i=1}^{20} p_{1,i}\left(d_1,\ldots,d_7\right)\mathcal{N}_i^{4,4,1} + \sum_{i=1}^{10} p_{2,i}\left(d_2,\ldots,d_7\right)\mathcal{N}_i^{4,4,2}$$
$$+\sum_{i=1}^{4} p_{3,i}\left(d_3,\ldots,d_7\right)\mathcal{N}_i^{4,4,3} + p_{4,1}\left(d_4,d_5,d_6,d_7\right)\mathcal{N}_1^{4,4,4},$$

and any (5)–covariant uniquely decomposes as:

$$\sum_{i=1}^{20} p_{1,i}\left(d_1,\ldots,d_7\right)\mathcal{N}_i^{4,5,1} + \sum_{i=1}^{20} p_{2,i}\left(d_2,\ldots,d_7\right)\mathcal{N}_i^{4,5,2}$$
$$+\sum_{i=1}^{12} p_{3,i}\left(d_3,\ldots,d_7\right)\mathcal{N}_i^{4,5,3} + \sum_{i=1}^{4} p_{4,i}\left(d_4,\ldots,d_7\right)\mathcal{N}_i^{4,5,4}.$$

7. Integrity bases for two, three, and four planar vectors under O(2)

The integrity bases for two, three, and four planar vectors under the O(2) group are directly deduced from the integrity bases under SO(2) given in table 2, table 3, and table 4. It suffices to remark that both the r_i and $s_{i,j}$ polynomials are invariant with respect to a reflection in any line, while the $t_{i,j}$ change sign. As a consequence, the r_i and $s_{i,j}$ belong to the A_1 irreducible representation while the $t_{i,j}$ transform as the A_2 irreducible representation of the O(2) group. The underscored numerator polynomials of the (0) representation of SO(2) in table 2, table 3 and table 4 are the numerator polynomials for the A_2 representation of O(2), while the remaining numerator polynomials of the (0) representation are the numerator polynomials for the A_1 Molien function of O(2).

The integrity basis for the E_m representations of O(2) are constructed with a similar pattern. The denominator polynomials are those of the (m) representation of SO(2). The numerator polynomials are those of the (m) representation of SO(2) and their complex conjugate.

8. Conclusion

The explicit expressions of the Molien function $M^G\left(\Gamma_f;\Gamma_n;\lambda\right)$ for the Γ_f–polynomials built up from the components of n planar vectors under $G = \mathrm{SO}(2), \mathrm{O}(2)$ were presented. The formulas obtained after a direct evaluation of the integral (10) do not admit any direct symbolic interpretation. Two other expressions of the Molien function, M_α and M_β, are presented.

For $0 \leq m \leq n-1$, the ring of invariants or the module of covariants features a Cohen–Macaulay structure that corresponds to a module over a ring of invariants. The M_α expression is a single rational function which admits a symbolic interpretation in term of integrity bases.

The modules of (m)–covariants are not free when $m \geq n$. Negative terms appear in the numerator of $M_\alpha^{\mathrm{SO}(2)}$ and indicate that relations exist between the denominator and numerator polynomials of $M_\alpha^{\mathrm{SO}(2)}$. Generalized integrity bases are proposed by considering the $M_\beta^{\mathrm{SO}(2)}$ form. A geometrical view of the non–free modules is obtained with the introduction of lattices of points. Each numerator covariant is multiplied by a polynomial function in k denominator polynomials and is attached to a k–dimensional lattice $\mathbb{N}^k$. Any covariant polynomial uniquely decomposes as a $\mathbb{C}$–linear combination of the elements in the $\mathbb{C}$–basis.

The integrity bases determined in this paper are summed up in table 2, table 3, and table 4. Two extensions of these work are possible. First, the empirical description of the non–free modules should be formalized in a more mathematical setting. Secondly, this presentation of the Molien function for non–free modules as a sum of rational functions probably occurs with an initial representation different from Γ_n and in other continuous groups of physical importance such as SO(3) or SU(2).

Acknowledgments

The authors thank Patrick Cassam–Chenaï and Frédéric Patras from Laboratoire J.A. Dieudonné, Université de Nice–Sophia Antipolis, France, for stimulating discussions. Financial support for the project *Application de la Théorie des Invariants à la Physique Moléculaire* via a CNRS grant *Projet Exploratoire Premier Soutien* (PEPS) *Physique Théorique et Interfaces* (PTI) is acknowledged.

B. Z. thanks the Isaac Newton Institute for Mathematical Sciences for financial support making possible his visit within the program "Mathematics for liquid crystals" and for fruitful discussions during this visit.

Appendix A. Expression of the $\mathcal{N}^{n,m,i}$ numerator polynomials

The sets $\mathcal{N}^{n,m,i}$ of numerator polynomials attached to the i^{th} rational fraction of $M_\beta^{\mathrm{SO}(2)}\left((m)\,;\Gamma_n;\lambda\right)$ are given in table A1, see (27) for a definition.

