INFINITELY MANY SOLUTIONS FOR A CLASS OF SUBLINEAR SCHRÖDINGER EQUATIONS WITH INDEFINITE POTENTIALS

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Abstract. In this paper we are concerned with qualitative properties of entire solutions to a Schrödinger equation with sublinear nonlinearity and sign-changing potentials. Our analysis considers three distinct cases and we establish sufficient conditions for the existence of infinitely many solutions.

1. Historical perspectives of the Schrödinger equation

The Schrödinger equation plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future behaviour of a dynamic system. The linear Schrödinger equation is a central tool of quantum mechanics, which provides a thorough description of a particle in a non-relativistic setting. Schrödinger's linear equation is

$$\Delta\psi + \frac{8\pi^2 m}{\hbar^2}\,(E(x) - V(x))\,\psi = 0\,,$$

where ψ is the Schrödinger wave function, m is the mass, $\hbar$ denotes Planck's constant, E is the energy, and V stands for the potential energy.

The structure of the nonlinear Schrödinger equation is much more complicated. This equation is a prototypical dispersive nonlinear partial differential equation that has been central for almost four decades now to a variety of areas in mathematical physics. The relevant fields of application vary from Bose-Einstein condensates and nonlinear optics (see Byeon and Wang [16]), propagation of the electric field in optical fibers (see Hasegawa and Kodama [26], Malomed [32]) to the self-focusing and collapse of Langmuir waves in plasma physics (see Zakharov [42]) and the behaviour of deep water waves and freak waves (the so-called rogue waves) in the ocean (see Onorato, Osborne, Serio and Bertone [34]). The nonlinear Schrödinger equation also describes various phenomena arising in the theory of Heisenberg ferromagnets and magnons, self-channelling of a high-power ultra-short laser in matter, condensed matter theory, dissipative quantum mechanics, electromagnetic fields (see Avron, Herbst and Simon [6]), plasma physics (e.g., the Kurihara superfluid film equation). We refer to Ablowitz, Prinari and Trubatch [1], Sulem [36] for a modern overview, including applications.

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Schrödinger also established the classical derivation of his equation, based upon the analogy between mechanics and optics, and closer to de Broglie's ideas. He developed a perturbation method, inspired by the work of Lord Rayleigh in acoustics, proved the equivalence between his wave mechanics and Heisenberg's matrix, and introduced the time dependent Schrödinger's equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\,\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \qquad \text{in } \mathbb{R}^N\ (N \geq 2), \tag{1}$$

where $p < 2N/(N-2)$ if $N \geq 3$ and $p < +\infty$ if $N = 2$. In physical problems, a cubic nonlinearity corresponding to $p = 3$ is common; in this case problem (1) is called the Gross-Pitaevskii equation. In the study of Eq. (1), Floer and Weinstein [24] and Oh [33] supposed that the potential V is bounded and possesses a non-degenerate critical point at $x = 0$. More precisely, it is assumed that V belongs to the class (V_a) (for some real number a) introduced in Kato [29]. Taking $\gamma > 0$ and $\hbar > 0$ sufficiently small and using a Lyapunov-Schmidt type reduction, Oh [33] proved the existence of standing wave solutions of problem (1), that is, a solution of the form

$$\psi(x,t) = e^{-iEt/\hbar}u(x)\,. \tag{2}$$

Using the ansatz (2), we reduce the nonlinear Schrödinger equation (1) to the semilinear elliptic equation

$$-\frac{\hbar^2}{2}\,\Delta u + (V(x) - E)\,u = |u|^{p-1}u\,.$$

The change of variable $y = \hbar^{-1}x$ (and replacing y by x) yields

$$-\Delta u + 2\,(V_\hbar(x) - E)\,u = |u|^{p-1}u \qquad \text{in } \mathbb{R}^N\,, \tag{3}$$

where $V_\hbar(x) = V(\hbar x)$.

