Integrable G-Strands on semisimple Lie groups

François Gay-Balmaz[1], Darryl D. Holm[2], and Tudor S. Ratiu[3]

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Abstract

The present paper derives systems of partial differential equations that admit a quadratic zero curvature representation for an arbitrary real semisimple Lie algebra. It also determines the general form of Hamilton's principles and Hamiltonians for these systems and analyzes the linear stability of their equilibrium solutions in the examples of $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$.

Contents

1 Introduction

The principal chiral model is a map $c : \mathbb{R}^{1,1} \to G$ from the Minkowski space-time $\mathbb{R}^{1,1}$ into a Lie group G. For its relation to harmonic maps into Lie groups $h : \mathbb{R}^2 \to G$, see [4, 18, 21]. Moreover, the principal chiral model is also an integrable system of

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partial differential equations (PDEs) whose doubly infinite sequence of conservation laws was found in [2] and whose soliton solutions can be generated using the *dressing method*, related to the classical Riemann-Hilbert problem in complex analysis. Both the conservation laws and the dressing method have been deduced from a *zero curvature representation* (ZCR) of the PDE system for the chiral model [4, 5, 7, 8, 19, 22, 23]. One of the key ideas behind the derivation of the conservation laws [2] is the use of Hamilton's principle.

Suppose G is a semisimple Lie group. Then the PDEs for the principal chiral model may be derived from Hamilton's principle $\delta S = 0$, with $S = \iint \ell(u, v) ds\, dt$, where the symmetry-reduced Lagrangian $\ell : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is defined using the ad-invariant Cartan-Killing form on the Lie algebra $\mathfrak{g}$ of the semisimple Lie group G. For semisimple Lie groups, the resulting PDEs may be expressed only in terms of Lie bracket operations, which in turn lead to the ZCR for these systems. The ZCR for the principal chiral model involves matrix differential operators with *rational* combinations of the spectral parameter that leads to the inverse scattering transform method and the dressing method for these PDEs.

It is also possible to obtain Hamiltonian structures and Lax equations for the principal chiral model generated by using matrix differential operators with polynomial dependence on a parameter [3]. For example, a recent paper [11] derives new types of integrable 4-wave equations via a Lie algebra approach for $\mathfrak{so}(5, \mathbb{C})$. These equations were derived directly from a zero curvature representation (ZCR) that is quadratic in a spectral parameters and were shown to admit solutions that may be constructed via the Riemann-Hilbert problem. Polynomial dependence of ZCRs on a parameter was also discussed in [10].

The present paper derives PDE systems that admit a quadratic ZCR for an arbitrary real semisimple Lie algebra and determines the general form of their Hamilton's principles and Hamiltonians. Several examples are given, including the cases of $\mathfrak{so}(3)$, $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{so}(4)$, and $\mathfrak{g}_2$. In all cases, the Hamiltonians are quadratic in the dependent variables, but sign-indefinite, so they admit instabilities. The effects of these instabilities on the full nonlinear solution behavior remains to be understood through numerical simulations.

Our approach is to derive the equations from Hamilton's principle for a Lagrangian defined on a Lie algebra $\mathfrak{g}$ that is invariant under transformations by the corresponding Lie group G. This approach produces G-Strand equations [14, 15]. A G-Strand is a map $g : \mathbb{R} \times \mathbb{R} \to G$ for a Lie group G that follows from Hamilton's principle for a certain class of G-invariant Lagrangians by using the Euler-Poincaré (EP) theory explained in [13]. The $SO(3)$-strand may be regarded physically as a continuous strand of rigid frames, as for a spin chain [13, 15]. The simplest example of the $SO(3)$-strand is the case in which the moment of inertia is proportional to the identity. This case yields the principal chiral model, which is known to admit a ZCR, although not of the quadratic type we study in this note [18]. Here, we use the EP theory to derive the G-Strand equations for any semisimple Lie algebra, in preparation for deriving the quadratic ZCR for such a system.

Plan of the paper. In section 2 we discuss the G-Strand equations for an arbitrary semisimple Lie algebra. In section 3 we discuss the chiral model for $SO(3)$. In section 4

we construct ZCRs that are quadratic in a spectral parameter for two classes of equations. These classes follow from either a normal real form in section 4.1, or a compact real form in section 4.2. These constructions impose additional relations among the dependent variables directly in the equations, in order that they admit the ZCR. This type of substitution of functional relations would not in general produce a Hamiltonian system. However, the expressions for the conserved energy in the cases derived from Hamilton's principle and EP theory provided clues for how to derive G-Strand PDEs that are manifestly Hamiltonian and admit the same ZCR. In particular, these equations possess an affine Lie–Poisson bracket defined on the dual of the corresponding Lie algebra $\mathfrak{g}$.

2 G-Strand equations

Lagrangian formulation. Let G be a real Lie group with Lie algebra $\mathfrak{g}$. The G-Strand equations are

$$\partial_t \frac{\delta\ell}{\delta\xi} - \mathrm{ad}^*_\xi \frac{\delta\ell}{\delta\xi} + \partial_s \frac{\delta\ell}{\delta\gamma} - \mathrm{ad}^*_\gamma \frac{\delta\ell}{\delta\gamma} = 0, \quad \partial_t\gamma - \partial_s\xi + [\xi,\gamma] = 0, \tag{2.1}$$

where $\ell : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ and $\xi, \eta : \mathbb{R} \times \mathbb{R} \to \mathfrak{g}$. This system of PDEs follows from Hamilton's principle

$$\delta \iint \ell(\xi(t,s), \gamma(t,s)) ds\, dt = 0$$

for the Lagrangian $\ell(\xi,\gamma)$ with $\xi = g^{-1}g_t$ and $\gamma = g^{-1}g_s$, for the map $g : \mathbb{R} \times \mathbb{R} \to G$. The first of these G-Strand equations depends on the choice of Lagrangian ℓ, while the second one does not. That is, the first one is dynamic and the second one is kinematic.

When $\mathfrak{g}$ admits a non-degenerate bi-invariant bilinear symmetric form, one can use it to identify the dual space $\mathfrak{g}^*$ with $\mathfrak{g}$. With this identification, we have $\mathrm{ad}^*_\xi = -\mathrm{ad}_\xi$ and the G-Strand equations (2.1) become in this case

$$\partial_t \frac{\delta\ell}{\delta\xi} + \mathrm{ad}_\xi \frac{\delta\ell}{\delta\xi} + \partial_s \frac{\delta\ell}{\delta\gamma} + \mathrm{ad}_\gamma \frac{\delta\ell}{\delta\gamma} = 0, \quad \partial_t\gamma - \partial_s\xi + [\xi,\gamma] = 0. \tag{2.2}$$

We will use the notations

$$\mu := \frac{\delta\ell}{\delta\xi}, \quad \pi := \frac{\delta\ell}{\delta\gamma},$$

so that the G-Strand equations (2.2) read

$$\partial_t\mu + [\xi,\mu] + \partial_s\pi + [\gamma,\pi] = 0, \quad \partial_t\gamma - \partial_s\xi + [\xi,\gamma] = 0. \tag{2.3}$$

Hamiltonian formulation. Let $\mathfrak{g}^*$ be the dual vector space to $\mathfrak{g}$ and $\langle\,,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the natural non-degenerate duality pairing. Define the Hamiltonian function $h : \mathcal{F}(\mathbb{R}, \mathfrak{g})^* \times \mathcal{F}(\mathbb{R}, \mathfrak{g}) \to \mathbb{R}$ by $h(\mu, \gamma) := \int (\langle\mu, \xi\rangle - \ell(\xi,\gamma))\, ds$, where $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$ is such that $\frac{\delta\ell}{\delta\xi} = \mu$. At this point, we assume that ℓ is non-degenerate in ξ which guarantees that $\frac{\delta\ell}{\delta\xi} = \mu$ has a unique solution $\xi(\mu)$.

