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Cascading gauge theory on dS_4 and String Theory Landscape

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Abstract

Placing anti-D3 branes at the tip of the conifold in Klebanov-Strassler geometry provides a generic way of constructing meta-stable de Sitter (dS) vacua in String Theory. A local geometry of such vacua exhibit gravitational solutions with a D3 charge measured at the tip opposite to the asymptotic charge. We discuss a restrictive set of such geometries, where anti-D3 branes are smeared at the tip. Such geometries represent holographic dual of cascading gauge theory in dS_4 with or without chiral symmetry breaking. We find that in the phase with unbroken chiral symmetry the D3 charge at the tip is always positive. Furthermore, this charge is zero in the phase with spontaneously broken chiral symmetry. We show that the effective potential of the chirally symmetric phase is lower than that in the symmetry broken phase, *i.e.,* there is no spontaneous chiral symmetry breaking for cascading gauge theory in dS_4. The positivity of the D3 brane charge in smooth de-Sitter deformed conifold geometries with fluxes presents difficulties in uplifting AdS vacua to dS ones in String Theory via smeared anti-D3 branes.

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1 Introduction and Summary

String Theory is expected to have a Landscape of (meta-stable) de-Sitter vacua [1]. A generic way to construct such vacua was presented in [2] (KKLT):

- first, turning on fluxes on Calabi-Yau compactifications of type IIB string theory produces highly warped geometry with stabilized complex structure (but not Kähler) moduli of the compactification [3];
- next, including non-perturbative effects (which are under control given the unbroken

supersymmetry), one obtains anti-de Sitter (AdS_4) vacua with *all* moduli fixed;

- finally, one uses anti-D3 branes of type IIB string theory to uplift AdS_4 to de Sitter (dS_4) vacua.

As the last step of the construction completely breaks supersymmetry, it is much less controlled. In fact, in [4–7] it was argued that putting anti-D3 branes at the tip of the Klebanov-Strassler (KS) [8] geometry (as done in KKLT construction) leads to a naked singularity. Whether or not the resulting singularity is physical is subject to debates[1]. In [10] it was shown that the singularity can not be cloaked by a regular event horizon, and thus must be unphysical [11]. This conclusion is reached analyzing local Klebanov-Tseytlin (KT) [12] or KS geometry with regular Schwarzschild horizon. Such geometry is dual to strongly coupled cascading gauge theory plasma with unbroken [13–17] (in KT case) or broken [18] (in KS case) chiral symmetry. It was shown that a D3-brane charge measured at the horizon is always positive, and thus can not cloak a physical negative-D3-charge singularity.

The *good* versus *bad* gravitational singularity criteria of Gubser [11] is based on a simple principle that singularities in gravitational backgrounds holographically dual to some strongly coupled gauge theories arise in the interior of the bulk space-time geometry, corresponding to the infrared (IR) in the dual gauge theories. Physical infrared singularities in gauge theories can be removed with an infrared cutoff. In the original paper, [11], this cutoff is provided by a temperature. However, the role of the cutoff can be served by a curvature scale of a boundary compactification manifold [19], or by a Hubble scale when the strongly coupled gauge theory is formulated in dS_4 [20]. In this paper we extend analysis of [10] considering[2] de Sitter deformation of the KT/KS geometries (holographically dual to cascading gauge theory in dS_4 with unbroken/broken chiral symmetry). As in [10], we ask the question whether it is possible to construct smooth geometries with a negative D3 charge in the interior of the space.

The analysis presented here closely follow [21]. In section 2 we review dual five-dimensional effective gravitational actions describing states of cascading gauge theory on $\mathcal{M}_4$ with (un-)broken chiral symmetry. In section 3 we construct states of cascading gauge theory in dS_4 with unbroken chiral symmetry. In section 4 we repeat the exercise for states of the theory with spontaneous broken chiral symmetry. In section 5 we

[1] See [9] for arguments in favour of this singularity.

[2] The early discussion of this problem was presented in [20].

compare effective potentials of the cascading gauge theory in dS_4 with broken and unbroken chiral symmetry and identify the true ground state of the theory. In section 6 we compute the D3 charge in the interior of the bulk of de Sitter deformed KT/KS geometries. Using results of [21], we compute the D3 charge in the interior of the bulk of S^3 deformed KT/KS geometries — in this last section we use the radius of the three-sphere ℓ_3 as an infrared cutoff to distinguish *good* versus *bad* gravitational singularities.

Our discussion is rather technical; so, for benefits of the readers who are interesting in results only, we collect them here. Recall that cascading gauge theory is a four-dimensional $\mathcal{N} = 1$ supersymmetric $SU(K+P) \times SU(K)$ gauge theory with two chiral superfields A_1, A_2 in the $(K+P, \overline{K})$ representation, and two fields B_1, B_2 in the $(\overline{K+P}, K)$. Perturbatively, this gauge theory has two gauge couplings g_1, g_2 associated with two gauge group factors, and a quartic superpotential

$$W \sim \mathrm{Tr}\,(A_i B_j A_k B_\ell)\,\epsilon^{ik}\epsilon^{j\ell}\,. \tag{1.1}$$

The theory has a global $SU(2) \times SU(2)$ (flavor) symmetry under which A_i and B_k (separately) transform as doublets. As this symmetry is always unbroken (both in the field theory and in the gravitational dual) all our conclusions concerning uplifting to de Sitter vacua with anti-D3 branes are strictly applicable when the anti-D3 branes are smeared on the tip of the conifold — it is only in this case that the dual gauge theory flavor symmetry is unbroken. To define a theory, one needs to specify the space-time four-manifold $\mathcal{M}_4$ in which the theory is formulated. In case when $\mathcal{M}_4 = R^{3,1}$, *i.e.,* Minkowski space-time, one finds that the sum of the gauge couplings does not run

$$\frac{d}{d\ln\mu}\left(\frac{\pi}{g_s} \equiv \frac{4\pi}{g_1^2(\mu)} + \frac{4\pi}{g_2^2(\mu)}\right) = 0\,, \tag{1.2}$$

while the difference between the two couplings is

$$\frac{4\pi}{g_2^2(\mu)} - \frac{4\pi}{g_1^2(\mu)} \sim P\ \left[3 + 2(1-\gamma_{ij})\right]\ \ln\frac{\mu}{\Lambda}\,, \tag{1.3}$$

where Λ is the strong coupling scale of the theory and γ_{ij} are anomalous dimensions[3] of operators $\mathrm{Tr}\,A_i B_j$. For generic $\mathcal{M}_4$, the sum of the gauge couplings runs; however, the theory is still determined by 2 parameters: the asymptotic value of the dilaton g_0,

$$g_0 \equiv \lim_{\mu\to\infty} g_s(\mu) = \lim_{\mu\to\infty}\left(\frac{4}{g_1^2(\mu)} + \frac{4}{g_2^2(\mu)}\right)^{-1}\,, \tag{1.4}$$

[3]When $K \gg P$, $\gamma_{ij} \approx -\frac{1}{2}$, see [8].

and the strong coupling scale Λ arising in the renormalization group running of the difference of two couplings (1.3). To summarize, cascading gauge theory is characterized by $\{P, g_0, \Lambda\}$ and the choice of a four-manifold $\mathcal{M}_4$. Relevant to the discussion here, when $\mathcal{M}_4 = dS_4$ or $R \times S^3$, the manifold provides one additional scale to the problem: the Hubble scale H (in case of dS_4) or the compactification scale ℓ_3^{-1} (in case of S^3 compactification). Depending on the ratio of the mass scale supplied by $\mathcal{M}_4$ and the strong coupling scale Λ, the cascading theory might undergo phase transition in the infrared associated with spontaneous breaking of the chiral symmetry[4] $\mathbb{Z}_{2P} \to \mathbb{Z}_2$. Ideally, we would like to explore the phase structure of the theory for arbitrary values of parameters — in practice, we are restricted to regions of parameter space where our numerical code used to generate $\mathcal{M}_4$ deformed KT/KS throat geometries is stable.

We now present the summary of our results:

- When $\mathcal{M}_4 = dS_4$ and the chiral symmetry is unbroken, the D3 brane charge at the tip of the conifold is always positive, as long as

$$\ln \frac{H^2}{\Lambda^2 P^2 g_0} \geq -0.4\,. \tag{1.5}$$

- When $\mathcal{M}_4 = dS_4$ and the chiral symmetry is broken, the D3 brane charge at the tip of the conifold is always zero; we managed to construct geometries of this type for

$$\ln \frac{H^2}{\Lambda^2 P^2 g_0} \geq -0.03\,. \tag{1.6}$$

- Comparing effective potential of the gauge theory in broken $\mathcal{V}^b_{eff}$ and unbroken $\mathcal{V}^s_{eff}$ phases we establish that in all cases, when we can construct the phase with spontaneously broken chiral symmetry,

$$\mathcal{V}^b_{eff} > \mathcal{V}^s_{eff}\,, \qquad \text{when} \qquad \ln \frac{H^2}{\Lambda^2 P^2 g_0} \geq -0.03\,, \tag{1.7}$$

i.e., spontaneous symmetry breaking does not happen for given values of the gauge theory parameters. To put these parameters in perspective, note that the (first-order) confinement/deconfinement and chiral symmetry breaking phase transition in cascading gauge theory plasma occurs at temperature T such that [16]

$$\ln \frac{T^2_{deconfinement,\chi\text{SB}}}{\Lambda^2 P^2 g_0} = 0.2571(2)\,, \tag{1.8}$$

[4]When $\mathcal{M}_4$ is Minkowski, the chiral symmetry is spontaneously broken, see [8].

and the (first-order) chiral symmetry breaking in cascading gauge theory on S^3 occurs for compactification scale $\mu_3 \equiv \ell_3^{-1}$ such that [21]

$$\ln \frac{\mu_{3,\chi\rm SB}^2}{\Lambda^2 P^2 g_0} = 0.4309(8)\,. \tag{1.9}$$

▪ When $\mathcal{M}_4 = R \times S^3$ and the chiral symmetry is unbroken, the D3 brane charge at the tip of the conifold is **negative** when

$$\ln \frac{\mu_3^2}{\Lambda^2 P^2 g_0} < \ln \frac{\mu_{3,negative}^2}{\Lambda^2 P^2 g_0} = 0.0318(3)\,. \tag{1.10}$$

However, since cascading gauge theory undergoes a first order phase transition with spontaneous breaking of the chiral symmetry at

$$\mu_{3,\chi\rm SB} > \mu_{3,negative}\,, \tag{1.11}$$

and the D3 brane charge at the tip of the conifold in broken phase is zero, the charge in the ground state is in fact zero whenever

$$\mu_3 \leq \mu_{3,\chi\rm SB}\,. \tag{1.12}$$

Furthermore, chirally symmetric states of cascading gauge theory on S^3 develop symmetry breaking tachyonic instabilities at $\mu_{3,tachyon}$ (below the first order chiral symmetry breaking scale $\mu_{3,\chi\rm SB}$)

$$\ln \frac{\mu_{3,tachyon}^2}{\Lambda^2 P^2 g_0} = 0.3297(3)\,. \tag{1.13}$$

which is again above $\mu_{3,negative}$.

Our results represented here, together with those reported in [10], point that the singularity of smeared anti-D3 branes at the tip of the conifold is unphysical: had it been otherwise, we should have been able to implement an infrared cutoff in the geometry with a D3 brane charge measured at the cutoff being negative. The role of the cutoff is played by the temperature (as discussed in [10]), by the compactification scale (when $\mathcal{M}_4 = R \times S^3$), or by the Hubble scale (when $\mathcal{M}_4 = dS_4$). Interesting, we find that the D3 brane charge can become negative when the KT throat geometry is S^3 deformed; however this occurs in the regime where this phase is unstable both via the first order phase transition and the tachyon condensation to S^3 deformed KS throat geometry — the latter geometry has zero D3 brane charge at the tip. All this raises questions about construction of generic de Sitter vacua in String Theory [2].

2 Dual effective actions of cascading gauge theory

Consider $SU(2) \times SU(2) \times \mathbb{Z}_2$ invariant states of cascading gauge theory on a 4-dimensional manifold $\mathcal{M}_4 \equiv \partial\mathcal{M}_5$. Effective gravitational action on a 5-dimensional manifold $\mathcal{M}_5$ describing holographic dual of such states was derived in [18]:

$$\begin{aligned} S_5\left[g_{\mu\nu}, \Omega_i, h_i, \Phi\right] =& \frac{108}{16\pi G_5}\int_{\mathcal{M}_5} \mathrm{vol}_{\mathcal{M}_5}\ \Omega_1\Omega_2^2\Omega_3^2 \left\{ R_{10} - \frac{1}{2}\left(\nabla\Phi\right)^2 \right. \\ & - \frac{1}{2}e^{-\Phi}\left(\frac{\left(h_1-h_3\right)^2}{2\Omega_1^2\Omega_2^2\Omega_3^2} + \frac{1}{\Omega_3^4}\left(\nabla h_1\right)^2 + \frac{1}{\Omega_2^4}\left(\nabla h_3\right)^2\right) \\ & - \frac{1}{2}e^{\Phi}\left(\frac{2}{\Omega_2^2\Omega_3^2}\left(\nabla h_2\right)^2 + \frac{1}{\Omega_1^2\Omega_2^4}\left(h_2 - \frac{P}{9}\right)^2 + \frac{1}{\Omega_1^2\Omega_3^4}h_2^2\right) \\ & \left. - \frac{1}{2\Omega_1^2\Omega_2^4\Omega_3^4}\left(4\Omega_0 + h_2\left(h_3-h_1\right) + \frac{1}{9}Ph_1\right)^2 \right\}, \end{aligned} \tag{2.1}$$

where Ω_0 is a constant, R_{10} is given by

$$\begin{aligned} R_{10} = R_5 + & \left(\frac{1}{2\Omega_1^2} + \frac{2}{\Omega_2^2} + \frac{2}{\Omega_3^2} - \frac{\Omega_2^2}{4\Omega_1^2\Omega_3^2} - \frac{\Omega_3^2}{4\Omega_1^2\Omega_2^2} - \frac{\Omega_1^2}{\Omega_2^2\Omega_3^2}\right) - 2\Box\ln\left(\Omega_1\Omega_2^2\Omega_3^2\right) \\ & - \left\{\left(\nabla\ln\Omega_1\right)^2 + 2\left(\nabla\ln\Omega_2\right)^2 + 2\left(\nabla\ln\Omega_3\right)^2 + \left(\nabla\ln\left(\Omega_1\Omega_2^2\Omega_3^2\right)\right)^2\right\}, \end{aligned} \tag{2.2}$$

and R_5 is the five-dimensional Ricci scalar of the metric

$$ds_5^2 = g_{\mu\nu}(y)dy^\mu dy^\nu\,, \tag{2.3}$$

that forms part of the ten dimensional full metric

$$ds_{10}^2 = ds_5^2 + ds_{T^{1,1}}^2\,, \qquad ds_{T^{1,1}}^2 = \Omega_1^2(y)g_5^2 + \Omega_2^2(y)(g_3^2+g_4^2) + \Omega_3^2(y)(g_1^2+g_2^2). \tag{2.4}$$

One-forms $\{g_i\}$ (for $i = 1, \cdots, 5$) are the usual forms defined in the warp-squashed $T^{1,1}$ and are given as in [18], for coordinates $0 \le \psi \le 4\pi$, $0 \le \theta_a \le \pi$ and $0 \le \phi_a \le 2\pi$ $(a = 1, 2)$.

All the covariant derivatives ∇_λ are with respect to the metric (2.3). Fluxes (and dilaton Φ) are parametrized in such a way that functions $h_1(y), h_2(y), h_3(y)$ appear as

$$\begin{aligned} B_2 =& h_1(y)g_1\wedge g_2 + h_3(y)g_3\wedge g_4, \\ F_3 =& \frac{1}{9}P\ g_5\wedge g_3\wedge g_4 + h_2(y)\ \left(g_1\wedge g_2 - g_3\wedge g_4\right)\wedge g_5 \\ & + \left(g_1\wedge g_3 + g_2\wedge g_4\right)\wedge d\left(h_2(y)\right)\,, \\ \Phi =& \Phi(y), \end{aligned} \tag{2.5}$$

where P corresponds to the number of fractional branes in the conifold.

Finally, G_5 is the five dimensional effective gravitational constant

$$G_5 \equiv \frac{729}{4\pi^3} G_{10}\,, \tag{2.6}$$

where G_{10} is a 10-dimensional gravitational constant of type IIB supergravity.

Chirally symmetric states of the cascading gauge theory are described by the gravitational configurations of (2.1) subject to constraints

$$h_1 = h_3\,, \qquad h_2 = \frac{P}{18}\,, \qquad \Omega_2 = \Omega_3\,. \tag{2.7}$$

In what follows, we find it convenient to introduce

$$\begin{aligned} h_1 =& \frac{1}{P}\left(\frac{K_1}{12} - 36\Omega_0\right)\,, \qquad h_2 = \frac{P}{18}\,K_2\,, \qquad h_3 = \frac{1}{P}\left(\frac{K_3}{12} - 36\Omega_0\right)\,, \\ \Omega_1 =& \frac{1}{3} f_c^{1/2} h^{1/4}\,, \qquad \Omega_2 = \frac{1}{\sqrt{6}} f_a^{1/2} h^{1/4}\,, \qquad \Omega_3 = \frac{1}{\sqrt{6}} f_b^{1/2} h^{1/4}\,, \end{aligned} \tag{2.8}$$

3 Chirally symmetric phase of cascading gauge theory on dS_4

We consider here $SU(2) \times SU(2) \times U(1) \times SO(4)$ (chirally-symmetric) states of the strongly coupled cascading gauge theory. We find it convenient to use a radial coordinate introduced in [23]:

$$ds_5^2 = g_{\mu\nu}(y) dy^\mu dy^\nu = h^{-1/2}\rho^{-2}\left(-dt^2 + \frac{1}{H^2}\cosh^2(Ht)\left(dS^3\right)^2\right) + h^{1/2}\rho^{-2}\,(d\rho)^2\,, \tag{3.1}$$

where $h = h(\rho)$. Furthermore, we use parametrization (2.8) and denote[5]

$$f_c = f_2\,, \qquad f_a = f_b = f_3\,, \qquad K_1 = K_3 = K\,, \qquad \Phi = \ln g\,, \tag{3.2}$$

with $f_i = f_i(\rho)$, and $K = K(\rho)$, $g = g(\rho)$.

