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Reconnection of skewed vortices

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Based on experimental evidence that vortex reconnection commences with the approach of nearly anti-parallel segments of vorticity, a linearised model is developed in which two Burgers-type vortices are driven together and stretched by an ambient irrotational strain field induced by more remote vorticity. When these Burgers vortices are exactly anti-parallel, they are annihilated on a time-scale that remains O(1) as the kinematic viscosity ν tends to zero. Axial flow can be included, giving helical vortex lines in each vortex whose pitch increases exponentially under the action of the ambient strain field. When the vortices are skew to each other, then under this action they are annihilated over a local extent that increases exponentially in the stretching direction, with again clear evidence of reconnection on an O(1) time-scale. The initial writhe helicity associated with the skewed geometry is eliminated during the process of reconnection, and is not compensated by creation of twist helicity.

1. Introduction

The reconnection of vortex filaments has been recently visualised in water in the brilliant experiment of Kleckner & Irvine (2013). These authors have succeeded in generating a vortex in the form of a trefoil knot, which is highly unstable. Figure 1 shows four clips from the video in the supplementary material to this paper. The first clip (a) shows a recently formed trefoil vortex, visualised by air bubbles which are drawn to the pressure minimum in the vortex core. The second clip (b) shows three regions where initially remote parts of the vortex are apparently swept into close proximity; here, the colours red and blue are used to show that where the vortex strands are close, they are in fact nearly anti-parallel. Clip (c) shows a close-up of one of these regions, and clip (d) shows the same an instant later, revealing that a rapid reconnection, which may fairly be described as 'explosive', has occurred, with complicated evolution at the ends of the reconnecting region.

Two length-scales characterise this type of flow, the geometric scale L of the initial trefoil knot, and the cross-sectional radius δ of the vortex core, i.e. the tubular region in which the vorticity is essentially concentrated; outside this region, the flow is effectively irrotational. If the circulation of the vortex is Γ, then the induced velocity at any point near the vortex is of order $U_0 \sim \Gamma/L$, and the rate of strain at any such point is of order $\gamma_0 \sim \Gamma/L^2$†. The vortex moves and is deformed under the action of the induced velocity and strain fields, locally like the familiar Burgers vortex; its cross-sectional scale is then given in order of magnitude by $\delta \sim (\nu/\gamma_0)^{1/2} \sim \mathrm{Re}^{-1/2}L$, where $\mathrm{Re} = \Gamma/\nu$ is the vortex Reynolds number, and ν the kinematic viscosity of the fluid We shall assume throughout that $\mathrm{Re} \gg 1$, so that $\delta \ll L$. This condition was satisfied in the above experiment, for

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† This ignores the swirl component of velocity $\Gamma/2\pi r$ at small distance r from any element of the vortex, which does not contribute to motion of that element.

which the Reynolds number of the initial flow was in the range $10^3 - 10^4$. With $\mathrm{Re} = 10^4$, and with $L \sim 10^{-1}$m and $\nu \sim 10^{-6}$m^2/s for water, the above estimates are $\delta \sim 1$ mm, $U_0 \sim 10^{-1}$m/s, $\Gamma \sim 10^{-2}$m^2/s, and $\gamma_0 \sim 1\,\mathrm{s}^{-1}$.

Now let a be any intermediate scale satisfying $\delta \ll a \ll L$. Then in any sphere of radius O(a), centred at or near any point of the vortex, the rate of strain may be treated as approximately uniform. If we refer to the principal axes of strain, this strain field may be written

$$\mathbf{U}_s = (-\alpha x, -\beta y, \gamma z), \qquad (1.1)$$

where, by virtue of incompressibility, $\alpha + \beta = \gamma$, and we may suppose that $\alpha > 0, \gamma > 0, -\alpha \leqslant -\beta \leqslant \gamma$.

If $\beta > 0$ (the case of 'extensive strain'), then any small material element of fluid is swept in towards the z-axis, and stretched in the $\pm z$-directions. In this situation, if any two initially remote elements of the vortex happen to come within a distance $\sim a$ of each other in such a region of extensive strain, they will be swept in towards the z-axis and progressively aligned with it. It is this type of process that appears to be taking place in figures 1(b) and 1(c), in which the colours red and blue are used to indicate opposite directions of vorticity and the length of the nearly anti-parallel segments revealed in this way is presumably $\sim a$. The explosive reconnection that occurs in passing from figure 1(c) to 1(d) can then be interpreted as due to the persistent sweeping of these anti-parallel segments towards the z-axis.

