FROM SEVEN TO ELEVEN: COMPLETELY POSITIVE MATRICES WITH HIGH CP-RANK

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Abstract. We study $n \times n$ completely positive matrices M on the boundary of the completely positive cone, namely those orthogonal to a copositive matrix S which generates a quadratic form with finitely many zeroes in the standard simplex. Constructing particular instances of S, we are able to construct counterexamples to the famous Drew-Johnson-Loewy conjecture (1994) for matrices of order seven through eleven.

Key words. copositive optimization, completely positive matrices, cp-rank, nonnegative factorization, circular symmetry

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1. Introduction. In this article we consider completely positive matrices M and their cp-rank. An $n \times n$ matrix M is said to be *completely positive* if there exists a nonnegative (not necessarily square) matrix V such that $\mathsf{M} = \mathsf{V}\mathsf{V}^\top$. Typically, a completely positive matrix M may have many such factorizations, and the *cp-rank* of M, cpr M, is the minimum number of columns in such a nonnegative factor V (for completeness, we define cpr $\mathsf{M} = 0$ if M is a square zero matrix and cpr $\mathsf{M} = \infty$ if M is not completely positive). Completely positive matrices form a cone dual to the cone of *copositive matrices*. An $n \times n$ matrix S is said to be copositive if $\mathbf{x}^\top \mathsf{S} \mathbf{x} \geq 0$ for every nonnegative vector $\mathbf{x} \in \mathbb{R}^n_+$. Both cones are central in the rapidly evolving field of *copositive optimization* which links discrete and continuous optimization, and has numerous real-world applications. For recent surveys and structured bibliographies, we refer to [5, 6, 8, 12], and for a fundamental text book to [2].

Determining the maximum possible cp-rank of $n \times n$ completely positive matrices,

$$p_n := \max\{\text{cpr } \mathsf{M} : \mathsf{M} \text{ is a completely positive } n \times n \text{ matrix}\},$$

is still an open problem for general n. It is known [2, Theorem 3.3] that $p_n = n$ if $n \leq 4$, whereas this equality does no longer hold for $n \geq 5$. Let $d_n := \left\lfloor \frac{n^2}{4} \right\rfloor$ and $s_n := \binom{n+1}{2} - 4$. For $n \geq 5$, it is known that [16]

$$d_n \leq p_n \leq s_n\,, \tag{1.1}$$

and that $d_n = p_n$ in case $n = 5$ [17]. It is still unknown whether $d_6 = p_6$ although the bracket (1.1) was reduced in the recent paper [16] where also the upper bound $p_n \leq s_n$ was established for the first time.

The famous Drew-Johnson-Loewy (DJL) conjecture [11] is by now twenty years old. It states that $d_n = p_n$ is true for all $n \geq 5$, and some evidence in support of the DJL conjecture is found in [1, 10, 11, 15], see also [2, Section 3.3]. However, we will show in this paper that the DJL conjecture does not hold for $n \in [7 : 11]$ by constructing examples which establish $p_n > d_n$.

The paper is organized as follows: In Section 2 we look at copositive matrices S which allow for finitely many (but many) zeroes $\mathbf{q}_i$ of the quadratic form $\mathbf{x}^\top \mathsf{S} \mathbf{x}$ over

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the standard simplex. Such matrices S lie on the boundary of the copositive cone, and elementary conic duality therefore tells us that there are nontrivial completely positive matrices M such that $\mathsf{M} \perp \mathsf{S}$ in the Frobenius inner product sense, and we will study the cp-rank of these M. Section 3 deals with a particular construction of above mentioned copositive matrices S (they will be cyclically symmetric) in a way that many $\mathbf{q}_i$ can coexist, and in Section 4 we present the second main result – counterexamples to the DJL conjecture for all $n \in [7:11]$. Let us mention here that such a counterexample for $n = 7$ with cp-rank 14 was announced in 2002, according to [2, p.177]. The matrix there (which never got public) should have rank 5; by contrast, our matrix M in Example 1 will have full rank 7, but also $\operatorname{cpr} \mathsf{M} = 14$ by mere coincidence.

Some notation and terminology: we abbreviate $[r:s] = \{r, r+1, \ldots, s\}$ for integers $r \leq s$, and by $|S|$ the number of elements of a finite set S. The nonnegative orthant is denoted by $\mathbb{R}^n_+$. For a vector $\mathbf{x} \in \mathbb{R}^n_+$, the index set

$$I(\mathbf{x}) = \{i \in [1:n] : x_i > 0\}$$

is the *support* of $\mathbf{x}$. Let $\mathbf{e}_i$ be the ith column vector of the $n \times n$ identity matrix I_n and $\mathbf{e} = \sum_{i=1}^n \mathbf{e}_i$. The zero vector and the zero matrix (of appropriate sizes) are denoted by $\mathbf{o}$ and O, respectively, and $\Delta = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{e}^\top \mathbf{x} = 1\}$ stands for the standard simplex. The space of real symmetric $n \times n$ matrices is denoted by $\mathcal{S}^n$, and the Frobenius inner product of two matrices $\{\mathsf{A}, \mathsf{B}\} \subset \mathcal{S}^n$ by $\langle \mathsf{A}, \mathsf{B} \rangle := \operatorname{trace}(\mathsf{AB})$. For an $n \times p$ matrix $\mathsf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_p]$, the relation $\mathsf{M} = \mathsf{V}\mathsf{V}^\top$ is equivalent to $\mathsf{M} = \sum_{i=1}^p \mathbf{v}_i \mathbf{v}_i^\top$. We will refer to this sum as a "cp decomposition" of M, if V has no negative entries. Given a square matrix S, we will, by slight abuse of language, use the phrase "zero(es) of S" as an abbreviation of "zero(es) of the quadratic form $\mathbf{x}^\top \mathsf{S} \mathbf{x}$ over $\mathbf{x} \in \Delta$"; this terminology differs slightly from that in [14].