Notice that parametrization (3.1) is not unique — the diffeomorphisms of the type

$$\begin{pmatrix} \rho \\ h \\ f_2 \\ f_3 \\ K \\ g \end{pmatrix} \Longrightarrow \begin{pmatrix} \hat\rho \\ \hat h \\ \hat f_2 \\ \hat f_3 \\ \hat K \\ \hat g \end{pmatrix} = \begin{pmatrix} \rho/(1+\alpha\,\rho) \\ (1+\alpha\,\rho)^4\,h \\ (1+\alpha\,\rho)^{-2}\,f_2 \\ (1+\alpha\,\rho)^{-2}\,f_3 \\ K \\ g \end{pmatrix}\,, \qquad \alpha = \text{const}\,, \tag{3.3}$$

[5]Recall that for the unbroken chiral symmetry we must set $K_2(\rho) \equiv 1$.

preserve the general form of the metric. We can completely fix (3.3), *i.e.,* parameter α in (3.3), requiring that for a geodesically complete $\mathcal{M}_5$ the radial coordinate ρ extends as

$$\rho \in [0, +\infty)\,. \tag{3.4}$$

3.1 Equations of motion

For a background ansatz (3.1), (3.2), the equations of motion obtained from (2.1) take form

$$\begin{aligned} 0 =& f_2'' + \frac{f_2(g')^2}{8g^2} - \frac{3f_2(K')^2}{16hf_3^2gP^2} + \frac{f_2(h')^2}{8h^2} - \frac{3f_2(f_3')^2}{4f_3^2} - \frac{f_2'^2}{2f_2} + \frac{f_2h'}{h\rho} + \left(\frac{3f_3'}{2f_3} - \frac{3}{\rho}\right) f_2' \\ &+ \frac{3gP^2}{4hf_3^2\rho^2} - \frac{K^2}{8h^2f_3^4\rho^2} + \frac{f_2(5f_3^2 - 9f_2 + 6f_3)}{f_3^2\rho^2} - 3hf_2H^2\,, \end{aligned} \tag{3.5}$$

$$\begin{aligned} 0 =& f_3'' + \frac{(K')^2}{16hf_3gP^2} + \frac{f_3(g')^2}{8g^2} + \frac{f_3(h')^2}{8h^2} + \frac{(f_3')^2}{4f_3} - \frac{3f_3'}{\rho} + \frac{f_3h'}{h\rho} - \frac{gP^2}{4f_2hf_3\rho^2} \\ &- \frac{K^2}{8f_2h^2f_3^3\rho^2} + \frac{5f_3^2 - 6f_3 + 3f_2}{f_3\rho^2} - 3hf_3H^2\,, \end{aligned} \tag{3.6}$$

$$\begin{aligned} 0 =& h'' + \frac{3(K')^2}{16gf_3^2P^2} - \frac{h(g')^2}{8g^2} - \frac{9(h')^2}{8h} + \frac{3h(f_3')^2}{4f_3^2} + \left(\frac{2f_3'}{f_3} + \frac{f_2'}{2f_2} - \frac{4}{\rho}\right) h' + \frac{hf_2'}{\rho f_2} \\ &+ \frac{9K^2}{8f_2hf_3^4\rho^2} + \left(\frac{4h}{f_3\rho} + \frac{hf_2'}{2f_3f_2}\right) f_3' + \frac{5gP^2}{4f_2f_3^2\rho^2} + \frac{h(f_2 - 13f_3^2 - 6f_3)}{f_3^2\rho^2} + 9h^2H^2\,, \end{aligned} \tag{3.7}$$

$$0 = K'' + \left(\frac{f_2'}{2f_2} - \frac{g'}{g} - \frac{h'}{h} - \frac{3}{\rho}\right) K' - \frac{2gKP^2}{hf_2f_3^2\rho^2}\,, \tag{3.8}$$

$$0 = g'' - \frac{(g')^2}{g} + \left(\frac{2f_3'}{f_3} + \frac{f_2'}{2f_2} - \frac{3}{\rho}\right) g' + \frac{(K')^2}{4hf_3^2P^2} - \frac{g^2P^2}{hf_2f_3^2\rho^2}\,. \tag{3.9}$$

Additionally we have the first order constraint

$$\begin{aligned} 0 =& (K')^2 + \frac{2hf_3^2P^2(g')^2}{g} + \frac{2f_3^2P^2g(h')^2}{h} - 12hP^2g(f_3')^2 - \frac{8f_3hgP^2(f_3'\rho - 2f_3)}{f_2\rho} f_2' \\ &+ \frac{16f_3gP^2(4f_3'h + f_3h')}{\rho} + \left(96hf_3 - 48hf_3^2 - 16hf_2 - \frac{4P^2g}{f_2} - \frac{2K^2}{hf_2f_3^2}\right) \frac{gP^2}{\rho^2} \\ &+ 48gP^2h^2f_3^2H^2\,. \end{aligned} \tag{3.10}$$

We explicitly verified that the constraint (3.10) is consistent with (3.5)-(3.9).

3.2 UV asymptotics

The general UV (as $\rho \to 0$) asymptotic solution of (3.5)-(3.10) describing the symmetric phase of cascading gauge theory takes form

$$\begin{aligned} f_2 =& 1 - \alpha_{1,0}\ (H\rho) + \left(-\frac{3}{8}P^2 g_0 - \frac{1}{4}K_0 + \frac{1}{4}(\alpha_{1,0})^2 + \frac{1}{2}P^2 g_0 \ln\rho \right)\ (H\rho)^2 \\ &+ \sum_{n=3}^{\infty}\sum_{k} a_{n,k}\ (H\rho)^n \ln^k\rho\,, \end{aligned} \tag{3.11}$$

$$\begin{aligned} f_3 =& 1 - \alpha_{1,0}\ (H\rho) + \left(-\frac{1}{2}P^2 g_0 - \frac{1}{4}K_0 + \frac{1}{4}(\alpha_{1,0})^2 + \frac{1}{2}P^2 g_0 \ln\rho \right)\ (H\rho)^2 \\ &+ \sum_{n=3}^{\infty}\sum_{k} b_{n,k}\ (H\rho)^n \ln^k\rho\,, \end{aligned} \tag{3.12}$$

$$\begin{aligned} h =& \frac{1}{8}P^2 g_0 + \frac{1}{4}K_0 - \frac{1}{2}P^2 g_0 \ln\rho + \alpha_{1,0}\left(\frac{1}{2}K_0 - P^2 g_0 \ln\rho\right)\ (H\rho) + \Bigg(\frac{119}{576}P^4 g_0^2 \\ &+ \frac{31}{96}K_0 P^2 g_0 - \frac{1}{4}P^2 g_0 \alpha_{1,0}^2 + \frac{1}{8}K_0^2 + \frac{5}{8}\alpha_{1,0}^2 K_0 - \frac{1}{96}P^2 g_0 (62 P^2 g_0 + 120\alpha_{1,0}^2 \\ &+ 48 K_0)\ln\rho + \frac{1}{2}P^4 g_0^2 \ln^2\rho\Bigg)\ (H\rho)^2 + \sum_{n=3}^{\infty}\sum_{k} h_{n,k}\ (H\rho)^n \ln^k\rho\,, \end{aligned} \tag{3.13}$$

$$\begin{aligned} K =& K_0 - 2P^2 g_0 \ln\rho - P^2 g_0 \alpha_{1,0}\ (H\rho) + \left(\frac{1}{16}P^2 g_0 (2K_0 + 9P^2 g_0 - 4\alpha_{1,0}^2)\right. \\ &\left. - \frac{1}{4}P^4 g_0^2 \ln\rho\right)\ (H\rho)^2 + \sum_{n=3}^{\infty}\sum_{k} K_{n,k}\ (H\rho)^n \ln^k\rho\,, \end{aligned} \tag{3.14}$$

$$g = g_0\left(1 - \frac{1}{2}P^2 g_0\ (H\rho)^2 + \sum_{n=3}^{\infty}\sum_{k} g_{n,k}\ (H\rho)^n \ln^k\rho\right)\,. \tag{3.15}$$

It is characterized by 7 parameters:

$$\{K_0\,,\ H\,,\ g_0\,,\ \alpha_{1,0}\,,\ a_{4,0}\,,\ a_{6,0}\,,\ a_{8,0}\,,\ g_{4,0}\}\,. \tag{3.16}$$

In what follows we developed the UV expansion to order $\mathcal{O}(\rho^{12})$ inclusive.

3.3 IR asymptotics

We use a radial coordinate ρ that extends to infinity, see (3.4). Introducing

$$y \equiv \frac{1}{\rho}\,, \qquad h^h \equiv y^{-2}\ h\,, \qquad f^h_{2,3} \equiv y\ f_{2,3}\,, \tag{3.17}$$

the general IR (as $y \to 0$) asymptotic solution of (3.5)-(3.10) describing the symmetric phase of cascading gauge theory takes form

$$\begin{aligned} f_2^h =& f_{2,0}^h - \frac{9H^2P^2(f_{3,0}^h)^2 g_0^h + 6H^4(K_0^h)^2 - 17(f_{2,0}^h)^2(f_{3,0}^h)^2 + 6f_{2,0}^h(f_{3,0}^h)^3}{5(f_{3,0}^h)^4}\, y \\ &+ \sum_{n=2} f_{2,n}^h y^n \,, \end{aligned} \tag{3.18}$$

$$\begin{aligned} f_3^h =& f_{3,0}^h - \frac{H^2P^2(f_{3,0}^h)^2 g_0^h + 6H^4(K_0^h)^2 + 7(f_{2,0}^h)^2(f_{3,0}^h)^2 - 18f_{2,0}^h(f_{3,0}^h)^3}{5f_{2,0}^h(f_{3,0}^h)^3}\, y \\ &+ \sum_{n=2} f_{3,n}^h y^n \,, \end{aligned} \tag{3.19}$$

$$\begin{aligned} h^h =& \frac{1}{4H^2}\Bigg(1 - \frac{2}{5}\frac{(3H^2P^2(f_{3,0}^h)^2 g_0^h + 10H^4(K_0^h)^2 + (f_{2,0}^h)^2(f_{3,0}^h)^2 - 6f_{2,0}^h(f_{3,0}^h)^3}{(f_{3,0}^h)^4 f_{2,0}^h}\, y \\ &+ \sum_{n=2} h_n^h y^n\Bigg) \,, \end{aligned} \tag{3.20}$$

$$K = K_0^h + \frac{16K_0^h g_0^h P^2 H^2}{5(f_{3,0}^h)^2 f_{2,0}^h}\, y + \sum_{n=2} K_n^h y^n \,, \tag{3.21}$$

$$g = g_0^h \left(1 + \frac{8g_0^h P^2 H^2}{5(f_{3,0}^h)^2 f_{2,0}^h}\, y + \sum_{n=2} g_n^h y^n\right) . \tag{3.22}$$

It is characterized by 4 additional parameters:

$$\{K_0^h \,,\ g_0^h \,,\ f_{2,0}^h \,,\ f_{3,0}^h\} \,. \tag{3.23}$$

In what follows we developed the IR expansion to order $\mathcal{O}(y^6)$ inclusive.

3.4 Symmetries

The background geometry (3.1), (3.2) enjoys 4 distinct scaling symmetries. We now discuss these symmetries and exhibit their action on the asymptotic parameters (3.16).

- First, we have:

$$P \to \lambda\, P \,,\ g \to \frac{1}{\lambda}\, g \,, \qquad \{\rho, f_i, h, K\} \to \{\rho, f_i, h, K\} \,, \qquad \{y, f_i^h, h^h\} \to \{y, f_i, h^h\} \,, \tag{3.24}$$

which acts on the asymptotic parameters as

$$\begin{aligned} & g_0 \to \frac{1}{\lambda}\, g_0 \,, \\ & \{K_0 \,, H \,, \alpha_{1,0} \,, a_{4,0} \,, a_{6,0} \,, a_{8,0} \,, g_{4,0}\} \to \{K_0 \,, H \,, \alpha_{1,0} \,, a_{4,0} \,, a_{6,0} \,, a_{8,0} \,, g_{4,0}\} \,, \end{aligned} \tag{3.25}$$

and

$$\{K_0^h\,,\ g_0^h\,,\ f_{2,0}^h\,,\ f_{3,0}^h\} \to \{K_0^h\,,\ \lambda^{-1}g_0^h\,,\ f_{2,0}^h\,,\ f_{3,0}^h\}\,. \tag{3.26}$$

We can use the exact symmetry (3.24) to set

$$g_0 = 1\,. \tag{3.27}$$

- Second, we have:

$$\begin{aligned} &P \to \lambda\ P\,,\ \rho \to \frac{1}{\lambda}\ \rho\,,\ h \to \lambda^2\ h\,,\ K \to \lambda^2 K\,, \qquad \{H, f_i, g\} \to \{H, f_i, g\}\,, \\ &\{y, f_2^h, f_3^h, h^h\} \to \{\lambda y, \lambda f_2^h, \lambda f_3^h, h^h\}\,, \end{aligned} \tag{3.28}$$

which acts on the asymptotic parameters as

$$g_0 \to g_0\,, \tag{3.29}$$

$$\alpha_{1,0} \to \lambda\alpha_{1,0}\,, \tag{3.30}$$

$$K_0 \to \lambda^2\bigg(K_0 - 2P^2 g_0\ \ln\lambda\bigg)\,, \tag{3.31}$$

$$a_{4,0} \to \lambda^4\bigg(a_{4,0} + \frac{1}{48}P^2 g_0(3K_0 - P^2 g_0)\ \ln\lambda - \frac{1}{16}P^4 g_0^2\ \ln^2\lambda\bigg)\,, \tag{3.32}$$

$$\begin{aligned} g_{4,0} \to \lambda^4\bigg(&g_{4,0} + \bigg(-\frac{3}{16}P^2\alpha_{1,0}^2 g_0 - \frac{5}{64}K_0 P^2 g_0 + \frac{37}{96}P^4 g_0^2 + 3a_{4,0}\bigg)\ \ln\lambda \\ &+ \frac{3}{64}P^2 g_0(P^2 g_0 + 2K_0)\ \ln^2\lambda - \frac{1}{16}P^4 g_0^2\ \ln^3\lambda\bigg)\,, \end{aligned} \tag{3.33}$$

$$\begin{aligned} a_{6,0} \to \lambda^6\bigg(&a_{6,0} + \bigg(\frac{89}{40}P^2 a_{4,0} g_0 - \frac{1}{5}P^2 g_0 g_{4,0} + \frac{1}{5}K_0 a_{4,0} + \frac{1491}{32000}K_0 P^4 g_0^2 \\ &+ \frac{689743}{3840000}P^6 g_0^3 + \frac{11}{320}K_0 P^2\alpha_{1,0}^2 g_0 - \frac{197}{640}P^4\alpha_{1,0}^2 g_0^2 + \frac{419}{38400}K_0^2 P^2 g_0\bigg)\ \ln\lambda \\ &+ \bigg(-\frac{1}{64}P^4\alpha_{1,0}^2 g_0^2 + \frac{1}{160}K_0^2 P^2 g_0 + \frac{171}{3200}K_0 P^4 g_0^2 - \frac{1}{2}P^2 a_{4,0} g_0 - \frac{1733}{16000}P^6 g_0^3\bigg)\ \ln^2\lambda \\ &+ \bigg(-\frac{463}{14400}P^6 g_0^3 - \frac{3}{160}K_0 P^4 g_0^2\bigg)\ \ln^3\lambda + \frac{3}{320}P^6 g_0^3\ \ln^4\lambda\bigg)\,, \end{aligned} \tag{3.34}$$

$$
\begin{aligned}
a_{8,0} \to \lambda^8 \Bigg(& a_{8,0} + \frac{1}{P^2 g_0 (70 K_0 - 141 P^2 g_0)} \Bigg(-140 P^4 a_{8,0} g_0^2 - \frac{11289869889229}{7468070400000} P^{12} g_0^6 \\
& + 18 K_0^2 a_{4,0}^2 + \frac{79241}{280} K_0 P^4 \alpha_{1,0}^2 a_{4,0} g_0^2 - \frac{67}{2} K_0 P^4 \alpha_{1,0}^2 g_0^2 g_{4,0} + \frac{131}{4} K_0^2 P^2 \alpha_{1,0}^2 a_{4,0} g_0 \\
& - 24 K_0 P^2 a_{4,0} g_0 g_{4,0} - \frac{17122502251}{790272000} K_0 P^8 \alpha_{1,0}^2 g_0^4 - \frac{1264903}{26880} K_0 P^6 \alpha_{1,0}^4 g_0^3 \\
& + \frac{3642629}{537600} K_0^2 P^6 \alpha_{1,0}^2 g_0^3 - \frac{3}{4} K_0^2 P^4 \alpha_{1,0}^4 g_0^2 - \frac{308363}{560} P^6 \alpha_{1,0}^2 a_{4,0} g_0^3 + \frac{135}{4} P^6 \alpha_{1,0}^2 g_0^3 g_{4,0} \\
& + \frac{16067}{6720} K_0^3 P^4 \alpha_{1,0}^2 g_0^2 - \frac{53709659}{3087000} K_0 P^6 a_{4,0} g_0^3 - \frac{15332}{1225} K_0 P^6 g_0^3 g_{4,0} - \frac{875}{4} P^4 \alpha_{1,0}^4 a_{4,0} g_0^2 \\
& + \frac{1923781}{33600} K_0^2 P^4 a_{4,0} g_0^2 - \frac{2001}{560} K_0^2 P^4 g_0^2 g_{4,0} + 350 P^4 \alpha_{1,0}^2 a_{6,0} g_0^2 - 12 P^4 a_{4,0} g_0^2 g_{4,0} \\
& + \frac{9013}{1120} K_0^3 P^2 a_{4,0} g_0 - \frac{5706}{35} K_0 P^2 a_{4,0}^2 g_0 + \frac{17699297459}{592704000} P^{10} \alpha_{1,0}^2 g_0^5 \\
& + \frac{1365178374361}{553190400000} K_0 P^{10} g_0^5 + \frac{4598761}{80640} P^8 \alpha_{1,0}^4 g_0^4 + \frac{2135}{192} P^6 \alpha_{1,0}^6 g_0^3 \\
& + \frac{48152049931}{189665280000} K_0^2 P^8 g_0^4 - \frac{33703011407}{148176000} P^8 a_{4,0} g_0^4 + \frac{14708381}{529200} P^8 g_0^4 g_{4,0} \\
& + \frac{402129463}{210739200} K_0^3 P^6 g_0^3 + \frac{3965783}{15052800} K_0^4 P^4 g_0^2 + \frac{1315}{6} P^6 a_{6,0} g_0^3 + \frac{49853}{70} P^4 a_{4,0}^2 g_0^2 \\
& - 8 P^4 g_0^2 g_{4,0}^2 \Bigg) \ln \lambda + \Bigg(-\frac{5436207853}{30732800000} P^8 g_0^4 - \frac{35277}{171500} P^6 \alpha_{1,0}^2 g_0^3 - \frac{1469772959}{31610880000} K_0 P^6 g_0^3 \\
& + \frac{489}{8960} P^4 \alpha_{1,0}^4 g_0^2 + \frac{8889}{89600} K_0 P^4 \alpha_{1,0}^2 g_0^2 + \frac{1953403}{105369600} K_0^2 P^4 g_0^2 + \frac{131}{8960} K_0^2 P^2 \alpha_{1,0}^2 g_0 \\
& - \frac{2780609}{1372000} P^4 a_{4,0} g_0^2 + \frac{859}{9800} P^4 g_0^2 g_{4,0} + \frac{9013}{2508800} K_0^3 P^2 g_0 - \frac{157}{140} P^2 \alpha_{1,0}^2 a_{4,0} g_0 \\
& - \frac{2949}{5600} K_0 P^2 a_{4,0} g_0 - \frac{3}{280} K_0 P^2 g_0 g_{4,0} + \frac{9}{560} K_0^2 a_{4,0} - \frac{36}{35} a_{4,0}^2 \Bigg) \ln^2 \lambda \\
& + \Bigg(\frac{2671073519}{47416320000} P^8 g_0^4 - \frac{180151}{2822400} P^6 \alpha_{1,0}^2 g_0^3 - \frac{3778787}{56448000} K_0 P^6 g_0^3 - \frac{27}{640} K_0 P^4 \alpha_{1,0}^2 g_0^2 \\
& - \frac{4513}{250880} K_0^2 P^4 g_0^2 + \frac{8879}{19600} P^4 a_{4,0} g_0^2 + \frac{1}{140} P^4 g_0^2 g_{4,0} + \frac{3}{8960} K_0^3 P^2 g_0 - \frac{3}{40} K_0 P^2 a_{4,0} g_0 \\
& \Bigg) \ln^3 \lambda + \Bigg(\frac{3590117}{112896000} P^8 g_0^4 + \frac{93}{4480} P^6 \alpha_{1,0}^2 g_0^3 + \frac{4537}{179200} K_0 P^6 g_0^3 - \frac{3}{1792} K_0^2 P^4 g_0^2 \\
& + \frac{3}{70} P^4 a_{4,0} g_0^2 \Bigg) \ln^4 \lambda + \Bigg(-\frac{4617}{448000} P^8 g_0^4 + \frac{3}{1600} K_0 P^6 g_0^3 \Bigg) \ln^5 \lambda - \frac{1}{1600} P^8 g_0^4 \ln^6 \lambda \Bigg) ,
\end{aligned}
\tag{3.35}
$$

and

$$
\{ K_0^h \,,\ g_0^h \,,\ f_{2,0}^h \,,\ f_{3,0}^h \} \to \{ \lambda^2 K_0^h \,,\ g_0^h \,,\ \lambda f_{2,0}^h \,,\ \lambda f_{3,0}^h \} \,. \tag{3.36}
$$

We can use the exact symmetry (3.28) to relate different sets of $\{K_0, P\}$. For the study of perturbative in P^2/K_0 expansion we find it convenient to set $K_0 = 1$ and vary P^2. To access the infrared properties of the theory we set $P = 1$ and vary K_0. Notice that the two approaches connect at $\{K_0 = 1, P = 1\}$.