Appendix B. Relations between the numerator and denominator polynomials for three planar vectors

Appendix B.1. Relations for the non–free module of (3)*–covariants*

$$\begin{aligned}
d_1\mathcal{N}_1^{3,3,2} &= d_2\mathcal{N}_1^{3,3,1} + (-d_2 - 2d_5)\mathcal{N}_2^{3,3,1} + 2d_2\mathcal{N}_3^{3,3,1} \\
&\quad + (d_1 + d_3 + 2d_4)\mathcal{N}_4^{3,3,1} - 2d_5\mathcal{N}_5^{3,3,1} + d_2\mathcal{N}_6^{3,3,1}, && (2.1a)\\
d_1\mathcal{N}_2^{3,3,2} &= -d_2\mathcal{N}_3^{3,3,1} - d_3\mathcal{N}_4^{3,3,1} + 2d_5\mathcal{N}_5^{3,3,1} - d_2\mathcal{N}_6^{3,3,1}, && (2.1b)\\
d_1\mathcal{N}_3^{3,3,2} &= -d_3\mathcal{N}_2^{3,3,1} + d_3\mathcal{N}_4^{3,3,1} + (2d_4 - 2d_5)\mathcal{N}_5^{3,3,1} + d_2\mathcal{N}_6^{3,3,1}, && (2.1c)\\
d_1\mathcal{N}_1^{3,3,3} &= -d_3\mathcal{N}_1^{3,3,1} + 2d_3\mathcal{N}_2^{3,3,1} + (-d_3 + 2d_4 - 2d_5)\mathcal{N}_3^{3,3,1} - d_3\mathcal{N}_4^{3,3,1}
\end{aligned}$$

Table A1. Expression of the $\mathcal{N}^{n,m,i}$ numerator polynomials, $2 \leq n \leq 4$ and $n \leq m \leq 5$.