It is these considerations that motivate the simple model considered in the following sections. For a more general discussion of the background to this problem, see Pullin & Saffman (1998). In the turbulence context, attempts to represent turbulence as a random distribution of vortex tubes and/or sheets go back to Burgers (1948), Townsend (1951) and Rott (1958). Concentrated vortex filaments have been identified in many direct numerical simulations (DNS) of turbulence (e.g. Vincent & Meneguzzi 1991), and in the experiment of Douady, Couder & Brachet (1991) in which vortices are visualised by small air bubbles. The analogous problem of magnetic tube reconnection is treated in the monograph of Priest & Forbes (2000), and simple models have been treated by Moffatt & Hunt (2002) and Hattori & Moffatt (2005).

2. Annihilation of Burgers vortices

For simplicity, we suppose that $\beta = \alpha$, and we first consider the action of the *axisymmetric* strain field

$$\mathbf{U} = (-\alpha x, -\alpha y, 2\alpha z), \quad \alpha > 0, \qquad (2.1)$$

on a vorticity distribution $(0, 0, \omega(x, y, t))$. The vorticity is swept towards the z-axis and stretched in the $\pm z$ directions. Within the context of 'rapid distortion theory' (RDT, Hunt & Carruthers 1990), the linearised vorticity equation is

$$\frac{\partial \omega}{\partial t} - \alpha x \frac{\partial \omega}{\partial x} - \alpha y \frac{\partial \omega}{\partial y} = 2\alpha \omega + \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right). \qquad (2.2)$$

Here, the 'self-interaction' of the vorticity field is neglected in comparison with the sweeping and stretching effect of the uniform strain field (2.1). Actually there is a potential conflict between this RDT approximation and the assumption $\mathrm{Re} \gg 1$ introduced above — see comment at the end of this section.

Equation (2.2) admits the well-known steady exact solution of the Navier-Stokes equa-
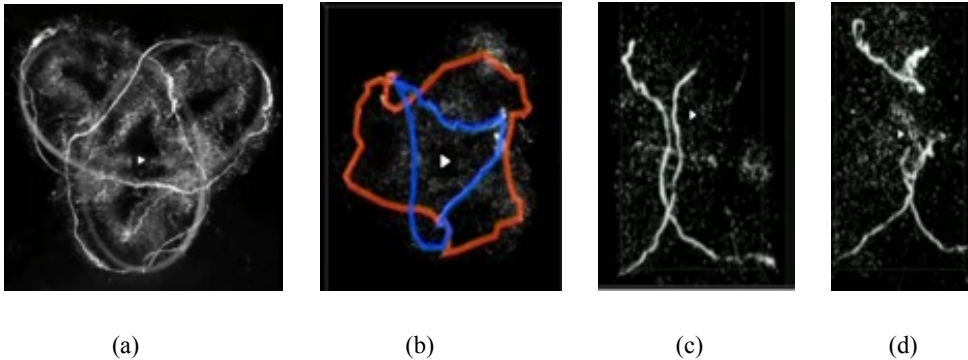
(a) (b) (c) (d)

FIGURE 1. Reconnection of a trefoil vortex in water (reproduced from Kleckner & Irvine 2013, with permission): (a) the recently formed unstable vortex visualised by air bubbles sucked into the low-pressure vortex core; (b) advanced stage of vortex evolution, in which opposite directions of vorticity are indicated by the colours red and blue; (c) a zoom-in to a region of nearly anti-parallel close vortex strands; (d) the same about 100ms later, after 'explosive' reconnection.

tions (Burgers 1948)

$$\omega_B(x,y) = (\Gamma/\pi\delta^2)e^{-r^2/\delta^2}, \quad \delta = \sqrt{2\nu/\alpha}, \tag{2.3}$$

where $r^2 = x^2 + y^2$, Γ is the circulation of the vortex, and δ is its radial scale. If we place a vortex of this structure with its centre-line at position $(0, y_0)$ at time $t = 0$, then it will be swept towards the line $(0,0)$ so that at time t its centre-line is at $(0, Y(t))$, where $Y(t) = y_0\,e^{-\alpha t}$. This moving vortex is still subject to the same uniform steady rate of strain α, and its vorticity field is therefore given by

$$\omega(x,y,t) = \omega_B(x, y - y_0\,e^{-\alpha t}) = (\Gamma/\pi\delta^2)e^{-r_1^2(t)/\delta^2}, \tag{2.4}$$

where $r_1^2(t) = x^2 + (y - y_0 e^{-\alpha t})^2$. It may be verified directly that (2.4) satisfies (2.2); this is still an exact solution, but now of the *unsteady* Navier-Stokes equations, in conjunction of course with the background uniform strain (2.1)†.