We have the relations

$$\frac{\delta h}{\delta\mu} = \xi \quad \text{and} \quad \frac{\delta h}{\delta\gamma} = -\frac{\delta\ell}{\delta\gamma},$$

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so equations (2.2) read

$$\partial_t \mu + \operatorname{ad}_{\frac{\delta h}{\delta \mu}} \mu - \partial_s \frac{\delta h}{\delta \gamma} - \operatorname{ad}_\gamma \frac{\delta h}{\delta \gamma} = 0 \quad \text{and} \quad \partial_t \gamma - \partial_s \frac{\delta h}{\delta \mu} + \left[\frac{\delta h}{\delta \mu}, \gamma \right] = 0 \,. \tag{2.4}$$

They belong to the class of the affine Lie–Poisson equations [9]. Note that equations (2.4) do not require the existence of a Lagrangian ℓ. As we shall see later on, this is an important remark in the formulation of the general G-Strand equations.

From the physical viewpoint, the G-Strand equations (2.4) are exactly Hamilton's equations for a static ideal complex fluid whose broken symmetry is the semisimple Lie group G [12, 9, 6]. Thus, an additional motivation for studying the G-Strands is the hope that integrable systems may be found that apply for the description of nonlinear waves on the order parameter spaces of complex fluids in the absence of dissipation.

3 Chiral model

One of the best studied G-Strand equations is given by the chiral model which we now briefly recall.

For a complex Lie algebra $\mathfrak{g}^\mathbb{C}$, let $u, v : \mathbb{R}^2 \to \mathfrak{g}^\mathbb{C}$ be smooth functions satisfying the *principal chiral field equations* (see, e.g., [18])

$$\partial_y u - \frac{1}{2}[u, v] = 0, \qquad \partial_x v + \frac{1}{2}[u, v] = 0. \tag{3.1}$$

This system is equivalent to

$$\partial_y u - \partial_x v - [u, v] = 0, \qquad \partial_y u + \partial_x v = 0. \tag{3.2}$$

The first equation implies that locally there exist a smooth map $g : U \subset \mathbb{R}^2 \to G^\mathbb{C}$, U open and simply connected in $\mathbb{R}^2$, $G^\mathbb{C}$ the connected simply connected complex Lie group with Lie algebra $\mathfrak{g}^\mathbb{C}$, such that $u = g^{-1}(\partial_x g)$ and $v = g^{-1}(\partial_y g)$ (see [20, Chapter V, Theorem 2.4]) and hence the second equation is equivalent to

$$\partial_y \left(g^{-1}(\partial_x g) \right) + \partial_x \left(g^{-1}(\partial_y g) \right) = 0.$$

If $\mathfrak{g}^\mathbb{C}$ admits a non-degenerate bi-invariant symmetric bilinear form κ, these equations admit a G-Strand formulation (2.2) with $\ell(u, v) := \kappa(u, v)$.

The spacetime coordinates for the chiral model are given by $t = \frac{1}{2}(x-y)$, $s = \frac{1}{2}(x+y)$. Then, $\xi := g^{-1}(\partial_t g)$ and $\gamma := g^{-1}(\partial_x g)$ are expressed in terms of u and v as $\xi = u - v$ and $\gamma = u + v$. Equations (3.2) become

$$\partial_t \gamma - \partial_s \xi + [\xi, \gamma] = 0, \qquad \partial_s \gamma - \partial_t \xi = 0. \tag{3.3}$$

They are G-Strand equations for $\ell(\xi, \gamma) := \frac{1}{4} \left(\kappa(\gamma, \gamma) - \kappa(\xi, \xi) \right)$. As it is well known, equations (3.2) are equivalent to the zero curvature condition for the connection one-form $\omega(\lambda) = \frac{u}{1-\lambda}dx + \frac{v}{1+\lambda}dy \in \Omega^1(\mathbb{R}^2; \mathfrak{g})$, for all $\lambda \in \overline{\mathbb{C}}\backslash\{\pm 1\}$ (see [18]). Of course, the same can be said about the equations (3.3).

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Example: Chiral model for $G = SE(3)$. The real Lie algebra $\mathfrak{g} = \mathfrak{se}(3)$ of the special Euclidean group has the following non-degenerate symmetric bi-invariant form:

$$\langle(\mathbf{\Omega}_1, \mathbf{\Gamma}_1), (\mathbf{\Omega}_2, \mathbf{\Gamma}_2)\rangle := \mathbf{\Omega}_1 \cdot \mathbf{\Gamma}_2 + \mathbf{\Gamma}_1 \cdot \mathbf{\Omega}_2, \qquad \mathbf{\Omega}_1, \mathbf{\Gamma}_1, \mathbf{\Omega}_2, \mathbf{\Gamma}_2 \in \mathbb{R}^3.$$

So, relative to this pairing, we get

$$\mathrm{ad}^*_{(\mathbf{\Omega}, \mathbf{\Gamma})}(\mathbf{\Pi}, \mathbf{\Sigma}) = -\mathrm{ad}_{(\mathbf{\Omega}, \mathbf{\Gamma})}(\mathbf{\Pi}, \mathbf{\Sigma}) = -(\mathbf{\Omega} \times \mathbf{\Pi}, \mathbf{\Omega} \times \mathbf{\Sigma} - \mathbf{\Pi} \times \mathbf{\Gamma}).$$

The chiral equations (3.2) become

$$\begin{cases} \partial_y \mathbf{u}_1 - \partial_x \mathbf{v}_1 - \mathbf{u}_1 \times \mathbf{v}_1 = \mathbf{0} \\ \partial_y \mathbf{u}_2 - \partial_x \mathbf{v}_2 - \mathbf{u}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{u}_2 = \mathbf{0} \\ \partial_y \mathbf{u}_i + \partial_x \mathbf{v}_i = \mathbf{0}, \quad i = 1, 2. \end{cases}$$

They are G-Strand equations for the Lagrangian $\ell(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2) := \mathbf{u}_1 \cdot \mathbf{v}_2 + \mathbf{u}_2 \cdot \mathbf{v}_1$.

In spacetime coordinates (t, s), these equations become

$$\begin{cases} \partial_t \boldsymbol{\gamma}_1 - \partial_s \boldsymbol{\xi}_1 - \boldsymbol{\xi}_1 \times \boldsymbol{\gamma}_1 = \mathbf{0} \\ \partial_t \boldsymbol{\gamma}_2 - \partial_s \boldsymbol{\xi}_2 + \boldsymbol{\xi}_1 \times \boldsymbol{\gamma}_2 - \boldsymbol{\gamma}_1 \times \boldsymbol{\xi}_2 = \mathbf{0} \\ \partial_s \boldsymbol{\gamma}_i - \partial_t \boldsymbol{\xi}_i = \mathbf{0}, \quad i = 1, 2. \end{cases}$$

These are G-Strand equations for $\ell(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2) := \frac{1}{2}(\boldsymbol{\gamma}_1 \cdot \boldsymbol{\gamma}_2 - \boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2)$. ◆