- Third, we have:

$$\begin{aligned} &\rho \to \lambda\, \rho\,,\ H \to \frac{1}{\lambda}\, H\,, \qquad \{P\,,\ f_2\,,\ f_3\,, h\,, K\,, g\} \to \{P\,,\ f_2\,,\ f_3\,, h\,, K\,, g\}\,, \\ &\{y, f_2^h, f_3^h, h^h\} \to \{\lambda^{-1} y, \lambda^{-1} f_2^h, \lambda^{-1} f_3^h, \lambda^2 h^h\}\,, \end{aligned} \tag{3.37}$$

This scaling symmetry acts on the asymptotic parameters as

$$\{g_0\,,\ \alpha_{1,0}\} \to \{g_0\,,\ \alpha_{1,0}\}\,, \tag{3.38}$$

$$K_0 \to K_0 + 2P^2 g_0 \ln\lambda\,, \tag{3.39}$$

$$a_{4,0} \to a_{4,0} + \left(\frac{1}{48} P^4 g_0^2 - \frac{1}{16} K_0 P^2 g_0\right)\ \ln\lambda - \frac{1}{16} P^4 g_0^2\ \ln^2\lambda\,, \tag{3.40}$$

$$\begin{aligned} g_{4,0} \to g_{4,0} &+ \left(\frac{3}{16} P^2 \alpha_{1,0}^2 g_0 + \frac{5}{64} K_0 P^2 g_0 - \frac{37}{96} P^4 g_0^2 - 3 a_{4,0}\right)\ \ln\lambda + \left(\frac{3}{64} P^4 g_0^2\right. \\ &\left. + \frac{3}{32} K_0 P^2 g_0\right)\ \ln^2\lambda + \frac{1}{16} P^4 g_0^2\ \ln^3\lambda\,, \end{aligned} \tag{3.41}$$

$$\begin{aligned} a_{6,0} \to a_{6,0} &+ \left(-\frac{89}{40} P^2 a_{4,0} g_0 + \frac{1}{5} P^2 g_0 g_{4,0} - \frac{1}{5} K_0 a_{4,0} - \frac{1491}{32000} K_0 P^4 g_0^2\right. \\ &\left. - \frac{689743}{3840000} P^6 g_0^3 - \frac{11}{320} K_0 P^2 \alpha_{1,0}^2 g_0 + \frac{197}{640} P^4 \alpha_{1,0}^2 g_0^2 - \frac{419}{38400} K_0^2 P^2 g_0\right)\ \ln\lambda + \Bigg(\\ &- \frac{1}{64} P^4 \alpha_{1,0}^2 g_0^2 + \frac{1}{160} K_0^2 P^2 g_0 + \frac{171}{3200} K_0 P^4 g_0^2 - \frac{1}{2} P^2 a_{4,0} g_0 - \frac{1733}{16000} P^6 g_0^3\Bigg)\ \ln^2\lambda \\ &+ \left(\frac{463}{14400} P^6 g_0^3 + \frac{3}{160} K_0 P^4 g_0^2\right)\ \ln^3\lambda + \frac{3}{320} P^6 g_0^3\ \ln^4\lambda\,, \end{aligned} \tag{3.42}$$

$$
\begin{aligned}
a_{8,0} \to a_{8,0} &+ \frac{1}{P^2 g_0(70K_0 - 141P^2 g_0)}\Bigg(\frac{11289869889229}{7468070400000}P^{12}g_0^6 + \Bigg(-\frac{17699297459}{592704000}\alpha_{1,0}^2 \\
&- \frac{1365178374361}{553190400000}K_0\Bigg)P^{10}g_0^5 + \Bigg(\frac{17122502251}{790272000}K_0\alpha_{1,0}^2 + \frac{33703011407}{148176000}a_{4,0} \\
&- \frac{14708381}{529200}g_{4,0} - \frac{48152049931}{189665280000}K_0^2 - \frac{4598761}{80640}\alpha_{1,0}^4\Bigg)P^8 g_0^4 + \Bigg(\frac{1264903}{26880}K_0\alpha_{1,0}^4 \\
&+ \frac{308363}{560}\alpha_{1,0}^2 a_{4,0} - \frac{135}{4}\alpha_{1,0}^2 g_{4,0} + \frac{53709659}{3087000}K_0 a_{4,0} + \frac{15332}{1225}K_0 g_{4,0} - \frac{402129463}{210739200}K_0^3 \\
&- \frac{2135}{192}\alpha_{1,0}^6 - \frac{1315}{6}a_{6,0} - \frac{3642629}{537600}K_0^2\alpha_{1,0}^2\Bigg)P^6 g_0^3 + \Bigg(\frac{3}{4}K_0^2\alpha_{1,0}^4 + 12a_{4,0}g_{4,0} \\
&+ \frac{2001}{560}K_0^2 g_{4,0} + \frac{67}{2}K_0\alpha_{1,0}^2 g_{4,0} + \frac{875}{4}\alpha_{1,0}^4 a_{4,0} - \frac{1923781}{33600}K_0^2 a_{4,0} + 140a_{8,0} - 350\alpha_{1,0}^2 a_{6,0} \\
&- \frac{79241}{280}K_0\alpha_{1,0}^2 a_{4,0} - \frac{16067}{6720}K_0^3\alpha_{1,0}^2 + 8g_{4,0}^2 - \frac{49853}{70}a_{4,0}^2 - \frac{3965783}{15052800}K_0^4\Bigg)P^4 g_0^2 \\
&+ \Bigg(-\frac{9013}{1120}K_0^3 a_{4,0} + \frac{5706}{35}K_0 a_{4,0}^2 - \frac{131}{4}K_0^2\alpha_{1,0}^2 a_{4,0} + 24K_0 a_{4,0}g_{4,0}\Bigg)P^2 g_0 \\
&- 18K_0^2 a_{4,0}^2\Bigg)\ \ln\lambda + \Bigg(-\frac{5436207853}{30732800000}P^8 g_0^4 + \Bigg(-\frac{1469772959}{31610880000}K_0 - \frac{35277}{171500}\alpha_{1,0}^2\Bigg)P^6 g_0^3 \\
&+ \Bigg(\frac{489}{8960}\alpha_{1,0}^4 + \frac{8889}{89600}K_0\alpha_{1,0}^2 + \frac{1953403}{105369600}K_0^2 - \frac{2780609}{1372000}a_{4,0} + \frac{859}{9800}g_{4,0}\Bigg)P^4 g_0^2 \\
&+ \Bigg(\frac{131}{8960}K_0^2\alpha_{1,0}^2 - \frac{2949}{5600}K_0 a_{4,0} - \frac{3}{280}K_0 g_{4,0} + \frac{9013}{2508800}K_0^3 - \frac{157}{140}\alpha_{1,0}^2 a_{4,0}\Bigg)P^2 g_0 \\
&+ \frac{9}{560}K_0^2 a_{4,0} - \frac{36}{35}a_{4,0}^2\Bigg)\ \ln^2\lambda + \Bigg(-\frac{2671073519}{47416320000}P^8 g_0^4 + \Bigg(\frac{180151}{2822400}\alpha_{1,0}^2 \\
&+ \frac{3778787}{56448000}K_0\Bigg)P^6 g_0^3 + \Bigg(-\frac{8879}{19600}a_{4,0} - \frac{1}{140}g_{4,0} + \frac{4513}{250880}K_0^2 + \frac{27}{640}K_0\alpha_{1,0}^2\Bigg)P^4 g_0^2 \\
&+ \Bigg(\frac{3}{40}K_0 a_{4,0} - \frac{3}{8960}K_0^3\Bigg)P^2 g_0\Bigg)\ \ln^3\lambda + \Bigg(\frac{3590117}{112896000}P^8 g_0^4 + \Bigg(\frac{93}{4480}\alpha_{1,0}^2 \\
&+ \frac{4537}{179200}K_0\Bigg)P^6 g_0^3 + \Bigg(\frac{3}{70}a_{4,0} - \frac{3}{1792}K_0^2\Bigg)P^4 g_0^2\Bigg)\ \ln^4\lambda + \Bigg(\frac{4617}{448000}P^8 g_0^4 \\
&- \frac{3}{1600}K_0 P^6 g_0^3\Bigg)\ \ln^5\lambda - \frac{1}{1600}P^8 g_0^4\ \ln^6\lambda\,,
\end{aligned}
\tag{3.43}
$$

and

$$
\{K_0^h\,,\ g_0^h\,,\ f_{2,0}^h\,,\ f_{3,0}^h\} \to \{K_0^h\,,\ g_0^h\,,\ \lambda^{-1}f_{2,0}^h\,,\ \lambda^{-1}f_{3,0}^h\}\,. \tag{3.44}
$$

We can use the exact symmetry (3.37) to set

$$
H = 1\,. \tag{3.45}
$$

▪ Forth, we have residual diffeomorphisms (3.3) of the metric parametrization (3.1). The latter transformations act on asymptotic parameters as

$$\{g_0\,,\ H\,,\ K_0\} \to \{g_0\,,\ H\,, K_0\}\,, \tag{3.46}$$

$$\alpha_{1,0} \to \alpha_{1,0} + 2\ \frac{\alpha}{H}\,, \tag{3.47}$$

$$a_{4,0} \to a_{4,0} + \frac{1}{4}P^2\alpha_{1,0}g_0\ \frac{\alpha}{H} + \frac{1}{4}P^2 g_0\ \frac{\alpha^2}{H^2}\,, \tag{3.48}$$

$$g_{4,0} \to g_{4,0} - \frac{3}{2}P^2\alpha_{1,0}g_0\ \frac{\alpha}{H} - \frac{3}{2}P^2 g_0\ \frac{\alpha^2}{H^2}\,, \tag{3.49}$$

$$\begin{aligned} a_{6,0} \to\ & a_{6,0} + \left(-\frac{11}{96}P^4 g_0^2\alpha_{1,0} - \frac{1}{8}P^2 g_0\alpha_{1,0}^3 + \frac{5}{32}P^2 g_0 K_0\alpha_{1,0} + 3\alpha_{1,0}a_{4,0}\right)\ \frac{\alpha}{H} \\ & + \left(-\frac{11}{96}P^4 g_0^2 + \frac{5}{32}K_0P^2 g_0 + 3a_{4,0}\right)\ \frac{\alpha^2}{H^2} + \frac{1}{4}P^2\alpha_{1,0}g_0\ \frac{\alpha^3}{H^3} + \frac{1}{8}P^2 g_0\frac{\alpha^4}{H^4}\,, \end{aligned} \tag{3.50}$$

$$\begin{aligned} a_{8,0} \to\ & a_{8,0} + \left(\frac{1791949}{2560000}P^6\alpha_{1,0}g_0^3 + \left(\frac{16839}{64000}K_0\alpha_{1,0} - \frac{10337}{11520}\alpha_{1,0}^3\right)P^4 g_0^2\right. \\ & + \left(-\frac{9}{10}\alpha_{1,0}g_{4,0} - \frac{473}{1920}K_0\alpha_{1,0}^3 + \frac{1417}{25600}K_0^2\alpha_{1,0} + \frac{1}{4}\alpha_{1,0}^5 + \frac{761}{80}\alpha_{1,0}a_{4,0}\right)P^2 g_0 \\ & \left. - 5\alpha_{1,0}^3 a_{4,0} + \frac{9}{10}K_0\alpha_{1,0}a_{4,0} + 10\alpha_{1,0}a_{6,0}\right)\ \frac{\alpha}{H} + \left(\frac{1791949}{2560000}P^6 g_0^3 + \left(-\frac{1793}{1280}\alpha_{1,0}^2\right.\right. \\ & \left.\left. + \frac{16839}{64000}K_0\right)P^4 g_0^2 + \left(\frac{761}{80}a_{4,0} - \frac{9}{10}g_{4,0} + \frac{1417}{25600}K_0^2 + \frac{99}{640}K_0\alpha_{1,0}^2\right)P^2 g_0 + 10a_{6,0}\right. \\ & \left. + \frac{9}{10}K_0 a_{4,0}\right)\ \frac{\alpha^2}{H^2} + \left(-\frac{145}{144}P^4 g_0^2\alpha_{1,0} + \left(-\frac{5}{12}\alpha_{1,0}^3 + \frac{77}{96}K_0\alpha_{1,0}\right)P^2 g_0\right. \\ & \left. + 10\alpha_{1,0}a_{4,0}\right)\ \frac{\alpha^3}{H^3} + \left(-\frac{145}{288}P^4 g_0^2 + \frac{77}{192}K_0P^2 g_0 + 5a_{4,0}\right)\ \frac{\alpha^4}{H^4} + \frac{1}{4}g_0P^2\alpha_{1,0}\ \frac{\alpha^5}{H^5} \\ & + \frac{1}{12}P^2 g_0\ \frac{\alpha^6}{H^6}\,, \end{aligned} \tag{3.51}$$

and

$$\{K_0^h\,,\ g_0^h\,,\ f_{2,0}^h\,,\ f_{3,0}^h\} \to \{K_0^h\,,\ g_0^h\,,\ f_{2,0}^h\,,\ f_{3,0}^h\}\,. \tag{3.52}$$

As mentioned earlier, the diffeomorphisms (3.3) can be completely fixed requiring that

$$\lim_{\rho\to+\infty} h^{-1/2}\rho^{-2} = 0\,, \tag{3.53}$$

i.e., in the holographic dual to the symmetric phase of cascading gauge theory the manifold $\mathcal{M}_5$ geodesically completes in the interior with smooth shrinking of dS_4 (see (3.1)) as $\rho \to +\infty$.

3.5 Keeping the physical parameters fixed

Cascading gauge theory on dS_4 has two dimensionfull physical parameters: the strong coupling scale Λ, the Hubble scale H, and a dimensionless physical parameter $P^2 g_0$. Recall that a symmetry transformation (3.37) rescales H, and a symmetry transformation (3.28) rescales P and affects K_0, while leaving the combination

$$\frac{K_0}{P^2 g_0} + 2\ln H + \ln P^2 g_0 \;=\; \text{invariant} \;\equiv -2\ln\Lambda + 2\ln H = \ln\frac{H^2}{\Lambda^2} \tag{3.54}$$

invariant. The latter invariant defines the strong coupling scale Λ of cascading gauge theory. In particular, using the symmetry choices (3.27) and (3.45) we identify

$$\frac{K_0}{P^2} = \ln\frac{1}{\Lambda^2 P^2} \equiv \frac{1}{\delta}\,. \tag{3.55}$$

Notice that (3.55) is not invariant under the symmetry transformation (3.28). This is because such transformation modifies $P^2 g_0$, and thus changes the theory; (3.55) is invariant under the residual diffeomorphisms (3.3).

As defined in (3.55), a new dimensionless parameter δ is small when the IR cutoff set by the dS_4 is much higher than the strong coupling scale Λ (and thus cascading gauge theory is close to be conformal). In section 3.7 we develop perturbative expansion in δ.

3.6 Numerical procedure

Although we would like to have an analytic control over the gravitational solution dual to a symmetric phase of cascading gauge theory, the relevant equations for $\{f_2,$ $f_3,$ $h,$ $K,$ $g\}$ (3.5)-(3.10) are rather complicated. Thus, we have to resort to numerical analysis. Recall that various scaling symmetries of the background equations of motion allowed us to set (see (3.27) and (3.45))

$$\lim_{\rho\to 0} g \equiv g_0 = 1\,, \qquad H = 1\,. \tag{3.56}$$

While the metric parametrization (3.1) has residual diffeomorphisms (3.3), the latter are fixed once we insist on the IR asymptotics at $y \equiv \frac{1}{\rho} \to 0$ (see (3.53)). Finally, a

scaling symmetry (3.28) relates different pairs $\{K_0, P\}$ so that only the ratio $\frac{K_0}{P^2} \equiv \frac{1}{\delta}$ is physically meaningful (see (3.55)). In the end, for a fixed δ, the gravitational solution is characterized by 5 parameters in the UV and 4 parameters in the IR:

$$\begin{aligned} &\text{UV}: \qquad \{\alpha_{1,0}\,,\ a_{4,0}\,,\ a_{6,0}\,,\ a_{8,0}\,,\ g_{4,0}\}\,, \\ &\text{IR}: \qquad \{K_0^h\,,\ g_0^h\,,\ f_{2,0}^h\,,\ f_{3,0}^h\}\,. \end{aligned} \tag{3.57}$$

Notice that $5+4=9$ is precisely the number of integration constants needed to specify a solution to (3.5)-(3.10) — we have 5 second order differential equations and a single first order differential constraint: $2\times 5-1=9$.

In practice, we replace the second-order differential equation for f_2 (3.5) with the constraint equation (3.10), which we use to algebraically eliminate f_2' from (3.6)-(3.9). The solution is found using the "shooting" method as detailed in [16].

Finding a "shooting" solution in 9-dimensional parameter space (3.57) is quite challenging. Thus, we start with (leading) analytic results for $\delta \ll 1$ (see section 3.7) and construct numerical solution for $(K_0=1, P^2)$ slowly incrementing P^2 from zero to one. Starting with the solution at $K_0=P^2=1$ we slowly decrease K_0 while keeping $P^2=1$.

3.7 Symmetric phase of cascading gauge theory at $\frac{H}{\Lambda} \gg 1$

In this section we describe perturbative solution in $\delta \ll 1$ (3.55) to (3.5)-(3.10). Such gravitational backgrounds describe cascading gauge theory on dS_4, which Hubble scale H is well above the strong coupling scale Λ of cascading gauge theory.

In the limit $\delta \to 0$ (or equivalently $P \to 0$) the gravitational background is simply that of the Klebanov-Witten model [22] on dS_4 [20]:

$$\begin{aligned} \delta=0: \qquad & f_2^{(0)}=f_3^{(0)}=1+\sqrt{\hat{K}_0}\rho\,, \qquad h^{(0)}=\frac{\hat{K}_0}{4(1+\sqrt{\hat{K}_0}\rho)^2}\,, \\ & K^{(0)}=\hat{K}_0\,, \qquad g^{(0)}=1\,, \end{aligned} \tag{3.58}$$

where $\hat{K}_0$ is a constant. Perturbatively, we find

$$\begin{aligned} f_i(\rho)&=f_i^{(0)}\times\sum_{j=0}^{\infty}\left(\frac{P^2}{\hat{K}_0}\right)^j\ f_{i,j}(\rho^2\hat{K}_0)\,, \qquad h(\rho)=h^{(0)}\times\sum_{j=0}^{\infty}\left(\frac{P^2}{\hat{K}_0}\right)^j\ h_j(\rho^2\hat{K}_0)\,, \\ K(\rho)&=\hat{K}^{(0)}\times\sum_{j=0}^{\infty}\left(\frac{P^2}{\hat{K}_0}\right)^j\ K_j(\rho^2\hat{K}_0)\,, \qquad g(\rho)=g^{(0)}\times\sum_{j=0}^{\infty}\left(\frac{P^2}{\hat{K}_0}\right)^j\ g_j(\rho^2\hat{K}_0)\,. \end{aligned} \tag{3.59}$$

Apart from technical complexity, there is no obstacle of developing perturbative solution to any order in $\frac{P^2}{\hat{K}_0}$. For our purposes it is sufficient to do so to order $\mathcal{O}\left(\frac{P^4}{\hat{K}_0^2}\right)$. Notice that explicit ρ dependence enters only in combination $\rho\sqrt{\hat{K}_0}$, thus, we can set $\hat{K}_0 = 1$ and reinstall explicit $\hat{K}_0$ dependence when necessary.