When $Y(t)(= y_0 e^{-\alpha t}) \ll \delta$, we may expand (2.4) in Taylor series:

$$\omega_B(x, y - Y(t)) = \omega_B(x,y) - Y(t)\frac{\partial\omega_B(x,y)}{\partial y} + \frac{1}{2}Y(t)^2\frac{\partial^2\omega_B(x,y)}{\partial y^2} + \dots. \tag{2.5}$$

Each term in this expansion is separately a solution of equation (2.2). In particular, the second term is

$$\omega_2(x,y,t) = (2y_0\Gamma/\pi\delta^4)e^{-\alpha t}\,y\,e^{-(x^2+y^2)/\delta^2}. \tag{2.6}$$

(Subsequent terms in the series, decaying as $e^{-n\alpha t}$ may be expressed in terms of Hermite polynomials $H_n(y)$.)

Suppose now that we have two vortices of equal and opposite circulations $\pm\Gamma$ with centre-lines initially at positions $(0, \pm y_0)$. The solutions are additive (vortex-vortex in-

† This solution is a particular case of the more general exact solution of the Navier-Stokes equations, $\omega(x,y,t) = \left(\Gamma/\pi\delta(t)^2\right)\exp\left[-[(x - x_0 e^{-\alpha t})^2 + (y - y_0 e^{-\alpha t})^2]/\delta(t)^2\right]$, with $\delta(t)^2 = (2\nu/\alpha)(1 - e^{-2\alpha t}) + \delta(0)^2 e^{-2\alpha t}$, which tends to the stable Burgers vortex as $t \to \infty$.

teraction being neglected) so that now the required solution is

$$\omega(x,y,t) = \omega_B(x, y - y_0\mathrm{e}^{-\alpha t}) - \omega_B(x, y + y_0\mathrm{e}^{-\alpha t}) = \frac{\Gamma}{\pi\delta^2}\left(\mathrm{e}^{-r_1^2(t)/\delta^2} - \mathrm{e}^{-r_2^2(t)/\delta^2}\right), \quad (2.7)$$

where $r_{1,2}^2(t) = x^2 + (y \mp y_0\mathrm{e}^{-\alpha t})^2$. Again, this is an exact solution of equation (2.2); it is not however an exact solution of the Navier-Stokes equations, because of the RDT neglect of vortex-vortex interactions.

When $Y(t) = y_0\mathrm{e}^{-\alpha t} \ll \delta$, we may again expand (2.7) as a Taylor series, in which now only the terms odd in y survive; at leading order,

$$\omega(x,y,t) \sim 2\omega_2(x,y,t) = (4y_0\Gamma/\pi\delta^4)\mathrm{e}^{-\alpha t}\, y\, \mathrm{e}^{-(x^2+y^2)/\delta^2}. \quad (2.8)$$

This describes the exponential decay of vorticity (and this despite the persistent stretching) on a time-scale $\sim \alpha^{-1}$. It should be noted that, although this decay is caused by viscosity $\nu > 0$, the time-scale of decay is independent of ν however small ν may be. It is in this sense that it may be considered to be a rapid, indeed explosive, process which it seems appropriate to describe as one of 'annihilation' of vorticity.

The neglected vortex-vortex interaction probably makes very little difference to this annihilation scenario. For so long as $Y(t)$ is still large compared with δ, this interaction merely provides an additional translational velocity $\Gamma/2Y(t)$ in the x-direction for the vortex pair. When $Y(t) \sim \delta$ or smaller, this translational velocity settles down to order Γ/δ; at this stage, the interaction presumably leads to some shedding of vorticity into a wake region in the manner described by Buntine & Pullin (1989), but the persistent inflow towards the (x,z)-plane will cause continuing rapid annihilation of this shed vorticity also; indeed it seems likely that the shedding of vorticity will, if anything, accelerate the overall annihilation process.

3. Effect of axial velocity in vortex cores

In a similar way, we may consider the addition to either vortex of an axial component of velocity $(0, 0, w(x,y,t))$, which evolves under the effect of the strain field (2.1) according to the equation

$$\frac{\partial w}{\partial t} - \alpha x\frac{\partial w}{\partial x} - \alpha y\frac{\partial w}{\partial y} = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \quad (3.1)$$

or equivalently

$$\frac{\partial w}{\partial t} - \alpha\left(\frac{\partial(xw)}{\partial x} + \frac{\partial(yw)}{\partial y}\right) + 2\alpha w = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right). \quad (3.2)$$

There is no pressure-gradient contribution in this equation because $\partial w/\partial z = 0$.