By $\mathcal{C}^{n*}$ we denote the cone of completely positive matrices,

$$\mathcal{C}^{n*} = \operatorname{conv}\{\mathbf{x}\mathbf{x}^\top : \mathbf{x} \in \mathbb{R}^n_+\}.$$

Both, $\mathcal{C}^{n*}$ and its dual, the cone of copositive matrices

$$\mathcal{C}^n = \left\{\mathsf{S} \in \mathcal{S}^n : \mathbf{x}^\top \mathsf{S} \mathbf{x} \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n_+\right\},$$

are pointed closed convex cones with nonempty interior. The copositive cone $\mathcal{C}^n$ and, in particular, its extremal rays, are important as any matrix on the boundary $\partial\mathcal{C}^{n*}$ of $\mathcal{C}^{n*}$ is orthogonal to an extremal ray of $\mathcal{C}^n$. So, studies of the extremal rays of $\mathcal{C}^n$ like in [9, 13, 14] lead to conclusions on all matrices on $\partial\mathcal{C}^{n*}$, which allow for inference on *upper* bounds on p_n. This was an essential ingredient of the arguments in [16, 17]. Here we employ a somewhat reverse approach: we start from (appropriate) matrices $\mathsf{S} \in \partial\mathcal{C}^n$ and construct $\mathsf{M} \in \partial\mathcal{C}^{n*}$ where we can calculate the cp-rank $\operatorname{cpr} \mathsf{M}$, improving upon *lower* bounds on p_n. Eventually, this will lead to examples refuting the DJL conjecture.

2. Iterative reduction of the cp-rank. Consider a copositive matrix $\mathsf{S} \in \partial\mathcal{S}^n$ and assume that $\{\mathbf{q}_1, \ldots, \mathbf{q}_m\}$ are all the zeroes of S. Since $\mathsf{S} \in \partial\mathcal{C}^n$, there is a matrix $\mathsf{M} \in \mathcal{C}^{n*} \setminus \{\mathsf{O}\}$ such that $\langle \mathsf{M}, \mathsf{S} \rangle = 0$, e.g., any matrix of the form

$$\mathsf{M} = \sum_{i=1}^m y_i \mathbf{q}_i \mathbf{q}_i^\top$$

for some $\mathbf{y} \in \mathbb{R}^m_+ \setminus \{\mathbf{o}\}$. The next result shows the converse of this statement, so that the set of possible cp decomposition of matrices orthogonal to S is quite restricted:

LEMMA 2.1. *Let $\{\mathbf{q}_1, \dots, \mathbf{q}_m\}$ be all the zeroes of $\mathsf{S} \in \partial\mathcal{C}^n$. Then any matrix $\mathsf{M} \in \mathcal{C}^{n*}$ orthogonal to S must be of the form*

$$\mathsf{M} = \sum_{j=1}^{m} y_j \mathbf{q}_j \mathbf{q}_j^\top \tag{2.1}$$

for some $\mathbf{y} \in \mathbb{R}^m_+$.

Proof. Let M have the cp decomposition $\mathsf{M} = \sum\limits_{i=1}^{p} \mathbf{v}_i \mathbf{v}_i^\top$ with $\mathbf{v}_i \in \mathbb{R}^n_+ \setminus \{\mathbf{o}\}$ for all $i \in [1:p]$. Then $\mathsf{M} \perp \mathsf{S}$ implies

$$0 = \langle \mathsf{M}, \mathsf{S} \rangle = \sum_{i=1}^{p} \mathbf{v}_i^\top \mathsf{S} \mathbf{v}_i \,,$$

and as S is copositive, every term in above sum must be zero. So all $\mathbf{q}'_i := \frac{1}{\mathbf{e}^\top \mathbf{v}_i} \mathbf{v}_i \in \Delta$ must be zeroes of S, therefore

$$\{\mathbf{q}'_i : i \in [1:p]\} \subseteq \{\mathbf{q}_j : j \in [1:m]\} \,.$$

Let $y_j := \sum\limits_{i \in [1:p] : \mathbf{q}'_i = \mathbf{q}_j} (\mathbf{e}^\top \mathbf{v}_i)^2 \geq 0$ with the usual rule $\sum\limits_{\emptyset} = 0$. Then (2.1) results easily. ◻

Although we have restricted the possible cp decompositions by above observation, there still could be infinitely many, but they can be obtained in a linear way. To be more precise, fix any $\mathbf{y} \in \mathbb{R}^m_+$ and consider

$$P_{\mathbf{y}} := \left\{ \mathbf{x} \in \mathbb{R}^m_+ : \sum_{i=1}^{m} x_i \mathbf{q}_i \mathbf{q}_i^\top = \sum_{j=1}^{m} y_j \mathbf{q}_j \mathbf{q}_j^\top \right\} . \tag{2.2}$$

A particular case is obtained if $P_{\mathbf{y}} = \{\mathbf{y}\}$, because then $\operatorname{cpr} \mathsf{M} = |I(\mathbf{y})|$ is immediate from Lemma 2.1. However, this may not always be the case but some variables x_k of points $\mathbf{x} \in P_{\mathbf{y}}$ may be fixed to y_k as follows:

LEMMA 2.2. *Consider $\mathsf{M} = \sum\limits_{j=1}^{m} y_j \mathbf{q}_j \mathbf{q}_j^\top \perp \mathsf{S}$ where $\{\mathbf{q}_1, \dots, \mathbf{q}_m\}$ are (all) the zeroes of $\mathsf{S} \in \partial\mathcal{C}^n$. Suppose that there is a $k \in [1:m]$ such that for two different indices r, s, we have*

$$\{r, s\} \subseteq I(\mathbf{q}_k) \quad \textit{but} \quad \{r, s\} \not\subseteq I(\mathbf{q}_i) \textit{ for all } i \neq k \,. \tag{2.3}$$

Then $x_k = \frac{\mathbf{e}_r^\top \mathsf{M} \mathbf{e}_s}{(\mathbf{e}_r^\top \mathbf{q}_k)(\mathbf{e}_s^\top \mathbf{q}_k)} = y_k$ holds for all $\mathbf{x} \in P_{\mathbf{y}}$.

Proof. Condition (2.3) implies $(\mathbf{e}_r^\top \mathbf{q}_k)(\mathbf{e}_s^\top \mathbf{q}_k) > 0$ and further that

$$x_k (\mathbf{e}_r^\top \mathbf{q}_k)(\mathbf{e}_s^\top \mathbf{q}_k) = \mathbf{e}_r^\top \mathsf{M} \mathbf{e}_s \quad \text{for all } \mathbf{x} \in P_{\mathbf{y}} \,.$$

Hence $x_k = \frac{\mathbf{e}_r^\top \mathsf{M} \mathbf{e}_s}{(\mathbf{e}_r^\top \mathbf{q}_k)(\mathbf{e}_s^\top \mathbf{q}_k)}$ is fixed. ◻

THEOREM 2.1. *Consider $\mathsf{M} = \sum\limits_{j=1}^{m} y_j \mathbf{q}_j \mathbf{q}_j^\top \perp \mathsf{S}$ where $\{\mathbf{q}_1, \dots, \mathbf{q}_m\}$ are (all) the zeroes of $\mathsf{S} \in \partial\mathcal{C}^n$. Suppose $y_m > 0$ and that condition (2.3) holds for $k = m$. Then for the reduced matrix*

$$\mathsf{M}' = \mathsf{M} - y_m \mathbf{q}_m \mathbf{q}_m^\top = \sum_{i<m} y_i \mathbf{q}_i \mathbf{q}_i^\top \,,$$

we have cpr $\mathsf{M} = 1 + $ *cpr* M'.

