A solution of the Gaussian optimizer conjecture

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Abstract

The long-standing conjectures of the optimality of Gaussian inputs for Gaussian channel and Gaussian additivity are solved for a broad class of covariant or contravariant Bosonic Gaussian channels (which includes in particular thermal, additive classical noise, and amplifier channels) restricting to the class of states with finite second moments. We show that the vacuum is the input state which minimizes the entropy at the output of such channels. This allows us to show also that the classical capacity of these channels (under the input energy constraint) is additive and is achieved by Gaussian encodings.

1 Introduction

The capacity of a communication channel is the maximum rate at which information (measured in bits per channel use) can be transmitted from sender to receiver with asymptotically vanishing error [1]. It provides an operationally defined measure of the communication efficiency of the channel by setting the ultimate limit at which messages can be transferred reliably. An explicit formula for this quantity in terms of an entropic functional of the conditional probability distribution that defines the noise model is given by the *Shannon noisy-channel coding theorem* [2]. For channels with additive Gaussian noise it amounts to the famous formula

$$C = \frac{1}{2} \log \left(1 + E/N\right), \tag{1}$$

where E/N is signal-to-noise ratio.

In the context of quantum information theory [3, 4], communication channels are described by linear, completely positive, trace preserving (CPTP) maps Φ which transform input density matrices to their output counterparts. For these models the analog of the Shannon noisy coding theorem was obtained in Refs. [5, 6] for finite-dimensional channels. It says that that the associated (classical) capacity is equal to

$$C(\Phi) = \lim_{n\to\infty} \frac{1}{n} C_\chi(\Phi^{\otimes n}), \tag{2}$$

where $\Phi^{\otimes n}$ is the map describing n channel uses (memoryless noise model), while C_χ is the χ-capacity defined by the expression

$$C_\chi(\Psi) = \max_{\mathcal{E}} \left\{ S(\Psi[\sum_j p_j \rho_j]) - \sum_j p_j S(\Psi[\rho_j]) \right\}, \tag{3}$$

where the maximization is performed over the set of (possibly constrained) input ensembles $\mathcal{E} = \{p_j; \rho_j\}$ (p_j being probabilities, while ρ_j being density matrices) and $S(\rho) = \mathrm{Tr}(\rho) \log(\rho)$ is the von Neumann entropy.

A special class of maps which play a fundamental role in quantum information theory constitute the so called Bosonic Gaussian Channels (BGCs) [7]. Among other noise models they describe thermal, attenuation, and amplification processes for all those communication setups where messages are encoded into the modes of the electromagnetic field (say optical fibers), i.e. the most common quantum communication architectures [8]. Computing the capacity of these channels is hence an important problem which has profound implications both from theoretical and technical point of view. Since BGCs act in infinite dimensions, the corresponding generalization of the coding theorem (2), (3), allowing for input constraint and continuous state ensembles is required, see [9, 10] and also [11], Ch.11. A long-standing conjecture, first proposed in [7], is that for these maps and energy constraints Gaussian encodings should provide the optimal communication rates, allowing to restrict the maximization of Eq. (3) over the set of Gaussian ensembles, a task which can be performed analytically (*optimal Gaussian ensemble conjecture*); moreover $C_\chi(\Phi)$ should be additive for BGC (*Gaussian additivity conjecture*) ensuring that no limit in (2) is necessary so that $C(\Phi) = C_\chi(\Phi)$. It turns out that in some important situations such statements can be reduced to similar conjectures on the output entropy of the channel Φ [12, 13, 14, 15]. In this formulation, Gaussian input states are supposed to provide the minimum for this quantity (*minimum output entropy conjecture*). Despite a number of indirect evidences of correctness (see e.g. Refs. [16, 17, 18]), up to date both the optimal Gaussian ensemble and the minimum output entropy conjectures remained open, except for a special class of quantum-limited attenuator (or lossy) channels [19]. Extending these results to a broader set of BGCs proved to be one of the major challenges in quantum information theory. In the present paper we give a solution to these problems by showing that vacuum input minimizes the entropy at the output of any multimode gauge-covariant or contravariant BGC, while the capacity constrained with an oscillator energy operator is attained on the corresponding Gaussian ensemble of coherent states. In our solution we restrict all optimizations to the class of states with finite second moments which natural when dealing with capacities of channels with energy-constrained inputs. As for the Gaussian minimal output entropy problem, this restriction can be relaxed but we postpone the solution to the future work [20].

The manuscript is organized as follows. In Sec. 2 we introduce the notation and define important classes of covariant quantum-limited attenuators, quantum-limited amplifiers, and quantum-limited contravariant channels. Furthermore we show (Proposition 1) that any Gaussian gauge-covariant channel can be expressed as a concatenation of a quantum-limited attenuator followed (in the Schrödinger picture) by a quantum-limited amplifier. In Sec. 3 we recall the additivity properties of entanglement-breaking channels, and observe that the quantum-limited contravariant channel is entanglement-breaking (Proposition 2) and shares the additivity properties with the complementary quantum-limited covariant amplifier (Proposition 3). In Sec. 4 we show that proving the minimal output entropy conjecture for an arbitrary covariant or contravariant channel reduces to proving it for a single-mode quantum-limited amplifier (Proposition 4). Also we show that proving the last fact allows one to compute the classical capacity of an arbitrary covariant (contravariant) channel under the energy constraint. In Sec. 5 we derive key identities that lead to the proof of the conjecture presented in Sec. 6. Specifically, Sec. 5 is devoted to characterization of the output states of the quantum-limited contravariant channel. In Proposition 5 we give an explicit measure-reprepare decomposition of these maps and present (Proposition 6) a covariant amplifier channel whose output states have the same spectrum as the original channel. Building up on these elements, in Sec. 6 we establish an important decomposition (Proposition 7) from which one can finally deduce that the minimal output entropy of a single-mode covariant amplifier is indeed achieved by the vacuum. In view of the results of Sec. 4, this proves both the Gaussian encoding and the minimal output entropy conjectures for the whole class of covariant and contravariant channels. The paper is concluded with Sec. 7 where we present specific examples which can be considered as counterparts of the Shannon formula (1) for quantum channels, and discuss further implications of our findings.