$\mathcal{N}^{n,m,i}$	Numerator polynomials
$\mathcal{N}^{2,2,1}$	$\pi_1^2,\ \pi_1\pi_2$
$\mathcal{N}^{2,2,2}$	π_2^2
$\mathcal{N}^{2,3,1}$	$\pi_1^3,\ \pi_1^2\pi_2$
$\mathcal{N}^{2,3,2}$	$\pi_1\pi_2^2,\ \pi_2^3$
$\mathcal{N}^{2,4,1}$	$\pi_1^4,\ \pi_1^3\pi_2$
$\mathcal{N}^{2,4,2}$	$\pi_1^2\pi_2^2,\ \pi_1\pi_2^3,\ \pi_2^4$
$\mathcal{N}^{2,5,1}$	$\pi_1^5,\ \pi_1^4\pi_2$
$\mathcal{N}^{2,5,2}$	$\pi_1^3\pi_2^2,\ \pi_1^2\pi_2^3,\ \pi_1\pi_2^4,\ \pi_2^5$
$\mathcal{N}^{3,3,1}$	$\pi_1^3,\ \pi_1^2\pi_2,\ \pi_1^2\pi_3,\ \pi_1\pi_2^2,\ \pi_1\pi_2\pi_3,\ \pi_1\pi_3^2$
$\mathcal{N}^{3,3,2}$	$\pi_2^3,\ \pi_2^2\pi_3,\ \pi_2\pi_3^2$
$\mathcal{N}^{3,3,3}$	π_3^3
$\mathcal{N}^{3,4,1}$	$\pi_1^4,\ \pi_1^3\pi_2,\ \pi_1^3\pi_3,\ \pi_1^2\pi_2^2,\ \pi_1^2\pi_2\pi_3,\ \pi_1^2\pi_3^2$
$\mathcal{N}^{3,4,2}$	$\pi_1\pi_2^3,\ \pi_1\pi_2^2\pi_3,\ \pi_1\pi_2\pi_3^2,\ \pi_2^4,\ \pi_2^3\pi_3,\ \pi_2^2\pi_3^2$
$\mathcal{N}^{3,4,3}$	$\pi_1\pi_3^3,\ \pi_2\pi_3^3,\ \pi_3^4$
$\mathcal{N}^{3,5,1}$	$\pi_1^5,\ \pi_1^4\pi_2,\ \pi_1^4\pi_3,\ \pi_1^3\pi_2^2,\ \pi_1^3\pi_2\pi_3,\ \pi_1^3\pi_3^2$
$\mathcal{N}^{3,5,2}$	$\pi_1^2\pi_2^3,\ \pi_1^2\pi_2^2\pi_3,\ \pi_1^2\pi_2\pi_3^2,\ \pi_1\pi_2^4,\ \pi_1\pi_2^3\pi_3,\ \pi_1\pi_2^2\pi_3^2,\ \pi_2^5,\ \pi_2^4\pi_3,\ \pi_2^3\pi_3^2$
$\mathcal{N}^{3,5,3}$	$\pi_1^2\pi_3^3,\ \pi_1\pi_2\pi_3^3,\ \pi_1\pi_3^4,\ \pi_2^2\pi_3^3,\ \pi_2\pi_3^4,\ \pi_3^5$
$\mathcal{N}^{4,4,1}$	$\pi_1^4,\ \pi_1^3\pi_2,\ \pi_1^3\pi_3,\ \pi_1^3\pi_4,\ \pi_1^2\pi_2^2,\ \pi_1^2\pi_2\pi_3,\ \pi_1^2\pi_2\pi_4,\ \pi_1^2\pi_3^2,\ \pi_1^2\pi_3\pi_4,\ \pi_1^2\pi_4^2,$ $\pi_1\pi_2^3,\ \pi_1\pi_2^2\pi_3,\ \pi_1\pi_2^2\pi_4,\ \pi_1\pi_2\pi_3^2,\ \pi_1\pi_2\pi_3\pi_4,\ \pi_1\pi_2\pi_4^2,\ \pi_1\pi_3^3,\ \pi_1\pi_3^2\pi_4,$ $\pi_1\pi_3\pi_4^2,\ \pi_1\pi_4^3$