4 Quadratic zero curvature representations

Let $\mathfrak{g}$ be a real Lie algebra with a bi-invariant bilinear symmetric non-degenerate form. We consider the Lie algebra valued functions on an open connected set $U \subset \mathbb{R}^2$ given by

$$L(t, s) := \lambda^2 a + \lambda \mu(t, s) - \gamma(t, s) \in \mathfrak{g}, \quad M(t, s) := \lambda^2 b - \lambda \pi(t, s) - \xi(t, s) \in \mathfrak{g},$$

where $a, b \in \mathfrak{g}$ are constant Lie algebra elements. The associated zero curvature representation (ZCR) is defined as [10, 11]

$$\partial_t L - \partial_s M + [L, M] = 0.$$

Imposing the ZCR yields the following system of relations

$$\begin{cases} \lambda^0 : \ \partial_t \gamma - \partial_s \xi + [\xi, \gamma] = 0 \\ \lambda^1 : \ \partial_t \mu + \partial_s \pi + [\xi, \mu] + [\gamma, \pi] = 0 \\ \lambda^2 : \ [a, \xi] + [\mu, \pi] + [\gamma, b] = 0 \\ \lambda^3 : \ [a, \pi] + [b, \mu] = 0 \\ \lambda^4 : \ [a, b] = 0. \end{cases} \tag{4.1}$$

Let us suppose that the Lie algebra $\mathfrak{g}$ is the normal real form of a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$; hence $\mathfrak{g}$ is split. The Killing form $\kappa : \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \to \mathbb{C}$ defined by

$$\kappa(x, y) := \mathrm{tr}(\mathrm{ad}_\xi \, \mathrm{ad}_\eta), \qquad \xi, \eta \in \mathfrak{g}^{\mathbb{C}},$$

is bilinear symmetric bi-invariant and non-degenerate. We shall use it to identify the dual space $\mathfrak{g}^*$ of the normal real form with $\mathfrak{g}$ itself. From bi-invariance, using this identification, we have $\mathrm{ad}_\xi^* = -\mathrm{ad}_\xi$, for all $\xi \in \mathfrak{g}$.

We recall some of the definitions and theorems from the theory of semisimple Lie algebras (see, e.g., [17]). Choose a Cartan subalgebra $\mathfrak{c}^\mathbb{C}$ of $\mathfrak{g}^\mathbb{C}$ and let

$$\mathfrak{g}^\mathbb{C} = \mathfrak{c}^\mathbb{C} \oplus \coprod_{\alpha \in \Delta_+} \left(\mathfrak{g}_\alpha^\mathbb{C} \oplus \mathfrak{g}_{-\alpha}^\mathbb{C} \right)$$

be the associated root space decomposition, where $\Delta \subset \left(\mathfrak{c}^\mathbb{C} \right)^*$ is the set of roots. Choose a base $\Pi := \{\alpha_1, \ldots, \alpha_r\}$ of Δ relative to which one defines the positive and negative roots Δ_+ and Δ_-. Recall that $\dim_\mathbb{C} \mathfrak{g}_{\pm\alpha}^\mathbb{C} = 1$. By non-degeneracy, each $\alpha \in \Delta$ defines a unique element $t_\alpha \in \mathfrak{c}^\mathbb{C}$ such that $\langle \alpha, \xi \rangle = \kappa(t_\alpha, \xi)$, for all $\xi \in \mathfrak{g}^\mathbb{C}$, which in turn induces a positive definite inner product, also denoted by κ, on $\mathrm{span}_\mathbb{R} \Delta$, i.e., $\kappa(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$ for all $\alpha, \beta \in \Delta$. Given $\alpha \in \Delta$, define $h_\alpha := \frac{2t_\alpha}{\kappa(\alpha,\alpha)}$ and let $h_i := h_{\alpha_i}$.

Many computations become easier if one chooses a Chevalley basis of $\mathfrak{g}^\mathbb{C}$. This is a vector space basis $\{h_i, e_\alpha \mid i = 1, \ldots, r, \; h_i \in \mathfrak{c}^\mathbb{C}, \; \alpha \in \Delta, \; e_\alpha \in \mathfrak{g}_\alpha\}$ of $\mathfrak{g}^\mathbb{C}$ satisfying ([17])

- $[e_\alpha, e_{-\alpha}] = h_\alpha$ for all $\alpha \in \Delta$;

- if $\alpha, \beta, \alpha + \beta \in \Delta$, then the constants $N_{\alpha,\beta} \in \mathbb{Z}$ defined by $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$, satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$.

In what follows we impose the convention $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \Delta$. Such Chevalley bases always exist.

Using the appropriate Chevalley basis, the equations for the G-Strand can be formulated explicitly for any semisimple Lie algebra.

4.1 Case 1: normal real form

Given a complex semisimple Lie algebra $\mathfrak{g}^\mathbb{C}$, its normal real form is defined in terms of a chosen Chevalley basis by $\mathfrak{g} := \mathrm{span}_\mathbb{R} \{h_i, e_\alpha \mid i = 1, \ldots, r, \; \alpha \in \Delta\}$, i.e., we have

$$\mathfrak{g} = \mathfrak{c} \oplus \coprod_{\alpha \in \Delta_+} \left(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \right),$$

where $\mathfrak{c} := \mathrm{span}_\mathbb{R} \{h_i \mid i = 1, \ldots, r\}$ and $\mathfrak{g}_\alpha = \mathbb{R} e_\alpha$.

Lagrangian formulation. Define the Lagrangian

$$\ell_{a,c,r}(\xi, \gamma) = -\frac{1}{2}\kappa \left(\varphi_{c,a}(\xi - r\gamma), \xi - r\gamma \right) - \kappa(c, \gamma), \quad r \in \mathbb{R}, \quad a, c \in \mathfrak{c}, \qquad (4.2)$$

where the *sectional operator* (see [8, Chapter 2]),

$$\varphi_{c,a} = \mathrm{ad}_c^{-1} \mathrm{ad}_a : \mathfrak{g} \to \mathfrak{n}_- \oplus \mathfrak{n}_+,$$

is given by

$$\varphi_{c,a}(\xi) = \sum_{\alpha \in \Delta_+} \frac{\langle \alpha, a \rangle}{\langle \alpha, c \rangle}(\xi_\alpha e_\alpha + \xi_{-\alpha} e_{-\alpha})$$

with c a regular semisimple element (i.e., $\langle \alpha, c \rangle \neq 0$ for all $\alpha \in \Delta$). The last three conditions in (4.1), that is,

$$[a, b] = 0, \quad \left[a, \frac{\delta \ell}{\delta \gamma} \right] + \left[b, \frac{\delta \ell}{\delta \xi} \right] = 0, \quad \text{and} \quad [a, \xi] + [\gamma, b] + \left[\frac{\delta \ell}{\delta \xi}, \frac{\delta \ell}{\delta \gamma} \right] = 0 \qquad (4.3)$$

are easily seen to hold. In order to satisfy the first equation, we choose $b = ra$. Using the expressions

$$\mu := \frac{\delta \ell}{\delta \xi} = -\varphi_{c,a}(\xi - r\gamma) \quad \text{and} \quad \pi := \frac{\delta \ell}{\delta \gamma} = r\varphi_{c,a}(\xi - r\gamma) - c$$

for the momenta, shows that the second and third equations in (4.3) are both verified.

The energy associated to the Lagrangian (4.2) is

$$e(\xi, \gamma) := \kappa \left(\frac{\delta \ell}{\delta \xi}, \xi \right) - \ell(\xi, \gamma) = -\frac{1}{2}\kappa \left(\varphi_{c,a}(\xi - r\gamma), \xi - r\gamma \right) - \kappa \left(\varphi_{c,a}(\xi - r\gamma), r\gamma \right) + \kappa(c, \gamma).$$

Note that the Lagrangian (4.2) is degenerate. This is easily seen by noting that the Legendre transformation $\xi \mapsto \mu := \frac{\delta \ell}{\delta \xi}$ is not surjective since the Cartan subalgebra is not contained in its range.