If for some $\xi \in \mathbb{R}^N \setminus \{0\}$, $V(x+s\xi) = V(x)$ for all $s \in \mathbb{R}$, equation (1) is invariant under the Galilean transformation

$$\psi(x,t) \longmapsto \psi(x-\xi t,t)\exp\left(i\xi\cdot x/\hbar - \frac{1}{2}i|\xi|^2t/\hbar\right)\psi(x-\xi t,t)\,.$$

Thus, in this case, standing waves reproduce solitary waves traveling in the direction of ξ. In other words, Schrödinger discovered that the standing waves are scalar waves rather than vector electromagnetic waves. This is an important difference, vector electromagnetic waves are mathematical waves which describe a direction (vector) of force, whereas the wave motions of space are scalar waves which are simply described by their wave-amplitude. The importance of this discovery was pointed out by Albert Einstein, who wrote: "The Schrödinger method, which has in a certain sense the character of a field theory, does indeed deduce the existence of only discrete states, in surprising agreement with empirical facts. It does so on the basis of differential equations applying a kind of resonance argument". (*On Quantum Physics*, 1954).

In a celebrated paper, Rabinowitz [35] proved that problem (3) has a ground-state solution (mountain-pass solution) for $\hbar > 0$ small, under the assumption that $\inf_{x\in\mathbb{R}^N} V(x) > E$. After making a standing wave ansatz, Rabinowitz reduces the problem to that of studying the semilinear elliptic equation

$$-\Delta u + V(x)u = f(x,u) \qquad \text{in } \mathbb{R}^N, \tag{4}$$

under suitable conditions on V and assuming that f is smooth, superlinear and has a subcritical growth.

2. Introduction and main results

In the present paper we are concerned with the existence of infinitely many solutions of the semilinear Schrödinger equation

$$-\Delta u + V(x)u = a(x)g(u) \qquad x \in \mathbb{R}^N \ (N \geq 3), \tag{5}$$

where V, a are functions changing sign and the nonlinearity g has a sublinear growth. Such problems in $\mathbb{R}^N$ arise naturally in various branches of physics and present challeging mathematical difficulties.

If problem (5) is considered in a bounded domain Ω, with the Dirichlet boundary condition, then there is a large literature on existence and a multiplicity of solutions (see [4, 15, 27, 28, 37, 39, 40]). In particular, Kajikiya [27] has considered such sublinear case with sign-changing nonlinearity and has proved the existence of infinitely many solutions.

If Ω is an unbounded domain and especially if $\Omega = \mathbb{R}^N$, the existence and multiplicity of nontrivial solutions for problem (5) have been extensively investigated in the literature over the past several decades, both for sublinear and superlinear nonlinearities.

In the superlinear case, we can cite the references [2, 5, 7, 9, 18, 20, 22, 23, 25, 35, 41]. In particular, Costa and Tehrani [18] have considered the following problem

$$-\Delta u - \lambda h(x)u = a(x)g(u), \quad u > 0 \ \text{ in } \ \mathbb{R}^N, \tag{6}$$

where $\lambda > 0, h$ is a positive function, a changes the sign in $\mathbb{R}^N$, $N \geq 3$, and g is a superlinear function. With furthermore assumptions on h, a and g, they proved the existence of $\lambda_1(h) > 0$ such that problem (6) admits one positive solution for $0 < \lambda < \lambda_1(h)$ and two positive solutions for $\lambda_1(h) < \lambda < \lambda_1(h) + \epsilon$, for some $\epsilon > 0$.

In recent years, many authors have studied the question of existence and multiplicity of solutions for problem (5) with sublinear nonlinearity, see [8, 11, 12, 13, 17, 19, 30, 38]. In most of the problems studied in the papers cited in the references above, V and a are considered to be positive. In particular, Brezis and Kamin [13] gave a sufficient and necessary condition for the existence of bounded positive solutions of problem (5) with $V = 0$ and $a > 0$.

Balabane, Dolbeault and Ounaies [8] proved that for each integer k, the equation (5) has a radially compactly supported solution that has k zeros in its support, provided that $V = a = -1$ and $g(u) = |u|^{-2\theta} u$, where $\theta \in]0, \frac{1}{2}[$.