2 Gaussian gauge-covariant and contravariant channels

The scenery for considering gauge-covariant states and channels is $s-$ dimensional complex Hilbert space $\mathbf{Z}$ which can be considered as $2s-$dimensional real space equipped with the symplectic form $\Delta(z,z') = \Im z^* z'$. To be specific, we consider vectors in $\mathbf{Z}$ as $s-$dimensional complex column vectors, in which case (complex-linear) operators in $\mathbf{Z}$ are represented by complex $s \times s-$matrices, and * denotes Hermitian conjugation. The gauge group acts in $\mathbf{Z}$ as multiplication by $e^{i\phi}$, where ϕ is real number called phase. The Weyl quantization is described by displacement operators $D(z)$ acting irreducibly in the representation space $\mathcal{H}$ and satisfying the relation

$$D(z)D(z') = \exp\left(-i\Im \bar{z} z'\right) D(z+z'). \tag{4}$$

Introducing the annihilation (resp. creation) operators of the system a_j and $a_j^\dagger$ which satisfy the commutation relations $\left[a_j, a_k^\dagger\right] = \delta_{jk} I$, we recall that $D(z)$ can be expressed as

$$D(z) = \exp[\mathbf{a}^\dagger z - z^* \mathbf{a}] = \exp \sum_{j=1}^{s} \left(z_j a_j^\dagger - \bar{z}_j a_j \right), \tag{5}$$

where $\mathbf{a} = [a_1, \dots, a_s]^t$ and $\mathbf{a}^\dagger = \left[a_1^\dagger, \dots, a_s^\dagger\right]$ are respectively column and row vectors. Next let Λ be the antiunitary operator of complex conjugation in $\mathbf{Z}$ which anticommutes with multiplication by i and satisfies $\Lambda^* = \Lambda$, $\Lambda^2 = I$. The associated transposition map T acting on operators in $\mathcal{H}$ can then be defined by the relation $\mathrm{T}[D(z)] = D(\Lambda z)$, $z \in \mathbf{Z}$, or equivalently $\mathrm{T}[a_j] = a_j^\dagger$ and $\mathrm{T}[\mathbf{a}] = \Lambda \mathbf{a} = \left[a_1^\dagger, \dots, a_s^\dagger\right]^t$ (notice that the last is a row vector different from $\mathbf{a}^\dagger$).

The gauge group has the unitary representation $\phi \to U_\phi = e^{i\phi N}$ in $\mathcal{H}$ where $N = \sum_{j=1}^s a_j^\dagger a_j$ is the total number operator. A state ρ is then said to be gauge-invariant if it commutes with all U_ϕ. or, equivalently, if its (symmetrically ordered) characteristic function $\mathcal{F}(z) = \mathrm{Tr}\rho D(z)$ [21] is invariant under the action of the gauge group. In particular Gaussian gauge-invariant states are described by the property

$$\mathrm{Tr}\rho D(z) = \exp\left(-z^* \alpha z\right), \tag{6}$$

where α is a complex-linear covariance operator satisfying $\alpha \geq I/2$. The vacuum state $\rho_{vac} = |0\rangle\langle 0|$ is an element of this set with $\alpha = I/2$.

A channel Φ with the input space $\mathcal{H}_A$ and the output space $\mathcal{H}_B$ satisfying

$$\Phi[e^{i\phi N_A} \rho e^{-i\phi N_A}] = e^{\pm i\phi N_B} \Phi[\rho] e^{\mp i\phi N_B},$$

is called *gauge-covariant* (*gauge-contravariant*). We denote by $s_A = \dim \mathbf{Z}_A$, $s_B = \dim \mathbf{Z}_B$ the numbers of modes of the input and output of the channel.

In the Heisenberg representation, a multimode bosonic Gaussian gauge-covariant channel Φ [22, 11] is described by the action of its adjoint Φ^* onto displacement operators as follows:

$$\Phi^*[D_B(z)] = D_A(K^* z) \exp\left(-z^* \mu z\right), \tag{7}$$

where K is complex-linear operator from $\mathbf{Z}_A$ to $\mathbf{Z}_B$ and μ is complex Hermitian operator in $\mathbf{Z}_B$ satisfying the inequality (cf. [22], eq. (24))

$$\mu \geq \pm \frac{1}{2}\left(I - KK^*\right). \tag{8}$$

Notice, that if (and only if) the operators K and μ can be simultaneously diagonalized in an orthonormal basis, then the channel decomposes into tensor product of one-mode channels (cf. [22]). In this case we call the channel *diagonalizable*. The gauge-covariant channel is *quantum-limited* if μ is a minimal solution of this inequality.

Special cases of the maps (7) are provided by the attenuators and amplifier channels, characterized by matrix K fulfilling the inequalities, $KK^* \leq I$ and $KK^* \geq I$ respectively. We are particularly interested in *quantum-limited attenuator* which corresponds to

$$KK^* \leq I, \qquad \mu = \frac{1}{2}\left(I - KK^*\right), \tag{9}$$

and *quantum-limited amplifier*

$$KK^* \geq I, \qquad \mu = \frac{1}{2}\left(KK^* - I\right). \tag{10}$$

These channels are diagonalizable: by using singular value decomposition $K = V_B K_c V_A$, where V_A, V_B are unitaries and K_c is (rectangular) diagonal matrix with nonnegative values on the diagonal, we have $KK^* = V_B K_c {K_c}^* V_B^*$, and

$$\Phi[\rho] = U_B \Phi_c[U_A^* \rho U_A] U_B^*, \tag{11}$$

where Φ_c is a tensor product of one-mode a (quantum limited) channels defined by the matrix K_c and where U_A, U_B are passive canonical unitary transformations acting on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, such that

$$U_B^* \mathbf{a} U_B = V_A \mathbf{a}, \qquad U_A^* \mathbf{a} U_A = {V_B}^t \mathbf{a},$$

with $\mathbf{a}$ being the column vector formed by the annihilation operators introduced in Eq. (4) (notice that in particular this implies $U_A|0\rangle = |0\rangle, U_B|0\rangle = |0\rangle$).