Equations. We can rewrite the G-Strand equations (2.3) in terms of the new variables $\zeta := \xi - r\gamma$ and γ as follows

$$\left(\partial_t - r\partial_s \right) \varphi_{c,a}(\zeta) + [\zeta, \varphi_{c,a}(\zeta)] + [\gamma, c] = 0, \qquad \left(\partial_t - r\partial_s \right) \gamma - \partial_s \zeta + [\zeta, \gamma] = 0. \quad (4.4)$$

Note that each term of the first equation is in $\mathfrak{n}_- \oplus \mathfrak{n}_+$, that the second equation in this system is independent of the chosen Lagrangian, and that it has non-trivial projections on the Cartan subalgebra $\mathfrak{c}$ and on $\mathfrak{n}_- \oplus \mathfrak{n}_+$. Change the spacetime coordinates (t, s) to $(\tau, \sigma) := (t, rt + s)$ and keep the same notations for the functions ζ and γ. Equations (4.4) become

$$\begin{bmatrix} \partial_\tau \varphi_{c,a} & 0 \\ -\partial_\sigma & \partial_\tau \end{bmatrix} \begin{bmatrix} \zeta \\ \gamma \end{bmatrix} + \begin{bmatrix} [\zeta, \varphi_{c,a}(\zeta)] + [\gamma, c] \\ [\zeta, \gamma] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad (4.5)$$

or

$$\partial_\tau \begin{bmatrix} \varphi_{c,a}(\zeta) \\ \gamma \end{bmatrix} + \partial_\sigma \begin{bmatrix} 0 \\ -\zeta \end{bmatrix} + \begin{bmatrix} [\zeta, \varphi_{c,a}(\zeta)] + [\gamma, c] \\ [\zeta, \gamma] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The unknowns in this system are (ζ, γ). However, the system above has only $2 \dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}$ equations. The missing $\operatorname{rank} \mathfrak{g}$ equations are due to the degeneracy of the Lagrangian (4.2).

Below, we shall define a Hamiltonian with the property that the associated G-Strand equations verify the zero curvature equations and the resulting system has $2 \dim \mathfrak{g}$ equations.

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Hamiltonian approach. Given $r \in \mathbb{R}$ and $a, c \in \mathfrak{c}$ with a regular (i.e., $\langle \alpha, a \rangle \neq 0$ for all $\alpha \in \Delta$), we consider the Hamiltonian

$$h(\mu, \gamma) = \int \left(-\frac{1}{2} \kappa(\varphi_{a,c}(\mu), \mu) + \kappa(r\mu + c, \gamma) \right) ds,$$

whose variational derivatives are

$$\xi = \frac{\delta h}{\delta \mu} = -\varphi_{a,c}(\mu) + r\gamma \quad \text{and} \quad \frac{\delta h}{\delta \gamma} = r\mu + c = -\pi.$$

The corresponding affine Lie–Poisson equations (2.4) are

$$\begin{cases} \partial_t \mu - r\partial_s \mu + [-\varphi_{a,c}(\mu) + r\gamma, \mu] - [\gamma, r\mu + c] = 0, \\ \partial_t \gamma - \partial_s(-\varphi_{a,c}(\mu) + r\gamma) + [-\varphi_{a,c}(\mu) + r\gamma, \gamma] = 0, \end{cases}$$

and these simplify to

$$\begin{cases} (\partial_t - r\partial_s)\mu - [\varphi_{a,c}(\mu), \mu] - [\gamma, c] = 0, \\ (\partial_t - r\partial_s)\gamma + \partial_s \varphi_{a,c}(\mu) - [\varphi_{a,c}(\mu), \gamma] = 0. \end{cases} \tag{4.6}$$

Upon setting $b = ra$, $\xi = -\varphi_{a,c}(\mu) + r\gamma$ and $\pi = -r\mu - c$, we see that the ZCR conditions (4.1) are satisfied. Note that this system has $2 \dim \mathfrak{g}$ equations.

Example: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. The standard Cartan subalgebra for $\mathfrak{sl}(2, \mathbb{R})$ is

$$\mathfrak{c} = \left\{ a = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \,\middle|\, \in \mathbb{R} \right\}$$

and the root vectors are

$$e_\alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{-\alpha} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

where the root $\alpha \in \mathfrak{c}^*$ is given by

$$\left\langle \alpha, \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \right\rangle = 2A.$$

Given $a, c \in \mathfrak{c}$ with $C \neq 0$, and writing an arbitrary Lie algebra element as

$$\xi = \begin{bmatrix} \xi_\mathfrak{c} & \xi_\alpha \\ \xi_{-\alpha} & -\xi_\mathfrak{c} \end{bmatrix}$$

we have

$$\varphi_{c,a} \begin{bmatrix} \xi_\mathfrak{c} & \xi_\alpha \\ \xi_{-\alpha} & -\xi_\mathfrak{c} \end{bmatrix} = \frac{A}{C} \begin{bmatrix} 0 & \xi_\alpha \\ \xi_{-\alpha} & 0 \end{bmatrix}, \quad \text{for all} \quad \xi \in \mathfrak{sl}(2, \mathbb{R}).$$

The Killing form is $\kappa(\xi, \eta) = 4 \operatorname{Tr}(\xi\eta)$. This yields the Lagrangian

$$\ell_{c,a,r}(\xi, \gamma) = -\frac{4A}{C}(\xi_\alpha - r\gamma_\alpha)(\xi_{-\alpha} - r\gamma_{-\alpha}) - 8C\gamma_\mathfrak{c}.$$

The first $SL(2, \mathbb{R})$-strand equation in (4.4) becomes

$$\frac{A}{C} \begin{bmatrix} 0 & \partial_t \zeta_\alpha \\ \partial_t \zeta_{-\alpha} & 0 \end{bmatrix} - \frac{rA}{C} \begin{bmatrix} 0 & \partial_s \zeta_\alpha \\ \partial_s \zeta_{-\alpha} & 0 \end{bmatrix} + 2\zeta_{\mathfrak{c}} \begin{bmatrix} 0 & \zeta_\alpha \\ -\zeta_{-\alpha} & 0 \end{bmatrix} + 2C \begin{bmatrix} 0 & -\gamma_\alpha \\ \gamma_{-\alpha} & 0 \end{bmatrix},$$

i.e.,

$$\begin{cases} \dfrac{A}{C}(\partial_t - r\partial_s)\zeta_\alpha + 2\zeta_{\mathfrak{c}}\zeta_\alpha - 2C\gamma_\alpha = 0 \\[2mm] \dfrac{A}{C}(\partial_t - r\partial_s)\zeta_{-\alpha} + 2\zeta_{\mathfrak{c}}\zeta_{-\alpha} + 2C\gamma_{-\alpha} = 0. \end{cases}$$

The second $SL(2, \mathbb{R})$-strand equation in (4.4) becomes

$$\begin{cases} (\partial_t - r\partial_s)\gamma_\alpha - \partial_s\zeta_\alpha + 2\left(\zeta_{\mathfrak{c}}\gamma_\alpha - \zeta_\alpha\gamma_{\mathfrak{c}}\right) = 0 \\[1mm] (\partial_t - r\partial_s)\gamma_{-\alpha} - \partial_s\zeta_{-\alpha} + 2\left(\zeta_{-\alpha}\gamma_{\mathfrak{c}} - \zeta_{\mathfrak{c}}\gamma_{-\alpha}\right) = 0 \\[1mm] (\partial_t - r\partial_s)\gamma_{\mathfrak{c}} - \partial_s\zeta_{\mathfrak{c}} + \zeta_\alpha\gamma_{-\alpha} - \gamma_\alpha\zeta_{-\alpha} = 0. \end{cases}$$

As noted in the general theory, the Lagrangian approach is degenerate and supplies only five equations for six unknowns: there is no equation for $\zeta_{\mathfrak{c}}$.