Zhang and Wang [43] proved the existence of infinitely many solutions for equation (5) with $g(u) = |u|^{p-1} u$, $0 < p < 1$ and the potentials $V > 0, a > 0$ satisfy the following assumptions:

(S_1) $V \in C\left(\mathbb{R}^N, \mathbb{R}\right)$ and there exists $r > 0$ such that

$$m\left\{x \in B\left(y, r\right); \ \ V(x) \leq M\right\} \to 0 \ \text{ as } \ |y| \to +\infty, \ \ \forall\, M > 0,$$

where m is the Lebesgue measure in $\mathbb{R}^N$.

(S_2) $a : \mathbb{R}^N \to \mathbb{R}$ is a continuous function and $a \in L^{\frac{2}{1-p}}\left(\mathbb{R}^N\right)$ $0 < p < 1$.

If V, a both change sign on $\mathbb{R}^N$, various difficulties arise. To authors' knowledge, few results are known in this case. On this subject, Costa and Tehrani [19] have proved the existence of at least one non trivial solution for the following equation:

$$-\Delta u+V(x)u=\lambda u+g(x,u),$$

under the following conditions:
(VC_1) $V\in C^{\beta}\left(\mathbb{R}^N\right)$ $(0<\beta<1)$ and $\lim\limits_{|x|\to+\infty}V(x)=0$;
(VC_2) $\displaystyle\int_{\mathbb{R}^N}\left(|\nabla\varphi|^2+V(x)\varphi^2\right)dx<0$ for some $\varphi\in C_0^1\left(\mathbb{R}^N\right)$;
(GC_1) $|g(x,s)|\le b_1(x)\,|s|^{\alpha}+b_2(x)$ for some $0<\alpha<1$ and a class of integrable functions b_1 and b_2;
(GC_2) $\lambda<0$ is an eigenvalue of the Schrödinger operator $L_V=-\Delta+V(x)$ in $\mathbb{R}^N$.
(GC_2) $\displaystyle\lim_{\substack{\|u_0\|\to+\infty\\ u_0\in\operatorname{Ker}(-\Delta+V-\lambda)}}\frac{1}{\|u_0\|^{2\alpha}}\int_{\mathbb{R}^N}G\left(x,u_0(x)\right)dx=\pm\infty.$

Tehrani [38] studied the following perturbed equation:

$$-\Delta u+V(x)u=a(x)g(u)+f, \tag{7}$$

where a, V change sign on $\mathbb{R}^N$, $f\in L^2\left(\mathbb{R}^N\right)$, and g is a sublinear function. With further assumptions on a, V, f, and g, he proved the existence of at least one non trivial solution.

Costa and Chabrowski [17] considered the following p-Laplacian equation:

$$-\Delta_p u-\lambda V(x)\,|u|^{p-2}\,u=a(x)\,|u|^{q-2}\,u,\quad x\in\mathbb{R}^N, \tag{8}$$

where $\lambda\in\mathbb{R}$ is a parameter, $1<q<p<p^*=\frac{Np}{N-p}$, $V\in L^{\frac{N}{p}}\left(\mathbb{R}^N\right)\cap L^{\infty}\left(\mathbb{R}^N\right)$, $a\in L^{\infty}\left(\mathbb{R}^N\right)$ and $\lim\limits_{|x|\to+\infty}a(x)=a_{\infty}<0$. With further assumptions on a and V, they proved the existence of $\lambda_1>0$ and $\lambda_{-1}<0$ such that problem (8) admits at least one positive solution for $\lambda_{-1}<\lambda<\lambda_1$ and two positive solutions for $\lambda>\lambda_1$ and $\lambda<\lambda_{-1}$.

Benrhouma [10] has proved the existence of at least three solutions for equation (7) with $g(u)=|u|^p\operatorname{sgn}(u)$, $0<p<1$, V changes the sign, and $a<0$.

In all works cited above, where a and V change sign, the authors proved the existence of at most three solutions. In this paper, we prove the existence of infinitely many solutions of problem (5) with a and V changing sign, under various assumptions on these potential functions.