A multimode *Gaussian gauge-contravariant channel* involving "phase inversion" acts as

$$\Phi^*[D_B(z)] = D_A(\Lambda K^* z) \exp\left(-z^* \mu z\right) = D_A(K^t \bar{z}) \exp\left(-z^* \mu z\right), \tag{12}$$

and μ is the Hermitian operator satisfying

$$\mu \geq \frac{1}{2}\left(I + KK^*\right), \tag{13}$$

(in the second identity of Eq. (12) we used the fact that for all z one has $\Lambda K^* z = K^t \bar{z}$ with K^t being the transpose of the matrix K and $\bar{z}$ being the column vector obtained by taking the complex conjugate of the elements of z). If the operators $K\Lambda$ and μ can be simultaneously diagonalized in an orthonormal basis, then this channel is equivalent in the sense of (11) to tensor product of one-mode channels and is called *diagonalizable*. These maps are quantum-limited if

$$\mu = \frac{1}{2}\left(I + KK^*\right) \tag{14}$$

and these are diagonalizable similarly to quantum-limited amplifiers.

All the examples of quantum-limited BGSs introduced above can also be expressed in terms of input-output equations for the column vectors of annihilation operators as

$$\begin{aligned}
\mathbf{a}_B &= K\mathbf{a}_A + \sqrt{I - KK^*}\mathbf{a}_E, \quad \textit{(attenuator)}, \\
\mathbf{a}_B &= K\mathbf{a}_A + \sqrt{KK^* - I}\Lambda\mathbf{a}_E, \quad \textit{(amplifier)}, \\
\mathbf{a}_B &= K\Lambda\mathbf{a}_A + \sqrt{I + KK^*}\mathbf{a}_E, \quad \textit{(gauge-contravariant)},
\end{aligned}$$

where the environment modes associated with $\mathbf{a}_E$ are in the vacuum state.

The concatenation $\Phi = \Phi_2 \circ \Phi_1$ of two channels Φ_1 and Φ_2 obeys the rule

$$K = K_2 K_1, \tag{15}$$

$$\mu = K_2 \mu_1 K_2^* + \mu_2. \tag{16}$$

The following proposition generalizes to many modes the decomposition of one-mode channels the usefulness of which was emphasized and exploited in the paper [15] (see also [23] on concatenations of one-mode channels):

Proposition 1 *Any bosonic Gaussian gauge-covariant channel Φ is a concatenation of quantum-limited attenuator Φ_1 and (diagonalizable) quantum-limited amplifier Φ_2.*

Any bosonic Gaussian gauge-contravariant channel Φ is a concatenation of quantum-limited attenuator Φ_1 and (diagonalizable) quantum-limited gauge-contravariant channel Φ_2.

Proof. By inserting

$$\mu_1 = \frac{1}{2}\left(I - K_1 K_1^*\right) = \frac{1}{2}\left(I - |K_1^*|^2\right), \quad \mu_2 = \frac{1}{2}\left(K_2 K_2^* - I\right) = \frac{1}{2}\left(|K_2^*|^2 - I\right)$$

into (16) and using (15) we obtain

$$\left|K_2^*\right|^2 = K_2 K_2^* = \mu + \frac{1}{2}(KK^* + I) \geq \begin{cases} I \\ KK^* \end{cases} \tag{17}$$

from the inequality (8). By using operator monotonicity of the square root, we have

$$|K_2^*| \geq I, \quad |K_2^*| \geq |K^*|.$$

The first inequality (17) implies that choosing $K_2 = |K_2^*| = \sqrt{\mu + \frac{1}{2}(KK^* + I)}$ and the corresponding $\mu_2 = \frac{1}{2}\left(\left|K_2^*\right|^2 - I\right)$, we obtain diagonalizable quantum-limited amplifier, since K_2 and μ_2 are commuting Hermitian operators. Notice, that in general we can take $K_2 = |K_2^*| V$, where V is arbitrary co-isometry, $VV^* = I$.

Then with $K_1 = |K_2^*|^{-1} K$ we obtain, taking into account the second inequality in (17)

$$K_1^* K_1 = K^* \left|K_2^*\right|^{-2} K = K^* \left[\mu + \frac{1}{2}(KK^* + I)\right]^{-1} K \leq K^*(KK^*)^- K \leq I,$$

where $^-$ means generalized inverse, which implies $K_1^* K_1 \leq I$, hence K_1 with the corresponding $\mu_1 = \frac{1}{2}(I - K_1 K_1^*)$ give the quantum-limited attenuator.

In the case of contravariant channel the equations (15) is replaced with

$$K\Lambda = K_2 \Lambda K_1.$$

By substituting this and

$$\mu_1 = \frac{1}{2}(I - K_1 K_1^*), \quad \mu_2 = \frac{1}{2}(K_2 K_2^* + I)$$

into (16) and using (13) we obtain

$$|K_2^*|^2 = K_2 K_2^* = \mu + \frac{1}{2}(KK^* - I) \geq KK^*.$$

Taking $K_2 = |K_2^*|$, $\mu_2 = \frac{1}{2}(|K_2^*|^2 + I)$ gives diagonalizable quantum-limited gauge-contravariant channel. With $K_1 = \Lambda^{-1} |K_2^*|^- K\Lambda = \Lambda |K_2^*|^- K\Lambda$ we obtain

$$\begin{aligned} K_1^* K_1 &= \Lambda K^* \left(|K_2^*|^-\right)^2 K\Lambda \\ &= \Lambda K^* \left[\mu + \frac{1}{2}(KK^* - I)\right]^- K\Lambda \leq K^*(KK^*)^- K \leq I, \end{aligned} \tag{18}$$

which implies $K_1 K_1^* \leq I$, hence K_1 with the corresponding μ_1 give the quantum-limited attenuator. ■

3 Entanglement breaking and additivity

The additivity properties of finite-dimensional entanglement-breaking channels [24] were generalized to infinite dimensions in [25]. Moreover it was shown in [25] that the convex closure of the output entropy for such channel Φ, defined as

$$\hat{S}_{\Phi}(\sigma) = \inf_{\pi:\bar{\rho}_{\pi}=\sigma} \int S(\Phi[\rho])\,\pi(d\rho), \tag{19}$$

where the infimum is taken over all probability distributions π on the state space with the baricenter $\bar{\rho}_{\pi} \equiv \int \rho\pi(d\rho) = \sigma$, is superadditive, i.e. for an entanglement-breaking channel Φ and arbitrary channel Ψ

$$\hat{S}_{\Phi\otimes\Psi}(\sigma_{12}) \geq \hat{S}_{\Phi}(\sigma_1) + \hat{S}_{\Psi}(\sigma_2) \tag{20}$$

for any state σ_{12}. This implies additivity of the minimal output entropy (see e.g. [11], Proposition 8.15)

$$\min_{\sigma_{12}} S(\Phi\otimes\Psi[\sigma_{12}]) = \min_{\sigma_1} S(\Phi[\sigma_1]) + \min_{\sigma_2} S(\Psi[\sigma_2]). \tag{21}$$