Proof. Of course also $\mathsf{M}' \perp \mathsf{S}$. If $\mathsf{M}' = \mathsf{O}$, then cpr $\mathsf{M} = 1$. Else, the result is not completely trivial since alternative cp decompositions of M' (i.e., those generated by $\mathbf{x} \in P_{\mathbf{y}'}$ where $\mathbf{y}' = \mathbf{y} - y_m \mathbf{e}_m \in \mathbb{R}^m_+$) may also involve $\mathbf{q}_m$. But this is ruled out as Lemma 2.2, applied to $\mathbf{y}'$, yields $x_m = y'_m = 0$ for all $\mathbf{x} \in P_{\mathbf{y}'}$. □

We can apply above theorem iteratively (for different k), of course. If we arrange the supports of (many) $\mathbf{q}_i$'s such that condition (2.3), or a similar one, continues to hold during the iterations, we can construct M with high cpr M. This will be done in the next section.

3. Zeroes of cyclically symmetric copositive matrices. We will employ symmetry transformations of the coordinates given by cyclic permutation, denoting by $a \oplus b$ the result of addition modulo n. To keep in line with previous and standard notation, we consider the remainders $[1 : n]$ instead of $[0 : n-1]$, e.g. $1 \oplus (n-1) = n$. To be more precise, let P_i be the square $n \times n$ permutation matrix which effects $\mathsf{P}_i \mathbf{x} = [x_{i \oplus j}]_{j \in [1:n]}$ for all $\mathbf{x} \in \mathbb{R}^n$ (for example, if $n = 3$ then $\mathsf{P}_2 \mathbf{x} = [x_3, x_1, x_2]^\top$). Obviously $\mathsf{P}_i = (\mathsf{P}_1)^i$ for all integers i (recall P^{-3} is the inverse matrix of PPP), and $\mathsf{P}_n = \mathsf{I}_n$. A *circulant matrix* $\mathsf{S} = \mathsf{C}(\mathbf{a})$ based on a vector $\mathbf{a} \in \mathbb{R}^n$ (as its last column rather than the first) is given by

$$\mathsf{S} = [\mathsf{P}_{n-1}\mathbf{a}, \mathsf{P}_{n-2}\mathbf{a}, \ldots, \mathsf{P}_1\mathbf{a}, \mathbf{a}] \,.$$

If $\mathsf{S} = \mathsf{C}(\mathbf{a}) \in \mathcal{S}^n$, i.e., if $\mathsf{C}(\mathbf{a})$ is symmetric, it is called *cyclically symmetric.*

LEMMA 3.1. *Any circulant matrix* $\mathsf{S} = \mathsf{C}(\mathbf{a})$ *satisfies* $\mathsf{P}_i^\top \mathsf{S} \mathsf{P}_i = \mathsf{S}$ *for all* $i \in [1 : n]$. *Furthermore, if*

$$a_i = a_{n-i} \quad \textit{for all } i \in [1 : n] \tag{3.1}$$

then $\mathsf{S} = \mathsf{C}(\mathbf{a})$ *is cyclically symmetric.*

Proof. The first relation is evident. To show the remaining assertion, assume (3.1) and let $\mathbf{e}_j^\top \mathsf{S} \mathbf{e}_i = \mathbf{e}_j^\top \mathsf{P}_{n-i} \mathbf{a} = a_k$ with $k \oplus i = j$ while $\mathbf{e}_i^\top \mathsf{S} \mathbf{e}_j = a_\ell$ with $\ell \oplus j = i$. Thus $i \oplus j = k \oplus \ell \oplus i \oplus j$ and $\{k, \ell\} \subseteq [1 : n]$, so we get $k + \ell \in \{n, 2n\}$ and therefore $a_k = a_\ell$. Hence $\mathsf{C}(\mathbf{a}) \in \mathcal{S}^n$. □

Cyclically symmetric matrices $\mathsf{S} = \mathsf{C}(\mathbf{a})$ can have many zeroes (for local minimizers of the quadratic form $\mathbf{x}^\top \mathsf{S} \mathbf{x}$, this has already been observed earlier, see [7] and references therein). To facilitate the argument, let us denote by $\mathsf{R} \in \mathcal{S}^n$ the *reflection matrix* which transforms every $\mathbf{x} \in \mathbb{R}^n$ into its *mirror image* $\mathsf{R}\mathbf{x} := [x_{n+1-i}]_{i \in [1:n]}$.

LEMMA 3.2. *Any cyclically symmetric matrix* $\mathsf{S} = \mathsf{C}(\mathbf{a})$ *satisfies* $\mathsf{R}^\top \mathsf{S} \mathsf{R} = \mathsf{S}$. *Fixing* $\mathbf{q}$, *for any shift* $\mathbf{q}' = \mathsf{P}_i \mathbf{q}$, *and for its mirror image* $\mathbf{q}'' = \mathsf{R}\mathbf{q}$, *we have*

$$(\mathbf{q}')^\top \mathsf{S} \mathbf{q}' = (\mathbf{q}'')^\top \mathsf{S} \mathbf{q}'' = \mathbf{q}^\top \mathsf{S} \mathbf{q} \,, \tag{3.2}$$

so that for any zero $\mathbf{q}$ *of* S *there are actually up to* $2n$ *zeroes: the shifts* $\mathsf{P}_i \mathbf{q}$ *for* $i \in [1 : n]$ *and their mirror images, if they are all different. Further, the supports are shifted cyclically,* $I(\mathsf{P}_i \mathbf{q}) = I(\mathbf{q}) \ominus i$. *However, the relative differences within the support of course remain: if* $\{r, s\} \subseteq I(\mathbf{q})$, *then* $r \ominus s = r' \ominus s'$ *if* $r' = r \oplus i$ *and* $s' = s \oplus i$.