We assume that $w(x,y,t)$ is exponentially small for large $r^2 = x^2 + y^2$. Let

$$Q(t) = \int w(x,y,t)\, dx\, dy, \quad (3.3)$$

the axial flux within the vortex. Integrating (3.2) over the (x,y) plane, we have immediately that $dQ/dt + 2\alpha Q = 0$, so that this axial flux decays exponentially:

$$Q(t) = Q_0\, e^{-2\alpha t}. \quad (3.4)$$

It is then not difficult to show that (3.2) admits a corresponding solution

$$w(x,y,t) = (Q_0/\pi\delta^2)e^{-2\alpha t}\mathrm{e}^{-r^2/\delta^2}, \quad (3.5)$$
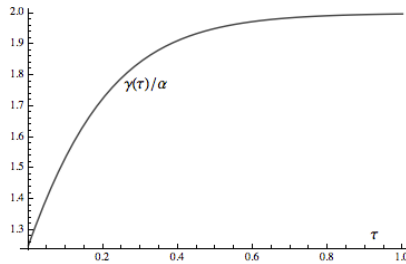
FIGURE 2. The effective stretch $\gamma(\tau)$ as a function of $\tau = \alpha t$, as given by (4.3) and (4.1), with initial condition $\beta_0 = \pi/6$. As τ increases, β decreases to zero, and $\gamma(\tau)$ rapidly asymptotes to 2α.

where still $\delta^2 = 2\nu/\alpha$. When combined with the Burgers solution (2.3), the vortex lines are now helices, which are stretched in the z-direction (with pitch increasing like $e^{2\alpha t}$) while being simultaneously subject to radial diffusion (cf the 'strained spiral vortex underlying Lundgren's 1982 model of turbulent fine structure). This combined solution provides a helical generalisation of the Burgers vortex, again an exact solution of the unsteady Navier-Stokes equation.

4. Skewed Burgers vortices

Suppose now that at time $t = 0$ we place a Burgers-type vortex with straight centre-line L_0 on the plane $y = y_0$ and tilted at an angle β_0 $(0 < \beta_0 < \pi/2)$ to the z-axis; L_0 is given in parametric form by $(x_0, y_0, z_0) = (p \sin \beta_0, y_0, p \cos \beta_0)$, where p is a parameter on the line running from $-\infty$ to $+\infty$. The gradient of L_0 is $m_0 = x_0/z_0 = \tan \beta_0$. We assume that the irrotational strain field (2.1) advects and stretches this line, sweeping it towards the z-axis. At time t, the point initially at (x_0, y_0, z_0) has moved to $(X, Y, Z) = (x_0 e^{-\alpha t}, y_0 e^{-\alpha t}, z_0 e^{2\alpha t})$, so that the gradient of the line, now $L(t)$, in the (x, z)-plane at time t is

$$m = \tan \beta = X/Z = (x_0/z_0)e^{-3\alpha t} = e^{-3\alpha t} \tan \beta_0. \tag{4.1}$$

Let

$$\mathbf{e}(t) = (\sin \beta(t), 0, \cos \beta(t)), \tag{4.2}$$

the unit vector directed along $L(t)$. Then the rate of stretch $\gamma(t)$ acting on the vortex at time t is

$$\gamma(t) = \mathbf{e} \cdot \nabla \mathbf{U} \cdot \mathbf{e} = -\alpha \sin^2 \beta + 2\alpha \cos^2 \beta = (3 \cos^2 \beta - 1)\alpha. \tag{4.3}$$

We may suppose that $\beta_0 < \cos^{-1}(1/\sqrt{3}) \approx 55^o$, so that $\gamma(t) > 0$ for all $t > 0$. Figure 2 shows the time variation of γ starting from an initial slope $\beta_0 = \pi/6$ (so $\gamma(0)/\alpha = 1.25$).

Under the basic assumption $\mathrm{Re} \gg 1$, the response of the vortex to this changing rate of stretch is quasi-static; the radial scale $\sigma(t)$ of the vortex adapts accordingly, and is given by

$$\frac{1}{\sigma(t)^2} = \frac{\gamma}{4\nu} = \frac{\alpha}{4\nu}(3 \cos^2 \beta(t) - 1). \tag{4.4}$$

Note that, although the strain field is not axisymmetric about the direction of $\mathbf{e}(t)$, the vortex itself at leading order remains axisymmetric about this direction, according to the asymptotic analysis of Moffatt, Kida & Ohkitani (1994).