It turns out that (20) implies additivity of the constrained $\chi-$capacity of channel Φ, e.g. see [26], Sec. 6. Namely, let H is positive selfadjoint operator (typically the energy operator), $H^{(n)} = H\otimes I\cdots\otimes I + \cdots + I\otimes\cdots\otimes I\otimes H$. Under mild regularity assumptions (see [25]) which are fulfilled in the Gaussian case the constrained $\chi-$capacity is equal to

$$C_{\chi}(\Phi,H,E) = \sup_{\rho:\mathrm{Tr}\rho H\leq E} \left[S(\Phi[\rho]) - \hat{S}_{\Phi}(\rho)\right]. \tag{22}$$

Then for any entanglement-breaking channel Φ

$$C_{\chi}(\Phi^{\otimes n}, H^{(n)}, nE)) = nC_{\chi}(\Phi,H,E), \tag{23}$$

implying

$$C(\Phi,H,E) = C_{\chi}(\Phi,H,E), \tag{24}$$

where $C(\Phi,H,E)$ is the constrained classical capacity of the channel Φ.

Coming to Bosonic system we restrict ourselves to the class of states $\mathfrak{S}_2$ with finite second moments, satisfying $\mathrm{Tr}\rho a_j^{\dagger}a_j < \infty$, $j = 1,\ldots,s$ (for rigorous definition of the moments see e.g. [11], Sec. 11.1). Notice that this class is invariant under the action of all Gaussian channels and all the entropy quantities are finite for states in this class. Moreover, one can check by inspection of proofs that the additivity properties listed above hold for Gaussian entanglement-breaking channels with optimizations restricted to the class $\mathfrak{S}_2$, provided Ψ is a Gaussian channel and H is a quadratic Hamiltonian in $a_j, a_j^\dagger,\ j = 1, \ldots, s$.

In what follows we adopt the restriction to the class $\mathfrak{S}_2$ without explicit introducing it into notations.

Proposition 2 *The Gaussian gauge-contravariant channel (12) is entanglement-breaking.*

The proof is based on the general criterion of entanglement breaking for Gaussian channels [11], Sec. 12.7.2, which in this case amounts to: there exist a decomposition $\mu = \mu_1 + \mu_2$ such that $\mu_1 \geq \frac{1}{2}KK^*, \mu_2 \geq \frac{1}{2}I$. This is indeed the case with $\mu_1 = \frac{1}{2}KK^*, \mu_2 = \frac{1}{2}I$. The decomposition corresponds to representation of the channel as 1) measurement of a Gaussian observable and 2) subsequent preparation of coherent states depending on the outcome of the measurement (see Sec. 5 for detail).

Proposition 2 implies that the additivity properties (20),(21),(23) hold for a Gaussian gauge-contravariant channels Φ and arbitrary channel Ψ.

Denote by $\tilde{\Phi}$ the complementary channel for Φ, then it is known that

$$\begin{aligned} \min_\sigma S\left(\Phi\left[\sigma\right]\right) &= \min_\sigma S\left(\tilde{\Phi}\left[\sigma\right]\right), \\ \hat{S}_\Phi\left(\sigma\right) &= \hat{S}_{\tilde{\Phi}}\left(\sigma\right), \end{aligned} \tag{25}$$

see [27], [28], or Sec.6.6.6 of [11].

Proposition 3 *The quantum-limited amplifier with diagonal matrix K and quantum-limited gauge-contravariant channel with diagonal matrix $\sqrt{KK^* - I}\Lambda$ are mutually complementary.*

In the diagonal case the channels split into tensor products of one-mode channels. For the proof in the case of one mode see [23] or [11], Sec. 12.6.1.

Via Proposition 3 and (25), the additivity properties (20), (21) then also hold for the diagonal quantum-limited amplifier Φ and arbitrary channel Ψ.

4 Reductions of the Gaussian optimizer conjecture

For the multi-mode quantum-limited attenuator Φ_1 the vacuum ρ_{vac} is invariant state, hence the minimal output entropy, equal to zero, is attained on the vacuum (as well as on any coherent state). Using Proposition 1 and the argument from [15, 29] allows to reduce solution of the minimal entropy conjecture for arbitrary channel satisfying the conditions of Proposition to the case of quantum-limited amplifier. Since quantum-limited amplifier Φ_2 is additive, this argument also implies the additivity property of the minimal output entropy. More precisely,

Proposition 4 *Under the hypothesis:*

(A) The minimal output entropy of any one-mode quantum-limited amplifier is attained on the vacuum state*,*

the following relation holds

$$\min_{\rho^{(n)}} S\left(\Phi^{\otimes n}\left[\rho^{(n)}\right]\right) = nS\left(\Phi\left[\rho_{vac}\right]\right), \tag{26}$$

for any Gaussian gauge-covariant or contravariant channel Φ.

Moreover, for any collection of channels $\Phi_1, \dots, \Phi_N$ satisfying the condition of the Proposition 1

$$\min_{\rho^{(n)}} S\left(\left(\otimes_{j=1}^{N}\Phi_j\right)\left[\rho^{(n)}\right]\right) = \sum_{j=1}^{N}\min_{\rho_j} S\left(\Phi_j\left[\rho_j\right]\right) = \sum_{j=1}^{N} S\left(\Phi_j\left[\rho_{j,vac}\right]\right). \tag{27}$$

Proof. First, let us notice that by the hypothesis (A) and the additivity property (21) we have for any multi-mode diagonal quantum-limited amplifier $\Phi = \otimes_{j=1}^{s}\Phi_j$, where Φ_j are one-mode quantum-limited amplifiers,

$$\min_{\rho} S\left(\Phi\left[\rho\right]\right) = \sum_{j=1}^{s}\min_{\rho_j} S\left(\Phi_j\left[\rho_j\right]\right) = \sum_{j=1}^{s} S\left(\Phi_j\left[\rho_{j,vac}\right]\right) = S\left(\Phi\left[\rho_{vac}\right]\right). \tag{28}$$

It follows that the analog of property (A) holds for any multi-mode diagonalizable quantum-limited amplifier because for such channel

$$\Phi\left[\rho\right] = U^{*}(\otimes_{j=1}^{N}\Phi_j)\left[U\rho U^{*}\right]U,$$

where U is the unitary operator in $\mathcal{H}^{\otimes N}$ implementing the unitary transformation R in Z which diagonalizes K. Notice also that $U\rho_{vac}U^{*}=\rho_{vac}$ because $U^{*}\mathbf{a}U=R\mathbf{a}$.