$\mathcal{N}^{4,4,2}$	$\pi_2^4,\ \pi_2^3\pi_3,\ \pi_2^3\pi_4,\ \pi_2^2\pi_3^2,\ \pi_2^2\pi_3\pi_4,\ \pi_2^2\pi_4^2,\ \pi_2\pi_3^3,\ \pi_2\pi_3^2\pi_4,\ \pi_2\pi_3\pi_4^2,\ \pi_2\pi_4^3$
$\mathcal{N}^{4,4,3}$	$\pi_3^4,\ \pi_3^3\pi_4,\ \pi_3^2\pi_4^2,\ \pi_3\pi_4^3$
$\mathcal{N}^{4,4,4}$	π_4^4
$\mathcal{N}^{4,5,1}$	$\pi_1^5,\ \pi_1^4\pi_2,\ \pi_1^4\pi_3,\ \pi_1^4\pi_4,\ \pi_1^3\pi_2^2,\ \pi_1^3\pi_2\pi_3,\ \pi_1^3\pi_2\pi_4,\ \pi_1^3\pi_3^2,\ \pi_1^3\pi_3\pi_4,\ \pi_1^3\pi_4^2,$ $\pi_1^2\pi_2^3,\ \pi_1^2\pi_2^2\pi_3,\ \pi_1^2\pi_2^2\pi_4,\ \pi_1^2\pi_2\pi_3^2,\ \pi_1^2\pi_2\pi_3\pi_4,\ \pi_1^2\pi_2\pi_4^2,\ \pi_1^2\pi_3^3,\ \pi_1^2\pi_3^2\pi_4,$ $\pi_1^2\pi_3\pi_4^2,\ \pi_1^2\pi_4^3$
$\mathcal{N}^{4,5,2}$	$\pi_1\pi_2^4,\ \pi_1\pi_2^3\pi_3,\ \pi_1\pi_2^3\pi_4,\ \pi_1\pi_2^2\pi_3^2,\ \pi_1\pi_2^2\pi_3\pi_4,\ \pi_1\pi_2^2\pi_4^2,\ \pi_1\pi_2\pi_3^3,\ \pi_1\pi_2\pi_3^2\pi_4,$ $\pi_1\pi_2\pi_3\pi_4^2,\ \pi_1\pi_2\pi_4^3,\ \pi_2^5,\ \pi_2^4\pi_3,\ \pi_2^4\pi_4,\ \pi_2^3\pi_3^2,\ \pi_2^3\pi_3\pi_4,\ \pi_2^3\pi_4^2,\ \pi_2^2\pi_3^3,$ $\pi_2^2\pi_3^2\pi_4,\ \pi_2^2\pi_3\pi_4^2,\ \pi_2^2\pi_4^3$
$\mathcal{N}^{4,5,3}$	$\pi_1\pi_3^4,\ \pi_1\pi_3^3\pi_4,\ \pi_1\pi_3^2\pi_4^2,\ \pi_1\pi_3\pi_4^3,\ \pi_2\pi_3^4,\ \pi_2\pi_3^3\pi_4,\ \pi_2\pi_3^2\pi_4^2,\ \pi_2\pi_3\pi_4^3,\ \pi_3^5,$ $\pi_3^4\pi_4,\ \pi_3^3\pi_4^2,\ \pi_3^2\pi_4^3$
$\mathcal{N}^{4,5,4}$	$\pi_1\pi_4^4,\ \pi_2\pi_4^4,\ \pi_3\pi_4^4,\ \pi_4^5$

$$+ (-2d_4 + 2d_5)\mathcal{N}_5^{3,3,1} + (-d_1 - d_2 + 2d_4)\mathcal{N}_6^{3,3,1}, \tag{2.1d}$$

$$d_2\mathcal{N}_1^{3,3,3} = d_3\mathcal{N}_4^{3,3,1} - d_2\mathcal{N}_6^{3,3,1} - d_3\mathcal{N}_1^{3,3,2} + (-d_3 - 2d_4 + 2d_5)\mathcal{N}_2^{3,3,2} + (-d_2 + 2d_5)\mathcal{N}_3^{3,3,2}. \tag{2.1e}$$