Substituting (3.59) in (3.5)-(3.10) we find to order $\mathcal{O}(\delta)$ the following equations

$$\begin{aligned} 0 =& f_{2,1}'' - \frac{\rho+6}{2\rho(\rho+1)} f_{2,1}' + \frac{\rho+2}{2\rho(\rho+1)} h_1' - \frac{3}{4}(K_1')^2 - \frac{3\rho^2-16\rho-16}{4\rho^2(\rho+1)^2} h_1 \\ & - \frac{4K_1 + 7f_{2,1} - 20 f_{3,1} - 3}{(\rho+1)\rho^2}\,, \end{aligned} \tag{3.60}$$

$$\begin{aligned} 0 =& f_{3,1}'' - \frac{\rho+6}{2\rho(\rho+1)} f_{3,1}' + \frac{1}{4}(K_1')^2 + \frac{\rho+2}{2\rho(\rho+1)} h_1' - \frac{3\rho^2-16\rho-16}{4\rho^2(\rho+1)^2} h_1 \\ & + \frac{5f_{2,1} + 8 f_{3,1} - 4K_1 - 1}{(\rho+1)\rho^2}\,, \end{aligned} \tag{3.61}$$

$$\begin{aligned} 0 =& h_1'' - \frac{\rho+4}{\rho(\rho+1)} h_1' + \frac{3}{4}(K_1')^2 + \frac{(\rho+2)(f_{2,1}' + 4 f_{3,1}')}{2\rho(\rho+1)} + \frac{9(\rho^2-16\rho-16)}{4\rho^2(\rho+1)^2} h_1 \\ & - \frac{17 f_{2,1} + 68 f_{3,1} - 36 K_1 - 5}{(\rho+1)\rho^2}\,, \end{aligned} \tag{3.62}$$

$$0 = K_1'' - \frac{\rho+6}{2\rho(\rho+1)} K_1' - \frac{8}{(\rho+1)\rho^2}\,, \tag{3.63}$$

$$0 = g_1'' - \frac{\rho+6}{2\rho(\rho+1)} g_1' + (K_1')^2 - \frac{4}{(\rho+1)\rho^2}\,, \tag{3.64}$$

along with the first order constraint

$$\begin{aligned} 0 =& f_{2,1}' + 4 f_{3,1}' + h_1' + \frac{(\rho+1)\rho}{2(\rho+2)}(K_1')^2 + \frac{(\rho+4)(3\rho+4)}{2\rho(\rho+2)(\rho+1)} h_1 \\ & + \frac{2(4 f_{3,1} + f_{2,1} - 4K_1 - 1)}{(\rho+2)\rho}\,. \end{aligned} \tag{3.65}$$

Above equations should be solved with $\mathcal{O}(\delta)$ UV and the IR boundary conditions prescribed in sections 3.2 and 3.3. We solve all the equations numerically. Parameterizing the asymptotics as follows

- UV, *i.e.*, $\rho \to 0$, (the independent coefficients being $\{\alpha_{1,1,0},\, k_{1,4,0},\, a_{1,6,0},\, a_{1,8,0},\, g_{1,4,0}\}$):

$$
\begin{aligned}
f_{2,1} = {} & \alpha_{1,1,0}\ \rho + \left(-\frac{3}{8} - \frac{1}{2}\alpha_{1,1,0} + \frac{1}{2}\ln\rho\right)\ \rho^2 + \left(\frac{1}{8} + \frac{1}{2}\alpha_{1,1,0} - \frac{1}{2}\ln\rho\right)\ \rho^3 \\
& + \left(-\frac{5}{24} - \frac{1}{2}\alpha_{1,1,0} + \frac{4}{3}k_{1,4,0} + \frac{9}{16}\ln\rho\right)\ \rho^4 + \left(\frac{29}{96} + \frac{1}{2}\alpha_{1,1,0} - \frac{8}{3}k_{1,4,0} - \frac{5}{8}\ln\rho\right)\ \rho^5 \\
& + \left(\frac{1}{160}\ln^2\rho + \frac{4}{15}\ln\rho k_{1,4,0} + \frac{25099}{38400}\ln\rho + a_{1,6,0}\right)\ \rho^6 + \left(-\frac{3}{160}\ln^2\rho - \frac{4}{5}\ln\rho k_{1,4,0}\right. \\
& \left. - \frac{8379}{12800}\ln\rho - \alpha_{1,1,0} - 3a_{1,6,0} + \frac{98}{15}k_{1,4,0} - \frac{17513}{25600}\right)\ \rho^7 + \left(\frac{3}{8960}\ln^3\rho + \frac{3}{140}\ln^2\rho k_{1,4,0}\right. \\
& \left. + \frac{16}{35}\ln\rho k_{1,4,0}^2 + \frac{89373}{2508800}\ln^2\rho + \frac{29791}{19600}\ln\rho k_{1,4,0} + \frac{223043661}{351232000}\ln\rho + a_{1,8,0}\right)\ \rho^8 \\
& + \mathcal{O}(\rho^9)\,,
\end{aligned}
\tag{3.66}
$$

$$
\begin{aligned}
f_{3,1} = {} & \alpha_{1,1,0}\ \rho + \left(-\frac{1}{2} - \frac{1}{2}\alpha_{1,1,0} + \frac{1}{2}\ln\rho\right)\ \rho^2 + \left(\frac{1}{4} + \frac{1}{2}\alpha_{1,1,0} - \frac{1}{2}\ln\rho\right)\ \rho^3 + \left(-\frac{41}{192}\right. \\
& \left. - \frac{1}{2}\alpha_{1,1,0} + \frac{1}{2}\ln\rho\right)\ \rho^4 + \left(\frac{7}{32} + \frac{1}{2}\alpha_{1,1,0} - \frac{1}{2}\ln\rho\right)\ \rho^5 + \left(-\frac{1}{640}\ln^2\rho - \frac{1}{15}\ln\rho k_{1,4,0}\right. \\
& \left. + \frac{19049}{38400}\ln\rho - \frac{5}{8}\alpha_{1,1,0} - \frac{1}{4}a_{1,6,0} + \frac{353}{480}k_{1,4,0} - \frac{381067}{1228800}\right)\ \rho^6 + \left(\frac{3}{640}\ln^2\rho + \frac{1}{5}\ln\rho k_{1,4,0}\right. \\
& \left. - \frac{6229}{12800}\ln\rho + \frac{7}{8}\alpha_{1,1,0} + \frac{3}{4}a_{1,6,0} - \frac{1043}{480}k_{1,4,0} + \frac{591377}{1228800}\right)\ \rho^7 + \left(\frac{3}{8960}\ln^3\rho\right. \\
& \left. + \frac{3}{140}\ln^2\rho k_{1,4,0} + \frac{16}{35}\ln\rho k_{1,4,0}^2 - \frac{28227}{2508800}\ln^2\rho - \frac{9409}{19600}\ln\rho k_{1,4,0} - \frac{4}{5}k_{1,4,0}^2\right. \\
& \left. + \frac{167306161}{351232000}\ln\rho - \frac{231}{64}\alpha_{1,1,0} - \frac{231}{32}a_{1,6,0} + a_{1,8,0} + \frac{129741}{6400}k_{1,4,0} - \frac{116879077}{49152000}\right)\ \rho^8 \\
& + \mathcal{O}(\rho^9)\,,
\end{aligned}
\tag{3.67}
$$

$$
\begin{aligned}
h_1 = {} & \frac{1}{2} - 2\ln\rho + \left(1 - 2\alpha_{1,1,0}\right)\,\rho + \left(\frac{19}{24} + \alpha_{1,1,0} - \ln\rho\right)\,\rho^2 + \left(-\frac{11}{24} - \alpha_{1,1,0}\right. \\
& \left. + \ln\rho\right)\,\rho^3 + \left(\frac{431}{1024} + \alpha_{1,1,0} + \frac{1}{6}k_{1,4,0} - \frac{127}{128}\ln\rho\right)\,\rho^4 + \left(-\frac{3359}{7680} - \alpha_{1,1,0} - \frac{1}{3}k_{1,4,0}\right. \\
& \left. + \frac{63}{64}\ln\rho\right)\,\rho^5 + \left(\frac{656813}{1536000} + \alpha_{1,1,0} + \frac{189}{200}k_{1,4,0} - \frac{12233}{12800}\ln\rho\right)\,\rho^6 + \left(-\frac{4213513}{10752000}\right. \\
& \left. - \alpha_{1,1,0} - \frac{1201}{600}k_{1,4,0} + \frac{11599}{12800}\ln\rho\right)\,\rho^7 + \left(-\frac{9}{7168}\ln^3\rho - \frac{9}{112}\ln^2\rho k_{1,4,0} - \frac{12}{7}\ln\rho k_{1,4,0}^2\right. \\
& \left. + \frac{12441}{2007040}\ln^2\rho + \frac{4147}{15680}\ln\rho k_{1,4,0} + \frac{212}{105}k_{1,4,0}^2 - \frac{238628771}{280985600}\ln\rho + \frac{637}{64}\alpha_{1,1,0}\right. \\
& \left. + \frac{693}{32}a_{1,6,0} - \frac{15}{4}a_{1,8,0} - \frac{17860741}{313600}k_{1,4,0} + \frac{110837461177}{16859136000}\right)\,\rho^8 + \mathcal{O}(\rho^9)\,;
\end{aligned}
\tag{3.68}
$$

$$
\begin{aligned}
K_1 = {} & -2\ln\rho + \rho - \frac{1}{8}\rho^2 - \frac{1}{24}\rho^3 + \left(k_{1,4,0} + \frac{3}{64}\ln\rho\right)\,\rho^4 + \left(\frac{33}{640} - 2k_{1,4,0} - \frac{3}{32}\ln\rho\right)\,\rho^5 \\
& + \left(-\frac{307}{3072} + \frac{35}{12}k_{1,4,0} + \frac{35}{256}\ln\rho\right)\,\rho^6 + \left(\frac{1031}{7168} - \frac{15}{4}k_{1,4,0} - \frac{45}{256}\ln\rho\right)\,\rho^7 + \left(-\frac{24077}{131072}\right. \\
& \left. + \frac{1155}{256}k_{1,4,0} + \frac{3465}{16384}\ln\rho\right)\,\rho^8 + \mathcal{O}(\rho^9)\,;
\end{aligned}
\tag{3.69}
$$

$$
\begin{aligned}
g_1 = {} & -\frac{1}{2}\rho^2 + \frac{1}{2}\rho^3 + \left(g_{1,4,0} + \left(-\frac{33}{64} + 4k_{1,4,0}\right)\ln\rho + \frac{3}{32}\ln^2\rho\right)\,\rho^4 + \left(-\frac{31}{128} - 2g_{1,4,0}\right. \\
& \left. - 2k_{1,4,0} + \left(\frac{15}{16} - 8k_{1,4,0}\right)\ln\rho - \frac{3}{16}\ln^2\rho\right)\,\rho^5 + \left(\frac{3671}{9216} + \frac{35}{12}g_{1,4,0} + \frac{161}{36}k_{1,4,0}\right. \\
& \left. + \left(-\frac{497}{384} + \frac{35}{3}k_{1,4,0}\right)\ln\rho + \frac{35}{128}\ln^2\rho\right)\,\rho^6 + \left(-\frac{533}{1024} - \frac{15}{4}g_{1,4,0} - \frac{83}{12}k_{1,4,0}\right. \\
& \left. + \left(\frac{103}{64} - 15k_{1,4,0}\right)\ln\rho - \frac{45}{128}\ln^2\rho\right)\,\rho^7 + \left(\frac{81683}{131072} - \frac{1}{2}k_{1,4,0}^2 + \frac{1155}{256}g_{1,4,0}\right. \\
& \left. + \frac{7117}{768}k_{1,4,0} + \left(-\frac{15499}{8192} + 18k_{1,4,0}\right)\ln\rho + \frac{27}{64}\ln^2\rho\right)\,\rho^8 + \mathcal{O}(\rho^9)\,;
\end{aligned}
\tag{3.70}
$$

- IR, *i.e.*, $y = \frac{1}{\rho} \to 0$, (the independent coefficients being $\{a_{1,0}^h, b_{1,0}^h, g_{1,0}^h, k_{1,0}^h\}$:

$$
\begin{aligned}
& f_{2,1} = a_{1,0}^h + \mathcal{O}(y)\,, \qquad f_{3,1} = b_{1,0}^h + \mathcal{O}(y)\,, \qquad g_1 = g_{1,0}^h + \mathcal{O}(y)\,, \\
& K_1 = k_{1,0}^h + \mathcal{O}(y)\,, \qquad h_1 = \left(-\frac{6}{5} + \frac{18}{5}a_{1,0}^h + \frac{72}{5}b_{1,0}^h - 8k_{1,0}^h\right)\,y + \mathcal{O}(y^2)\,,
\end{aligned}
\tag{3.71}
$$

we find

$$\begin{aligned}
&\alpha_{1,1,0} = 0.43427(8)\,, \qquad k_{1,4,0} = 0.04829(9)\,, \qquad a_{1,6,0} = -0.40703(7) \\
&a_{1,8,0} = -0.42707(1)\,, \qquad g_{1,4,0} = -0.26443(7)\,, \qquad a^h_{1,0} = -0.15661(4) \\
&b^h_{1,0} = -0.37883(6)\,, \qquad g^h_{1,0} = -0.72222(2)\,, \qquad k^h_{1,0} = -1.10592(2)
\end{aligned} \tag{3.72}$$

In an analogous way, it is possible to go to second order in δ by taking eqs. (3.5)-(3.10) and evaluate them with the expansion (3.59) to second order in δ. Then, we will get equations for functions $f_{2,2}, f_{3,2}, h_2, K_2, g_2$. As with the first order equations, one uses the UV and IR boundary conditions prescribed in sections 3.2 and 3.3. Setting $H = 1$, we get that the independent coefficients in the UV are $\{\alpha_{2,1,0},\, k_{2,4,0},\, a_{2,6,0},\, a_{2,8,0},\, g_{2,4,0}\}$, while those in the IR are $\{a^h_{2,0}, b^h_{2,0}, g^h_{2,0}, k^h_{2,0}\}$. Solving numerically, we find the values of these constants to be

$$\begin{aligned}
&\alpha_{2,1,0} = 0.35729(1)\,, \qquad k_{2,4,0} = 0.18423(1)\,, \qquad a_{2,6,0} = -0.48877(2)\,, \\
&a_{2,8,0} = -0.60853(7)\,, \qquad g_{2,4,0} = -0.64457(3)\,, \qquad a^h_{2,0} = 0.54009(5)\,, \\
&b^h_{2,0} = 0.63805(4)\,, \qquad g^h_{2,0} = 0.31165(0)\,, \qquad k^h_{2,0} = 1.65246(0)\,.
\end{aligned} \tag{3.73}$$

We can now identify the leading $\mathcal{O}(\delta^2)$ values of general UV and IR parameters (see (3.57)):

$$\begin{aligned}
&\alpha_{1,0} = -1 - \alpha_{1,1,0}\,\delta - \alpha_{2,1,0}\,\delta^2\,, \\
&a_{4,0} = \left(-\frac{1}{12} + \frac{4}{3}k_{1,4,0}\right)\delta + \left(-\frac{139}{1152} + \frac{a_{1,1,0}}{24} + \frac{2g_{1,4,0}}{3} - \frac{22k_{1,4,0}}{9} + \frac{4k_{2,4,0}}{3}\right)\delta^2, \\
&g_{4,0} = g_{1,4,0}\,\delta + g_{2,4,0}\,\delta^2\,, \\
&a_{6,0} = \left(a_{1,6,0} + \frac{29}{96} - \frac{8}{3}k_{1,4,0} + \frac{1}{2}\alpha_{1,1,0}\right)\,\delta\, + \\
&+\left(\frac{145}{576} - \frac{5a_{1,1,0}}{32} - \frac{a^2_{1,1,0}}{4} + \frac{a_{2,1,0}}{2} - \frac{4g_{1,4,0}}{3} + \frac{44k_{1,4,0}}{9} - \frac{4a_{1,1,0}k_{1,4,0}}{3} - \frac{8k_{2,4,0}}{3}\right. \\
&\left.+a_{2,6,0}\right)\delta^2\,, \\
&a_{8,0} = \left(a_{1,8,0} - 3a_{1,6,0} - \frac{17513}{25600} - \alpha_{1,1,0} + \frac{98}{15}k_{1,4,0}\right)\delta + \\
&+\left(-\frac{87973}{192000} - \frac{15353a_{1,1,0}}{25600} - \frac{a^2_{1,1,0}}{2} - 2a_{1,1,0}a_{1,6,0} - a_{2,1,0} - 3a_{2,6,0} + a_{2,8,0} + \right. \\
&\left.+\frac{101g_{1,4,0}}{30} - \frac{2423k_{1,4,0}}{180} + \frac{178a_{1,1,0}k_{1,4,0}}{15} + \frac{98k_{2,4,0}}{15}\right)\delta^2\,,
\end{aligned} \tag{3.74}$$

$$K_0^h = 1 + k_{1,0}^h\ \delta + k_{2,0}^h\ \delta^2\,, \qquad g_0^h = 1 + g_{1,0}^h\ \delta + g_{2,0}^h\ \delta^2\,,$$
$$f_{2,0}^h = 1 + a_{1,0}^h\ \delta + a_{2,0}^h\ \delta^2\,, \qquad f_{3,0}^h = 1 + b_{1,0}^h\ \delta + b_{2,0}^h\ \delta^2\,, \tag{3.75}$$

where we set $K_0 = 1$.

Figure 1 compares the values of general UV and IR parameters $\alpha_{1,0}$, $a_{4,0}$, $a_{6,0}$, $a_{8,0}$, $g_{4,0}$, K_0^h, g_0^h, $f_{2,0}^h$, $f_{3,0}^h$ (see (3.57)), with their perturbative predictions at linear and quadratic order. The results for first and second order will help to correctly initialize the fully non-linear calculation and at the same time provide a verification of the results, at least for small enough δ.