Therefore by complementarity (Proposition 3 and (25)) similar property holds for diagonalizable quantum-limited gauge-contravariant channel.

Let us prove (27) since (26) is a particular case. For every channel Φ_j we have decomposition $\Phi_j=\Phi_{2,j}\circ\Phi_{1,j}$, where $\Phi_{1,j}$ is quantum-limited attenuator and $\Phi_{2,j}$ has the property (28). Then

$$\sum_{j=1}^{N}S\left(\Phi_j\left[\rho_{j,vac}\right]\right)=\sum_{j=1}^{N}S\left(\Phi_{2,j}\left[\rho_{j,vac}\right]\right),$$

where the invariance of vacuum state under $\Phi_{1,j}$ was used. By (28) this is equal to

$$\sum_{j=1}^{N}\min_{\sigma_j}S\left(\Phi_{2,j}\left[\sigma_j\right]\right).$$

By the additivity (21) of the minimal output entropy for Gaussian gauge-contravariant channels and complementary quantum-limited amplifiers, the last sum is equal to

$$\begin{aligned}\min_{\sigma^{(n)}}S\left(\left(\otimes_{j=1}^{N}\Phi_{2,j}\right)\left[\sigma^{(n)}\right]\right) &\leq \min_{\rho^{(n)}}S\left(\left(\otimes_{j=1}^{N}\Phi_{j}\right)\left[\rho^{(n)}\right]\right)\\ &\leq \sum_{j=1}^{N}S\left(\Phi_j\left[\rho_{j,vac}\right]\right),\end{aligned}$$

hence (27) follows. ■

We now turn to the classical capacity. In [11] Proposition 12.39 it is shown that for a Gaussian gauge-covariant channel Φ, the validity of the minimal output entropy conjecture implies positive solution of the optimal Gaussian ensemble conjecture for the $\chi-$capacity under the energy constraint $\mathrm{Tr}\rho H\leq E$ with gauge-invariant oscillator Hamiltonian $H=\sum_{j,k=1}^{s}a_j^{\dagger}\epsilon_{jk}a_k$, where $\epsilon=[\epsilon_{jk}]$ is a positive definite matrix. The resulting expression is

$$\begin{aligned}C_{\chi}(\Phi,H,E) &= \max_{\rho:\mathrm{Tr}\rho H\leq E}S\left(\Phi\left[\rho\right]\right)-\min_{\rho}S\left(\Phi\left[\rho\right]\right)\\ &= \max_{\rho:\mathrm{Tr}\rho H\leq E}S\left(\Phi\left[\rho\right]\right)-S\left(\Phi\left[\rho_{vac}\right]\right). \qquad (29)\end{aligned}$$

Then, under the hypothesis (A), one shows the additivity of $C_\chi(E)$ similarly to the case of one-mode pure loss channel in [19]:

$$
\begin{aligned}
n\left[\max_{\rho:\mathrm{Tr}\rho H\leq E} S\left(\Phi\left[\rho\right]\right) - S\left(\Phi\left[\rho_{vac}\right]\right)\right] &\leq nC_\chi(\Phi,H,E) \leq C_\chi(\Phi^{\otimes n},H^{(n)},nE) \\
&\leq \max_{\rho^{(n)}:\mathrm{Tr}\rho^{(n)}H^{(n)}\leq nE} S\left(\Phi^{\otimes n}\left[\rho^{(n)}\right]\right) - \min_{\rho^{(n)}} S\left(\Phi^{\otimes n}\left[\rho^{(n)}\right]\right) \\
&\leq n\left[\max_{\rho:\mathrm{Tr}\rho H\leq E} S\left(\Phi\left[\rho\right]\right) - S\left(\Phi\left[\rho_{vac}\right]\right)\right],
\end{aligned}
$$

where in the last inequality we used Lemma 11.20 [11] and (26).

Thus $C_\chi(\Phi^{\otimes n},H^{(n)},nE) = nC_\chi(\Phi,H,E)$ and hence the constrained classical capacity $C(\Phi,H,E) = \lim_{n\to\infty}\frac{1}{n}C_\chi(\Phi^{\otimes n},H^{(n)},nE)$ of the Gaussian gauge-covariant channel is given by the same expression (29).

Now let us use the familiar formula for the entropy of Gaussian state (6) $S\left(\rho\right) = \mathrm{tr}\, g(\alpha - I/2)$, where $g(x) = (x+1)\log(x+1) - x\log x$, and tr denotes trace of operators in $\mathbf{Z}$. Applying the transformation rule $\alpha \to K\alpha K^* + \mu$ of the covariance matrix α under the action of the channel (7), see e.g. [11] Ch. 12, to the vacuum state with $\alpha = I/2$, we obtain explicitly the minimal output entropy

$$\min_\rho S(\Phi^{\otimes n}[\rho]) = n\,\mathrm{tr}\, g(\mu + \left(KK^* - I\right)/2). \tag{30}$$

For the classical capacity the relation (29) gives

$$
\begin{aligned}
C(\Phi;H,E) &= C_\chi(\Phi;H,E) \\
&= \max_{\nu\in\Sigma_E} \mathrm{tr}\, g(K\nu K^* + \mu + \left(KK^* - I\right)/2) - g(\mu + \left(KK^* - I\right)/2),
\end{aligned}
$$

where $\Sigma_E = \{\nu : \nu \geq 0,\ \mathrm{tr}\nu\epsilon \leq E\}$ is the set of covariance matrices ν of the Gaussian ensemble with the energy constraint. This reduces to a finite-dimensional optimization problem which is a quantum analog of "water-filling" problem in classical information theory, see e.g. [1, 9, 30, 31, 32, 33]. It can be solved explicitly only in some special cases, e.g. when K,μ,ϵ commute, and it is a subject of separate study.