Appendix B.2. Relations for the non–free module of (4)*–covariants*

$$d_1\mathcal{N}_1^{3,4,2} = d_2\mathcal{N}_1^{3,4,1} + (-d_2 - 2d_5)\mathcal{N}_2^{3,4,1} + 2d_2\mathcal{N}_3^{3,4,1} + (d_1 + d_3 + 2d_4)\mathcal{N}_4^{3,4,1} - 2d_5\mathcal{N}_5^{3,4,1} + d_2\mathcal{N}_6^{3,4,1}, \tag{2.2a}$$

$$d_1\mathcal{N}_2^{3,4,2} = -d_2\mathcal{N}_3^{3,4,1} - d_3\mathcal{N}_4^{3,4,1} + 2d_5\mathcal{N}_5^{3,4,1} - d_2\mathcal{N}_6^{3,4,1}, \tag{2.2b}$$

$$
\begin{aligned}
d_1\mathcal{N}_3^{3,4,2} &= -d_3\mathcal{N}_2^{3,4,1} + d_3\mathcal{N}_4^{3,4,1} + (2d_4 - 2d_5)\mathcal{N}_5^{3,4,1} + d_2\mathcal{N}_6^{3,4,1}, && (2.2c)\\
d_1\mathcal{N}_4^{3,4,2} &= d_2\mathcal{N}_1^{3,4,1} - 2d_5\mathcal{N}_2^{3,4,1} + 2d_2\mathcal{N}_3^{3,4,1}\\
&\quad + (d_1 - d_2 + d_3 + 2d_4 - 2d_5)\mathcal{N}_4^{3,4,1} + (2d_2 - 2d_5)\mathcal{N}_5^{3,4,1} +\\
&\quad d_2\mathcal{N}_6^{3,4,1} + (d_3 + 2d_4)\mathcal{N}_1^{3,4,2} - 2d_5\mathcal{N}_2^{3,4,2} + d_2\mathcal{N}_3^{3,4,2}, && (2.2d)\\
d_1\mathcal{N}_5^{3,4,2} &= -d_2\mathcal{N}_5^{3,4,1} - d_3\mathcal{N}_1^{3,4,2} + 2d_5\mathcal{N}_2^{3,4,2} - d_2\mathcal{N}_3^{3,4,2}, && (2.2e)\\
d_1\mathcal{N}_6^{3,4,2} &= -d_3\mathcal{N}_4^{3,4,1} + d_3\mathcal{N}_1^{3,4,2} + (2d_4 - 2d_5)\mathcal{N}_2^{3,4,2} + d_2\mathcal{N}_3^{3,4,2}, && (2.2f)\\
d_1\mathcal{N}_1^{3,4,3} &= -d_3\mathcal{N}_1^{3,4,1} + 2d_3\mathcal{N}_2^{3,4,1} + (-d_3 + 2d_4 - 2d_5)\mathcal{N}_3^{3,4,1}\\
&\quad - d_3\mathcal{N}_4^{3,4,1} + (-2d_4 + 2d_5)\mathcal{N}_5^{3,4,1} + (-d_1 - d_2 + 2d_4)\mathcal{N}_6^{3,4,1}, && (2.2g)\\
d_1\mathcal{N}_2^{3,4,3} &= d_3\mathcal{N}_4^{3,4,1} - d_3\mathcal{N}_5^{3,4,1} - d_2\mathcal{N}_6^{3,4,1} - d_3\mathcal{N}_1^{3,4,2}\\
&\quad + (-2d_4 + 2d_5)\mathcal{N}_2^{3,4,2} + (-d_2 + 2d_4)\mathcal{N}_3^{3,4,2}, && (2.2h)\\
d_1\mathcal{N}_3^{3,4,3} &= d_3\mathcal{N}_1^{3,4,1} - 2d_3\mathcal{N}_2^{3,4,1} + (-2d_4 + 2d_5)\mathcal{N}_3^{3,4,1}\\
&\quad + (2d_3 + 2d_4 - 2d_5)\mathcal{N}_5^{3,4,1} + (d_1 + 2d_2 - d_3 - 2d_5)\mathcal{N}_6^{3,4,1}\\
&\quad + d_3\mathcal{N}_1^{3,4,2} + (2d_4 - 2d_5)\mathcal{N}_2^{3,4,2} + (d_2 - 2d_4)\mathcal{N}_3^{3,4,2}\\
&\quad + 2d_4\mathcal{N}_1^{3,4,3}, && (2.2i)\\
d_2\mathcal{N}_1^{3,4,3} &= d_3\mathcal{N}_4^{3,4,1} - d_2\mathcal{N}_6^{3,4,1} - d_3\mathcal{N}_1^{3,4,2} + (-d_3 - 2d_4 + 2d_5)\mathcal{N}_2^{3,4,2}\\
&\quad + (-d_2 + 2d_5)\mathcal{N}_3^{3,4,2}, && (2.2j)\\
d_2\mathcal{N}_2^{3,4,3} &= d_3\mathcal{N}_1^{3,4,2} - d_2\mathcal{N}_3^{3,4,2} - d_3\mathcal{N}_4^{3,4,2} + (-d_3 - 2d_4 + 2d_5)\mathcal{N}_5^{3,4,2}\\
&\quad + (-d_2 + 2d_5)\mathcal{N}_6^{3,4,2}, && (2.2k)\\
d_2\mathcal{N}_3^{3,4,3} &= -d_3\mathcal{N}_4^{3,4,1} + d_2\mathcal{N}_6^{3,4,1} + (2d_3 + 2d_4 - 2d_5)\mathcal{N}_2^{3,4,2}\\
&\quad + (2d_2 - 2d_5)\mathcal{N}_3^{3,4,2} + d_3\mathcal{N}_4^{3,4,2} + (2d_4 - 2d_5)\mathcal{N}_5^{3,4,2}\\
&\quad + (d_2 - d_3 - 2d_4)\mathcal{N}_6^{3,4,2} + 2d_5\mathcal{N}_2^{3,4,3}. && (2.2l)
\end{aligned}
$$