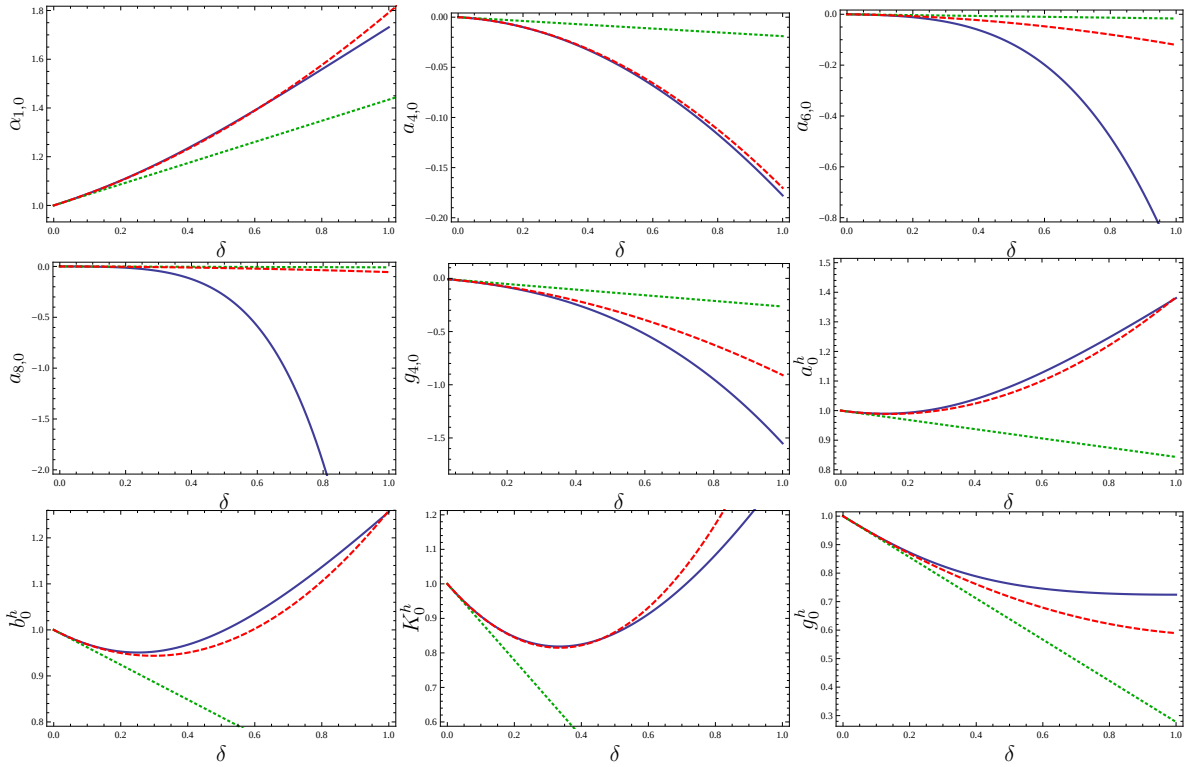

Figure 1: (Colour online) Comparison of values of UV parameters $\{\alpha_{1,0}, a_{4,0}, a_{6,0}, \alpha_{8,0}, g_{4,0}\}$ and IR parameters $\{a_0^h, b_0^h, K_0^h, g_0^h\}$ (see (3.57)) in the range $\delta \in [0,1]$ (blue curves) with their perturbative predictions (3.74)-(3.75) at first (green dotted) and second order (red dashed) in δ.

4 Cascading gauge theory on dS_4 with spontaneously broken chiral symmetry

4.1 $R^{1,3} \to dS_4$ deformation of Klebanov-Strassler state of cascading gauge theory

$\mathcal{N} = 1$ supersymmetric ground state of cascading gauge theory on $R^{3,1}$ — referred to as Klebanov-Strassler state — spontaneously breaks chiral symmetry [8]. A natural route to construct a χSB state of the theory on dS_4 is to "deform" Klebanov-Strassler state: $R^{1,3} \to dS_4$. We explain now how to achieve this in a "continuous" fashion.

Consider the five-dimensional metric of the type:

$$ds_5^2 = g_{\mu\nu}(y)dy^\mu dy^\nu = c_1^2 \left(-dt^2 + \frac{1}{H^2}\cosh^2(Ht)\left(dS^3\right)^2 \right) + c_3^2 \, (d\rho)^2 \,, \tag{4.1}$$

where $c_i = c_i(\rho)$. We will be interested in χSB states of cascading gauge theory on dS_4 with a Hubble scale H . One can derive equations of motion from (2.1). Alternatively, we can construct an effective 1-dimensional action[6] from (2.1), by restricting to the metric ansatz (4.1), and the ρ-only dependence of the scalar fields $\{\Phi, h_i, \Omega_i\}$:

$$S_5 \left[g_{\mu\nu}, \Omega_i, h_i, \Phi \right] \implies S_1 \left[c_i, \Omega_i, h_i, \Phi \right] . \tag{4.2}$$

It can be verified that equations of motion obtained from S_1 coincide with those obtained from (2.1), provided we vary[7] S_1 with respect to c_3, treating it as an unconstrained field. The 1-dimensional effective action approach makes it clear that the only place where the information about dS_4 enters is through the evaluation of R_5 in (2.2):

$$R_5 = -\frac{8c_1''}{c_3^2 c_1} + \frac{8c_1' c_3'}{c_3^3 c_1} - \frac{12(c_1')^2}{c_3^2 c_1^2} + \frac{12\kappa}{c_1^2} \,, \tag{4.3}$$

where derivatives are with respect to ρ, and $\kappa = H^2$.

[6] Effectively, in obtaining S_1 we perform Kaluza-Klein-like reduction of the effective action S_5 on dS_4.

[7] This produces the first order constraint similar to (3.10).

4.2 Equations of motion

As in (3.1) and (2.8) we denote

$$\begin{aligned}
c_1 =& h^{-1/4}\rho^{-1}\,, \qquad c_3 = h^{1/4}\rho^{-1}\,, \qquad \Phi = \ln g\,,\\
h_1 =& \frac{1}{P}\left(\frac{K_1}{12} - 36\Omega_0\right)\,, \qquad h_2 = \frac{P}{18}\,K_2\,, \qquad h_3 = \frac{1}{P}\left(\frac{K_3}{12} - 36\Omega_0\right)\,,\\
\Omega_1 =& \frac{1}{3}f_c^{1/2}h^{1/4}\,, \qquad \Omega_2 = \frac{1}{\sqrt{6}}f_a^{1/2}h^{1/4}\,, \qquad \Omega_3 = \frac{1}{\sqrt{6}}f_b^{1/2}h^{1/4}\,.
\end{aligned} \tag{4.4}$$

The equations of motion obtained from $S_1\left[c_i, \Omega_i, h_i, \Phi\right]$ are

$$\begin{aligned}
0 =& f_c'' - \frac{3f_c'}{\rho} - 3hf_c\kappa - \frac{(f_c')^2}{2f_c} + \frac{5f_c}{\rho^2} + \frac{f_c(g')^2}{8g^2} + \frac{3f_b'f_c'}{4f_b} + \frac{63f_a}{16f_b\rho^2} + \frac{63f_b}{16f_a\rho^2} + \frac{3f_c}{f_a\rho^2}\\
&- \frac{f_c(f_a')^2}{8f_a^2} + \frac{3f_a'f_c'}{4f_a} + \frac{f_c(h')^2}{8h^2} - \frac{f_c(f_b')^2}{8f_b^2} + \frac{3f_c}{f_b\rho^2} - \frac{63}{8\rho^2} - \frac{K_1^2}{8f_a^2h^2f_b^2\rho^2} + \frac{3gP^2}{2f_a^2h\rho^2}\\
&- \frac{f_cf_a'f_b'}{2f_af_b} - \frac{27K_1K_3}{32f_ahf_bgP^2\rho^2} - \frac{K_2^2K_1^2}{32f_a^2h^2f_b^2\rho^2} + \frac{K_2K_1^2}{8f_a^2h^2f_b^2\rho^2} - \frac{K_2^2K_3^2}{32f_a^2h^2f_b^2\rho^2} - \frac{3f_c(K_1')^2}{32hf_b^2gP^2}\\
&- \frac{3f_c(K_3')^2}{32f_a^2hgP^2} + \frac{3gP^2K_2^2}{8hf_b^2\rho^2} + \frac{3gP^2K_2^2}{8f_a^2h\rho^2} - \frac{3gP^2K_2}{2f_a^2h\rho^2} - \frac{9f_c^2}{f_af_b\rho^2} + \frac{f_ch'}{h\rho} + \frac{K_2^2K_1K_3}{16f_a^2h^2f_b^2\rho^2}\\
&- \frac{K_2K_1K_3}{8f_a^2h^2f_b^2\rho^2} - \frac{gP^2f_c(K_2')^2}{12f_ahf_b} + \frac{27K_1^2}{64f_ahf_bgP^2\rho^2} + \frac{27K_3^2}{64f_ahf_bgP^2\rho^2}\,,
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
0 =& f_a'' - \frac{45f_a^2}{16f_cf_b\rho^2} + \frac{f_ah'}{h\rho} + \frac{gP^2(K_2')^2}{36hf_b} + \frac{5(K_3')^2}{32f_ahgP^2} - \frac{f_af_b'f_c'}{4f_cf_b} - \frac{(f_a')^2}{8f_a} + \frac{5f_a}{\rho^2} - \frac{3f_a'}{\rho}\\
&- \frac{K_2^2K_1^2}{32f_cf_ah^2f_b^2\rho^2} + \frac{K_2K_1^2}{8f_cf_ah^2f_b^2\rho^2} - \frac{K_2^2K_3^2}{32f_cf_ah^2f_b^2\rho^2} - \frac{3gP^2K_2}{2f_cf_ah\rho^2} + \frac{3gP^2K_2^2}{8f_cf_ah\rho^2}\\
&- \frac{9K_3^2}{64f_chf_bgP^2\rho^2} - \frac{9K_1^2}{64f_chf_bgP^2\rho^2} + \frac{3f_a}{f_b\rho^2} + \frac{3f_c}{f_b\rho^2} + \frac{9K_1K_3}{32f_chf_bgP^2\rho^2} + \frac{K_2^2K_1K_3}{16f_cf_ah^2f_b^2\rho^2}\\
&- \frac{K_2K_1K_3}{8f_cf_ah^2f_b^2\rho^2} - \frac{5f_agP^2K_2^2}{8f_chf_b^2\rho^2} - \frac{K_1^2}{8f_cf_ah^2f_b^2\rho^2} + \frac{3gP^2}{2f_cf_ah\rho^2} - \frac{3f_a(K_1')^2}{32hf_b^2gP^2} - \frac{9}{\rho^2} + \frac{f_a(g')^2}{8g^2}\\
&- 3f_ah\kappa + \frac{f_a'f_b'}{2f_b} + \frac{f_c'f_a'}{4f_c} - \frac{f_a(f_b')^2}{8f_b^2} + \frac{9f_a}{8f_c\rho^2} + \frac{f_a(h')^2}{8h^2} + \frac{27f_b}{16f_c\rho^2}\,,
\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
0 = {} & f_b'' - \frac{3f_b'}{\rho} - \frac{(f_b')^2}{8f_b} + \frac{5f_b}{\rho^2} - \frac{45f_b^2}{16f_cf_a\rho^2} + \frac{f_bh'}{h\rho} - \frac{K_1^2}{8f_ch^2f_a^2f_b\rho^2} - \frac{3f_b(K_3')^2}{32hgf_a^2P^2} \\
& - \frac{K_2^2K_1^2}{32f_ch^2f_a^2f_b\rho^2} + \frac{K_2K_1^2}{8f_ch^2f_a^2f_b\rho^2} - \frac{K_2^2K_3^2}{32f_ch^2f_a^2f_b\rho^2} - \frac{9K_1^2}{64f_chgf_aP^2\rho^2} + \frac{3gP^2K_2^2}{f_chf_b\rho^2} \\
& - \frac{9K_3^2}{64f_chgf_aP^2\rho^2} - \frac{5gf_bP^2}{2f_chf_a^2\rho^2} + \frac{3f_b}{f_a\rho^2} + \frac{3f_c}{f_a\rho^2} - \frac{f_bf_c'f_a'}{4f_cf_a} + \frac{5(K_1')^2}{32hgf_bP^2} + \frac{gP^2(K_2')^2}{36hf_a} \\
& - \frac{9}{\rho^2} + \frac{27f_a}{16f_c\rho^2} + \frac{9f_b}{8f_c\rho^2} + \frac{K_2^2K_1K_3}{16f_ch^2f_a^2f_b\rho^2} - \frac{K_2K_1K_3}{8f_ch^2f_a^2f_b\rho^2} + \frac{5gf_bP^2K_2}{2f_chf_a^2\rho^2} - \frac{5gf_bP^2K_2^2}{8f_chf_a^2\rho^2} \\
& + \frac{9K_1K_3}{32f_chgf_aP^2\rho^2} + \frac{f_b(g')^2}{8g^2} - 3hf_b\kappa + \frac{f_a'f_b'}{2f_a} - \frac{f_b(f_a')^2}{8f_a^2} + \frac{f_b'f_c'}{4f_c} + \frac{f_b(h')^2}{8h^2}\,,
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
0 = {} & h'' + \frac{K_2^2K_1^2}{4f_cf_a^2f_b^2h\rho^2} - \frac{K_2K_1^2}{f_cf_a^2f_b^2h\rho^2} + \frac{K_2^2K_3^2}{4f_cf_a^2f_b^2h\rho^2} + \frac{9K_1^2}{16f_cf_af_b\rho^2gP^2} + \frac{9K_3^2}{16f_cf_af_b\rho^2gP^2} \\
& + \frac{2hf_c'}{f_c\rho} + \frac{4hf_b'}{f_b\rho} + \frac{4hf_a'}{f_a\rho} + \frac{(K_1')^2}{8f_b^2gP^2} + \frac{(K_3')^2}{8f_a^2gP^2} + \frac{gP^2K_2^2}{2f_cf_b^2\rho^2} + \frac{gP^2K_2^2}{2f_cf_a^2\rho^2} - \frac{2gP^2K_2}{f_cf_a^2\rho^2} + \frac{f_c'h'}{2f_c} \\
& + \frac{h'f_b'}{f_b} + \frac{h'f_a'}{f_a} - \frac{16h}{\rho^2} - \frac{(h')^2}{h} + 12h^2\kappa - \frac{K_2^2K_1K_3}{2f_cf_a^2f_b^2h\rho^2} + \frac{K_2K_1K_3}{f_cf_a^2f_b^2h\rho^2} + \frac{K_1^2}{f_cf_a^2f_b^2h\rho^2} \\
& + \frac{2gP^2}{f_cf_a^2\rho^2} + \frac{gP^2(K_2')^2}{9f_af_b)} - \frac{9K_1K_3}{8f_cf_af_b\rho^2gP^2} - \frac{3h'}{\rho}\,,
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
0 = {} & K_1'' - \frac{gK_2^2K_1P^2}{f_cf_a^2h\rho^2} + \frac{gK_2^2K_3P^2}{f_cf_a^2h\rho^2} + \frac{4gK_2K_1P^2}{f_cf_a^2h\rho^2} - \frac{2gK_2K_3P^2}{f_cf_a^2h\rho^2} - \frac{9f_bK_1}{2f_cf_a\rho^2} + \frac{9f_bK_3}{2f_cf_a\rho^2} \\
& - \frac{4gK_1P^2}{f_cf_a^2h\rho^2} + \frac{K_1'f_c'}{2f_c} - \frac{K_1'g'}{g} - \frac{K_1'h'}{h} + \frac{f_a'K_1'}{f_a} - \frac{3K_1'}{\rho} - \frac{K_1'f_b'}{f_b}\,,
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
0 = {} & K_3'' + \frac{gK_2^2K_1P^2}{f_cf_b^2h\rho^2} - \frac{gK_2^2K_3P^2}{f_cf_b^2h\rho^2} - \frac{2gK_2K_1P^2}{f_cf_b^2h\rho^2} + \frac{9f_aK_1}{2f_cf_b\rho^2} - \frac{9f_aK_3}{2f_cf_b\rho^2} + \frac{K_3'f_c'}{2f_c} \\
& - \frac{K_3'g'}{g} + \frac{f_b'K_3'}{f_b} - \frac{K_3'h'}{h} - \frac{3K_3'}{\rho} - \frac{K_3'f_a'}{f_a}\,,
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
0 = {} & K_2'' - \frac{9f_bK_2}{2f_cf_a\rho^2} - \frac{9f_aK_2}{2f_cf_b\rho^2)} + \frac{9f_b}{f_cf_a\rho^2} - \frac{9K_2K_1^2}{8f_cgP^2hf_bf_a\rho^2} + \frac{9K_2K_1K_3}{4f_cgP^2hf_bf_a\rho^2} \\
& - \frac{9K_2K_3^2}{8f_cgP^2hf_bf_a\rho^2} + \frac{9K_1^2}{4f_cgP^2hf_bf_a\rho^2} - \frac{9K_1K_3}{4f_cgP^2hf_bf_a\rho^2} + \frac{K_2'f_c'}{2f_c} + \frac{K_2'g'}{g} - \frac{K_2'h'}{h} \\
& - \frac{3K_2'}{\rho}\,,
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
0 = {} & g'' - \frac{g^2P^2K_2^2}{2f_cf_a^2h\rho^2} - \frac{g^2P^2K_2^2}{2f_cf_b^2h\rho^2} + \frac{2g^2P^2K_2}{f_cf_a^2h\rho^2} + \frac{9K_1^2}{16f_cf_af_bh\rho^2P^2} + \frac{9K_3^2}{16f_cf_af_bh\rho^2P^2} \\
& - \frac{(g')^2}{g} - \frac{9K_1K_3}{8f_cf_af_bh\rho^2P^2} + \frac{(K_3')^2}{8f_a^2hP^2} + \frac{(K_1')^2}{8f_b^2hP^2} - \frac{2g^2P^2}{f_cf_a^2h\rho^2} - \frac{g^2P^2(K_2')^2}{9f_af_bh} + \frac{g'f_c'}{2f_c} \\
& + \frac{g'f_a'}{f_a} + \frac{g'f_b'}{f_b} - \frac{3g'}{\rho}\,.
\end{aligned}
\tag{4.12}
$$

Additionally, we have the first order constraint

$$
\begin{aligned}
0 = {} & \frac{8}{9}g^2(K_2')^2f_bf_aP^4 + (K_3')^2f_b^2 + (K_1')^2f_a^2 - \frac{4g^2K_2^2f_a^2P^4}{f_c\rho^2} + \frac{4gf_a^2f_b^2P^2(h')^2}{h} \\
& + \frac{4h(g')^2f_a^2f_b^2P^2}{g} + \frac{96hgf_a^2f_bP^2}{\rho^2} + \frac{96hgf_af_b^2P^2}{\rho^2} - \frac{96hgf_a^2f_b^2P^2}{\rho^2} - \frac{4gK_1^2P^2}{f_ch\rho^2} \\
& + 96h^2gf_a^2f_b^2P^2\kappa + \frac{9K_1K_3f_bf_a}{f_c\rho^2} + \frac{32gf_a^2f_b^2P^2h'}{\rho} + \frac{16g^2K_2f_b^2P^4}{f_c\rho^2} - \frac{4g^2K_2^2f_b^2P^4}{f_c\rho^2} \\
& - \frac{gK_2^2K_1^2P^2}{f_ch\rho^2} + \frac{4gK_2K_1^2P^2}{f_ch\rho^2} - \frac{gK_2^2K_3^2P^2}{f_ch\rho^2} + \frac{64hgf_a^2f_bP^2f_b'}{\rho} + \frac{64hgf_af_b^2P^2f_a'}{\rho} \\
& - 16hgf_af_bP^2f_a'f_b' - \frac{32f_chgf_af_bP^2}{\rho^2} - \frac{18hgf_af_b^3P^2}{f_c\rho^2} - \frac{18hgf_a^3f_bP^2}{f_c\rho^2} + \frac{36hgf_a^2f_b^2P^2}{f_c\rho^2} \\
& - \frac{9K_3^2f_bf_a}{f_c\rho^2} - 4hgf_b^2P^2(f_a')^2 - 4hgf_a^2P^2(f_b')^2 - \frac{16g^2f_b^2P^4}{f_c\rho^2} + \frac{2gK_2^2K_1K_3P^2}{f_ch\rho^2} \\
& - \frac{4gK_2K_1K_3P^2}{f_ch\rho^2} - \frac{8hgf_b^2f_aP^2f_c'f_a'}{f_c} + \frac{32hgf_a^2f_b^2P^2f_c'}{f_c\rho} - \frac{8hgf_a^2f_bP^2f_b'f_c'}{f_c} - \frac{9K_1^2f_bf_a}{2f_c\rho^2}\,.
\end{aligned}
\tag{4.13}
$$

We explicitly verified that for any value κ the constraint (4.13) is consistent with (4.5)-(4.12). Moreover, with

$$
f_c = f_2\,, \qquad f_a = f_b = f_3\,, \qquad K_1 = K_3 = K\,, \qquad K_2 = 1\,, \tag{4.14}
$$

equations (4.5)-(4.13) are equivalent to (3.5)-(3.10).