Similar argument applies to Gaussian gauge-contravariant channel (12).

5 Some useful properties of contravariant channels

From the previous sections it follows that minimal output entropy problem for an arbitrary gauge-covariant channel Φ can be solved by proving that the minimal output entropy conjecture for a single-mode quantum-limited amplifier. Alternatively, due to the identity (25) and to Proposition 3, this is equivalent to show that single-mode quantum-limited contravariant map $\tilde{\Phi}$ has the vacuum as minimizer of output entropy. We start hence by providing a characterization of the output states of these maps. Even though for the proof of the conjecture we need only the single-mode case, for the sake of generality we will present them in the multi-mode scenario. Denote by $|z\rangle = D(z)|0\rangle$, $z \in \mathbf{Z}$, the multimode coherent states. From now on we skip the subscripts A, B occasionally, since it should be clear from the context.

Proposition 5 *Given a quantum-limited Gaussian gauge-contravariant channel Φ described by the matrix K via Eqs. (12) and (14), its action on an input state ρ can be expressed as the following measure-reprepare mapping*

$$\rho \mapsto \Phi[\rho] = \int \frac{d^{2s}z}{\pi^s} | - K\bar{z}\rangle\langle -K\bar{z}| \; \langle z|\rho|z\rangle. \tag{31}$$

Proof. The relation (31) follows from the general measure-reprepare representation of Gaussian entanglement-breaking channels, [11], Sec. 12.7.2. For a direct verification of (31) it is sufficient to show that Husimi functions (diagonal values in the coherent-state representation) coincide for operator (12) and the dual of (31). This amounts to the identity (where we redenoted some variables)

$$\langle u|D(\Lambda K^* w)|u\rangle \; e^{-\frac{1}{2}w^*(I+KK^*)w} = \int \frac{d^{2s}z}{\pi^s} \langle -K\bar{z}|D(w)| - K\bar{z}\rangle \; |\langle u|z\rangle|^2 ,$$

which is verified by using the formulas $\langle u|D(w)|u\rangle = \exp\left(2i\Im \bar{u}w - |w|^2/2\right)$ and $|\langle u|z\rangle|^2 = \exp\left(-|w-z|^2\right)$. ■

For any channel (31) we introduce the following *skewed* counterparts defined as

$$\begin{aligned} \rho \mapsto \Psi_+[\rho] &= \int \frac{d^{2s}z}{\pi^s} |Kz\rangle\langle Kz| \; \langle z|\rho|z\rangle, & (32) \\ \rho \mapsto \Psi_-[\rho] &= \int \frac{d^{2s}z}{\pi^s} | - Kz\rangle\langle -Kz| \; \langle z|\rho|z\rangle. & (33) \end{aligned}$$

These are again measure-reprepare channels where, differently from Φ of Eq. (31), after a measurement outcome z, the output system is initialized into the coherent states $|\pm Kz\rangle$. Similarly to Proposition 5 one can verify that in the Heisenberg representation the channels $\Psi_\pm$ are described by the gauge-covariant mappings

$$\Psi_\pm^*[D(z)] = D(\pm Kz)\exp\left(-z^*(KK^*+I)z/2\right). \tag{34}$$

However these channels are no longer quantum-limited.

Proposition 6 *The output states $\Psi_+[\rho]$, $\Psi_-[\rho]$, and $\Phi[\rho]$ have the same eigenvalues and hence the same entropy.*

Proof. Noticing that $T[|z\rangle\langle z|] = |\bar{z}\rangle\langle\bar{z}|$, it follows that $\Psi_-[\rho] = T[\Phi[\rho]]$. Therefore $\Psi_-[\rho]$ and $\Phi[\rho]$ must have the same spectrum. To prove that also $\Psi_+[\rho]$ shares the same property, notice that we can transform such state into $\Psi_-[\rho]$ by a phase transformation $e^{-i\pi N}$. Indeed, $e^{-i\pi N}|z\rangle = |e^{-i\pi}z\rangle = |-z\rangle$, hence

$$e^{-i\pi N}\Psi_+[\rho]e^{i\pi N} = \Psi_-[\rho].$$

■

We conclude that given a quantum-limited contravariant channel Φ and an input state ρ, there exists a unitary transformation U (possibly dependent upon ρ) such that

$$\Phi[\rho] = U\Psi_+[\rho]U^*.$$

We have now all the elements we need to prove the minimum output entropy conjecture. For this purpose we have to take a step back and re-introduce the complementary counterpart of the contravariant channel.

6 Proof of the Gaussian optimizer conjecture

In this Section we prove the hypothesis (A) in Proposition 4.

Here Φ and $\tilde{\Phi}$ are the single-mode quantum-limited covariant amplifier, resp. contravariant channel defined by the parameter $K \geq 1$ via the relations

$$\Phi^*[D(z)] = D(Kz)\exp\left(-z^*(K^2-1)z/2\right), \tag{35}$$

$$\tilde{\Phi}^*[D(z)] = D(G\bar{z})\exp\left(-z^*(G^2+1)z/2\right), \tag{36}$$

where $G = \sqrt{K^2 - 1}$. In what follows we can assume that $K > 1$ (the case $K = 1$ is trivial, corresponding to the identity channel). Accordingly G is strictly positive and we can apply to the contravariant channel $\tilde{\Phi}$ all the results we have derived in the previous Sections. From Proposition 3 we know that Φ and $\tilde{\Phi}$ are mutually complementary hence the density operators $\Phi(|\psi\rangle\langle\psi|)$ and $\tilde{\Phi}(|\psi\rangle\langle\psi|)$ have the same nonzero spectrum, hence there exists a partial isometry $\tilde{V}$ (possibly dependent upon the input state $|\psi\rangle$), mapping the support of one operator onto the support of another, such that

$$\Phi\left[|\psi\rangle\langle\psi|\right] = \tilde{V}\tilde{\Phi}\left[|\psi\rangle\langle\psi|\right]\tilde{V}^*.$$

In such a case we will call the partial isometry *connecting* the relevant density operators. Remind that the connected operators have equal entropies. Furthermore from Proposition 6 it also follows that an analogous relation connects $\tilde{\Phi}$ and Ψ_+. Therefore for any $|\psi\rangle$ there exists a connecting partial isometry V (possibly dependent upon $|\psi\rangle$) such that

$$\Phi\left[|\psi\rangle\langle\psi|\right] = V\Psi_+\left[|\psi\rangle\langle\psi|\right]V^*, \tag{37}$$

where Ψ_+ is the channel (34) associated with the quantum-limited contravariant channel $\tilde{\Phi}$, i.e.