Now the perpendicular distance r_p from any point $\mathbf{x} = (x, y, z)$ to the line $L(t)$ is given
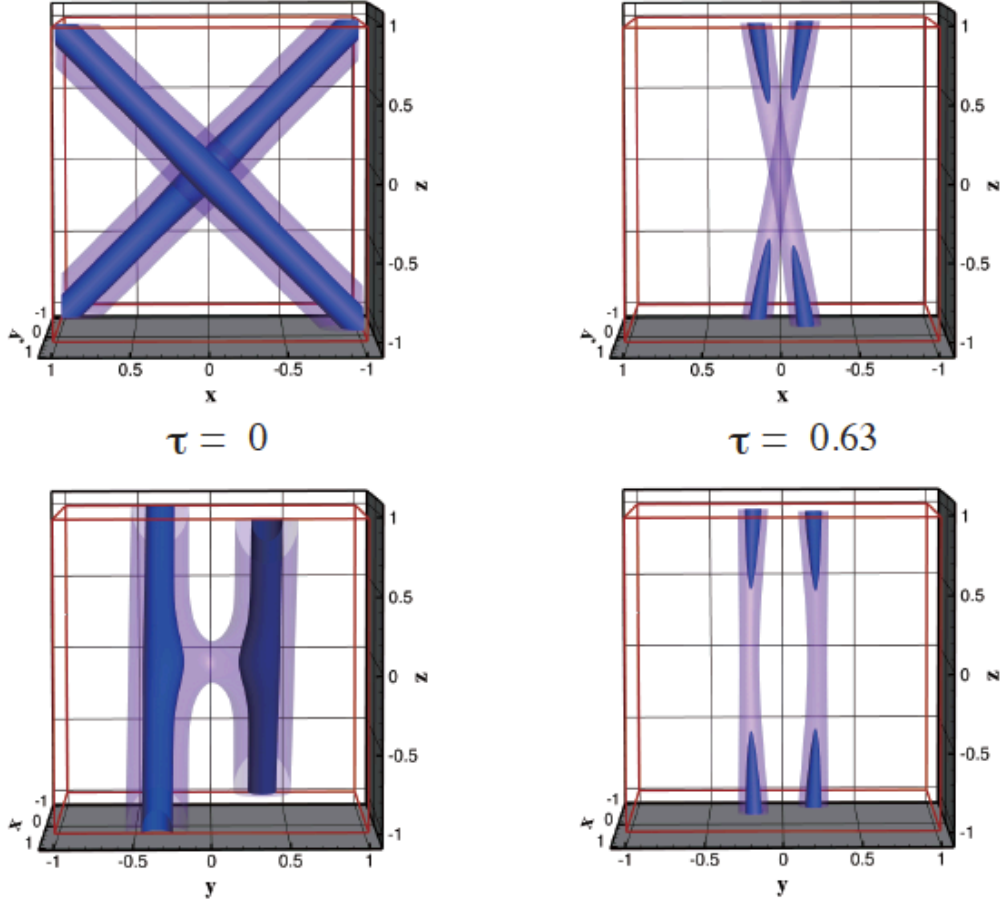
FIGURE 3. Skewed vortices offset in the y-direction (viewed in the y-direction above and in the x-direction below) subjected to the straining flow (2.1) that aligns them onto the z axis, showing unmistakable evidence of reconnection by the time $\tau = 0.63$; contour surfaces are $|\boldsymbol{\omega}|/|\boldsymbol{\omega}|_{max} = 0.95$ (blue) and 0.85 (purple); parameter values: $\beta_0 = \pi/4$, $\tilde{y}_0 = y_0/\delta = 1.356$. Note the 'bridge' in the lower view at $\tau = 0$, evidence of incipient reconnection.

by

$$r_p^2(\mathbf{x}, t) = (x \cos \beta(t) - z \sin \beta(t))^2 + (y - Y(t))^2, \qquad (4.5)$$

where still $Y(t) = y_0 e^{-\alpha t}$. The Burgers-type vortex centred on $L(t)$ may then be expected to have time-dependent vorticity field

$$\boldsymbol{\omega}(\mathbf{x}, t) = \frac{\Gamma}{\pi \sigma(t)^2} \exp\left[-r_p^2(\mathbf{x}, t)/\sigma(t)^2\right] \mathbf{e}(t) = \frac{\gamma(t)\Gamma}{4\pi\nu} \exp\left[-\gamma(t) r_p^2(\mathbf{x}, t)/4\nu\right] \mathbf{e}(t), \quad (4.6)$$

where $\mathbf{e}(t)$, $\gamma(t)$, and $r_p(\mathbf{x}, t)$ are given by (4.2), (4.3) and (4.5) respectively.