Denote by s the best Sobolev constant,

$$s=\inf\left\{\|\nabla u\|_2^2,\quad u\in W^{1,2}\left(\mathbb{R}^N\right),\quad\int_{\mathbb{R}^N}|u(x)|^{\frac{2N}{N-2}}dx=1\right\},\qquad N\ge 3.$$

We suppose the following hypotheses on g:
(G_1) $g\in C\left(\mathbb{R},\mathbb{R}\right)$, g is odd and there exist $c>0$, $q\in]0,1[$ such that

$$|g(x)|\le c\,|x|^q,\quad\text{for all }x\in\mathbb{R};$$

(G_2) $\lim_{x\to0}\frac{G(x)}{|x|^2}=+\infty$, where $G\left(x\right)=\displaystyle\int_0^x g(t)dt,\quad\forall\,x\in\mathbb{R}$;
(G_3) G is positive on $\mathbb{R}\setminus\{0\}$.

We give three theorems on the existence of infinitely many solutions to the nonlinear problem (5).

Theorem 2.1. *Assume that* $g(x) = |x|^{q-1} x$, $0 < q < 1$, *and* V *satisfies:*
(V_1) $V \in L^{\infty}\left(\mathbb{R}^N\right)$, $\lim\limits_{|x| \to +\infty} V(x) = v_{\infty} > 0$ *and*

$$\left\| V^- \right\|_{\frac{N}{2}} < s,$$

where $u^{\mp}(x) = \max\{\mp u(x), 0\}$, *for all* $x \in \mathbb{R}^N$ *and for all* $u \in E$ *and* a *satisfies:*
(A_1) $a \in L^{\infty}\left(\mathbb{R}^N\right)$, $\lim\limits_{|x| \to +\infty} a(x) = a_{\infty} < 0$ *and there exist* $y = (y_1, \cdots, y_N) \in \mathbb{R}^N$, $R_0 > 0$ *such that*

$$a(x) > 0, \quad for\ all \quad x \in B(y, R_0).$$

Then problem (5) *possesses a sequence of nontrivial solutions converging to* 0.

In the next two theorems, we change the assumption of boundedness of a by the integrability condition. The last assumption was supported to make the energy functional associated to problem (5) well defined and to guarantee that the functional $F(u) = \int_{\mathbb{R}^N} a(x) G(u(x))\, dx$ has a compact gradient. This compactness property in turn was used to prove the required Palais-Smale condition, which is essential in the application of the critical point theory. Then we have the following two multiplicity properties.

Theorem 2.2. *Suppose that* g *satisfies* $(G_1), (G_2), (G_3)$, *and the potentials* V *and* a *satisfy the following hypotheses:*
(V_2) $V \in L^{\frac{N}{2}}\left(\mathbb{R}^N\right)$ *and*

$$\left\| V^- \right\|_{\frac{N}{2}} < s;$$

(A_2) $a \in L^{\frac{2^*}{2^* - (q+1)}}\left(\mathbb{R}^N\right)$ *and there exist* $y \in \mathbb{R}^N$ *and* $R_0 > 0$ *such that*

$$a(x) > 0, \quad \forall\, x \in B(y, R_0).$$

Then problem (5) *possesses a bounded sequence of nontrivial solutions.*

Theorem 2.3. *Assume that* g *satisfies* $(G_1), (G_2), (G_3)$, V *satisfies* (V_1) *and* a *satisfies*
(A_3) $a \in L^{\frac{2}{1-q}}\left(\mathbb{R}^N\right)$, *and there exist* $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$ *and* $R_0 > 0$ *such that*

$$a(x) > 0, \quad \forall\, x \in B(y, R_0).$$

Then problem (5) *possesses a bounded sequence of nontrivial solutions.*

This paper is organized as follows. In section 2, we give some notations, we present the variational framework and we recall some definitions, and standard results. Next, the sections 3, 4 and 5 are dedicated to the proof of Theorems 2.1, 2.2 and 2.3.