We now pass to the Hamiltonian approach. The equations (4.6) become in this case

$$\begin{cases} (\partial_t - r\partial_s)\mu_{\mathfrak{c}} = 0 \\[1mm] (\partial_t - r\partial_s)\mu_\alpha + \dfrac{2C}{A}\mu_{\mathfrak{c}}\mu_\alpha + 2C\gamma_\alpha = 0 \\[2mm] (\partial_t - r\partial_s)\mu_{-\alpha} - \dfrac{2C}{A}\mu_{\mathfrak{c}}\mu_{-\alpha} - 2C\gamma_{-\alpha} = 0 \end{cases}$$

and

$$\begin{cases} (\partial_t - r\partial_s)\gamma_{\mathfrak{c}} - \dfrac{C}{A}\left(\mu_\alpha\gamma_{-\alpha} - \gamma_\alpha\mu_{-\alpha}\right) = 0 \\[2mm] (\partial_t - r\partial_s)\gamma_\alpha + \dfrac{C}{A}\partial_s\mu_\alpha - \dfrac{2C}{A}\mu_\alpha\gamma_{\mathfrak{c}} = 0 \\[2mm] (\partial_t - r\partial_s)\gamma_{-\alpha} + \dfrac{C}{A}\partial_s\mu_{-\alpha} - \dfrac{2C}{A}\mu_{-\alpha}\gamma_{\mathfrak{c}} = 0. \qquad \blacklozenge \end{cases}$$

Example: $\mathfrak{g} = \mathfrak{g}_2(\mathbb{R})$. To illustrate the versatility of our approach, we compute the G-Strand equations for the real normal form $\mathfrak{g}_2(\mathbb{R})$ of the 14 dimensional exceptional complex Lie algebra $\mathfrak{g}_2(\mathbb{C})$. Using the standard root space decomposition in the construction of $\mathfrak{g}_2(\mathbb{C})$ from its Dynkin diagram (see, e.g., [17, §19.3]), the fact that $\mathfrak{g}_2(\mathbb{C})$ is a Lie subalgebra of $\mathfrak{so}(7, \mathbb{C})$, and the isomorphism realizing the elements of $\mathfrak{so}(7, \mathbb{C})$ as skew-symmetric matrices (see [1, Chapter VIII]), one finds that the general $\mathfrak{g}_2(\mathbb{C})$ matrix has the following expression

$$\begin{bmatrix} A^{\mathrm{skew}} - \frac{1}{2}\widehat{\mathbf{u}} & \mathrm{i}\mathbf{v} & -\mathrm{i}\left(A^{\mathrm{sym}} + \frac{1}{2}\widehat{\mathbf{v}}\right) \\ -\mathrm{i}\mathbf{v}^\mathsf{T} & 0 & -\mathbf{u}^\mathsf{T} \\ \mathrm{i}\left(A^{\mathrm{sym}} - \frac{1}{2}\widehat{\mathbf{v}}\right) & \mathbf{u} & A^{\mathrm{skew}} + \frac{1}{2}\widehat{\mathbf{u}} \end{bmatrix} \in \mathfrak{g}_2(\mathbb{C}) \subset \mathfrak{so}(7, \mathbb{C}), \qquad (4.7)$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{C}^3$, $A \in \mathfrak{sl}(3, \mathbb{C})$, $A^{\mathrm{skew}} := (A - A^\mathsf{T})/2$, and $A^{\mathrm{sym}} := (A + A^\mathsf{T})/2$. The Cartan subalgebra consists of matrices of this form with $A = \mathrm{diag}(a_1, a_2 - a_1, -a_2)$ and

$\mathbf{u} = \mathbf{v} = \mathbf{0}$. Note that if we have two elements of the form (4.7) corresponding to $(A_i, \mathbf{u}_i, \mathbf{v}_i)$, $i = 1, 2$, then the Lie bracket is of the form (4.7), where

$$
\begin{cases}
A = [A_1, A_2] + \dfrac{3}{4}\left(\mathbf{u}_2\mathbf{v}_1^\mathsf{T} + \mathbf{v}_1\mathbf{u}_2^\mathsf{T} - \mathbf{v}_2\mathbf{u}_1^\mathsf{T} - \mathbf{u}_1\mathbf{v}_2^\mathsf{T}\right) + \dfrac{3}{4}\widehat{\mathbf{u}_1 \times \mathbf{u}_2} - \dfrac{3}{4}\widehat{\mathbf{v}_1 \times \mathbf{v}_2} \\[2mm]
\qquad + \dfrac{1}{2}(\mathbf{v}_2 \cdot \mathbf{u}_1 - \mathbf{v}_1 \cdot \mathbf{u}_2)\mathbf{I}_3 \\[2mm]
\mathbf{u} = \mathbf{u}_1 \times \mathbf{u}_2 + \mathbf{v}_1 \times \mathbf{v}_2 - A_1^{\text{sym}}\mathbf{v}_2 + A_1^{\text{skew}}\mathbf{u}_2 + A_2^{\text{sym}}\mathbf{v}_1 - A_2^{\text{skew}}\mathbf{u}_1 \\[2mm]
\mathbf{v} = \mathbf{u}_2 \times \mathbf{v}_1 + \mathbf{v}_2 \times \mathbf{u}_1 + A_1^{\text{skew}}\mathbf{v}_2 - A_1^{\text{sym}}\mathbf{u}_2 - A_2^{\text{skew}}\mathbf{v}_1 + A_2^{\text{sym}}\mathbf{u}_1
\end{cases}
\tag{4.8}
$$

and the trace of their product equals

$$
-3\mathbf{u}_1 \cdot \mathbf{u}_2 + 3\mathbf{v}_1 \cdot \mathbf{v}_2 + 2\operatorname{Tr}(A_1 A_2). \tag{4.9}
$$

The associated real normal form is obtained from this expression by setting all matrices and vectors real (but the factors of $\mathrm{i} = \sqrt{-1}$ in (4.7) remain). If a, c are in the Cartan subalgebra and c is regular semisimple, the sectional operator has the following effect on an element of the form (4.7)

$$
A\,[a_{ij}] \longmapsto A_{c,a} := \begin{bmatrix} 0 & \dfrac{2a_1 - a_2}{2c_1 - c_2}a_{12} & \dfrac{a_1 + a_2}{c_1 + c_2}a_{13} \\[3mm] \dfrac{2a_1 - a_2}{2c_1 - c_2}a_{21} & 0 & \dfrac{2a_2 - a_1}{2c_2 - c_1}a_{23} \\[3mm] \dfrac{a_1 + a_2}{c_1 + c_2}a_{31} & \dfrac{2a_2 - a_1}{2c_2 - c_1}a_{32} & 0 \end{bmatrix}
$$

$$
\mathbf{u} \longmapsto \mathbf{u}_{c,a} := \begin{bmatrix} \dfrac{a_1}{c_1}u_1 \\[3mm] \dfrac{a_2 - a_1}{c_2 - c_1}u_2 \\[3mm] \dfrac{a_2}{c_2}u_3 \end{bmatrix}, \quad \mathbf{v} \longmapsto \mathbf{v}_{c,a} := \begin{bmatrix} \dfrac{a_1}{c_1}v_1 \\[3mm] \dfrac{a_2 - a_1}{c_2 - c_1}v_2 \\[3mm] \dfrac{a_2}{c_2}v_3 \end{bmatrix}.
$$

The denominators do not vanish precisely because c is a regular semisimple element. Since the Killing form κ for $\mathfrak{g}_2(\mathbb{R})$ is a constant multiple of the trace of the product of matrices of the form (4.7), we shall take this coefficient to be one.