Proof. The relation $\mathsf{R}^\top \mathsf{S} \mathsf{R} = \mathsf{S}$ can be checked in a straightforward manner while the equations in (3.2) follow from

$$(\mathbf{q}')^\top \mathsf{S} \mathbf{q}' = \mathbf{q}^\top \mathsf{P}_i^\top \mathsf{S} \mathsf{P}_i \mathbf{q} = \mathbf{q}^\top \mathsf{S} \mathbf{q}$$

and from

$$(\mathbf{q}'')^\top \mathsf{S}\mathbf{q}'' = \mathbf{q}^\top \mathsf{R}^\top \mathsf{S}\mathsf{R}\mathbf{q} = \mathbf{q}^\top \mathsf{S}\mathbf{q}\,.$$

The assertions about the supports are evident. □

LEMMA 3.3. *Consider a zero* $\mathbf{q}$ *of* $\mathsf{S} = \mathsf{C}(\mathbf{a})$ *and suppose that there is exactly one pair* $\{r,s\} \subseteq I(\mathbf{q})$ *such that* $r \ominus s = d$*, i.e., all other pairs* $\{a,b\} \subseteq I(\mathbf{q})$ *satisfy* $a \ominus b \neq d$*, in particular* $s \ominus r \neq d$*, which rules out* $d = \frac{n}{2}$ *in the case of even* n*. Further assume that for any other zero* $\mathbf{q}'$ *which is not a cyclic permutation of* $\mathbf{q}$*, this difference never occurs: whenever* $\{a,b\} \subseteq I(\mathbf{q}')$*, then* $a \ominus b \neq d$*. Let* $\{\mathbf{q}_i : i \in [1:m]\}$ *denote all zeroes of* S *and put* $\mathbf{q}_m = \mathbf{q}$*. Then condition (2.3) holds for* $k = m$*.*

Proof. By the assumptions it is clear that $\{r,s\} \subseteq I(\mathbf{q}')$ can never hold for any zero $\mathbf{q}' \notin \{\mathsf{P}_i\mathbf{q} : i \in [1:n]\}$. So consider instead $\mathbf{q}' = \mathsf{P}_i\mathbf{q}$ for $i \in [1:n-1]$. We argue by contradiction: if $\{r,s\} \subseteq I(\mathbf{q}')$, then $\{r \oplus i, s \oplus i\} \subseteq I(\mathbf{q})$ but differs from the pair $\{r,s\}$ (note that $r = s \oplus i$ and simultaneously $s = r \oplus i$ is impossible since $d \neq \frac{n}{2}$). Obviously the difference would be the same, namely d, which by assumption is absurd. □

THEOREM 3.1. *Let* $\mathbf{q}$ *satisfy the hypothesis of Lemma 3.3 and let* $Q := \{\mathsf{P}_i\mathbf{q} : i \in [1:n]\}$*. Let* $\widetilde{\mathsf{M}} := \sum_{\mathbf{f}\in Q} x_{\mathbf{f}}\,\mathbf{f}\mathbf{f}^\top$*. If* $x_{\mathbf{f}} > 0$ *for all* $\mathbf{f} \in Q$*, then the minimal cp decomposition of* $\widetilde{\mathsf{M}}$ *is unique and cpr* $\widetilde{\mathsf{M}} = |Q|$*.*

Proof. We iterate the reduction step of Theorem 2.1, applying it to M' instead of M, and repeat the construction. Lemma 3.3 guarantees that we end with a zero matrix, so we show cpr $\mathsf{M} = |Q|$. Moreover, Lemma 2.2 guarantees that all variables are fixed, so that the minimal cop decomposition is unique. □

The next two results deal with instances where there is more than one minimal cp decomposition of a similarly constructed matrix:

LEMMA 3.4. *Consider* $\mathbf{q} \in \mathbb{R}^n_+$ *such that* $Q := \{\mathsf{P}_i\mathbf{q} : i \in [1:n]\}$ *satisfies* $|Q| = n$ *and* $\mathsf{R}\mathbf{q} \notin Q$*. Let*

$$U_{\mathbf{q}} := \{d \in [1:n-1] : \ d = r \ominus s \text{ has exactly one solution with } \{r,s\} \subseteq I(\mathbf{q})\}\,.$$

Suppose there are $d_1, d_2 \in U_{\mathbf{q}}$ *with* $d_1 = r \ominus s$ *and* $d_2 = \rho \ominus \sigma$*, such that* $\rho + \sigma - r - s$ *and* n *are coprime. We consider the following subset of the vector space* $\mathcal{C}^{n*}$*:*

$$\mathcal{F} := \left\{\mathbf{f}\mathbf{f}^\top : \mathbf{f} \in Q\right\} \cup \left\{\mathsf{R}\mathbf{f}(\mathsf{R}\mathbf{f})^\top : \mathbf{f} \in Q\right\}.$$

Then every $(2n-1)$*-element subset of* $\mathcal{F}$ *is linearly independent, moreover* $\mathcal{F}$ *itself has rank* $2n-1$*.*

Proof. We first observe that our assumptions on Q imply $|\mathcal{F}| = 2n$. Moreover, $U_{\mathsf{R}\mathbf{q}} = U_{\mathbf{q}}$. Because of

$$\sum_{\mathbf{f}\in Q} \mathsf{R}\mathbf{f}(\mathsf{R}\mathbf{f})^\top = \mathsf{R}\left(\sum_{\mathbf{f}\in Q} \mathbf{f}\mathbf{f}^\top\right)\mathsf{R}^\top = \sum_{\mathbf{f}\in Q} \mathbf{f}\mathbf{f}^\top\,,$$

the rank of $\mathcal{F}$ can be at most $2n-1$. Let $\mathbf{q}_i := \mathsf{P}_i\mathbf{q}$ for $i \in [1:n]$. Then $\{r \ominus i, s \ominus i\} \subseteq I(\mathbf{q}_i)$. Further define $\mathbf{q}_i' := \mathsf{R}\mathbf{q}_j$ where j is chosen such that also $\{r \ominus i, s \ominus i\} \subseteq I(\mathbf{q}_i')$. Next consider the equation

$$\sum_{i=1}^n x_i \mathbf{q}_i \mathbf{q}_i^\top + \sum_{i=1}^n x_i' \mathbf{q}_i' {\mathbf{q}_i'}^\top = \mathsf{O}. \tag{3.3}$$