$$\Psi_+^*[D(z)] = D(Gz)\exp\left(-z^*(G^2+1)z/2\right) = D(\sqrt{K^2-1}\,z)\exp\left(-z^*K^2z/2\right). \tag{38}$$

As already noticed the channel Ψ_+ is in general not quantum-limited. Nevertheless, following Proposition 1, we can express it as a concatenation of a quantum-limited attenuator Φ_1 followed by a quantum-limited covariant amplifier Φ_2, i.e.$\Psi_+ = \Phi_2 \circ \Phi_1$. The parameters K_2 and K_1 which define these maps can be computed as

$$K_2 = \sqrt{K^2/2 + \frac{G^2+1}{2}} = K, \tag{39}$$

$$K_1 = G/K = \frac{\sqrt{K^2-1}}{K}. \tag{40}$$

From Eq. (39) it follows that Φ_2 is nothing but the channel Φ we started from, hence $\Psi_+ = \Phi \circ \Phi_1$. Therefore substituting this into Eq. (37) we get

$$\Phi\left[|\psi\rangle\langle\psi|\right] = V\left(\Phi\circ\Phi_1\right)\left[|\psi\rangle\langle\psi|\right]V^*, \tag{41}$$

which applies for all pure inputs $|\psi\rangle$ (we remind that the connecting partial isometry V can in principle depend upon $|\psi\rangle$).

It is worth observing that from Eq. (41) it follows that the minimal output entropy of the quantum-minimal amplifier Φ coincides with the minimal output entropy of $\Phi \circ \Phi_1$ with Φ_1 being the attenuator (40), a fact which is fully consistent with the conjecture since Φ_1 admits the vacuum as fixed point. We can however say more.

Proposition 7 *Let Φ the quantum-limited covariant amplifier of Eq. (35) and Φ_1 the attenuator channel associated to it through Eq. (40). Then given a pure input state $\rho = |\psi\rangle\langle\psi|$ and integer n, there exists an ensemble $\mathcal{E} = \{p_i; |\psi_i\rangle\}$ and connecting partial isometries U_i satisfying the relations*

$$\Phi_1^n(\rho) = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{42}$$

$$\Phi[\rho] = \sum_i p_i U_i \Phi[|\psi_i\rangle\langle\psi_i|] U_i^*. \tag{43}$$

Proof. We prove this by induction. Let

$$\Phi_1(|\psi\rangle\langle\psi|) = \sum_j p_j |\psi_j\rangle\langle\psi_j|$$

be the spectral decomposition of the state $\Phi_1(|\psi\rangle\langle\psi|)$, where p_j are strictly positive (zero eigenvalues can be omitted as they do not contribute to the sum). Inserting this into (41) we get

$$\Phi(|\psi\rangle\langle\psi|) = \sum_j p_j V \Phi(|\psi_j\rangle\langle\psi_j|) V^*,$$

proving the statement for $n = 1$.

Assume now that the statement is valid for some n. Then from (42)

$$\begin{aligned} \Phi_1^{n+1}(|\psi\rangle\langle\psi|) &= \sum_i p_i \Phi_1 \left[|\psi_i\rangle\langle\psi_i|\right] \\ &= \sum_{i,j} p_i p_{j|i} |\psi_{j|i}\rangle\langle\psi_{j|i}|, \end{aligned}$$

where $\Phi_1\left[|\psi_i\rangle\langle\psi_i|\right] = \sum_{,j} p_{j|i}|\psi_{j|i}\rangle\langle\psi_{j|i}|$ is the spectral decomposition. By using (43), (41) we obtain

$$\begin{aligned}
\Phi(|\psi\rangle\langle\psi|) &= \sum_i p_i U_i(\Phi[|\psi_i\rangle\langle\psi_i|])U_i^* \\
&= \sum_i p_i U_i V_i \left(\Phi\circ\Phi_1\right)\left([|\psi_i\rangle\langle\psi_i|]\right)[U_iV_i]^* \\
&= \sum_{i,j} p_j p_{j|i}\left[U_iV_i\right]\Phi(|\psi_{j|i}\rangle\langle\psi_{j|i}|)\left[U_iV_i\right]^*,
\end{aligned}$$

proving the statement for $n+1$. ■

We can then use the concavity of the von Neumann entropy to write the inequality

$$S(\Phi[\rho]) \geq \sum_i p_i S(\Phi(|\psi_i\rangle\langle\psi_i|)), \tag{44}$$

where the ensemble satisfies (42). Notice also for $n\to\infty$ the channel Φ_1^n brings all the states into the fixed point, i.e. the vacuum state,

$$\lim_{n\to\infty}\Phi_1^n(|\psi\rangle\langle\psi|) = |0\rangle\langle 0|,$$

the convergence being in trace-norm. Accordingly as $n\to\infty$ the only state surviving in the decomposition (42) is the vacuum state. It seems then reasonable to conclude that in the limit $n\to\infty$ the right-hand side of Eq. (44) should reduce to $S(\Phi(|\psi\rangle\langle\psi|)) \geq S(\Phi(|0\rangle\langle 0|))$ hence proving the thesis. To make this precise we use the monotonicity of the relative entropy [11], i.e.