The Lagrangian (4.2) becomes in this case

$$
\ell_{c,a,r}(\xi, \gamma) = \frac{3}{2}\left(\frac{a_1}{c_1}u_1^2 + \frac{a_2 - a_1}{c_2 - c_1}u_2^2 + \frac{a_2}{c_2}u_3^2\right) - \frac{3}{2}\left(\frac{a_1}{c_1}v_1^2 + \frac{a_2 - a_1}{c_2 - c_1}v_2^2 + \frac{a_2}{c_2}v_3^2\right)
$$
$$
- \frac{2a_1 - a_2}{2c_1 - c_2}a_{21}a_{12} - \frac{a_1 + a_2}{c_1 + c_2}a_{31}a_{13} - \frac{2a_2 - a_1}{2c_2 - c_1}a_{32}a_{23}
$$
$$
- 2\left(c_1 A_{11}^\gamma + (c_2 - c_1)A_{22}^\gamma - c_2 A_{33}^\gamma\right)
$$

where $\xi - r\gamma$ has the form (4.7) and the matrices A in (4.7) for γ and c are denoted by $[A_{ij}^\gamma]$ and $[A_{ij}^c] \in \mathfrak{sl}(3, \mathbb{R})$, respectively. Using (4.8) and noting that the last summand

of the first equation vanishes in this case, the first equation in (4.4) becomes

$$
(\partial_t - r\partial_s) A_{c,a} + [A, A_{c,a}] + \frac{3}{4}\left(\mathbf{u}_{c,a}\mathbf{v}^\mathsf{T} + \mathbf{v}\mathbf{u}_{c,a}^\mathsf{T} - \mathbf{v}_{c,a}\mathbf{u}^\mathsf{T} - \mathbf{u}\mathbf{v}_{c,a}^\mathsf{T}\right)
$$

$$
+ \frac{3}{4}\left(\widehat{\mathbf{u} \times \mathbf{u}_{c,a}} - \widehat{\mathbf{v} \times \mathbf{v}_{c,a}}\right) + [A^\gamma, A^c] = 0
$$

$$
(\partial_t - r\partial_s)\mathbf{u}_{c,a} + \mathbf{u} \times \mathbf{u}_{c,a} + \mathbf{v} \times \mathbf{v}_{c,a} - A^{\mathrm{sym}}\mathbf{v}_{c,a} + A^{\mathrm{skew}}\mathbf{u}_{c,a} + A^{\mathrm{sym}}_{c,a}\mathbf{v} - A^{\mathrm{skew}}_{c,a}\mathbf{u} = 0
$$

$$
(\partial_t - r\partial_s)\mathbf{v}_{c,a} + \mathbf{u}_{c,a} \times \mathbf{v} + \mathbf{v}_{c,a} \times \mathbf{u} + A^{\mathrm{skew}}\mathbf{v}_{c,a} - A^{\mathrm{sym}}\mathbf{u}_{c,a} - A^{\mathrm{skew}}_{c,a}\mathbf{v} + A^{\mathrm{sym}}_{c,a}\mathbf{u} = 0.
$$

The explicit form of the second equation in (4.4) can be easily obtained by decomposing each of the terms according to (4.7) and using the bracket formula (4.8).

As noted in the general theory, the Lagrangian approach to the G-Strand equations misses two equations. The Hamiltonian approach corrects this problem. The equations can be obtained, as above, by decomposing μ and γ according to (4.7) and using the expressions of the Lie bracket in (4.8).   ♦

4.2   Case 2: compact real form

Fix a Chevalley basis of $\mathfrak{g}^\mathbb{C}$. Define the *compact real form* of $\mathfrak{g}^\mathbb{C}$ by

$$
\mathfrak{l} := \left\{ \mathrm{i}\sum_{j=1}^r a_j h_j + \sum_{\alpha\in\Delta_+} x_\alpha(e_\alpha - e_{-\alpha}) + \mathrm{i}\sum_{\alpha\in\Delta_+} y_\alpha(e_\alpha + e_{-\alpha}) \,\middle|\, a_j, x_\alpha, y_\alpha \in \mathbb{R} \right\} \tag{4.10}
$$

$$
=: \mathrm{i}\mathfrak{c} \oplus \mathfrak{u} \oplus \mathfrak{v}.
$$

The Lie algebra $\mathfrak{l}$ is real and compact (i.e., the Killing form κ is negative definite on $\mathfrak{l}$), $\mathrm{i}\mathfrak{c}$ is its Cartan subalgebra, $\mathfrak{l}\otimes_\mathbb{R}\mathbb{C} = \mathfrak{g}^\mathbb{C}$, and $\mathfrak{l}$ is the fixed point set of the anticomplex involution $\sigma : \mathfrak{g}^\mathbb{C} \to \mathfrak{g}^\mathbb{C}$ given in the Chevalley basis by $\sigma(h_j) = -h_j$ and $\sigma(e_\alpha) = -e_{-\alpha}$ for all $j = 1,\ldots,r$, $\alpha\in\Delta$. For example, if $\mathfrak{g}^\mathbb{C} = \mathfrak{sl}(r+1,\mathbb{C})$, then $\sigma(\xi) = -(\bar{\xi})^\mathsf{T}$ for all $\xi \in \mathfrak{sl}(r+1,\mathbb{C})$, and $\mathfrak{l} = \mathfrak{su}(r+1)$.

Given $a \in \mathfrak{c}$ and writing $\xi = \sum_{\alpha\in\Delta_+}(x_\alpha u_\alpha + y_\alpha v_\alpha)$, we have the formulas

$$
\mathrm{ad}_{\mathrm{i}a}\,\xi = \sum_{\alpha\in\Delta_+}\langle\alpha, a\rangle(-y_\alpha u_\alpha + x_\alpha v_\alpha) \quad\text{and}\quad \mathrm{ad}_{\mathrm{i}a}^{-1}\,\xi = \sum_{\alpha\in\Delta_+}\frac{1}{\langle\alpha, a\rangle}(y_\alpha u_\alpha - x_\alpha v_\alpha),
$$

where, in the second equality, $a \in \mathfrak{c}$ is regular semisimple.

Given $a, b \in \mathfrak{c}$, we have

$$
\mathrm{ad}_{\mathrm{i}a}^{-1}\,\mathrm{ad}_{\mathrm{i}b}\left(\mathrm{i}h + \sum_{\alpha\in\Delta_+}(x_\alpha u_\alpha + y_\alpha v_\alpha)\right) = \sum_{\alpha\in\Delta_+}\frac{\langle\alpha, b\rangle}{\langle\alpha, a\rangle}(x_\alpha u_\alpha + y_\alpha v_\alpha)
$$

Let us define the Lagrangian

$$
\ell_{a,c,r}(\xi, \gamma) = -\frac{1}{2}\kappa\left(\varphi_{c,a}(\xi - r\gamma), \xi - r\gamma\right) - \kappa(\mathrm{i}c, \gamma), \quad r\in\mathbb{R}, \quad a, c \in \mathfrak{c},
$$

with the sectional operator (see [8, Chapter 2]),

$$
\varphi_{c,a} = \mathrm{ad}_{\mathrm{i}c}^{-1}\,\mathrm{ad}_{\mathrm{i}a} : \mathfrak{u}\oplus\mathfrak{v} \to \mathfrak{u}\oplus\mathfrak{v},
$$

where c is regular semisimple. Using the expressions

$$\mu = \frac{\delta\ell}{\delta\xi} = -\varphi_{c,a}(\xi - r\gamma), \quad \pi = \frac{\delta\ell}{\delta\gamma} = r\varphi_{c,a}(\xi - r\gamma) - \mathrm{i}c,$$

it is easy to check that ℓ verifies the conditions

$$[a, b] = 0, \quad \left[a, \frac{\delta\ell}{\delta\gamma}\right] + \left[b, \frac{\delta\ell}{\delta\xi}\right] = 0, \quad [a, \xi] + [\gamma, b] + \left[\frac{\delta\ell}{\delta\xi}, \frac{\delta\ell}{\delta\gamma}\right] = 0,$$

with $b = ra$.