Multiplying with $\mathbf{e}_{r\oplus j}^{\top}$ from the left and with $\mathbf{e}_{s\oplus j}$ from the right, we obtain

$$x_j + x_j' = 0 \quad \text{for all } j \in [1:n]\,.$$

Multiplying with $\mathbf{e}_{\rho\oplus j}^{\top}$ from the left and with $\mathbf{e}_{\sigma\oplus j}$ from the right, we obtain

$$x_j + x'_{j\oplus\rho\oplus\sigma\ominus r\ominus s} = 0 \quad \text{for all } j \in [1:n].$$

From these equations we conclude that $x_j' = x'_{j\oplus\rho\oplus\sigma\ominus r\ominus s}$ for all $j \in [1:n]$, and fixing $x_1' = \xi$, this means that our system of $2n$ equations has the unique solution $x_i = -x_i' = -\xi$ for $i \in [1:n]$. So there is a one parameter family of solutions parameterized by ξ, showing that if any of the coefficients in (3.3) is zero, all others also must be zero, so indeed every $(2n-1)$-element subset of $\mathcal{F}$ has to be linearly independent, as asserted. ☐

THEOREM 3.2. *Let $\mathbf{q}$ satisfy the hypothesis of Lemma 3.4, let $Q := \{\mathsf{P}_i\mathbf{q} : i \in [1:n]\}$ and $Q' := \{\mathsf{R}\mathbf{q} : \mathbf{q} \in Q\}$. Select $x_{\mathbf{f}} \geq 0$ for all $\mathbf{f} \in Q \cup Q'$ and consider the matrix $\overline{\mathsf{M}} = \sum_{\mathbf{f}\in Q\cup Q'} x_{\mathbf{f}}\,\mathbf{f}\mathbf{f}^{\top}$. Then we have:*

(a) *If all $x_{\mathbf{f}} > 0$ and if $|\mathrm{argmin}\,\{x_{\mathbf{f}} : \mathbf{f} \in Q\}| = |\mathrm{argmin}\,\{x_{\mathbf{f}} : \mathbf{f} \in Q'\}| = 1$, then there are exactly two different minimal cp decompositions of $\overline{\mathsf{M}}$ and cpr $\overline{\mathsf{M}} = 2|Q| - 1$.*

(b) *If $x_{\mathbf{f}} = 0$ for at least one $\mathbf{f} \in Q$ and at least one $\mathbf{f} \in Q'$, then the minimal cp decomposition of $\overline{\mathsf{M}}$ is unique and cpr $\overline{\mathsf{M}} = |I(\mathbf{x})|$.*

Proof. Define $u_{\mathbf{f}} := 1$ for all $\mathbf{f} \in Q$ and $u_{\mathbf{f}} := -1$ for all $\mathbf{f} \in Q'$. Then, by the proof of Lemma 3.4, the solutions $\mathbf{y}$ of the equation $\overline{\mathsf{M}} = \sum_{\mathbf{f}\in Q\cup Q'} y_{\mathbf{f}}\,\mathbf{f}\mathbf{f}^{\top}$ are given by $\mathbf{y} = \mathbf{x} + \xi\mathbf{u}$. In case (a), the solutions $\mathbf{y} \geq \mathbf{o}$ additionally require $\xi \in [-\min\{x_{\mathbf{f}} : \mathbf{f} \in Q\}, \min\{x_{\mathbf{f}} : \mathbf{f} \in Q'\}]$, with $|I(\mathbf{y})| = 2|Q| - 1$ (resp. $|I(\mathbf{y})| = 2|Q|$) for ξ on the boundary (resp. in the interior) of that interval. In case (b), the condition $\mathbf{y} \geq \mathbf{o}$ is violated for any $\xi \neq 0$, so $\mathbf{y} = \mathbf{x}$ is unique. ☐

4. Counterexamples to the Drew-Johnson-Loewy conjecture.

For the examples to follow, we selected matrices S with integer entries, where we could determine all minimizers of the quadratic form $\mathbf{x}^{\top}\mathsf{S}\mathbf{x}$ by exact arithmetic, solving the first-order conditions and checking the values for nonnegativity with the help of (3.2), cf. also [3, 4].

Example 1 ($p_7 \geq 14$)**:** Let $\mathsf{S} = \mathsf{C}([-153, 127, -27, -27, 127, -153, 162]^{\top})$. Then the set of zeroes of S in $\Delta \in \mathbb{R}^7$ consists of 14 vectors: $\mathbf{q}_i = \mathsf{P}_i\mathbf{u}, i \in [1:7]$, where $\mathbf{u} = \frac{1}{7}[3,3,0,0,1,0,0]^{\top}$, and $\mathbf{q}_i = \mathsf{P}_i\mathbf{v}, i \in [8:14]$, where $\mathbf{v} = \frac{1}{35}[9,17,9,0,0,0,0]^{\top}$. Let

$$\mathsf{M} := \sum_{i=1}^{14} \mathbf{q}_i\mathbf{q}_i^{\top} = \tfrac{1}{1225}\,\mathsf{C}([531, 81, 150, 150, 81, 531, 926]^{\top}).$$

The differences $r \ominus s$ which can be computed from subsets $\{r, s\} \subseteq I(\mathbf{u}) = \{1, 2, 5\}$, or from subsets $\{r, s\} \subseteq I(\mathbf{v}) = \{1, 2, 3\}$, can be arranged in two matrices,

$$D_{\mathbf{u}} = \begin{bmatrix} 0 & 6 & 3 \\ 1 & 0 & 4 \\ 4 & 3 & 0 \end{bmatrix} \quad \text{and} \quad D_{\mathbf{v}} = \begin{bmatrix} 0 & 6 & 5 \\ 1 & 0 & 6 \\ 2 & 1 & 0 \end{bmatrix}.$$