Unlike (2.4), this is not an exact solution of the Navier-Stokes equation, but it provides a leading-order approximation for the evolution of the vortex when $\mathrm{Re} \gg 1$. Note that for large t, when $\beta \to 0$ and $Y \to 0$, the solution approaches the steady exact Burgers solution (2.3), so presumably the field (4.6) becomes increasingly accurate as a solution of the Navier-Stokes vorticity equation, as time increases.

Now, just as in §2, we can superpose two solutions. We place a second vortex with circulation $-\Gamma$, centre-line on the plane $y = -Y(t)$ and directed along $\mathbf{e}' = (-\sin\beta, 0, \cos\beta)$, and with the same Burgers-type cross-sectional structure; this is in effect the 'chopsticks model' of Kimura & Koikari (2004). For the second vortex, the vorticity field is

$$\boldsymbol{\omega}'(\mathbf{x}, t) = -(\gamma\Gamma/4\pi\nu)\exp\left(-\gamma r_p'^2/4\nu\right)\mathbf{e}', \qquad (4.7)$$

with

$$r_p'^2 = (x\cos\beta(t) + z\sin\beta(t))^2 + (y + Y(t))^2. \qquad (4.8)$$

The two vortices are now swept towards each other and overlap in a slicing scissor movement on the plane $y = 0$. The z-components of vorticity annihilate in an ever-increasing neighbourhood of $z = 0$ (actually $\sim e^{3\alpha t}$), just as in §2 above, resulting in reconnection of the vortex tubes. Figure 3 show two views of the vorticity field (in the y and x directions) given by the combined solution

$$\boldsymbol{\omega}(\mathbf{x}, t) + \boldsymbol{\omega}'(\mathbf{x}, t) = \frac{\gamma\Gamma}{4\pi\nu}\left[\exp\left(-\frac{\gamma r_p^2}{4\nu}\right)\mathbf{e}(t) - \exp\left(-\frac{\gamma r_p'^2}{4\nu}\right)\mathbf{e}'(t)\right]; \qquad (4.9)$$

these views are at the initial instant $\tau = \alpha t = 0$ and at a later dimensionless time $\tau = 0.63$. The value of the dimensionless initial separation $\tilde{y}_0 = y_0/\delta = 1.356$ is chosen so that reconnection is just beginning (as indicated by the presence of the 'bridge' of vorticity in the lower view – see Kida & Takaoka 1987, 1994). By the time $\tau = 0.63$, reconnection is already well advanced.

The neglected vortex-vortex interaction effect is a serious complicating factor, because each vortex tends to crank the other into a double spiral in the region of closest approach of the vortices. It seems likely however that, under the persistent action of the ambient strain, this interactive effect will, if anything, simply accelerate the reconnection process.

5. Helicity evolution during vortex reconnection

We can now address the interesting question of how the helicity of the vorticity distribution changes during the above type of reconnection process. The linkage helicity is certainly changed as a result of reconnection (as evident in the experiment of Kleckner & Irvine 2013), but it is possible that some or all of this linkage helicity is converted to internal twist helicity during reconnection. For the analogous problem of magnetic flux-tube reconnection, it has been argued by Wright & Berger (1989) that (magnetic) helicity is converted in this way during reconnection, the total net helicity remaining nearly constant; this requires an appropriately ordered reconnection of 'sub-tubes', but it has never been convincingly established that this ordering is favoured by a natural diffusive process. The issue is important because the idea that magnetic fields relax in such a way as to minimise energy subject to conserved helicity (Taylor 1974) is one of the cornerstones of the MHD of turbulent fusion plasmas.

We start from the consideration that, for the two skewed vortices considered above, even without axial flow in either vortex, there is an interaction (or 'writhe') helicity arising from the fact that the velocity induced by either vortex has a non-zero component parallel to the other. When the vortices are well-separated, the resulting helicity is easily calculated: consider a steady Burgers vortex B_1 (with vorticity $\boldsymbol{\omega}_1$ and associated velocity $\mathbf{u_1}$) as given by (2.3), and a second Burgers vortex B_2 ($\boldsymbol{\omega}_2, \mathbf{u_2}$) as described by (4.6). Suppose that at time $t = 0$, the vortices are well separated, i.e. $y_0 \gg \delta$, and that the vortices have circulations $\pm\Gamma$ respectively. With $r^2 = x^2 + y^2$, for $r \gg \delta$ the velocity field

due to B_1 is just that due to a concentrated line vortex, i.e.