Equations. As in the case of the real normal form, we can rewrite the equations (2.3) in terms of the new variables $\zeta := \xi - r\gamma$ and γ as follows

$$(\partial_t - r\partial_s)\,\varphi_{c,a}(\zeta) + [\zeta, \varphi_{c,a}(\zeta)] + [\gamma, \mathrm{i}c] = 0, \qquad (\partial_t - r\partial_s)\,\gamma - \partial_s\zeta + [\zeta, \gamma] = 0. \quad (4.11)$$

Note that each term of the first equation is in $\mathfrak{u} \oplus \mathfrak{v}$. Note that the second equation in this system is independent of the chosen Lagrangian but contains, as in the normal real form, components in the Cartan algebra.

Hamiltonian approach. Given $r \in \mathbb{R}$ and $\mathrm{i}a, \mathrm{i}c \in \mathfrak{c}$ with a regular, we consider the Hamiltonian

$$h(\mu, \gamma) = \int \left(-\frac{1}{2}\kappa(\varphi_{a,c}(\mu), \mu) + \kappa(r\mu + \mathrm{i}c, \gamma)\right) ds. \tag{4.12}$$

We have

$$\xi = \frac{\delta h}{\delta\mu} = -\varphi_{a,c}(\mu) + r\gamma \quad \text{and} \quad \frac{\delta h}{\delta\gamma} = r\mu + \mathrm{i}c.$$

The affine Lie–Poisson equations are

$$\begin{cases} \partial_t\mu - r\partial_s\mu + [-\varphi_{a,c}(\mu) + r\gamma, \mu] - [\gamma, r\mu + \mathrm{i}c] = 0, \\ \partial_t\gamma - \partial_s(-\varphi_{a,c}(\mu) + r\gamma) + [-\varphi_{a,c}(\mu) + r\gamma, \gamma] = 0, \end{cases}$$

and simplify to

$$\begin{cases} (\partial_t - r\partial_s)\mu - [\varphi_{a,c}(\mu), \mu] - [\gamma, \mathrm{i}c] = 0, \\ (\partial_t - r\partial_s)\gamma + \partial_s\varphi_{a,c}(\mu) - [\varphi_{a,c}(\mu), \gamma] = 0. \end{cases} \tag{4.13}$$

With $b = ra$, $\xi = -\varphi_{a,c}(\mu) + r\gamma$ and $\pi = -r\mu - \mathrm{i}c$, we see that (4.1) are satisfied.

Example: $\mathfrak{g} = \mathfrak{so}(3)$. The linear map $\mathbb{R}^3 \ni \mathbf{u} \longmapsto \widehat{\mathbf{u}} \in \mathfrak{so}(3)$, where $\widehat{\mathbf{u}}\mathbf{v} := \mathbf{u} \times \mathbf{v}$, for any $\mathbf{v} \in \mathbb{R}^3$, is a Lie algebra isomorphism between $(\mathbb{R}^3, \times)$ and $(\mathfrak{so}(3), [\,,])$, i.e., $[\widehat{\mathbf{u}}, \widehat{\mathbf{v}}] = \widehat{\mathbf{u} \times \mathbf{v}}$. Note that $\mathrm{tr}\,(\widehat{\mathbf{u}}\widehat{\mathbf{v}}) = -2\mathbf{u} \cdot \mathbf{v}$. Below we shall identify these two Lie algebras.

The Lie algebra $\mathfrak{so}(3) \cong \mathbb{R}^3$ is the compact real form of the complex simple Lie algebra $\mathfrak{so}(3, \mathbb{C}) \simeq \mathbb{C}^3$. Recall that the complex dimension of any Cartan subalgebra is one, i.e., $r = 1$ in the general theory. From the definition (4.10) of the compact real form of $\mathfrak{so}(3, \mathbb{C})$, once the real Cartan subalgebra $\mathfrak{c}$ is chosen, there exists a unique, up

to real scaling, non-zero $\mathbf{A} \in \mathbb{R}^3$, such that $i\mathfrak{c} = \text{span}_{\mathbb{R}} \mathbf{A}$. Therefore, any $\mathbf{a}, \mathbf{c} \in \mathfrak{c}$ are of the form $\mathbf{a} = ia\mathbf{A}$ and $\mathbf{c} = ic\mathbf{A}$, $a, c \in \mathbb{R}$. The associated sectional operator is hence

$$\varphi_{\mathbf{a},\mathbf{c}}(\boldsymbol{\zeta}) = \text{ad}_{i\mathbf{a}}^{-1} \text{ad}_{i\mathbf{c}}(\boldsymbol{\zeta}) = \frac{c}{a}\left(\boldsymbol{\zeta} - \left(\boldsymbol{\zeta} \cdot \frac{\mathbf{A}}{\|\mathbf{A}\|}\right)\frac{\mathbf{A}}{\|\mathbf{A}\|}\right) = \frac{c}{a}\boldsymbol{\zeta}_{\perp},$$

where $\boldsymbol{\zeta}_{\perp}$ is the projection of $\boldsymbol{\zeta}$ on the subspace $\mathbf{A}^{\perp}$. The Hamiltonian (4.12) becomes

$$\begin{aligned} h(\boldsymbol{\mu}, \boldsymbol{\gamma}) &= \int \left(\frac{c}{a}\boldsymbol{\mu}_{\perp} \cdot \boldsymbol{\mu} - 2(r\boldsymbol{\mu} + c\mathbf{A}) \cdot \boldsymbol{\gamma}\right) ds \\ &= \int \left(\frac{c}{a}\|\boldsymbol{\mu}_{\perp}\|^2 - 2r\boldsymbol{\mu} \cdot \boldsymbol{\gamma} - 2c\mathbf{A} \cdot \boldsymbol{\gamma}\right) ds \end{aligned} \tag{4.14}$$

and hence, relative to the Killing form (so minus twice the dot product)

$$\frac{\delta h}{\delta \boldsymbol{\mu}} = -\frac{c}{a}\boldsymbol{\mu}_{\perp} + r\boldsymbol{\gamma}, \qquad \frac{\delta h}{\delta \boldsymbol{\gamma}} = r\boldsymbol{\mu} + c\mathbf{A}.$$

The affine Lie–Poisson equations (4.13) are

$$(\partial_t - r\partial_s)\boldsymbol{\mu} - \frac{c}{a}\boldsymbol{\mu}_{\perp} \times \boldsymbol{\mu} - \boldsymbol{\gamma} \times c\mathbf{A} = 0 \quad \text{and} \quad (\partial_t - r\partial_s)\boldsymbol{\gamma} + \frac{c}{a}\partial_s\boldsymbol{\mu}_{\perp} - \frac{c}{a}\boldsymbol{\mu}_{\perp} \times \boldsymbol{\gamma} = 0. \tag{4.15}$$

Physical interpretation of ZCR $SO(3)$-strands. For $\mathfrak{so}(3)$, the quadratic ZCR G-Strand equations describe nonlinear unidirectional vector waves on a type of spin chain that is analogous to a strand of heavy tops strung along a filament with arc length s. The quadratic Hamiltonian (4.14) is not positive definite, which means that, in general, its equilibria will not be stable. In particular, it affords no control over either $\|\boldsymbol{\gamma}\|$ or $\|\boldsymbol{\mu}\|$.

An alternative form of the affine Lie–Poisson equations is

$$\frac{\partial}{\partial t}\begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\gamma} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}\times & \partial_s + \boldsymbol{\gamma}\times \\ \partial_s + \boldsymbol{\gamma}\times & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta \boldsymbol{\mu} \\ \delta h/\delta \boldsymbol{\gamma} \end{bmatrix}, \tag{4.16}$$

which makes their analogy with the heavy top very clear. Namely, we are dealing with heavy tops of angular frequency $\boldsymbol{\xi} = \frac{c}{a}\boldsymbol{\mu}_{\perp}$ at each location s, spread out along a filament and interacting with their neighbors through their gradients in s. Equations (4.15) give two unidirectional vector wave equations.