We note that the difference $d = 2$ appears only once, so we may apply Lemma 3.3 and Lemma 2.2, to conclude that in any cp decomposition $\mathsf{M} = \sum_{i=1}^{14} x_i \mathbf{q}_i \mathbf{q}_i^\top$ we must have $x_i = 1$ for $i \in [8:14]$. Next, consider the matrix $\mathsf{M}' := \mathsf{M} - \sum_{i=8}^{14} \mathbf{q}_i \mathbf{q}_i^\top = \sum_{i=1}^{7} x_i \mathbf{q}_i \mathbf{q}_i^\top$. As the only vectors that may occur in a cp decomposition of M' are shifted versions of $\mathbf{u}$, the fact that the difference $d = 1$ appears only once in $D_\mathbf{u}$ again allows to invoke Lemma 3.3 and Lemma 2.2. We conclude that $x_i = 1$ also for $i \in [1:7]$, that M has a unique minimal cp decomposition, and that $\operatorname{cpr} \mathsf{M} = 14$. Another matrix of this sort, having small integer entries, is

$$\tilde{M}_7 := \tfrac{2}{3} 7^2 \sum_{i=1}^{7} \mathbf{q}_i \mathbf{q}_i^\top + \tfrac{1}{3} 35^2 \sum_{i=8}^{14} \mathbf{q}_i \mathbf{q}_i^\top = \begin{bmatrix} 163 & 108 & 27 & 4 & 4 & 27 & 108 \\ 108 & 163 & 108 & 27 & 4 & 4 & 27 \\ 27 & 108 & 163 & 108 & 27 & 4 & 4 \\ 4 & 27 & 108 & 163 & 108 & 27 & 4 \\ 4 & 4 & 27 & 108 & 163 & 108 & 27 \\ 27 & 4 & 4 & 27 & 108 & 163 & 108 \\ 108 & 27 & 4 & 4 & 27 & 108 & 163 \end{bmatrix}.$$

Note that both, above matrix and M, have no zero entries and full rank.

Example 2 ($p_9 \geq 26$)**:** Let

$$\mathsf{S} = \mathsf{C}([-1056, 959, -484, 231, 231, -484, 959, -1056, 1089]^\top).$$

Then the set of zeroes of S in $\Delta \in \mathbb{R}^9$ consists of 27 vectors: indeed, let

$$\left.\begin{array}{rcl} \mathbf{u} & = & \frac{1}{26}[11, 12, 0, 0, 3, 0, 0, 0, 0]^\top \\ \mathbf{v} & = & \frac{1}{26}[12, 11, 0, 0, 0, 0, 3, 0, 0]^\top \\ \mathbf{w} & = & \frac{1}{130}[33, 64, 33, 0, 0, 0, 0, 0, 0]^\top \end{array}\right\} \text{ and define } \mathbf{q}_i := \begin{cases} \mathsf{P}_i \mathbf{u}\,, & \text{if } i \in [1:9]\,, \\ \mathsf{P}_i \mathbf{v}\,, & \text{if } i \in [10:18]\,, \\ \mathsf{P}_i \mathbf{w}\,, & \text{if } i \in [19:27]\,. \end{cases}$$

The set of zeroes of S is $\{\mathbf{q}_i : i \in [1:27]\}$ and $\mathsf{P}_2 \mathbf{v} = \mathsf{R}\mathbf{u} \notin \{\mathsf{P}_i \mathbf{u} : i \in [1:9]\}$. Put

$$\begin{aligned} \mathsf{M} :=& 2\sum_{i=1}^{18} \mathbf{q}_i \mathbf{q}_i^\top - \mathbf{q}_9 \mathbf{q}_9^\top - \mathbf{q}_{11} \mathbf{q}_{11}^\top + \sum_{i=19}^{27} \mathbf{q}_i \mathbf{q}_i^\top \\ =& \frac{1}{16900} \begin{bmatrix} 30649 & 14124 & 1089 & 3600 & 2475 & 3300 & 3600 & 1089 & 17424 \\ 14124 & 30074 & 17424 & 1089 & 2700 & 3300 & 3300 & 3600 & 1089 \\ 1089 & 17424 & 33674 & 17424 & 1089 & 3600 & 3300 & 3300 & 3600 \\ 3600 & 1089 & 17424 & 33674 & 17424 & 1089 & 3600 & 3300 & 3300 \\ 2475 & 2700 & 1089 & 17424 & 33224 & 17424 & 1089 & 2700 & 2475 \\ 3300 & 3300 & 3600 & 1089 & 17424 & 33674 & 17424 & 1089 & 3600 \\ 3600 & 3300 & 3300 & 3600 & 1089 & 17424 & 33674 & 17424 & 1089 \\ 1089 & 3600 & 3300 & 3300 & 2700 & 1089 & 17424 & 30074 & 14124 \\ 17424 & 1089 & 3600 & 3300 & 2475 & 3600 & 1089 & 14124 & 30649 \end{bmatrix}. \end{aligned}$$

We have $I(\mathbf{u}) = \{1, 2, 5\}$, $I(\mathbf{v}) = \{1, 2, 7\}$, $I(\mathbf{w}) = \{1, 2, 3\}$, and with the notation of Example 1 we compute

$$D_\mathbf{u} = \begin{bmatrix} 0 & 8 & 5 \\ 1 & 0 & 6 \\ 4 & 3 & 0 \end{bmatrix}, \quad D_\mathbf{v} = \begin{bmatrix} 0 & 8 & 3 \\ 1 & 0 & 4 \\ 6 & 5 & 0 \end{bmatrix}, \quad D_\mathbf{w} = \begin{bmatrix} 0 & 8 & 7 \\ 1 & 0 & 8 \\ 2 & 1 & 0 \end{bmatrix}.$$

We note that the difference $d = 2$ appears only once, so we may apply Lemma 3.3 and Lemma 2.2 to conclude that in any cp decomposition $\mathsf{M} = \sum_{i=1}^{27} x_i \mathbf{q}_i \mathbf{q}_i^\top$ we must have $x_i = 1$ for $i \in [19:27]$. Next, consider the matrix $\overline{\mathsf{M}} := \mathsf{M} - \sum\limits_{i=19}^{27} \mathbf{q}_i\mathbf{q}_i^\top$. As the only vectors that may occur in a cp decomposition of $\overline{\mathsf{M}}$ are shifts and mirror images of shifts of $\mathbf{u}$, the fact that the differences $d_1 = 1 = 2-1$ and $d_2 = 3 = 5-2$ appear only once in $D_{\mathbf{u}}$, and $5+2-2-1 = 4$ and 9 are coprime allows to invoke Lemma 3.4 and Theorem 3.2. We conclude that there are exactly two vectors $\mathbf{x} \in \mathbb{R}^{18}_+$ of support of size 17, (and no such vectors of smaller support,) that give rise to minimal cp decompositions of $\overline{\mathsf{M}}$, and that cpr $\mathsf{M} = 26$. Another matrix of this sort, having small integer entries, is