Appendix B.3. Relations for the non–free module of (5)*–covariants*

$$
\begin{aligned}
d_1\mathcal{N}_1^{3,5,2} &= d_2\mathcal{N}_1^{3,5,1} + (-d_2 - 2d_5)\mathcal{N}_2^{3,5,1} + 2d_2\mathcal{N}_3^{3,5,1}\\
&\quad + (d_1 + d_3 + 2d_4)\mathcal{N}_4^{3,5,1} - 2d_5\mathcal{N}_5^{3,5,1} + d_2\mathcal{N}_6^{3,5,1}, && (2.3a)\\
d_1\mathcal{N}_2^{3,5,2} &= -d_2\mathcal{N}_3^{3,5,1} - d_3\mathcal{N}_4^{3,5,1} + 2d_5\mathcal{N}_5^{3,5,1} - d_2\mathcal{N}_6^{3,5,1}, && (2.3b)\\
d_1\mathcal{N}_3^{3,5,2} &= -d_3\mathcal{N}_2^{3,5,1} + d_3\mathcal{N}_4^{3,5,1} + (2d_4 - 2d_5)\mathcal{N}_5^{3,5,1} + d_2\mathcal{N}_6^{3,5,1}, && (2.3c)\\
d_1\mathcal{N}_4^{3,5,2} &= d_2\mathcal{N}_1^{3,5,1} - 2d_5\mathcal{N}_2^{3,5,1} + 2d_2\mathcal{N}_3^{3,5,1}\\
&\quad + (d_1 - d_2 + d_3 + 2d_4 - 2d_5)\mathcal{N}_4^{3,5,1} + (2d_2 - 2d_5)\mathcal{N}_5^{3,5,1}\\
&\quad + d_2\mathcal{N}_6^{3,5,1} + (d_3 + 2d_4)\mathcal{N}_1^{3,5,2} - 2d_5\mathcal{N}_2^{3,5,2} + d_2\mathcal{N}_3^{3,5,2}, && (2.3d)\\
d_1\mathcal{N}_5^{3,5,2} &= -d_2\mathcal{N}_5^{3,5,1} - d_3\mathcal{N}_1^{3,5,2} + 2d_5\mathcal{N}_2^{3,5,2} - d_2\mathcal{N}_3^{3,5,2}, && (2.3e)\\
d_1\mathcal{N}_6^{3,5,2} &= -d_3\mathcal{N}_4^{3,5,1} + d_3\mathcal{N}_1^{3,5,2} + (2d_4 - 2d_5)\mathcal{N}_2^{3,5,2} + d_2\mathcal{N}_3^{3,5,2}, && (2.3f)\\
d_1\mathcal{N}_7^{3,5,2} &= d_2\mathcal{N}_1^{3,5,1} - 2d_5\mathcal{N}_2^{3,5,1} + 2d_2\mathcal{N}_3^{3,5,1} + (d_1 + d_3 + 2d_4 - 2d_5)\mathcal{N}_4^{3,5,1}\\
&\quad + (2d_2 - 2d_5)\mathcal{N}_5^{3,5,1} + d_2\mathcal{N}_6^{3,5,1}\\
&\quad + (-d_2 + d_3 + 2d_4 - 2d_5)\mathcal{N}_1^{3,5,2} + (2d_2 - 2d_5)\mathcal{N}_2^{3,5,2}
\end{aligned}
$$