One may verify directly that these equations imply the following characteristic-derivative relations

$$\begin{cases} (\partial_t - r\partial_s)\left(\frac{c}{a}\|\boldsymbol{\mu}_{\perp}\|^2 - 2c\mathbf{A} \cdot \boldsymbol{\gamma}\right) = 0, \\ (\partial_t - r\partial_s)(\boldsymbol{\mu} \cdot \boldsymbol{\gamma}) + \frac{c}{2a}\partial_s\|\boldsymbol{\mu}_{\perp}\|^2 = 0, \\ (\partial_t - r\partial_s)(\boldsymbol{\mu} \cdot \mathbf{A}) = \frac{c}{a}(\boldsymbol{\mu}_{\perp} \times \boldsymbol{\mu}) \cdot \mathbf{A} = 0, \\ (\partial_t - r\partial_s)(\boldsymbol{\gamma} \cdot \mathbf{A}) = \frac{c}{a}(\boldsymbol{\mu}_{\perp} \times \boldsymbol{\gamma}) \cdot \mathbf{A} \neq 0, \\ (\partial_t - r\partial_s)\|\boldsymbol{\gamma}\|^2 = -\frac{2c}{a}\boldsymbol{\gamma}_{\perp} \cdot \partial_s\boldsymbol{\mu}_{\perp} \neq 0. \end{cases} \tag{4.17}$$

Consequently, we have three independent conservation laws

$$C_1 = \int \left(\frac{c}{a} \|\boldsymbol{\mu}_\perp\|^2 - 2c\mathbf{A} \cdot \boldsymbol{\gamma} \right) ds \,, \quad C_2 = \int \boldsymbol{\mu} \cdot \boldsymbol{\gamma} \, ds \,, \quad C_3 = \int \boldsymbol{\mu} \cdot \mathbf{A} \, ds \,. \qquad (4.18)$$

Using the expression of the affine Lie–Poisson Hamiltonian vector field in (4.16), it follows that $X_{C_2}(\boldsymbol{\mu}, \boldsymbol{\gamma}) = (\partial_s \boldsymbol{\mu}, \partial_s \boldsymbol{\gamma})$, so its flow is uniform translations in the space variable s. Similarly, since $X_{C_3}(\boldsymbol{\mu}, \boldsymbol{\gamma}) = (\boldsymbol{\mu} \times \mathbf{A}, \boldsymbol{\gamma} \times \mathbf{A})$, one concludes that its flow is rigid rotations about $\mathbf{A}$. A direct verification shows that C_1, C_2, C_3 are in involution under the affine Lie–Poisson bracket defined by the operator in the right hand side of (4.16). In addition, note that the Hamiltonian (4.14) can be expressed as $h = C_1 - 2rC_2$.

Linear instability for ZCR $SO(3)$- and $SL(2, \mathbb{R})$-strands. By the third equation in (4.17), the parallel component $\mu_\parallel := \mathbf{A} \cdot \boldsymbol{\mu}$ is conserved along traveling wave characteristics, $(\partial_t - r\partial_s)\mu_\parallel = 0$, so its linear dispersion relation is $\omega = -rk$. We write the equations (4.15) in the form

$$\left(\partial_t - r\partial_s - \frac{c}{a} \boldsymbol{\mu}_\perp \times \right) \boldsymbol{\mu} - c\boldsymbol{\gamma} \times \mathbf{A} = 0 \quad \text{and} \quad \left(\partial_t - r\partial_s - \frac{c}{a} \boldsymbol{\mu}_\perp \times \right) \boldsymbol{\gamma} + \frac{c}{a} \partial_s \boldsymbol{\mu}_\perp = 0.$$

Linearizing around the equilibrium solutions $\boldsymbol{\mu}_e = m\mathbf{A}$ and $\boldsymbol{\gamma}_e = n\mathbf{A}$ with constants m and n yields the dispersion relation

$$(\omega + rk)^2 \Big((\omega + rk)^4 - (m^2 - 2an)(\omega + rk)^2 - a^2(k^2 - n^2) \Big) = 0 \qquad (4.19)$$

from which we find the branches of the dispersion relations for the parallel and transverse components. The parallel components are stable, and the transverse components have a low wavenumber band of stable solutions for $(m^2 - 2an)^2 \geqslant 4a^2(n^2 - k^2) \geqslant 0$. It remains to determine how the solutions of the full nonlinear system will interact with the unstable transverse linear modes at higher wave numbers for this class of equilibrium solutions. However, we expect that nonlinearity cannot saturate the linear instability, because the conserved quadratic form provided by the Hamiltonian (4.14) is sign-indefinite, which precludes Lyapunov stability.

An analogous computation for the non-compact ZCR $SL(2, \mathbb{R})$-strand for the equilibria with $m = 0 = n$ removes the middle term in (4.19) and changes the sign of the remaining a^2k^2 term. In that case, only the parallel components of the linear modes can propagate stably and the linear instability is not controlled by nonlinear terms. Thus, one may conjecture that the equilibrium solutions of all G-Strand equations on semisimple Lie algebras that possess a quadratic ZCR will be linearly unstable, for Lie algebras with either compact or non-compact real forms.   ◆

Example: $\mathfrak{g} = \mathfrak{so}(4)$.   Implementing the Lie algebra isomorphism

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (\mathbf{x}, \mathbf{y}) \longmapsto \frac{1}{2} \begin{bmatrix} \widehat{\mathbf{x}} + \widehat{\mathbf{y}} & \mathbf{x} - \mathbf{y} \\ -(\mathbf{x} - \mathbf{y})^\mathsf{T} & 0 \end{bmatrix} \in \mathfrak{so}(4)$$

it immediately follows that the ZCR equations for $\mathfrak{so}(4)$ decouple into two ZCR equations for $SO(3)$.   ◆

5 Conclusion and outlook

Motivated partly by previous work on the zero curvature representation (ZCR) of completely integrable chiral models and partly by the underlying Hamiltonian structures of ideal complex fluids corresponding to (2.4), we have studied the class of G-Strand equations (2.2) on semisimple Lie algebras that admit a ZCR that is quadratic in the spectral parameter. The ZCR conditions impose functional relations among the variables which reduce the dimension of the dynamics, but still preserve the Hamiltonian structure of the original equations. The main results from this investigation are the following.

1. Using the root space decomposition, the equations for the G-Strand were formulated explicitly for any semisimple Lie algebra.

2. ZCRs were constructed that are quadratic in a spectral parameter for two classes of equations. These classes follow from either a normal real form discussed in section 4.1, or a compact real form discussed in section 4.2.

3. The equilibria for these equations were studied for $\mathfrak{so}(3)$ and $\mathfrak{sl}(2,\mathbb{R})$, and were found to be linearly unstable for both classes of equations at sufficiently high wave numbers. The effect of these instabilities on the full nonlinear solution behavior in physically reasonable situations remains to be understood through numerical simulations to be reported elsewhere.

As mentioned earlier, this work was partly motivated by the universal form of the affine Lie–Poisson bracket for ideal complex fluids [12, 9] which, for semisimple Lie groups, leads to a quadratic ZCR. However, the usual physical applications of the theory of complex fluids tend to be rather dissipative, so inertial (Hamiltonian) terms are usually neglected. We hope the present result that the equilibria for the underlying ideal theory are typically unstable for semisimple symmetry groups will stimulate additional investigations of the interactions between inertial and dissipative terms in complex fluid equations.

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