4.3 UV asymptotics

The general UV (as $\rho \to 0$) asymptotic solution of (4.5)-(4.13) describing the phase of cascading gauge theory with spontaneously broken chiral symmetry takes the form

$$
\begin{aligned}
f_c = {} & 1 - \alpha_{1,0}\rho + \left(-\frac{3}{8}g_0P^2 - \frac{1}{4}K_0 + \frac{1}{4}\alpha_{1,0}^2 + \frac{1}{2}P^2g_0\ln\rho\right)\rho^2 \\
& + \frac{1}{4}P^2\alpha_{1,0}g_0\rho^3 + \sum_{n=4}^{\infty}\sum_k f_{c,n,k}\;\rho^n\ln^k\rho\,,
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
f_a =& 1-\alpha_{1,0}\rho+\left(-\frac{1}{2}g_0P^2-\frac{1}{4}K_0+\frac{1}{4}\alpha_{1,0}^2+\frac{1}{2}P^2g_0\ln\rho\right)\rho^2+f_{a,3,0}\rho^3 \\
&+\sum_{n=4}^{\infty}\sum_k f_{a,n,k}\ \rho^n\ln^k\rho\,,
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
f_b =& 1-\alpha_{1,0}\rho+\left(-\frac{1}{2}g_0P^2-\frac{1}{4}K_0+\frac{1}{4}\alpha_{1,0}^2+\frac{1}{2}P^2g_0\ln\rho\right)\rho^2 \\
&+\left(\frac{1}{2}P^2\alpha_{1,0}g_0-f_{a,3,0}\right)\rho^3+\sum_{n=4}^{\infty}\sum_k f_{b,n,k}\ \rho^n\ln^k\rho\,,
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
h =& \frac{1}{8}g_0P^2+\frac{1}{4}K_0-\frac{1}{2}P^2g_0\ln\rho+\left(-P^2g_0\ln\rho+\frac{1}{2}K_0\right)\alpha_{1,0}\rho+\Bigg(\Bigg(-\frac{1}{4}g_0P^2 \\
&-\frac{5}{4}P^2g_0\ln\rho+\frac{5}{8}K_0\Bigg)\alpha_{1,0}^2+\frac{119}{576}P^4g_0^2+\frac{31}{96}P^2g_0K_0+\frac{1}{8}K_0^2+\frac{1}{2}P^4g_0^2\ln\rho^2 \\
&-\frac{31}{48}P^4g_0^2\ln\rho-\frac{1}{2}\ln\rho P^2g_0K_0\Bigg)\rho^2+\Bigg(\Bigg(-\frac{5}{4}P^2g_0\ln\rho-\frac{11}{24}g_0P^2+\frac{5}{8}K_0\Bigg)\alpha_{1,0}^3 \\
&+\left(\frac{3}{2}P^4g_0^2\ln\rho^2-\frac{23}{16}P^4g_0^2\ln\rho+\frac{19}{64}P^4g_0^2-\frac{3}{2}\ln\rho P^2g_0K_0+\frac{23}{32}P^2g_0K_0\right. \\
&\left.+\frac{3}{8}K_0^2\right)\alpha_{1,0}\Bigg)\rho^3+\sum_{n=4}^{\infty}\sum_k h_{n,k}\ \rho^n\ln^k\rho\,,
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
K_1 =& K_0-2P^2g_0\ln\rho-P^2\alpha_{1,0}g_0\rho+\left(-\frac{1}{4}P^2\alpha_{1,0}^2g_0-\frac{1}{4}P^4g_0^2\ln\rho+\frac{9}{16}P^4g_0^2\right. \\
&\left.+\frac{1}{8}P^2g_0K_0\right)\rho^2+\Bigg(-\frac{1}{12}\alpha_{1,0}^3g_0P^2+\frac{1}{48}g_0P^2\Bigg(-36P^2g_0\ln\rho+13P^2g_0 \\
&+6K_0\Bigg)\alpha_{1,0}+\frac{1}{48}g_0P^2\Big(96f_{a,3,0}\ln\rho+32f_{a,3,0}+32k_{2,3,0}\Big)\Bigg)\rho^3 \\
&+\sum_{n=4}^{\infty}\sum_k k_{1,n,k}\ \rho^n\ln^k\rho\,,
\end{aligned}
\tag{4.19}
$$

$$
K_2 = 1+\left(k_{2,3,0}-\frac{3}{4}\alpha_{1,0}P^2g_0\ln\rho+3f_{a,3,0}\ln\rho\right)\rho^3+\sum_{n=4}^{\infty}\sum_k k_{2,n,k}\ \rho^n\ln^k\rho\,, \tag{4.20}
$$

$$
\begin{aligned}
K_3 =& K_0 - 2P^2 g_0 \ln\rho - P^2\alpha_{1,0} g_0 \rho + \bigg(-\frac{1}{4}P^2\alpha_{1,0}^2 g_0 - \frac{1}{4}P^4 g_0^2 \ln\rho + \frac{9}{16}P^4 g_0^2 \\
&+ \frac{1}{8}P^2 g_0 K_0\bigg)\rho^2 + \bigg(-\frac{1}{12}\alpha_{1,0}^3 g_0 P^2 + \frac{1}{48} g_0 P^2\bigg(12P^2 g_0 \ln\rho + 29P^2 g_0 \\
&+ 6K_0\bigg)\alpha_{1,0} - \frac{1}{48} g_0 P^2\bigg(96 f_{a,3,0}\ln\rho + 32 f_{a,3,0} + 32 k_{2,3,0}\bigg)\bigg)\rho^3 \\
&+ \sum_{n=4}^{\infty}\sum_k k_{3,n,k}\ \rho^n \ln^k\rho\,,
\end{aligned}
\tag{4.21}
$$

$$
g = g_0\bigg(1 - \frac{1}{2}P^2 g_0\rho^2 - \frac{1}{2}\alpha_{1,0}P^2 g_0\rho^3 + \sum_{n=4}^{\infty}\sum_k g_{n,k}\ \rho^n \ln^k\rho\bigg)\,. \tag{4.22}
$$

It is characterized by 11 parameters:

$$
\{K_0\,,\ H\,,\ g_0\,,\ \alpha_{1,0}\,,\ k_{2,3,0}\,,\ f_{c,4,0}\,,\ f_{a,3,0}\,,\ f_{a,6,0}\,,\ f_{a,7,0}\,,\ f_{a,8,0}\,,\ g_{4,0}\}\,. \tag{4.23}
$$

In what follows we developed the UV expansion to order $\mathcal{O}(\rho^{10})$ inclusive.

4.4 IR asymptotics

As in section 3.3, we use a radial coordinate ρ that extends to infinity, see (3.4). The crucial difference between the IR boundary conditions for a chirally symmetric phase discussed in section 3.3 and the IR boundary conditions for a χSB phase discussed here is that in the former case the manifold $\mathcal{M}_5$ geodesically completes with (a smooth) shrinking to zero size of $dS_4 \subset \mathcal{M}_5$, while in the latter case, much like in supersymmetric Klebanov-Strassler state of cascading gauge theory [8], the 10-dimensional uplift of $\mathcal{M}_5$,

$$
\mathcal{M}_5\ \to\ \mathcal{M}_{10} = \mathcal{M}_5 \times X_5\,, \tag{4.24}
$$

geodesically completes with (a smooth) shrinking of a 2-cycle in the compact manifold X_5 [8]. Introducing

$$
y \equiv \frac{1}{\rho}\,, \qquad h^h \equiv y^{-4}\ h\,, \qquad f^h_{a,b,c} \equiv y^2\ f_{a,b,c}\,, \tag{4.25}
$$

the general IR (as $y \to 0$) asymptotic solution of (4.5)-(4.13) describing the χSB phase of cascading gauge theory takes form

$$\begin{aligned} f_c^h =&\frac{3}{4}f_{a,0}^h + \left(-\frac{19(k_{2,2}^h)^2P^2g_0^h}{540h_0^h} - \frac{3}{4}f_{a,0}^h h_0^h\kappa - \frac{3f_{a,0}^h k_{2,4}^h}{2k_{2,2}^h} - \frac{13P^2g_0^h}{15(f_{a,0}^h)^2h_0^h} + \frac{6}{5}\right. \\ &\left. + \frac{f_{a,0}^h(k_{1,3}^h)^2}{64P^2g_0^hh_0^h} - \frac{27}{5f_{a,0}^hk_{2,2}^h} + \frac{19(k_{3,1}^h)^2}{320P^2f_{a,0}^hg_0^hh_0^h} + \frac{3k_{1,3}^hk_{3,1}^h}{20k_{2,2}^hP^2f_{a,0}^hg_0^hh_0^h}\right) y^2 \\ &+ \sum_{n=2}^{\infty} f_{c,n}^h\, y^{2n}\,, \end{aligned} \tag{4.26}$$

$$\begin{aligned} f_a^h =&f_{a,0}^h + \left(\frac{17(k_{2,2}^h)^2P^2g_0^h}{405h_0^h} + 2f_{a,0}^hh_0^h\kappa + \frac{f_{a,0}^hk_{2,4}^h}{k_{2,2}^h} - \frac{4P^2g_0^h}{45(f_{a,0}^h)^2h_0^h} + \frac{11}{5} + \frac{f_{a,0}^h(k_{1,3}^h)^2}{48P^2g_0^hh_0^h}\right. \\ &\left. + \frac{18}{5f_{a,0}^hk_{2,2}^h} - \frac{17(k_{3,1}^h)^2}{240P^2f_{a,0}^hg_0^hh_0^h} - \frac{k_{1,3}^hk_{3,1}^h}{10k_{2,2}^hP^2f_{a,0}^hg_0^hh_0^h}\right) y^2 + \sum_{n=2}^{\infty} f_{a,n}^h\, y^{2n}\,, \end{aligned} \tag{4.27}$$

$$f_b = 3\, y^2 + \sum_{n=2}^{\infty} f_{b,n}^h\, y^{2n}\,, \tag{4.28}$$

$$\begin{aligned} h^h =&h_0^h + \left(-\frac{g_0^hP^2(k_{2,2}^h)^2}{27f_{a,0}^h} - 2\kappa(h_0^h)^2 - \frac{4g_0^hP^2}{9(f_{a,0}^h)^3} - \frac{(k_{1,3}^h)^2}{48g_0^hP^2} - \frac{(k_{3,1}^h)^2}{16g_0^hP^2(f_{a,0}^h)^2}\right) y^2 \\ &+ \sum_{n=2}^{\infty} h_n^h\, y^{2n}\,, \end{aligned} \tag{4.29}$$

$$K_1 = k_{1,3}^h\, y^3 + \sum_{n=2}^{\infty} k_{1,n}^h\, y^{2n+1}\,, \tag{4.30}$$

$$K_2 = k_{2,2}^h\, y^2 + k_{2,4}^h\, y^4 + \sum_{n=3}^{\infty} k_{2,n}^h\, y^{2n}\,, \tag{4.31}$$

$$\begin{aligned} K_3 =&k_{3,1}^h\, y + \left(\frac{41P^2g_0^h(k_{2,2}^h)^2k_{3,1}^h}{810f_{a,0}^hh_0^h} + \frac{4P^2g_0^hk_{2,2}^hk_{1,3}^h}{135f_{a,0}^hh_0^h} + \frac{7}{10}h_0^hk_{3,1}^h\kappa - \frac{1}{5}k_{1,3}^h + \frac{k_{2,4}^hk_{3,1}^h}{k_{2,2}^h}\right. \\ &+ \frac{2P^2g_0^hk_{3,1}^h}{15(f_{a,0}^h)^3h_0^h} + \frac{4k_{3,1}^h}{5f_{a,0}^h} + \frac{(k_{1,3}^h)^2k_{3,1}^h}{480P^2g_0^hh_0^h} + \frac{18k_{3,1}^h}{5(f_{a,0}^h)^2k_{2,2}^h} - \frac{41(k_{3,1}^h)^3}{480P^2(f_{a,0}^h)^2g_0^hh_0^h} \\ &\left. - \frac{k_{1,3}^h(k_{3,1}^h)^2}{10P^2(f_{a,0}^h)^2g_0^hh_0^hk_{2,2}^h}\right) y^3 + + \sum_{n=2}^{\infty} k_{3,n}^h\, y^{2n+1}\,, \end{aligned} \tag{4.32}$$

$$
\begin{aligned}
g =& g_0^h \Bigg(1 + \left(\frac{P^2 g_0^h (k_{2,2}^h)^2}{27 f_{a,0}^h h_0^h} + \frac{4P^2 g_0^h}{9 (f_{a,0}^h)^3 h_0^h} - \frac{(k_{1_3}^h)^2}{48 P^2 h_0^h g_0^h} - \frac{(k_{3,1}^h)^2}{16 P^2 (f_{a,0}^h)^2 h_0^h g_0^h} \right) y^2 \\
&+ \sum_{n=2}^{\infty} g_n^h \, y^{2n} \Bigg) \,.
\end{aligned}
\tag{4.33}
$$

Notice that the prescribed IR boundary conditions imply

$$
\lim_{y \to 0} \, \Omega_3^2 = \lim_{y \to 0} \, \frac{1}{6} \, f_b \, h^{1/2} = \lim_{y \to 0} \, \frac{y^2}{6} \, f_b \, (h^h)^{1/2} = 0 \,, \tag{4.34}
$$

with all the other warp factors in (2.4) being finite. Moreover, see (2.4),

$$
\lim_{y \to 0} \left(\Omega_1^2 \, g_5^2 + \Omega_2^2 \, [g_3^2 + g_4^2] \right) = \frac{1}{6} f_{a,0}^h (h_0^h)^{1/2} \left(\frac{1}{2} g_5^2 + g_3^2 + g_4^2 \right) \,, \tag{4.35}
$$

which is the metric of the round S^3 which stays of finite size in the deep infrared as the 2-cycle fibered over it (smoothly) shrinks to zero size (4.34). Asymptotic solution (4.26)-(4.33) is characterized by 7 additional parameters:

$$
\{ f_{a,0}^h \,, \; h_0^h \,, \; k_{1,3}^h \,, \; k_{2,2}^h \,, \; k_{2,4}^h \,, \; k_{3,1}^h \,, g_0^h \} \,. \tag{4.36}
$$

In what follows we developed the IR expansion to order $\mathcal{O}(y^{10})$ inclusive.

4.5 Symmetries and numerical procedure

The background geometry (4.4) dual to a phase of cascading gauge theory with spontaneously broken chiral symmetry on dS_4 enjoys all the symmetries, properly generalized, discussed in section 3.4:

- $$
  P \to \lambda P \,, \; g \to \frac{1}{\lambda} g \,, \; \{ \rho, f_{a,b,c}, h, K_{1,2,3} \} \to \{ \rho, f_{a,b,c}, h, K_{1,2,3} \} \,, \tag{4.37}
  $$

- $$
  P \to \lambda P \,, \; \rho \to \frac{1}{\lambda} \rho \,, \; \{ h, K_{1,3} \} \to \lambda^2 \{ h, K_{1,3} \} \,, \; \{ f_{a,b,c}, K_2, g \} \to \{ f_{a,b,c}, K_2, g \} \,, \tag{4.38}
  $$

- $$
  \rho \to \lambda \rho \,, \; H \to \frac{1}{\lambda} H \,, \; \{ P, f_{a,b,c}, h, K_{1,2,3}, g \} \to \{ P, f_{a,b,c}, h, K_{1,2,3}, g \} \,, \tag{4.39}
  $$

■

$$\begin{pmatrix} P \\ \rho \\ h \\ f_{a,b,c} \\ K_{1,2,3} \\ g \end{pmatrix} \Longrightarrow \begin{pmatrix} \hat{P} \\ \hat{\rho} \\ \hat{h} \\ \hat{f}_{a,b,c} \\ \hat{K}_{1,2,3} \\ \hat{g} \end{pmatrix} = \begin{pmatrix} P \\ \rho/(1+\alpha\ \rho) \\ (1+\alpha\ \rho)^4\ h \\ (1+\alpha\ \rho)^{-2}\ f_{a,b,c} \\ K_{1,2,3} \\ g \end{pmatrix}, \qquad \alpha = \text{const}\,. \tag{4.40}$$

Thus, much like in section 3.4, we can set

$$g_0 = 1\,, \qquad H = 1\,, \qquad \frac{K_0}{P^2} = \ln\ \frac{1}{\Lambda^2 P^2} \equiv \frac{1}{\delta}\,, \tag{4.41}$$

The residual diffeomorphisms (4.40) are actually completely fixed once we insist on the IR asymptotics as in (4.26)-(4.33).

The numerical procedure for solving the background equations (4.5)-(4.13), subject to the boundary conditions (4.15)-(4.22) and (4.26)-(4.33) is identical to the one described earlier, see section 3.6. Given (4.41), for a fixed δ, the gravitational solution is characterized by 8 parameters in the UV and 7 parameters in the IR:

$$\begin{aligned} &\text{UV}: \qquad \{\alpha_{1,0}\,,\ k_{2,3,0}\,,\ f_{c,4,0}\,,\ f_{a,3,0}\,,\ f_{a,6,0}\,,\ f_{a,7,0}\,,\ f_{a,8,0}\,,\ g_{4,0}\}\,, \\ &\text{IR}: \qquad \{f^h_{a,0}\,,\ h^h_0\,,\ k^h_{1,3}\,,\ k^h_{2,2}\,,\ k^h_{2,4}\,,\ k^h_{3,1}\,, g^h_0\}\,. \end{aligned} \tag{4.42}$$

Notice that $8+7=15$ is precisely the number of integration constants needed to specify a solution to (4.5) -(4.13) — we have 8 second order differential equations and a single first order differential constraint: $2\times 8-1=15$.

In practice, we replace the second-order differential equation for f_c (4.5) with the constraint equation (4.13), which we use to algebraically eliminate f_c' from (4.6)-(4.12). The solution is found using the "shooting" method as detailed in [16].

Ultimately, we are interested in the solution at $\kappa = H^2 = 1$. Finding such a "shooting" solution in 15-dimensional parameter space (4.42) is quite challenging. Thus, we start with the analytic result for $\kappa = 0$ (the Klebanov-Strassler state of cascading gauge theory), and a fixed value of δ, and slowly increase κ to $\kappa = 1$. We further use the obtained solution as a starting point to explore other values of δ.

4.6 κ-deformation of Klebanov-Strassler state

We begin with mapping the Klebanov-Strassler solution [8] to a $\kappa = 0$ solution of (4.5)-(4.13). We set

$$g_0 = 1\,, \qquad P = 1\,. \tag{4.43}$$

$\mathcal{N} = 1$ supersymmetric Klebanov-Strassler solution takes form[8]:

$$\begin{aligned} &ds_5^2 = H_{KS}^{-1/2}\ \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H_{KS}^{1/2}\ \omega_{1,KS}^2\ dr^2\,, \\ &\Omega_i = \omega_{i,KS}\ H_{KS}^{1/2}\,, \qquad h_i = h_{i,KS}\,, \end{aligned} \tag{4.44}$$

$$\begin{aligned} &h_{1,KS} = \frac{\cosh r - 1}{18\sinh r}\left(\frac{r\cosh r}{\sinh r} - 1\right)\,, \qquad h_{2,KS} = \frac{1}{18}\left(1 - \frac{r}{\sinh r}\right)\,, \\ &h_{3,KS} = \frac{\cosh r + 1}{18\sinh r}\left(\frac{r\cosh r}{\sinh r} - 1\right)\,, \qquad g = 1\,, \\ &\omega_{1,KS} = \frac{\epsilon^{2/3}}{\sqrt{6}\hat{K}_{KS}}\,, \qquad \omega_{2,KS} = \frac{\epsilon^{2/3}\hat{K}_{KS}^{1/2}}{\sqrt{2}}\cosh\frac{r}{2}\,, \qquad \omega_{3,KS} = \frac{\epsilon^{2/3}\hat{K}_{KS}^{1/2}}{\sqrt{2}}\sinh\frac{r}{2}\,, \end{aligned} \tag{4.45}$$

with

$$\hat{K}_{KS} = \frac{(\sinh(2r) - 2r)^{1/3}}{2^{1/3}\sinh r}\,,\ H'_{KS} = \frac{16((9h_{2,KS} - 1)h_{1,KS} - 9h_{3,KS}h_{2,KS})}{9\epsilon^{8/3}\hat{K}_{KS}^2\sinh^2 r}\,,\ \Omega_0 = 0\,, \tag{4.46}$$

where now $r \to \infty$ is the boundary and $r \to 0$ is the IR. Above solution is parametrized by a single constant ϵ which will be mapped to K_0, and which in turn will determine all the parameters in (4.42) once $\kappa = 0$.