$$
\begin{aligned}
&+ d_2\mathcal{N}_3^{3,5,2} + (d_3+2d_4)\mathcal{N}_4^{3,5,2} - 2d_5\mathcal{N}_5^{3,5,2} + d_2\mathcal{N}_6^{3,5,2}, && (2.3g)\\
d_1\mathcal{N}_8^{3,5,2} =\ & - d_2\mathcal{N}_2^{3,5,2} - d_3\mathcal{N}_4^{3,5,2} + 2d_5\mathcal{N}_5^{3,5,2} - d_2\mathcal{N}_6^{3,5,2}, && (2.3h)\\
d_1\mathcal{N}_9^{3,5,2} =\ & - d_3\mathcal{N}_1^{3,5,2} + d_3\mathcal{N}_4^{3,5,2} + (2d_4-2d_5)\mathcal{N}_5^{3,5,2} + d_2\mathcal{N}_6^{3,5,2}, && (2.3i)\\
d_1\mathcal{N}_1^{3,5,3} =\ & - d_3\mathcal{N}_1^{3,5,1} + 2d_3\mathcal{N}_2^{3,5,1} + (-d_3+2d_4-2d_5)\mathcal{N}_3^{3,5,1} - d_3\mathcal{N}_4^{3,5,1}\\
&+ (-2d_4+2d_5)\mathcal{N}_5^{3,5,1} + (-d_1-d_2+2d_4)\mathcal{N}_6^{3,5,1}, && (2.3j)\\
d_1\mathcal{N}_2^{3,5,3} =\ & d_3\mathcal{N}_4^{3,5,1} - d_3\mathcal{N}_5^{3,5,1} - d_2\mathcal{N}_6^{3,5,1} - d_3\mathcal{N}_1^{3,5,2}\\
&+ (-2d_4+2d_5)\mathcal{N}_2^{3,5,2} + (-d_2+2d_4)\mathcal{N}_3^{3,5,2}, && (2.3k)\\
d_1\mathcal{N}_3^{3,5,3} =\ & d_3\mathcal{N}_1^{3,5,1} - 2d_3\mathcal{N}_2^{3,5,1} + (-2d_4+2d_5)\mathcal{N}_3^{3,5,1}\\
&+ (2d_3+2d_4-2d_5)\mathcal{N}_5^{3,5,1}\\
&+ (d_1+2d_2-d_3-2d_5)\mathcal{N}_6^{3,5,1} + d_3\mathcal{N}_1^{3,5,2} + (2d_4-2d_5)\mathcal{N}_2^{3,5,2}\\
&+ (d_2-2d_4)\mathcal{N}_3^{3,5,2} + 2d_4\mathcal{N}_1^{3,5,3}, && (2.3l)\\
d_1\mathcal{N}_4^{3,5,3} =\ & d_3\mathcal{N}_1^{3,5,2} - d_3\mathcal{N}_2^{3,5,2} - d_2\mathcal{N}_3^{3,5,2} - d_3\mathcal{N}_4^{3,5,2} + (-2d_4+2d_5)\mathcal{N}_5^{3,5,2}\\
&+ (-d_2+2d_4)\mathcal{N}_6^{3,5,2}, && (2.3m)\\
d_1\mathcal{N}_5^{3,5,3} =\ & - d_3\mathcal{N}_4^{3,5,1} + d_2\mathcal{N}_6^{3,5,1} + (2d_3+2d_4-2d_5)\mathcal{N}_2^{3,5,2}\\
&+ (2d_2-d_3-2d_5)\mathcal{N}_3^{3,5,2} + d_3\mathcal{N}_4^{3,5,2} + (2d_4-2d_5)\mathcal{N}_5^{3,5,2}\\
&+ (d_2-2d_4)\mathcal{N}_6^{3,5,2} + 2d_4\mathcal{N}_2^{3,5,3}, && (2.3n)\\
d_1\mathcal{N}_6^{3,5,3} =\ & - d_3\mathcal{N}_1^{3,5,1} + 2d_3\mathcal{N}_2^{3,5,1} + (2d_4-2d_5)\mathcal{N}_3^{3,5,1} + d_3\mathcal{N}_4^{3,5,1}\\
&+ (-2d_3-2d_4+2d_5)\mathcal{N}_5^{3,5,1} + (-d_1-3d_2+2d_5)\mathcal{N}_6^{3,5,1}\\
&- d_3\mathcal{N}_1^{3,5,2} + (-2d_3-4d_4+4d_5)\mathcal{N}_2^{3,5,2}\\
&+ (-3d_2+2d_3+2d_4+2d_5)\mathcal{N}_3^{3,5,2} - d_3\mathcal{N}_4^{3,5,2}\\
&+ (-2d_4+2d_5)\mathcal{N}_5^{3,5,2} + (-d_2+2d_4)\mathcal{N}_6^{3,5,2}\\
&+ (-d_3-2d_5)\mathcal{N}_1^{3,5,3} - 2d_4\mathcal{N}_2^{3,5,3} + 2d_4\mathcal{N}_3^{3,5,3}, && (2.3o)\\
d_2\mathcal{N}_1^{3,5,3} =\ & d_3\mathcal{N}_4^{3,5,1} - d_2\mathcal{N}_6^{3,5,1} - d_3\mathcal{N}_1^{3,5,2} + (-d_3-2d_4+2d_5)\mathcal{N}_2^{3,5,2}\\
&+ (-d_2+2d_5)\mathcal{N}_3^{3,5,2}, && (2.3p)\\
d_2\mathcal{N}_2^{3,5,3} =\ & d_3\mathcal{N}_1^{3,5,2} - d_2\mathcal{N}_3^{3,5,2} - d_3\mathcal{N}_4^{3,5,2} + (-d_3-2d_4+2d_5)\mathcal{N}_5^{3,5,2}\\
&+ (-d_2+2d_5)\mathcal{N}_6^{3,5,2}, && (2.3q)\\
d_2\mathcal{N}_3^{3,5,3} =\ & - d_3\mathcal{N}_4^{3,5,1} + d_2\mathcal{N}_6^{3,5,1} + (2d_3+2d_4-2d_5)\mathcal{N}_2^{3,5,2}\\
&+ (2d_2-2d_5)\mathcal{N}_3^{3,5,2} + d_3\mathcal{N}_4^{3,5,2} + (2d_4-2d_5)\mathcal{N}_5^{3,5,2}\\
&+ (d_2-d_3-2d_4)\mathcal{N}_6^{3,5,2} + 2d_5\mathcal{N}_2^{3,5,3}, && (2.3r)\\
d_2\mathcal{N}_4^{3,5,3} =\ & d_3\mathcal{N}_4^{3,5,2} - d_2\mathcal{N}_6^{3,5,2} - d_3\mathcal{N}_7^{3,5,2} + (-d_3-2d_4+2d_5)\mathcal{N}_8^{3,5,2}\\
&+ (-d_2+2d_5)\mathcal{N}_9^{3,5,2}, && (2.3s)\\
d_2\mathcal{N}_5^{3,5,3} =\ & - d_3\mathcal{N}_1^{3,5,2} + d_2\mathcal{N}_3^{3,5,2} + (2d_3+2d_4-2d_5)\mathcal{N}_5^{3,5,2}\\
&+ (2d_2-2d_5)\mathcal{N}_6^{3,5,2} + d_3\mathcal{N}_7^{3,5,2} + (2d_4-2d_5)\mathcal{N}_8^{3,5,2}\\
&+ (d_2-d_3-2d_4)\mathcal{N}_9^{3,5,2} + 2d_5\mathcal{N}_4^{3,5,3}, && (2.3t)\\
d_2\mathcal{N}_6^{3,5,3} =\ & d_3\mathcal{N}_4^{3,5,1} - d_2\mathcal{N}_6^{3,5,1} + d_3\mathcal{N}_1^{3,5,2} + (-2d_3-2d_4+2d_5)\mathcal{N}_2^{3,5,2}
\end{aligned}
$$