$$\widetilde{\mathsf{M}}_9 := \tfrac{5}{6}\, 26^2 \left(\sum_{i=1}^{18} \mathbf{q}_i\mathbf{q}_i^\top - \tfrac{3}{5} \left(\mathbf{q}_7\mathbf{q}_7^\top + \mathbf{q}_{13}\mathbf{q}_{13}^\top \right) \right) + \tfrac{1}{3}\, 130^2 \sum_{i=19}^{27} \mathbf{q}_i\mathbf{q}_i^\top$$
$$= \begin{bmatrix} 2548 & 1628 & 363 & 60 & 55 & 55 & 60 & 363 & 1628 \\ 1628 & 2548 & 1628 & 363 & 60 & 55 & 55 & 60 & 363 \\ 363 & 1628 & 2483 & 1562 & 363 & 42 & 22 & 55 & 60 \\ 60 & 363 & 1562 & 2476 & 1628 & 363 & 42 & 55 & 55 \\ 55 & 60 & 363 & 1628 & 2548 & 1628 & 363 & 60 & 55 \\ 55 & 55 & 42 & 363 & 1628 & 2476 & 1562 & 363 & 60 \\ 60 & 55 & 22 & 42 & 363 & 1562 & 2483 & 1628 & 363 \\ 363 & 60 & 55 & 55 & 60 & 363 & 1628 & 2548 & 1628 \\ 1628 & 363 & 60 & 55 & 55 & 60 & 363 & 1628 & 2548 \end{bmatrix}.$$

Note that neither of these matrices of cp-rank 26 are cyclically symmetric, they have no zero entries and full rank.

Example 3 ($p_8 \geq 18$)**:** Continuing Example 2, we observe that the upper left 8×8-submatrix of S has 18 zeroes. These are obtained by taking the first 8 coordinates of those zeroes $\mathbf{q}$ of S satisfying $\mathbf{e}_9^\top \mathbf{q} = 0$. Define the set $S_8 := \{\mathbf{q} \in \{\mathbf{q}_1, \ldots, \mathbf{q}_{27}\} : \mathbf{e}_9^\top \mathbf{q} = 0\}$. Then the matrix $\mathsf{M} := \sum\limits_{\mathbf{q}\in S_8} \mathbf{q}\mathbf{q}^\top$ satisfies cpr $\mathsf{M} = 18$, by Lemma 3.3, Lemma 3.4 and Theorem 3.2. Moreover all entries in the last row and the last column of M are zero, therefore also M_8, the upper left 8×8-submatrix of M, has cpr $\mathsf{M}_8 = 18$. Again, by adjusting weights, we came up with a matrix with small integer entries:

$$\widetilde{\mathsf{M}}_8 := \begin{bmatrix} 541 & 880 & 363 & 24 & 55 & 11 & 24 & 0 \\ 880 & 2007 & 1496 & 363 & 48 & 22 & 22 & 24 \\ 363 & 1496 & 2223 & 1452 & 363 & 24 & 22 & 11 \\ 24 & 363 & 1452 & 2325 & 1584 & 363 & 48 & 55 \\ 55 & 48 & 363 & 1584 & 2325 & 1452 & 363 & 24 \\ 11 & 22 & 24 & 363 & 1452 & 2223 & 1496 & 363 \\ 24 & 22 & 22 & 48 & 363 & 1496 & 2007 & 880 \\ 0 & 24 & 11 & 55 & 24 & 363 & 880 & 541 \end{bmatrix}.$$

Note that $\widetilde{\mathsf{M}}_8$ is, again, not cyclically symmetric, and that it has full rank.

Example 4 ($p_{10} \geq 27$)**:** Continuing Example 2, let $\mathsf{M} \in \mathcal{C}^{10*}$ be the matrix obtained from $\widetilde{\mathsf{M}}_9$ by appending a zero column $\mathbf{o} \in \mathbb{R}^9$ and completing this to a symmetric 10×10 matrix by adding one row $\mathbf{e}_{10}^\top$ as the last one. Then, by [17, Prop.2.2], we get

$$\text{cpr } \mathsf{M} = \text{cpr } \widetilde{\mathsf{M}}_9 + 1 = 27\,.$$

Example 5 ($p_{11} \geq 32$)**:** Consider

$$\mathsf{S} = \mathsf{C}([32, 18, 4, -24, -31, -31, -24, 4, 18, 32, 32]^\top).$$

There are 33 zeroes of S; indeed, let

$$\left.\begin{array}{rcl} \mathbf{u} &=& \frac{1}{21}[8,0,3,0,0,0,10,0,0,0,0]^\top \\ \mathbf{v} &=& \frac{1}{21}[10,0,0,0,3,0,8,0,0,0,0]^\top \\ \mathbf{w} &=& \frac{1}{7}[2,0,0,2,0,0,0,3,0,0,0]^\top \end{array}\right\} \text{ and define } \mathbf{q}_i := \begin{cases} \mathsf{P}_i\mathbf{u}\,, & \text{if } i \in [1:11]\,, \\ \mathsf{P}_i\mathbf{v}\,, & \text{if } i \in [12:22]\,, \\ \mathsf{P}_i\mathbf{w}\,, & \text{if } i \in [23:33]\,, \end{cases}$$

then the set of zeroes can be written as $\{\mathbf{q}_i : i \in [1:33]\}$. Now put

$$\begin{aligned} \mathsf{M} :=& 2\sum_{i=1}^{22} \mathbf{q}_i\mathbf{q}_i^\top - \mathbf{q}_{11}\mathbf{q}_{11}^\top - \mathbf{q}_{13}\mathbf{q}_{13}^\top + \sum_{i=23}^{33} \mathbf{q}_i\mathbf{q}_i^\top \\ =& \frac{1}{441}\begin{bmatrix} 781 & 0 & 72 & 36 & 228 & 320 & 240 & 228 & 36 & 96 & 0 \\ 0 & 845 & 0 & 96 & 36 & 228 & 320 & 320 & 228 & 36 & 96 \\ 72 & 0 & 827 & 0 & 72 & 36 & 198 & 320 & 320 & 198 & 36 \\ 36 & 96 & 0 & 845 & 0 & 96 & 36 & 228 & 320 & 320 & 228 \\ 228 & 36 & 72 & 0 & 781 & 0 & 96 & 36 & 228 & 240 & 320 \\ 320 & 228 & 36 & 96 & 0 & 845 & 0 & 96 & 36 & 228 & 320 \\ 240 & 320 & 198 & 36 & 96 & 0 & 745 & 0 & 96 & 36 & 228 \\ 228 & 320 & 320 & 228 & 36 & 96 & 0 & 845 & 0 & 96 & 36 \\ 36 & 228 & 320 & 320 & 228 & 36 & 96 & 0 & 845 & 0 & 96 \\ 96 & 36 & 198 & 320 & 240 & 228 & 36 & 96 & 0 & 745 & 0 \\ 0 & 96 & 36 & 228 & 320 & 320 & 228 & 36 & 96 & 0 & 845 \end{bmatrix}, \end{aligned}$$