Comparing the metric ansatz in (4.44) and (4.1), (4.4) we identify

$$\frac{(d\rho)^2}{\rho^4} = (w_{1,KS}(r))^2(dr)^2\,. \tag{4.47}$$

Introducing

$$z \equiv e^{-r/3}\,, \tag{4.48}$$

we find from (4.47)

$$\frac{1}{\rho} = \frac{\sqrt{6}\ (2\epsilon)^{2/3}}{4}\ \int_1^z\ du\ \frac{u^6 - 1}{u^2(1 - u^{12} + 12u^6\ln u)^{1/3}}\,. \tag{4.49}$$

[8]See eqs. (2.22) and (2.34) in [18].

In the UV, $r \to \infty$, $z \to 0$ and $\rho \to 0$ we have

$$
\begin{aligned}
e^{-r/3} \equiv z = \frac{\sqrt{6}\ (2\epsilon)^{2/3}}{4}\rho\Bigg(1 + \mathcal{Q}\rho + \mathcal{Q}^2\rho^2 + \mathcal{Q}^3\rho^3 + \mathcal{Q}^4\rho^4 + \mathcal{Q}^5\rho^5 + \Bigg(\frac{27}{80}\epsilon^4\ln 3 + \mathcal{Q}^6 \\
+ \frac{27}{800}\epsilon^4 - \frac{9}{16}\epsilon^4\ln 2 + \frac{9}{20}\epsilon^4\ln\epsilon + \frac{27}{40}\epsilon^4\ln\rho\Bigg)\rho^6 + \Bigg(-\frac{63}{16}\epsilon^4\mathcal{Q}\ln 2 + \frac{189}{80}\epsilon^4\mathcal{Q}\ln 3 + \mathcal{Q}^7 \\
+ \frac{729}{800}\mathcal{Q}\epsilon^4 + \frac{63}{20}\epsilon^4\mathcal{Q}\ln\epsilon + \frac{189}{40}\mathcal{Q}\epsilon^4\ln\rho\Bigg)\rho^7 + \Bigg(\frac{2403}{400}\epsilon^4\mathcal{Q}^2 - \frac{63}{4}\epsilon^4\mathcal{Q}^2\ln 2 + \frac{189}{20}\epsilon^4\mathcal{Q}^2\ln 3 \\
+ \frac{63}{5}\epsilon^4\mathcal{Q}^2\ln\epsilon + \mathcal{Q}^8 + \frac{189}{10}\epsilon^4\mathcal{Q}^2\ln\rho\Bigg)\rho^8 + \Bigg(\frac{189}{5}\epsilon^4\mathcal{Q}^3\ln\epsilon + \frac{9729}{400}\epsilon^4\mathcal{Q}^3 - \frac{189}{4}\epsilon^4\mathcal{Q}^3\ln 2 \\
+ \frac{567}{20}\epsilon^4\mathcal{Q}^3\ln 3 + \mathcal{Q}^9 + \frac{567}{10}\epsilon^4\mathcal{Q}^3\ln\rho\Bigg)\rho^9 + \mathcal{O}(\rho^{10}\ln\rho)\Bigg)\,,
\end{aligned}
\tag{4.50}
$$

where

$$
\begin{aligned}
\mathcal{Q} =& \frac{\sqrt{6}\ (2\epsilon)^{2/3}}{4}\left\{\int_0^1 du\,\left(\frac{1-u^6}{u^2(1-u^{12}+12u^6\ln u)^{1/3}} - \frac{1}{u^2}\right) - 1\right\} \\
=& -\frac{\sqrt{6}\ (2\epsilon)^{2/3}}{4}\ \times\ 0.839917(9)\,.
\end{aligned}
\tag{4.51}
$$

In the IR, $r \to 0$, $z \to 1_-$ and $\frac{1}{\rho} \to 0$ we have

$$
r = \frac{\sqrt{6}\ 2^{1/3}}{3^{1/3}\ \epsilon^{2/3}}\ y\ \left(1 - \frac{2^{2/3}\ 3^{1/3}}{15\ \epsilon^{4/3}}\ y^2 + \frac{71\ 3^{2/3}\ 2^{1/3}}{2625\ \epsilon^{8/3}}\ y^4 + \mathcal{O}(y^6)\right). \tag{4.52}
$$

Using (4.50) and (4.52), and the exact analytic solution describing the Klebanov-Strassler state of cascading gauge theory (4.45), (4.46) we can identify parameters[9] (4.42)

$$
\begin{aligned}
&K_0 = -\ln 3 + \frac{5}{3}\ \ln 2 - \frac{4}{3}\ \ln\epsilon - \frac{2}{3}\,, \qquad a_{1,0} = 2\mathcal{Q}\,, \\
&k_{2,3,0} = \frac{3\sqrt{6}}{8}\epsilon^2(3\ln 3 - 5\ln 2 + 4\ln\epsilon)\,, \qquad f_{c,4,0} = 0\,, \qquad f_{a,3,0} = \frac{3\sqrt{6}}{4}\ \epsilon^2\,, \\
&f_{a,6,0} = \left(-\frac{27}{16}\ln 2 + \frac{81}{50} + \frac{81}{80}\ln 3 + \frac{27}{20}\ln\epsilon\right)\epsilon^4 + \frac{3\sqrt{6}}{4}\mathcal{Q}^3\epsilon^2\,, \\
&f_{a,7,0} = \frac{3}{800}\mathcal{Q}(2268 - 1800\ln 2 + 1440\ln\epsilon + 1080\ln 3)\epsilon^4 + \frac{3\sqrt{6}}{4}\epsilon^2\mathcal{Q}^4\,, \\
&f_{a,8,0} = \frac{3}{32}\mathcal{Q}^2(270 - 180\ln 2 + 108\ln 3 + 144\ln\epsilon)\epsilon^4 + \frac{3\sqrt{6}}{4}\mathcal{Q}^5\epsilon^2\,, \qquad g_{4,0} = 0\,,
\end{aligned}
\tag{4.53}
$$

[9]We matched the asymptotic expansions (4.15)-(4.22) and (4.26)-(4.33) with the exact solution (4.45) to the order we developed them: $\mathcal{O}(\rho^{10})$ and $\mathcal{O}(y^{10})$ correspondingly.

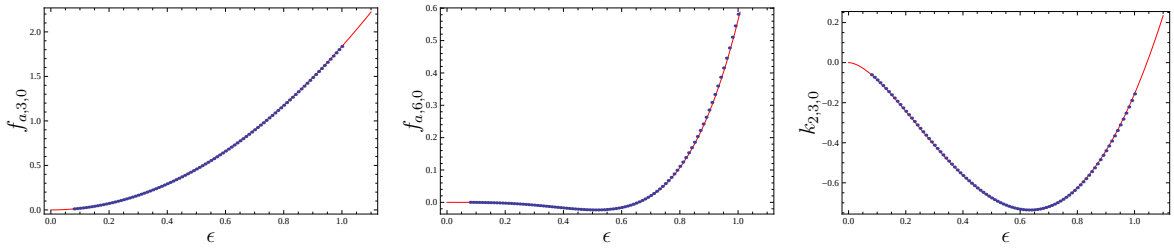

Figure 2: (Colour online) Comparison of values of select UV parameters $\{f_{a,3,0}, f_{a,6,0}, k_{2,3,0}\}$ of Klebanov-Strassler state obtained numerically (blue dots) with the analytic prediction (red curves), see (4.53).

in the UV, and

$$
\begin{aligned}
& f^h_{a,0} = 2^{1/3}\ 3^{2/3}\ \epsilon^{4/3}\,, \qquad h^h_0 = \epsilon^{-8/3}\ \times\ 0.056288(0)\,, \\
& k^h_{1,3} = \frac{4\sqrt{6}}{9\ \epsilon^2}\,, \qquad k^h_{2,2} = \frac{2^{2/3}}{3^{2/3}\ \epsilon^{4/3}}\,, \qquad k^h_{2,4} = -\frac{11\ 2^{1/3}\ 3^{2/3}}{45\ \epsilon^{8/3}}\,, \\
& k^h_{3,1} = \frac{4\sqrt{6}\ 2^{1/3}\ 3^{2/3}}{27\ \epsilon^{2/3}}\,, \qquad g^h_0 = 1\,,
\end{aligned}
\tag{4.54}
$$

in the IR. Notice that inverting the first identification in (4.53), $\epsilon = \epsilon(K_0)$, we obtain a prediction for all the parameters (4.42) as a function of K_0.

Figures 2 and 3 compare the results of select UV and IR parameters in (4.42) obtained numerically (blue dots) with analytic predictions (red curves) (4.53) and (4.54) for the supersymmetric Klebanov-Strassler state. In this numerical computation we must set $\kappa = 0$. Notice that in Klebanov-Strassler state the string coupling is identically constant, *i.e.,* $g = 1$. The latter in particular implies that $g_{4,0} = 0$ and $g^h_0 = 1$. To find our numerical solutions, we set those values as constants and eliminate the second order equation (4.12) for g, finding excellent agreement between the expected and the numerical result.

As we mentioned earlier, we are after the states of cascading gauge theory with broken chiral symmetry on dS_4, *i.e.,* the deformations of Klebanov-Strassler states at $\kappa = 1$. In practice we start with numerical Klebanov-Strassler state at $K_0 = 0.25$ ($P = 1$) and increase κ in increments of $\delta\kappa = 10^{-3}$ up to $\kappa = 1$. The resulting state is then used as a starting point to explore the states of cascading gauge theory on dS_4 with χSB for other values of $K_0 \neq 0.25$.

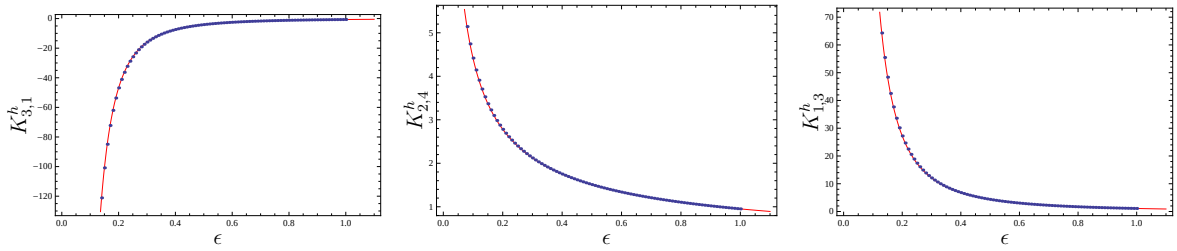

Figure 3: (Colour online) Comparison of values of select IR parameters $\{K^h_{3,1}, K^h_{2,4}, K^h_{1,3}\}$ of Klebanov-Strassler state obtained numerically (blue dots) with the analytic prediction (red curves), see (4.54).

5 Ground state of cascading gauge theory on dS_4

Recall that effective potential $\mathcal{V}_{eff}$ of a theory on dS_4 is defined (by analogy with the free energy density in thermodynamics) via

$$e^{-V_4^E\ \mathcal{V}_{eff}} = \mathcal{Z}_E\,, \tag{5.1}$$

where $\mathcal{Z}_E$ is a Euclidean partition function of the theory on dS_4, and V_4^E is a volume of the analytically continued de Sitter, $dS_4 \to S^4$,

$$V_4^E = \frac{8\pi^2}{3H^4}\,. \tag{5.2}$$

For a cascading gauge theory with a dual gravitational action given by (2.1), the effective potential is

$$\mathcal{V}_{eff} = \int_{\rho_{UV}}^{\infty} d\rho\ \mathcal{L}_E\,, \tag{5.3}$$

where $\mathcal{L}_E$ is the Euclidean one-dimensional Lagrangian density corresponding to the state, and ρ_{UV} is the UV cut-off, regularizing the Euclidean gravitational action in (5.3). Briefly, holographic renormalization of the theory modifies the effective potential

$$\int_{\rho_{UV}}^{\infty} d\rho\ \mathcal{L}_E \to \int_{\rho_{UV}}^{\infty} d\rho\ \mathcal{L}_E + S_{GH}^{\rho_{UV}} + S_{counterterms}^{\rho_{UV}}\,, \tag{5.4}$$

to include the Gibbons-Hawking and the local counterterms at the cut-off boundary $\rho = \rho_{UV}$ in a way that would render the renormalized effective potential finite in the limit $\rho_{UV} \to 0$.

Here, we have to distinguish two states of cascading gauge theory: with broken (we use the superscript ^{b}) and the unbroken (we use the superscript ^{s}) chiral symmetry.

These states are constructed (numerically) in sections 4 and 3 correspondingly. Given a cascading gauge theory on dS_4, *i.e.,* having fixed its strong coupling scale Λ, the dilaton asymptotic value g_0, the rank offset parameter P, and the Hubble scale H, the true ground state of the theory minimizes the effective potential $\mathcal{V}_{eff}$.

We now present some computational details of $\mathcal{V}^b_{eff}$ — the effective potential of the state of cascading gauge theory on dS_4 with (spontaneously) broken chiral symmetry. Using the equations of motion (4.5)-(4.13), it is possible to show that the on-shell gravitational Lagrangian (2.1) takes form

$$\begin{aligned}\mathcal{L}^b_E =&\frac{108}{16\pi G_5}\times\left(\frac{d}{d\rho}\left(\frac{2c_1^3c_1'\Omega_1\Omega_2^2\Omega_3^2}{c_3}\right)-6\kappa\ c_1^2c_3\Omega_1\Omega_2^2\Omega_3^2\ \right)\\ =&-\frac{108}{16\pi G_5}\times\left(\ \frac{d}{d\rho}\left(\frac{f_c^{1/2}f_af_b(\rho h'+4h)}{216h\rho^4}\right)+\frac{\kappa}{18}\ \frac{hf_af_bf_c^{1/2}}{\rho^3}\ \right),\end{aligned}\tag{5.5}$$

leading to

$$\begin{aligned}\frac{16\pi G_5}{108}\ \mathcal{V}^b_{eff} =&-\left(\frac{f_c^{1/2}f_af_b(\rho h'+4h)}{216h\rho^4}\right)\Big|^{\infty}_{\rho=\rho_{UV}}-\frac{\kappa}{18}\ \int_{\rho_{UV}}^{\infty}d\rho\ \frac{hf_af_bf_c^{1/2}}{\rho^3}\\ =&-\left(\frac{f_c^{1/2}f_af_b(\rho h'+4h)}{216h\rho^4}\right)\Big|_{\rho=\rho_{UV}}-\frac{\kappa}{18}\ \int_{\rho_{UV}}^{\infty}d\rho\ \frac{hf_af_bf_c^{1/2}}{\rho^3},\end{aligned}\tag{5.6}$$

where we used the fact that (see (4.26)-(4.29))

$$\lim_{\rho\to\infty}\ \frac{f_c^{1/2}f_af_b(\rho h'+4h)}{216h\rho^4}=-\lim_{y\to0}\frac{(f_c^h)^{1/2}f_a^hf_b^h(h^h)'}{216h^h}=0\,.\tag{5.7}$$

Both terms in (5.6) are divergent as $\rho_{UV}\to0$. First, using the asymptotic expansion (4.15)-(4.18), we isolate the divergence of the integral in (5.6):

$$\mathcal{I}^b_{\rho_{UV}}\equiv-6\kappa\ \int_{\rho_{UV}}^{1}d\rho\ \frac{f_af_bf_c^{1/2}h}{\rho^3}\equiv\mathcal{I}^b_{finite}+\mathcal{I}^b_{\rho_{UV},divergent}+\mathcal{O}\left(\rho_{UV}\ln^2\rho_{UV}\right)\,,\tag{5.8}$$

$$\begin{aligned}&\mathcal{I}^b_{finite}=-6\kappa\int_0^1d\rho\ \left(\frac{f_af_bf_c^{1/2}h}{\rho^3}-\mathcal{J}^b_{divergent}\right),\\ &\mathcal{J}^b_{divergent}=\frac{1}{\rho^3}\left(\frac{1}{8}g_0P^2+\frac{1}{4}K_0-\frac{1}{2}P^2g_0\ln\rho\right)+\frac{1}{\rho^2}\left(\frac{1}{4}\alpha^b_{1,0}g_0P^2\ln\rho\right.\\ &\left.-\frac{1}{16}\alpha^b_{1,0}(5g_0P^2+2K_0)\right)+\frac{1}{\rho}\left(-\frac{1}{8}P^4\kappa g_0^2\ln^2\rho+\left(\frac{1}{8}K_0P^2\kappa g_0+\frac{5}{48}P^4\kappa g_0^2\right)\ln\rho\right.\\ &\left.+\frac{1}{16}(\alpha^b_{1,0})^2P^2g_0+\frac{67}{1152}P^4\kappa g_0^2-\frac{5}{96}K_0P^2\kappa g_0-\frac{1}{32}K_0^2\kappa\right),\end{aligned}\tag{5.9}$$

$$
\begin{aligned}
&\mathcal{I}^b_{\rho_{UV},divergent} = -6\kappa \int_{\rho_{UV}}^{1} d\rho\ \mathcal{J}^b_{divergent} \\
=&\frac{1}{\rho_{UV}^2}\left(\frac{3}{2}\kappa g_0 P^2 \ln\rho_{UV} - \frac{3}{8}\kappa(-g_0P^2+2K_0)\right) \\
&+\frac{1}{\rho_{UV}}\left(-\frac{3}{2}\kappa\alpha^b_{1,0} g_0 P^2 \ln\rho_{UV} + \frac{3}{8}\kappa\alpha^b_{1,0}(g_0P^2+2K_0)\right) \\
&-\frac{1}{4}\kappa^2 P^4 g_0^2 \ln^3\rho_{UV} - \frac{1}{192}\kappa(-72K_0P^2\kappa g_0 - 60P^4\kappa g_0^2)\ln^2\rho_{UV} \\
&-\frac{1}{192}\kappa(-72(\alpha^b_{1,0})^2P^2g_0 - 67P^4\kappa g_0^2 + 60K_0P^2\kappa g_0 + 36K_0^2\kappa)\ln\rho_{UV} \\
&+\left\{-\frac{1}{192}\kappa(-72\alpha^b_{1,0}g_0P^2 - 144K_0 - 144\alpha^b_{1,0}K_0 + 72g_0P^2)\right\},
\end{aligned}
\tag{5.10}
$$

where in the last line we separated the finite piece coming from the upper limit of integration in $\mathcal{I}^b_{\rho_{UV},divergent}$. The superscript ^{b} in the UV parameter $\alpha_{1,0}$ is used to indicate that it is computed in the phase with broken chiral symmetry. Combining the divergent terms in (5.10) with divergences of the boundary term in (5.6) we find