$$S(|\psi_i\rangle\langle\psi_i|||\sigma) \geq S(\Phi(|\psi_i\rangle\langle\psi_i|)||\Phi(\sigma))$$

where $|\psi_i\rangle$ is one of the vectors of the ensemble for Φ_1^n, and σ a state to be defined later. By reorganizing various terms this gives

$$S(\Phi(|\psi_i\rangle\langle\psi_i|)) \geq -\mathrm{Tr}\Phi(|\psi_i\rangle\langle\psi_i|)\log\Phi(\sigma) + \mathrm{Tr}|\psi_i\rangle\langle\psi_i|\log\sigma,$$

which, substituted into (44), yields

$$\begin{aligned}
S(\Phi(|\psi\rangle\langle\psi|)) &\geq -\mathrm{Tr}\Phi(\sum_i p_i|\psi_i\rangle\langle\psi_i|)\log\Phi(\sigma) + \mathrm{Tr}\sum_i p_i|\psi_i\rangle\langle\psi_i|\log\sigma \\
&= -\mathrm{Tr}\left(\Phi\circ\Phi_1^n\right)(|\psi\rangle\langle\psi|)\log\Phi(\sigma)] + \mathrm{Tr}\Phi_1^n(|\psi\rangle\langle\psi|)\log\sigma \\
&= S((\Phi\circ\Phi_1^n)(|\psi\rangle\langle\psi|)) + S((\Phi\circ\Phi_1^n)(|\psi\rangle\langle\psi|)||\Phi(\sigma)) \\
&\quad +\mathrm{Tr}\Phi_1^n(|\psi\rangle\langle\psi|)\log\sigma \\
&\geq S((\Phi\circ\Phi_1^n)(|\psi\rangle\langle\psi|)) + \mathrm{Tr}\Phi_1^n(|\psi\rangle\langle\psi|)\log\sigma.
\end{aligned} \tag{45}$$

Assume next that σ is a Gibbs state, i.e.

$$\sigma = (1-\gamma) \sum_{k=0}^{\infty} \gamma^k |k\rangle\langle k|,$$

with $\gamma > 0$. With this choice the second term of Eq. (45) can be well defined for all input states $|\psi\rangle\langle\psi|$ having finite second moments. Indeed by repeated use of the relation

$$\mathrm{Tr}\Phi_1(\rho) a^\dagger a = \left[\frac{K^2-1}{K^2}\right] \mathrm{Tr}\rho a^\dagger a$$

valid for states from $\mathfrak{S}_2$, we have

$$\mathrm{Tr}[\Phi_1^n(|\psi\rangle\langle\psi|)\log\sigma] = \log(1-\gamma) + \log\gamma \; \left[\frac{K^2-1}{K^2}\right]^n \langle\psi|a^\dagger a|\psi\rangle \; .$$

Substituting this into the right-hand-side of Eq. (45) gives

$$S(\Phi(|\psi\rangle\langle\psi|)) \geq S((\Phi\circ\Phi_1^n)\,(|\psi\rangle\langle\psi|)) + \log(1-\gamma) + \log\gamma \; \left[\frac{K^2-1}{K^2}\right]^n \langle\psi|a^\dagger a|\psi\rangle \; .$$

Taking the limit $n \to \infty$ and using lower semicontinuity of the quantum entropy in the first term we obtain

$$S(\Phi(|\psi\rangle\langle\psi|)) \geq S(\Phi(|0\rangle\langle 0|)) + \log(1-\gamma).$$

Taking the limit $\gamma \to 0$ we finally have

$$S(\Phi(|\psi\rangle\langle\psi|)) \geq S(\Phi(|0\rangle\langle 0|))$$

for all input states $|\psi\rangle\langle\psi|$ with finite second moments.

7 Implications and perspectives

In this work we have proven that the minimal entropy at the output of a (possibly multimode) BGC covariant (or contravariant) channel Φ is achieved by the vacuum input state, restricting our analysis to the class of state with finite second moments. As detailed in Sec. 4 this implies both the additivity of the minimal output entropy functional and of the classical capacity (under energy constraint) whose value is achieved via Gaussian encodings. To be specific, let us apply the formulas (30), (31) to single-mode ($s = 1$) quantum channels [23] to obtain the quantum counterparts of the Shannon formula (1). In this case one identifies three classes of covariant maps:

- The *thermal noise channels* describing a passive exchange (beam splitter) interaction with an external Gibbs thermal state. Following the notation of [12] they are characterized by two real parameters $\eta \in [0,1]$ and $N \in [0,\infty[$, associated respectively to the intensity of exchange coupling and to the temperature of the system environment, and which enter into Eq. (7) as $K = \sqrt{\eta}$ and $\mu = (1-\eta)(N+1/2)$. For these channels our result shows that

$$\begin{aligned} \min_\rho S(\Phi^{\otimes n}[\rho]) &= n\, g((1-\eta)N), \\ C(\Phi; H, E) &= C_\chi(\Phi; H, E) \\ &= g(\eta E + (1-\eta)N) - g((1-\eta)N). \end{aligned}$$

- The *additive classical noise channels* which randomly displace the input states in phase space. They can be fully characterized via a single parameter $N \in [0,\infty[$ which represents the variance of the Gaussian probability distribution governing the displacement transformation, and which enters into Eq. (7) via the identities $K = 1$, $\mu = N$. In this case we have

$$\begin{aligned} \min_\rho S(\Phi^{\otimes n}[\rho]) &= n\, g(N), \\ C(\Phi; H, E) &= C_\chi(\Phi; H, E) = g(E+N) - g(N). \end{aligned}$$

- The *noisy amplifier channels* characterized by two real parameters $\kappa \in [1,\infty[$ and $N \in [0,\infty[$ entering Eq. (7) via the identities $K = \sqrt{\kappa}$ and $\mu = (\kappa-1)(N+1/2)$. In this case we have

$$\begin{aligned} \min_\rho S(\Phi^{\otimes n}[\rho]) &= n\, g((\kappa-1)(N+1)), \\ C(\Phi; H, E) &= C_\chi(\Phi; H, E) \\ &= g(\kappa E + (\kappa-1)(N+1)) - g((\kappa-1)(N+1)). \end{aligned}$$

In the single mode scenario, all BGCs with nondegenerate K are unitarily equivalent to covariant (or contravariant) channels via metaplectic unitary transformations, while for degenerate channels the proof is even simpler [13]. Thus one can easily verify that the optimal states which minimize the output entropy are squeezed vacuum and the corresponding coherent states, a result which in certain regimes allows also to compute the constrained classical capacity. Most importantly, with Proposition 7 it is possible to prove [20] that, for arbitrary covariant (contravariant) channels the vacuum input produces the output which majorizes all other output states (a result which implies the minimal output conjecture).

In the multimode case, the argument of this paper concerning the minimal output entropy applies to BGCs gauge-covariant with respect to any "squeezed" complex structure in the underlying real symplectic space, since it can be always reduced to the standard one considered in this paper. However the problem with the classical capacity arises already in the case (perhaps artificial from the point of view of applications) where the complex structures associated with the gauge-covariant BGC and the energy operator do not agree.

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