$$\mathbf{u_1}(\mathbf{x}) = (\Gamma/2\pi r^2)(-y, x, 0),\tag{5.1}$$

and this is approximately constant on any cross-section $p = $ cst. on B_2. So, on B_2,

$$\mathbf{u_1} = (\Gamma/2\pi)(p^2\sin^2\beta + y_0^2)^{-1}(-y_0, p\sin\beta, 0).\tag{5.2}$$

Hence, on B_2, with $\mathbf{e} = (\sin\beta, 0, \cos\beta)$,

$$\mathbf{u_1}\cdot\mathbf{e} = (-\Gamma y_0\sin\beta/2\pi)(p^2\sin^2\beta + y_0^2)^{-1},\tag{5.3}$$

and so, integrating first over the cross-section of the vortex, then along its axis,

$$\int_{B_2}\mathbf{u_1}\cdot\boldsymbol{\omega}_2\,dV = \frac{\Gamma^2 y_0\sin\beta}{2\pi}\int_{-\infty}^{\infty}\frac{dp}{p^2\sin^2\beta + y_0^2} = \Gamma^2/2.\tag{5.4}$$

By symmetry, we have a similar result for the integral over B_2, and hence the total initial helicity is

$$\mathcal{H} = \int\mathbf{u}\cdot\boldsymbol{\omega}\,dV = \Gamma^2,\tag{5.5}$$

the integral now being over all space. Note that this helicity is determined solely by the instantaneous vorticity distribution and is unaffected by the presence of the background irrotational strain. It admits interpretation as the writhe helicity (Moffatt & Ricca 1992); if the two vortices are closed by semi-circles at infinity, then there is an additional helicity contribution $\pm\Gamma^2$ from these semi-circles, the total being then $2\Gamma^2$ or zero according as the vortices, closed in this way, are linked or unlinked† (Moffatt 1969). If the sign of circulation of one of the vortices is changed, then of course the helicity changes sign also.

  As the vortex B_2 is swept towards B_1 by the strain field, this helicity remains constant until the separation $Y(t)$ reduces to $O(\delta)$. We may calculate the helicity as a function of the dimensionless time $\tau = \alpha t$ as follows. The velocity $\mathbf{u_1}$ induced by the vortex B_1 is

$$\mathbf{u_1}(\mathbf{x}) = (\Gamma/2\pi r^2)(1 - \mathrm{e}^{-r^2/\delta^2})(-y, x, 0),\tag{5.6}$$

(which asymptotes to (5.1) for $r \gg \delta$) and the vorticity field $\boldsymbol{\omega}_2(\mathbf{x}, t)$ of B_2 is (from (4.6)

$$\boldsymbol{\omega}_2(\mathbf{x}, t) = \frac{\gamma(t)\Gamma}{4\pi\nu}\exp\left[-\gamma(t)r_p^2(\mathbf{x}, t)/4\nu\right]\mathbf{e}(t),\tag{5.7}$$

where $\mathbf{e(t)}$, $\gamma(t)$, and $r_p(\mathbf{x}, t)$ are still given by (4.2), (4.3) and (4.5) respectively. The helicity is given by

$$\mathcal{H}(\tau) = \int\mathbf{u}\cdot\boldsymbol{\omega}\,\mathrm{d}V = \int(\mathbf{u_1}\cdot\boldsymbol{\omega}_2 + \mathbf{u_2}\cdot\boldsymbol{\omega}_1)\,\mathrm{d}V = 2\int\mathbf{u_1}\cdot\boldsymbol{\omega}_2\,\mathrm{d}V,\tag{5.8}$$

so, substituting (5.6) and (5.7) and rearranging, we obtain,

$$\mathcal{H}(\tau) = \frac{\Gamma^2}{\pi^2}s(\tau)\sin\beta\iiint\frac{y}{r^2}(1 - \mathrm{e}^{-r^2/\delta^2})\mathrm{e}^{-s(\tau)r_p^2/\delta^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\tag{5.9}$$

where $s(\tau) = (3\cos^2\beta - 1)/2$.

  It is natural now to introduce dimensionless variables $(\hat{x}, \hat{y}, \hat{z}, \hat{Y}, \hat{y}_0) = \delta^{-1}(x, y, z, Y, y_0)$; moreover, let $Z = \hat{z}\sin\beta$ (a change of variable indicating that, after the onset of reconnection, the scale of the associated region increases in the z-direction like $\mathrm{cosec}\,\beta \sim \mathrm{e}^{3\alpha t}$).