3. Notations and preliminary results

In this section we present some notations and preliminaries that will be useful in the sequel. We denote:

(∗) $\|u\|_m = \left(\int_{\mathbb{R}^N} |u(x)|^m\, dx \right)^{\frac{1}{m}}$, $1 \le m < +\infty$;
(∗) $2^* = \frac{2N}{N-2}$ if $N \ge 3$ and $2^* = +\infty$ if $n \in \{1, 2\}$;
(∗) B_R denotes the ball centered at the origin of radius $R > 0$ in $\mathbb{R}^N$ and $B_R^c =$

$\mathbb{R}^N \backslash B_R$;
$(*)$ $F'(u)$: the Fréchet derivative of F at u.

Let F_1, F_2 be Banach spaces and $T : F_1 \to F_2$. T is said to be a sequentially compact operator if given any bounded sequence (x_n) in F_1 then $(T(x_n))$ has a convergent subsequence in F_2.

Let $E = H^1(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ $(0 < q < 1)$ be the reflexive Banach space endowed with the norm

$$\|u\| = \|\nabla u\|_2 + \|u\|_{q+1} \, .$$

Let $X = D^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*}(\mathbb{R}^N) ; \ \nabla u \in \left(L^2(\mathbb{R}^N) \right)^N \right\}$, endowed with the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} |\nabla u|^2 (x) dx \right)^{\frac{1}{2}} .$$

Then X is a reflexive Banach space.

Let

$$Y = \left\{ u \in H^1(\mathbb{R}^N) ; \ \int_{\mathbb{R}^N} V^+(x) u^2(x) dx < +\infty \right\},$$

under the hypotheses $V \in L^\infty(\mathbb{R}^N)$ and $\mathrm{esslim}_{x \to +\infty} V(x) > 0$. We endow Y with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + \int_{\mathbb{R}^N} V^+(x) uv dx$$

and the associated norm $\| \ \|_Y$, which is equivalent to the usual norm

$$\|u\|_{H^1} = \|\nabla u\|_2 + \|u\|_2 \, .$$

Consider the following functionals

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 (x) + V(x) u^2(x) \right) dx - \int_{\mathbb{R}^N} a(x) G(u(x)) \, dx,$$

$$\varphi(u) = \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2(x) dx - \int_{\mathbb{R}^N} a(x) G(u(x)) \, dx,$$

$$\psi(u) = -\frac{1}{2} \int_{\mathbb{R}^N} V^-(x) u^2(x) dx - \int_{\mathbb{R}^N} a(x) G(u(x)) \, dx,$$

Under suitable assumptions on a, G, V (to be fixed later), I, φ and ψ are well defined and of class C^1 on X, Y or E. A critical point of I is a weak solution of problem (5).

Next, let us recall that a Palais-Smale sequence for the functional I, for short we write (PS), is a sequence (u_n) such that

$$I(u_n) \ \text{ is bounded and } \ \|I'(u_n)\| \to 0.$$

The functional I is said to satisfy the Palais-Smale condition if any (PS)-sequence possesses a convergent subsequence.

A first main difficulty that appears in the study of problem (5) is the loss of compactness. In order to overcome this difficulty, we use the Lions compactness principle [31]. A second main difficulty is to satisfy the geometric conditions required by the Ambrosetti-Rabinowitz theorem [4]. We use a geometrical construction of subset to overcome this difficulty. Let us give a based definition and recall the mountain pass theorem of Ambrosetti and Rabinowitz.

Definition 3.1. *Let E be a Banach space. A subset A of E is said to be symmetric if $u \in E$ implies $-u \in E$. For a closed symmetric set A which does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k, we define $\gamma(A) = +\infty$. We set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$.*

Theorem 3.2 (Theorem of Ambrosetti-Rabinowitz [4])**.** *Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfy:*
(1) *I is even, bounded from below, $I(0) = 0$ and I satisfies the Palais-Smale condition.*
(2) *For each $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that*

$$\sup_{u \in A_k} I(u) < 0.$$

Under assumptions (1) − (2), we define c_k by

$$c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} I(u).$$

Then each c_k is a critical value of I, $c_k \leq c_{k+1} < 0$ for $k \in \mathbb{N}$ and (c_k) converges to zero. Moreover, if $c_k = c_{k+1} = \cdots = c_{k+p} = c$, then $\gamma(K_c) \geq p+1$. The critical set K_c is defined by

$$K_c = \{u \in E;\ I'(u) = 0,\ I(u) = c\}.$$

4. Proof of Theorem 2.1

In this section we consider the case where a is bounded and we define I on the function space $E = H^1(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$.