and again M has full rank. We get $I(\mathbf{u}) = \{1, 3, 7\}$, $I(\mathbf{v}) = \{1, 5, 7\}$, $I(\mathbf{w}) = \{1, 4, 8\}$, and we calculate

$$D_{\mathbf{u}} = \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 7 \\ 6 & 4 & 0 \end{bmatrix}, \quad D_{\mathbf{v}} = \begin{bmatrix} 0 & 7 & 5 \\ 4 & 0 & 9 \\ 6 & 2 & 0 \end{bmatrix}, \quad D_{\mathbf{w}} = \begin{bmatrix} 0 & 8 & 4 \\ 3 & 0 & 7 \\ 7 & 4 & 0 \end{bmatrix}.$$

Analogously to Example 2 we now show that the minimal cp rank is 32. Since the difference $d = 3$ appears only once (in $D_{\mathbf{w}}$), we must have $x_i = 1$ for $i \in [22:33]$ by Lemma 3.3 and Lemma 2.2. Therefore consider $\overline{\mathsf{M}} := \mathsf{M} - \sum\limits_{i=22}^{33} \mathbf{q}_i\mathbf{q}_i^\top$. We can see that the differences $d_1 = 6 = 7 - 1$ and $d_2 = 4 = 7 - 3$ appear only once in $D_{\mathbf{u}}$, and knowing that $7 + 3 - 7 - 1 = 2$ and 11 are coprime allows to invoke Lemma 3.4 and Theorem 3.2. Hence there are exactly two vectors $\mathbf{x} \in \mathbb{R}^{22}_+$ of support of size 21 for $\overline{\mathsf{M}}$ and this leads to a total of 32 for cpr M.

TABLE 4.1
(Ranges for) maximal cp-rank p_n of cp matrices of order n.

n	5	6	7	8	9	10	11
d_n	6	9	12	16	20	25	30
p_n	6	≤ 15	≥ 14	≥ 18	≥ 26	≥ 27	≥ 32
s_n	11	17	24	32	41	51	62

Table 4.1 summarizes the known bracket and consequences from above examples. A tighter upper bound $p_6 \leq 15$ was proved in [16, Thm.6.1], but up to now no $\mathsf{M} \in \mathcal{C}^{6*}$ with cpr $\mathsf{M} > 9 = d_6$ is known.

REFERENCES

[1] Abraham Berman and Naomi Shaked-Monderer. Remarks on completely positive matrices. *Linear and Multilinear Algebra*, 44:149–163, 1998.

[2] Abraham Berman and Naomi Shaked-Monderer. *Completely positive matrices*. World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[3] Immanuel M. Bomze. Detecting all evolutionarily stable strategies. *J. Optim. Theory Appl.*, 75(2):313–329, 1992.

[4] Immanuel M. Bomze. Regularity versus degeneracy in dynamics, games, and optimization: a unified approach to different aspects. *SIAM Rev.*, 44(3):394–414, 2002.

[5] Immanuel M. Bomze. Copositive optimization – recent developments and applications. *European J. Oper. Res.*, 216:509–520, 2012.

[6] Immanuel M. Bomze, Werner Schachinger, and Gabriele Uchida. Think co(mpletely)positive ! – matrix properties, examples and a clustered bibliography on copositive optimization. *J. Global Optim.*, 52:423–445, 2012.

[7] Mark Broom, Chris Cannings, and Glenn T. Vickers. On the number of local maxima of a constrained quadratic form. *Proc. R. Soc. Lond. A*, 443:573–584, 1993.

[8] Samuel Burer. Copositive programming. In Miguel F. Anjos and Jean Bernard Lasserre, editors, *Handbook of Semidefinite, Cone and Polynomial Optimization: Theory, Algorithms, Software and Applications*, International Series in Operations Research and Management Science, pages 201–218. Springer, New York, 2012.

[9] Peter J. C. Dickinson, Mirjam Dür, Luuk Gijben, and Roland Hildebrand. Irreducible elements of the copositive cone. *Linear Algebra Appl.*, 439(6):1605–1626, 2013.

[10] John H. Drew and Charles R. Johnson. The no long odd cycle theorem for completely positive matrices. In *Random discrete structures*, volume 76 of *IMA Vol. Math. Appl.*, pages 103–115. 1996.

[11] John H. Drew, Charles R. Johnson, and Raphael Loewy. Completely positive matrices associated with M-matrices. *Linear Multilinear Algebra*, 37(4):303–310, 1994.

[12] Mirjam Dür. Copositive programming — a survey. In Moritz Diehl, Francois Glineur, Elias Jarlebring, and Wim Michiels, editors, *Recent Advances in Optimization and its Applications in Engineering*, pages 3–20. Springer, Berlin Heidelberg New York, 2010.

[13] Roland Hildebrand. The extremal rays of the 5×5 copositive cone. *Linear Algebra Appl.*, 437(7):1538–1547, 2012.

[14] Roland Hildebrand. Minimal zeros of copositive matrices. Preprint, `http://arxiv.org/abs/1401.0134`, 2014.

[15] Raphael Loewy and Bit-Shun Tam. CP rank of completely positive matrices of order 5. *Linear Algebra Appl.*, 363:161–176, 2003.

[16] Naomi Shaked-Monderer, Abraham Berman, Immanuel M. Bomze, Florian Jarre, and Werner Schachinger. New results on the cp rank and related properties of co(mpletely)positive matrices. *Linear Multilinear Algebra*, to appear. Also available at: arxiv.org/abs/1305.0737, 2013.

[17] Naomi Shaked-Monderer, Immanuel M. Bomze, Florian Jarre, and Werner Schachinger. On the cp-rank and minimal cp factorizations of a completely positive matrix. *SIAM J. Matrix Anal. Appl.*, 34(2):355–368, 2013.