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† For every linked configuration, there is a corresponding unlinked configuration obtained by rotating one of the semi-circles through an angle π about its diameter.
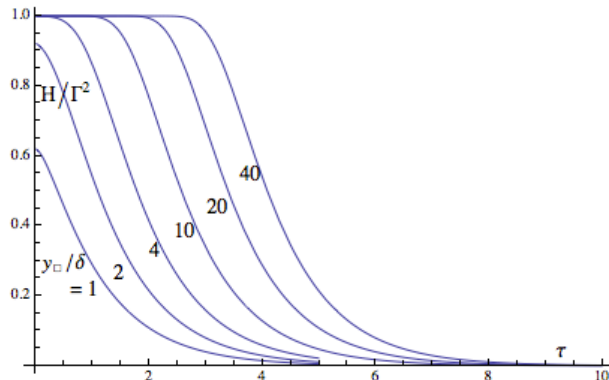
FIGURE 4. Dimensionless helicity $\mathcal{H}/\Gamma^2$ as a function of dimensionless time $\tau = \alpha t$ for the skewed vortices B_1 and B_2 driven together by the strain field (2.1), as evaluated from (5.12) for $\beta_0 = \pi/6$ and six values of the initial dimensionless separation $\hat{y}_0 = y_0/\delta$. For $\hat{y}_0 \gg 1$, the helicity remains constant until the vortices overlap, and then decays to zero exponentially in a time of order α^{-1}.

Then (5.9) becomes

$$\mathcal{H}(\tau) = \frac{\Gamma^2}{\pi^2} s(\tau) \iiint \frac{\hat{y}}{\hat{x}^2 + \hat{y}^2} (1 - e^{-(\hat{x}^2+\hat{y}^2)}) e^{-s(\tau)[(\hat{x}\cos\beta - Z)^2 + (\hat{y}-\hat{Y})^2]} \, d\hat{x} \, d\hat{y} \, dZ . \quad (5.10)$$

We first integrate with respect to Z (from $-\infty$ to $+\infty$) giving

$$\mathcal{H}(\tau) = \frac{s^{1/2}\Gamma^2}{\pi^{3/2}} \iint \frac{\hat{y}}{\hat{x}^2 + \hat{y}^2} (1 - e^{-(\hat{x}^2+\hat{y}^2)}) e^{-s(\hat{y}-\hat{Y})^2} \, d\hat{x} \, d\hat{y} . \quad (5.11)$$

We may now integrate with respect to $\hat{x}$ (again from $-\infty$ to $+\infty$) giving after some simplification†

$$\mathcal{H}(\tau) = \frac{s^{1/2}\Gamma^2}{\pi^{1/2}} \int_0^\infty \operatorname{erf} \hat{y} \left(e^{-s(\hat{y}-\hat{Y})^2} - e^{-s(\hat{y}+\hat{Y})^2} \right) d\hat{y} . \quad (5.12)$$

With $\hat{Y}(\tau) = \hat{y}_0 e^{-\tau}$, and $s(\tau)$ as defined below (5.9), this integral is now easily evaluated using *Mathematica* (note that for large $\hat{Y}$, the dominant contribution comes from a neighbourhood of $\hat{y} = \hat{Y}$). Figure 4 shows the result for an initial skewness angle $\beta_0 = \pi/6$, and for six values of the initial dimensionless separation $\hat{y}_0$. As expected, for $\hat{y}_0 \gg 1$, $\mathcal{H}/\Gamma^2$ remains equal to 1 (as anticipated in (5.5)) for so long as the vortices are well separated; however, as $\hat{Y}(\tau)$ decreases to O(1) and smaller, the helicity decays exponentially to zero in a time of order α^{-1}.

Unfortunately therefore, we find no evidence for helicity conservation during reconnection within the framework of our present model. On the contrary, linkage helicity is indeed destroyed during the reconnection process, but without any compensating generation of internal twist helicity (and we note that the model is even more reliable in the magnetic context, in which the magnetic analogue of equation (2.2) is exact in the weak magnetic field limit when Lorentz forces are negligible).

Again, of course, one may ask 'what if the vortex-vortex interaction terms are retained in this model problem?'. To answer this would require a full numerical simulation of the

† Care is needed to treat separately the cases for which $\hat{y}$ is positive or negative, the latter being then transformed by the change of variable $\hat{y} \to -\hat{y}$.

3D time-dependent Navier-Stokes equations which would inevitably run into the unsolved and deeply challenging finite-time singularity problem. We hope nevertheless to address this in future work. In the meantime all that can be said is that the linearised model described in this paper provides no evidence for conservation of helicity during viscous reconnection of vortex tubes.

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