Lemma 4.1. *Assume that (A_1) and (V_1) hold. Then any (PS)-sequence of I is bounded in E.*

Proof. By standard arguments, I is well defined and of class C^1 on E.

Let (u_n) be a (PS)-sequence of I. Then there exists $\alpha > 0$ such that $I(u_n) \leq \alpha$. Applying Hölder inequality, conditions (A_1) and (V_1), we have

$$\begin{aligned}
\alpha \quad & \geq I(u_n) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2(x) + V(x)u_n(x)^2\right)dx - \frac{1}{q+1}\int_{\mathbb{R}^N} a(x)\,|u_n(x)|^{q+1}\,dx \\
& \geq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_n(x)|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)u_n(x)^2 dx - \frac{1}{q+1}\int_{\mathbb{R}^N} a^+(x)\,|u_n|^{q+1}(x)dx \\
& + \frac{1}{q+1}\int_{\mathbb{R}^N} a^-(x)\,|u_n|^{q+1}(x)dx \\
& \geq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2(x)dx - \frac{\|V^-\|_{\frac{N}{2}}}{2s}\|\nabla u_n\|_2^2 - \frac{1}{q+1}\int_{\mathbb{R}^N} a^+(x)\,|u_n|^{q+1}(x)dx.
\end{aligned}$$

By (A_1), there exists $R > 0$ such that

$$(9)\quad -\|a\|_\infty \leq a(x) \leq \frac{a_\infty}{2} < 0,\quad \forall |x| \geq R \ \text{ and } \ a^+ \in L^m(\mathbb{R}^N),\quad \forall 1 \leq m \leq +\infty.$$

Combining (3.8) and (9), we infer that

$$\begin{aligned}\alpha \geq I(u_n) &\geq \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{\|V^-\|_{\frac{N}{2}}}{2s}\|\nabla u_n\|_2^2 - s^{\frac{-q-1}{2}}\left\|a^+\right\|_{\frac{2^*}{2^*-(q+1)}}\|\nabla u_n\|_2^{q+1}\\ &\geq \left(\frac{1}{2} - \frac{\|V^-\|_{\frac{N}{2}}}{2s}\right)\|\nabla u_n\|_2^2 - s^{\frac{-q-1}{2}}\left\|a^+\right\|_{\frac{2^*}{2^*-(q+1)}}\|\nabla u_n\|_2^{q+1},\end{aligned}$$

hence there exists $\beta > 0$ such that

$$\|\nabla u_n\|_2 \leq \beta \qquad \forall n \in \mathbb{N}. \tag{10}$$

On the other hand, there exists $c > 0$ such that

$$\begin{aligned}c + \frac{\|u_n\|}{2} &\geq -\frac{1}{2}\langle I'(u_n), u_n\rangle + I(u_n) = \left(\frac{1}{2} - \frac{1}{q+1}\right)\int_{\mathbb{R}^N} a(x)\,|u_n|^{q+1}(x)dx\\ &= \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N} a^-(x)\,|u_n|^{q+1}(dx - \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N} a^+(x)\,|u_n|^{q+1}dx\\ &= \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N}\left(a^-(x) + \chi_{B_R}(x)\right)|u_n|^{q+1}dx\\ &- \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N}\left(a^+(x) + \chi_{B_R}(x)\right)|u_n|^{q+1}dx\\ &\geq \left(\frac{1}{q+1} - \frac{1}{2}\right)\min\left\{\frac{-a_\infty}{2}, 1\right\}\int_{\mathbb{R}^N}|u_n|^{q+1}(x)dx\\ &- s^{\frac{-q-1}{2}}\left(\frac{1}{q+1} - \frac{1}{2}\right)\left\|a^+ + \chi_{B_R}\right\|_{\frac{2^*}{2^*-(q+1)}}\|\nabla u_n\|_2^{q+1}.\end{aligned}$$