$$
\begin{aligned}
&+(-3d_2+2d_5)\mathcal{N}_3^{3,5,2}-d_3\mathcal{N}_4^{3,5,2}+(-2d_3-4d_4+4d_5)\mathcal{N}_5^{3,5,2}\\
&+(-3d_2+2d_3+2d_4+2d_5)\mathcal{N}_6^{3,5,2}-d_3\mathcal{N}_7^{3,5,2}\\
&+(-2d_4+2d_5)\mathcal{N}_8^{3,5,2}+(-d_2+2d_4)\mathcal{N}_9^{3,5,2}-2d_5\mathcal{N}_2^{3,5,3}\\
&+(-d_3-2d_4)\mathcal{N}_4^{3,5,3}+2d_5\mathcal{N}_5^{3,5,3}. \qquad (2.3u)
\end{aligned}
$$

Appendix C. First–order syzygies among the generators, four vector case

The following list of relations contains 38 syzygies of degree four in variables x_i, y_i between the generators of all the invariants in the four vector case. They are linearly dependent. A set of 36 linearly independent relations suggested by $N_{4,0}(\lambda)$ can be obtained by removing σ_{34} and σ_{38} of the list. Other choices are possible, for example σ_{16} and σ_{25} can be removed instead.

$$
\begin{array}{rclrcl}
\sigma_1 &=& t_{1,2}^2-r_1r_2+s_{1,2}^2, & \sigma_2 &=& t_{1,3}^2-r_1r_3+s_{1,3}^2,\\
\sigma_3 &=& t_{1,4}^2-r_1r_4+s_{1,4}^2, & \sigma_4 &=& t_{2,3}^2-r_2r_3+s_{2,3}^2,\\
\sigma_5 &=& t_{2,4}^2-r_2r_4+s_{2,4}^2, & \sigma_6 &=& t_{3,4}^2-r_3r_4+s_{3,4}^2,\\
\sigma_7 &=& t_{1,2}t_{1,3}-r_1s_{2,3}+s_{1,2}s_{1,3}, & \sigma_8 &=& t_{1,2}t_{2,3}-s_{1,2}s_{2,3}+r_2s_{1,3},\\
\sigma_9 &=& t_{1,2}t_{1,4}-r_1s_{2,4}+s_{1,2}s_{1,4}, & \sigma_{10} &=& t_{1,2}t_{2,4}-s_{1,2}s_{2,4}+r_2s_{1,4},\\
\sigma_{11} &=& t_{1,2}t_{3,4}-s_{1,3}s_{2,4}+s_{2,3}s_{1,4}, & \sigma_{12} &=& t_{1,3}t_{1,4}-r_1s_{3,4}+s_{1,3}s_{1,4},\\
\sigma_{13} &=& t_{1,3}t_{2,3}-r_3s_{1,2}+s_{1,3}s_{2,3}, & \sigma_{14} &=& t_{1,3}t_{2,4}-s_{1,2}s_{3,4}+s_{2,3}s_{1,4},\\
\sigma_{15} &=& t_{1,3}t_{3,4}-s_{1,3}s_{3,4}+r_3s_{1,4}, & \sigma_{16} &=& t_{1,4}t_{2,3}-s_{1,2}s_{3,4}+s_{2,4}s_{1,3},\\
\sigma_{17} &=& t_{1,4}t_{2,4}-r_4s_{1,2}+s_{1,4}s_{2,4}, & \sigma_{18} &=& t_{1,4}t_{3,4}-r_4s_{1,3}+s_{1,4}s_{3,4},\\
\sigma_{19} &=& t_{2,3}t_{2,4}-r_2s_{3,4}+s_{2,3}s_{2,4}, & \sigma_{20} &=& t_{2,3}t_{3,4}-s_{2,3}s_{3,4}+r_3s_{2,4},\\
\sigma_{21} &=& t_{2,4}t_{3,4}-r_4s_{2,3}+s_{2,4}s_{3,4}, & \sigma_{22} &=& s_{1,3}t_{1,2}+r_1t_{2,3}-s_{1,2}t_{1,3},\\
\sigma_{23} &=& s_{2,3}t_{1,2}+s_{1,2}t_{2,3}-r_2t_{1,3}, & \sigma_{24} &=& r_3t_{1,2}+s_{1,3}t_{2,3}-s_{2,3}t_{1,3},\\
\sigma_{25} &=& s_{3,4}t_{1,2}+s_{1,4}t_{2,3}-s_{2,4}t_{1,3}, & \sigma_{26} &=& s_{1,4}t_{1,2}+r_1t_{2,4}-s_{1,2}t_{1,4},\\
\sigma_{27} &=& s_{2,4}t_{1,2}+s_{1,2}t_{2,4}-r_2t_{1,4}, & \sigma_{28} &=& s_{3,4}t_{1,2}+s_{1,3}t_{2,4}-s_{2,3}t_{1,4},\\
\sigma_{29} &=& r_4t_{1,2}+s_{1,4}t_{2,4}-s_{2,4}t_{1,4}, & \sigma_{30} &=& s_{1,4}t_{1,3}+r_1t_{3,4}-s_{1,3}t_{1,4},\\
\sigma_{31} &=& s_{2,4}t_{1,3}+s_{1,2}t_{3,4}-s_{2,3}t_{1,4}, & \sigma_{32} &=& s_{3,4}t_{1,3}+s_{1,3}t_{3,4}-r_3t_{1,4},\\
\sigma_{33} &=& r_4t_{1,3}+s_{1,4}t_{3,4}-s_{3,4}t_{1,4}, & \sigma_{34} &=& s_{1,4}t_{2,3}+s_{1,2}t_{3,4}-s_{1,3}t_{2,4},\\
\sigma_{35} &=& s_{2,4}t_{2,3}+r_2t_{3,4}-s_{2,3}t_{2,4}, & \sigma_{36} &=& s_{3,4}t_{2,3}+s_{2,3}t_{3,4}-r_3t_{2,4},\\
\sigma_{37} &=& r_4t_{2,3}+s_{2,4}t_{3,4}-s_{3,4}t_{2,4}, & \sigma_{38} &=& t_{1,4}t_{2,3}-t_{1,3}t_{2,4}+t_{1,2}t_{3,4}.
\end{array}
$$

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