$$
\begin{aligned}
&\frac{16\pi G_5}{108}\ \mathcal{V}^b_{eff} = \left\{\mathcal{V}^b_{eff,-4}\ \frac{1}{\rho^4} + \mathcal{V}^b_{eff,-3}\ \frac{1}{\rho^3} + \mathcal{V}^b_{eff,-2}\frac{1}{\rho^2} + \mathcal{V}^b_{eff,-1}\ \frac{1}{\rho} + \mathcal{V}^b_{eff,0}\right. \\
&\left.+\mathcal{O}(\rho^0)\right\}\Big|_{\rho=\rho_{UV}},
\end{aligned}
\tag{5.11}
$$

with

$$
\mathcal{V}^b_{eff,-4} = \frac{K_0 - 2\ln\rho}{27(1+2K_0-4\ln\rho)}\,, \tag{5.12}
$$

$$
\mathcal{V}^b_{eff,-3} = \frac{\alpha^b_{1,0}}{27(1+2K_0-4\ln\rho)^2}\left(16\ln\rho^2 - (4(1+4K_0))\ln\rho + 1 + 2K_0 + 4K_0^2\right), \tag{5.13}
$$

$$
\begin{aligned}
\mathcal{V}^b_{eff,-2} =& -\frac{1}{3888(1+2K_0-4\ln\rho)^3}\left(6912\ln\rho^4 - (192(37+72K_0-36(\alpha^b_{1,0})^2))\ln\rho^3\right. \\
&+(32(43+333K_0-108(\alpha^b_{1,0})^2+324K_0^2-324K_0(\alpha^b_{1,0})^2))\ln\rho^2 - (4(-97 \\
&+344K_0-360(\alpha^b_{1,0})^2+1332K_0^2-864K_0(\alpha^b_{1,0})^2+864K_0^3 \\
&-1296K_0^2(\alpha^b_{1,0})^2))\ln\rho - 99 - 194K_0 + 36(\alpha^b_{1,0})^2 + 344K_0^2 - 720K_0(\alpha^b_{1,0})^2 \\
&\left.+888K_0^3 - 864K_0^2(\alpha^b_{1,0})^2 + 432K_0^4 - 864K_0^3(\alpha^b_{1,0})^2\right),
\end{aligned}
\tag{5.14}
$$

$$\begin{aligned}\mathcal{V}^b_{eff,-1} =&\frac{\alpha^b_{1,0}}{3888(1+2K_0-4\ln\rho)^4}\Big(-27648\ln\rho^5+(1536(32+45K_0-6(\alpha^b_{1,0})^2))\ln\rho^4\\&-(64(413+1536K_0-108(\alpha^b_{1,0})^2+1080K_0^2-288K_0(\alpha^b_{1,0})^2))\ln\rho^3\\&+(48(161+826K_0-88(\alpha^b_{1,0})^2+1536K_0^2-216K_0(\alpha^b_{1,0})^2+720K_0^3\\&-288K_0^2(\alpha^b_{1,0})^2))\ln\rho^2-(16(134+483K_0+21(\alpha^b_{1,0})^2+1239K_0^2\\&-264K_0(\alpha^b_{1,0})^2+1536K_0^3-324K_0^2(\alpha^b_{1,0})^2+540K_0^4-288K_0^3(\alpha^b_{1,0})^2))\ln\rho\\&+301+1072K_0-300(\alpha^b_{1,0})^2+1932K_0^2+168K_0(\alpha^b_{1,0})^2+3304K_0^3\\&-1056K_0^2(\alpha^b_{1,0})^2+3072K_0^4-864K_0^3(\alpha^b_{1,0})^2+864K_0^5-576K_0^4(\alpha^b_{1,0})^2\Big)\,,\end{aligned}\tag{5.15}$$

$$\begin{aligned}\mathcal{V}^b_{eff,0} =&-\frac{1}{432}\ln\rho^3+\frac{1}{3456}(13+12K_0)\ln\rho^2-\frac{1}{82944}(103+312K_0-576(\alpha^b_{1,0})^2\\&+144K_0^2)\ln\rho\,,\end{aligned}\tag{5.16}$$

where we set $P=1$, $g_0=1$, $\kappa=1$, and used (4.15)-(4.18). Turns out that all the divergences are removed once we include the generalized[10] Gibbons-Hawking term, see [23],

$$S^{\rho_{UV}}_{GH}=\frac{108}{8\pi G_5}\,\frac{1}{c_3}\left(c_1^4\Omega_1\Omega_2^2\Omega_3^2\right)'\Big|_{\rho=\rho_{UV}}=\frac{1}{8\pi G_5}\,\frac{\rho}{h^{1/4}}\left(\frac{h^{1/4}f_c^{1/2}f_af_b}{\rho^4}\right)'\Big|_{\rho=\rho_{UV}}\,,\tag{5.17}$$

and the local counter-terms obtained in [23] with the following obvious modifications:

$$K^{KT}=\frac{1}{2}K_1+\frac{1}{2}K_3\,,\qquad \Omega_1^{KT}=3\Omega_1\,,\qquad \Omega_2^{KT}=\frac{\sqrt{6}}{2}\left(\Omega_2+\Omega_3\right)\,.\tag{5.18}$$

We find

$$\begin{aligned}&16\pi G_5\;\mathcal{V}^b_{eff}=3f_{c,4,0}+\frac{9}{32}(\alpha^b_{1,0})^2+\frac{3}{16}K_0(\alpha^b_{1,0})^2+\frac{59}{48}K_0+\frac{805}{1152}-\frac{3}{4}\alpha^b_{1,0}K_0\\&-\frac{3}{8}\alpha^b_{1,0}-\frac{1}{8}K_0^2+\mathcal{I}^b_{finite}+\int_0^1 dy\;(-6h^hf_a^hf_b^h(f_c^h)^{1/2})+\mathcal{V}^b_{ambiguity}\,,\\&\mathcal{V}^b_{ambiguity}=-36\kappa_1^bK_0^2-36\kappa_2^bK_0-36\kappa_3^b\,,\end{aligned}\tag{5.19}$$

where $\mathcal{V}^b_{ambiguity}$ comes from the renormalization scheme ambiguities $\{\kappa_i^b\}$, see [23]. Note that the ambiguities are completely specified by the gauge theory parameters,

[10] "Generalized" five-dimensional Gibbons-Hawking term is just a dimensional reduction of the 10-dimensional Gibbons-Hawking term corresponding to (2.4).

i.e., $\{K_0, P, g_0\}$ and the Hubble scale H, (the non-normalizable coefficients of the holographic gravitational dual).

Identical analysis for the symmetric phase leads to

$$
\begin{aligned}
&16\pi G_5\ \mathcal{V}^s_{eff} = 3a_{4,0} + \frac{805}{1152} - \frac{3}{8}\alpha^s_{1,0} + \frac{59}{48}K_0 + \frac{9}{32}(\alpha^s_{1,0})^2 - \frac{3}{4}\alpha^s_{1,0}K_0 + \frac{3}{16}(\alpha^s_{1,0})^2K_0 \\
&- \frac{1}{8}K_0^2 + \mathcal{I}^s_{finite} + \int_0^1 dy\ (-6h^h(f^h_3)^2(yf^h_2)^{1/2}) + \mathcal{V}^s_{ambiguity}\,, \\
&\mathcal{I}^s_{finite} = -6\int_0^1 d\rho\ \left(\frac{f_3^2 f_2^{1/2} h}{\rho^3} - \mathcal{J}^s_{divergent}\right), \\
&\mathcal{J}^s_{divergent} = \frac{1}{\rho^3}\left(\frac{1}{8} + \frac{1}{4}K_0 - \frac{1}{2}\ln\rho\right) + \frac{1}{\rho^2}\left(\frac{1}{4}\alpha^s_{1,0}\ln\rho - \frac{1}{16}\alpha^s_{1,0}(5+2K_0)\right) \\
&+ \frac{1}{\rho}\left(-\frac{1}{8}\ln^2\rho + \left(\frac{1}{8}K_0 + \frac{5}{48}\right)\ln\rho + \frac{1}{16}(\alpha^s_{1,0})^2 + \frac{67}{1152} - \frac{5}{96}K_0 - \frac{1}{32}K_0^2\right), \\
&\mathcal{V}^s_{ambiguity} = -36\kappa^s_1 K_0^2 - 36\kappa^s_2 K_0 - 36\kappa^s_3\,.
\end{aligned}
\tag{5.20}
$$

We can now compare the effective potentials of a chirally symmetric state and a state spontaneously breaking chiral symmetry for a cascading gauge theory on dS_4 (we restored the full $\{P, g_0, H\}$ dependence)

$$
\begin{aligned}
&16\pi G_5\left(\mathcal{V}^b_{eff} - \mathcal{V}^s_{eff}\right) = 3(f_{c,4,0} - H^4 a_{4,0}) + \frac{3}{16}(-3P^2\alpha^b_{1,0}g_0 + 2P^2 g_0 - 2K_0\alpha^b_{1,0} \\
&+ 4K_0)H^2(H\alpha^s_{1,0} - \alpha^b_{1,0}) - \frac{3}{32}(3P^2 g_0 + 2K_0)H^2(H\alpha^s_{1,0} - \alpha^b_{1,0})^2 \\
&+ \left(\mathcal{I}^b_{finite} - \mathcal{I}^s_{finite}\right) + H^2\left(\int_0^1 dy\ (-6h^h f^h_a f^h_b (f^h_c)^{1/2}) - \int_0^1 dy\ (-6h^h(f^h_3)^2(yf^h_2)^{1/2})\right),
\end{aligned}
\tag{5.21}
$$

where we used the *same* renormalization scheme for computing both $\mathcal{V}^b_{eff}$ and $\mathcal{V}^s_{eff}$, *i.e.,* we set

$$
H^{-4}\kappa_i\Big|^b = H^{-4}\kappa_i\Big|^s, \qquad i = 1, 2, 3\,. \tag{5.22}
$$

Figure 4 presents effective potentials (and their difference) between the state with spontaneously broken chiral symmetry, $\mathcal{V}^b_{eff}$, and the chirally symmetric state, $\mathcal{V}^s_{eff}$, of cascading gauge theory on dS_4 as a function of $\ln\frac{H^2}{\Lambda^2}$. Over the range of $\frac{H}{\Lambda}$ studied[11],

$$
\frac{16\pi G_5}{P^4 g_0^2} \times \frac{\mathcal{V}^b_{eff} - \mathcal{V}^s_{eff}}{H^4} > 0\,, \qquad \ln\frac{H^2}{\Lambda^2} \geq -0.03\,, \tag{5.23}
$$

[11]It is difficult to keep our current numerical procedure stable for smaller values of $\frac{H}{\Lambda}$.

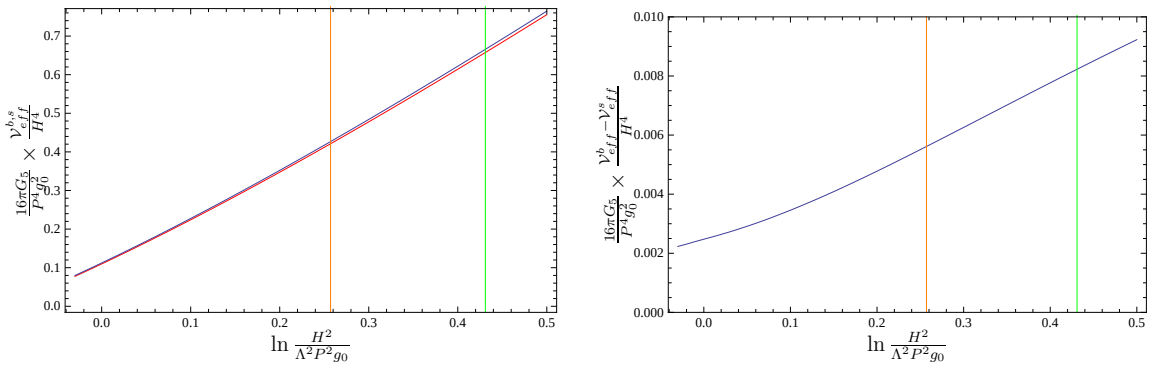

Figure 4: (Colour online) Left Panel: effective potentials of the chirally symmetric (V^s_{eff}, red) and the broken phase (V^b_{eff}, blue) of the cascading gauge theory on dS_4. Right Panel: the difference ($V^b_{eff} - V^s_{eff}$). The vertical lines represent the first order chiral symmetry breaking phase transitions of cascading gauge theory on S^3 [21] (green line) and at finite temperature [16] (orange line).

implying that chirally symmetric phase is a true ground state of cascading gauge theory on dS_4. For comparison, the vertical green and orange lines indicate the first order chiral symmetry breaking phase transitions of cascading gauge theory on S^3 [21] and at finite temperature [16].

6 Properties of dS_4 deformed KT/KS geometries

Given numerical constructions of dS_4 deformed KT/KS geometries as in section 3, we can compute the D3 brane charge at the tip of the conifold. Following [10], we find (see (3.21))

$$Q^{D3,s} = \frac{1}{27\pi} \lim_{y\to 0} K(y) = \frac{K_0^h}{27\pi}\,, \tag{6.1}$$

and (see (4.30)-(4.32))

$$Q^{D3,b} = \frac{1}{54\pi} \lim_{y\to 0} \Big(K_1(y)(2 - K_2(y)) + K_2(y)K_3(y) \Big) = 0\,, \tag{6.2}$$

where we use superscripts ^{b} and ^{s} to denote chiral symmetry broken (deformed KS) and chiral symmetry unbroken (deformed KT) phases.

Figure 5 presents D3 brane charge at the tip of the conifold of the dS_4 deformed KT throat geometry, $Q^{D3,s}$, as a function of $\frac{H}{\Lambda}$. Note that over all the range of parameters accessible with our numerical code $Q^{D3,s} > 0$.

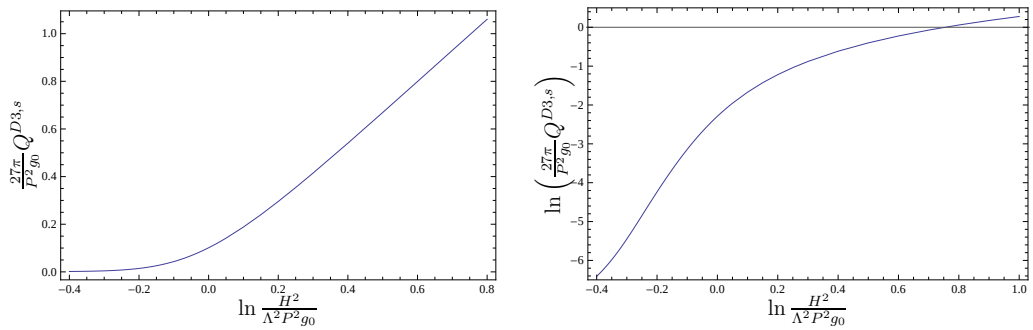

Figure 5: (Colour online) Left Panel: D3 brane charge at the tip of the conifold of the dS_4 deformed KT throat geometry, $Q^{D3,s}$, as a function of $\frac{H}{\Lambda}$. Right Panel: logarithm of D3 brane charge at the tip of the conifold of the dS_4 deformed KT throat geometry, $Q^{D3,s}$, as a function of $\frac{H}{\Lambda}$.

7 Properties of S^3 deformed KT/KS geometries

Using numerical constructions of S^3 deformed KT/KS geometries presented in [21], we can compute the D3 brane charge at the tip of the conifold. Following [10], we find (see eq.(3.24) of [21])

$$Q^{D3,s} = \frac{1}{27\pi} \lim_{y\to 0} K(y) = \frac{K_0^h}{27\pi}\,, \tag{7.1}$$

and (see eqs.(5.34)-(5.36) of [21])

$$Q^{D3,b} = \frac{1}{54\pi} \lim_{y\to 0}\Big(K_1(y)(2-K_2(y)) + K_2(y)K_3(y)\Big) = 0\,. \tag{7.2}$$

where we use superscripts ^{b} and ^{s} to denote chiral symmetry broken (deformed KS) and chiral symmetry unbroken (deformed KT) phases.

Figure 6 presents D3 brane charge at the tip of the conifold of the S^3 deformed KT throat geometry, $Q^{D3,s}$, as a function of $\frac{\mu_3}{\Lambda}$. Here, unlike the dS_4 deformed KT throat geometry, we find that $Q^{D3,s}$ can become negative! This happens whenever

$$\mu_3 < \mu_{3,negative}\,, \qquad \ln\frac{\mu^2_{3,negative}}{\Lambda^2 P^2 g_0} = 0.0318(3)\,, \tag{7.3}$$

which is represented by black vertical lines in figure 6. However, these negative values of $Q^{D3,s}$ are not physical. The issue is that prior we reach the compactification scale $\mu_{3,negative}$, namely at $\mu_{3,\chi\rm SB}$ [21]

$$\mu_{3,\chi\rm SB} > \mu_{3,negative}\,, \qquad \ln\frac{\mu^2_{3,\chi\rm SB}}{\Lambda^2 P^2 g_0} = 0.4309(8)\,, \tag{7.4}$$

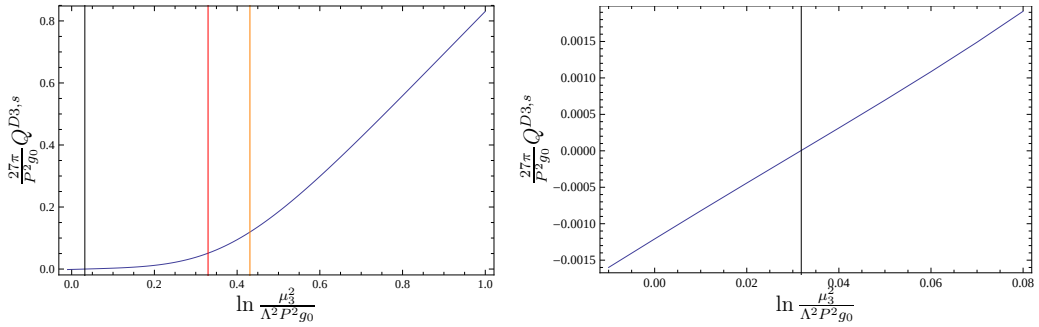

Figure 6: (Colour online) D3 brane charge at the tip of the conifold of the S^3 deformed KT throat geometry, $Q^{D3,s}$, as a function of $\frac{\mu_3}{\Lambda}$. The vertical orange line represents the value of the compactification scale $\mu_{3,\chi\text{SB}}$ below which it becomes energetically favourable to tunnel to S^3 deformed KS throat geometry, with spontaneous breaking of chiral symmetry. The vertical red line represents the value of the compactification scale $\mu_{3,tachyon}$ below which some of the linearized fluctuations (spontaneously breaking the chiral symmetry) become tachyonic. The vertical black lines denote the value of the compactification scale $\mu_{3,negative}$ below which $Q^{D3,s} < 0$.

chirally symmetric phase of cascading gauge theory on S^3 undergoes a first order phase transition to a symmetry broken phase (deformed KS geometry), where $Q^{D3,b} = 0$, see (7.2). This first order transition is further enhanced by perturbative tachyonic instabilities in chirally symmetric phase which arise at a slightly lower value of μ_3, namely at $\mu_{3,tachyon}$ [21]

$$\mu_{\chi\text{SB}} > \mu_{3,tachyon} > \mu_{3,negative}\,, \qquad \ln \frac{\mu_{3,tachyon}^2}{\Lambda^2 P^2 g_0} = 0.3297(3)\,. \tag{7.5}$$

Thus, a correct behaviour of the D3 charge at the tip of the conifold in S^3 deformed throat geometries is

$$Q^{D3} = \begin{cases} Q^{D3,s} > 0\,, & \mu_3 > \mu_{3,\chi\text{SB}}\,; \\ Q^{D3,b} = 0\,, & \mu_3 \leq \mu_{3,\chi\text{SB}}\,. \end{cases} \tag{7.6}$$

Once again, the D3 charge at the tip of the conifold is never negative.

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