Thus, there is a constant $c > 0$ such that

$$\int_{\mathbb{R}^N}|u_n|^{q+1}dx \leq c\left(\|\nabla u_n\|_2 + \|u_n\|_{q+1} + \left\|a^+\right\|_{\frac{2^*}{2^*+(q+1)}}\|\nabla u_n\|_2^{q+1}\right).$$

Relation (10) yields

$$\|u_n\|_{q+1}^{q+1} \leq c + c\,\|u_n\|_{q+1} \quad \text{for all} \quad n \in \mathbb{N}. \tag{11}$$

Combining (10) and (11), we get

$$\|u_n\| \leq c \quad \forall n \in \mathbb{N}.$$

The proof is complete. □

We need the following lemma to prove that the Palais-Smale condition is satisfied for I on E.

Lemma 4.2. *There exists a constant $c > 0$ such that for all real numbers x, y,*

$$\left||x+y|^{q+1} - |x|^{q+1} - |y|^{q+1}\right| \leq c\,|x|^q\,|y|. \tag{12}$$

Proof. If $x = 0$, the inequality (12) is trivial.

Suppose that $x \neq 0$. We consider the continous function f defined on $\mathbb{R}\backslash\{0\}$ by

$$f(t) = \frac{|1+t|^{q+1} - |t|^{q+1} - 1}{|t|}.$$

Then $\lim_{|t|\to+\infty} f(t) = 0$ and $\lim_{t\to 0\pm} f(t) = \pm(q+1)$. Thus, there exists a constant $c > 0$ such that $|f(t)| \le c, \quad \forall\, t \in \mathbb{R}\backslash\{0\}$. In particular $\left|f\left(\frac{y}{x}\right)\right| \le c$, so

$$\left|\left|1+\frac{y}{x}\right|^{q+1} - \left|\frac{y}{x}\right|^{q+1} - 1\right| \le c\left|\frac{y}{x}\right|.$$

Multiplying by $|x|^{q+1}$, we obtain the desired result. $\square$

Lemma 4.3. *Assume (A_1) and (V_1) hold. Then I satisfies the Palais-Smale condition on E.*

Proof. Let (u_n) be a (PS)-sequence. By Lemma 4.1, (u_n) is bounded in E. Then there exists a subsequence $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$ for all $1 \le p \le 2^*$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.

Fix $\varphi \in D(\mathbb{R}^N)$. By the weak convergence of (u_n) to u, we get

$$\int_{\mathbb{R}^N} \nabla u_n \nabla\varphi(x) + V(x)u_n\varphi(x)dx \to \int_{\mathbb{R}^N} \nabla u\nabla\varphi + V(x)u\varphi(x)dx. \tag{13}$$

By compactness Sobolev embedding, $u_n \to u$ in $L^{q+1}(\text{supp }(\varphi))$, hence there exists a function $h \in L^{q+1}(\mathbb{R}^N)$ such that

$$a(x)\,|u_n|^{q-1}\,u_n\varphi \to a(x)\,|u|^{q-1}\,u\varphi \ \text{ a.e. } \ \text{in } \mathbb{R}^N$$

and

$$|a|\,|u_n|^q\,|\varphi| \le \|a\|_\infty\,|h|\,|\varphi| \quad \text{in } \ \mathbb{R}^N.$$

Using the Lebesgue dominated convergence theorem, we deduce that

$$\int_{\mathbb{R}^N} a(x)\,|u_n|^{q-1}\,u_n\varphi(x)dx \to \int_{\mathbb{R}^N} a(x)\,|u|^{q-1}\,u\varphi(x)dx. \tag{14}$$

Combining relations (13) and (14), we obtain

$$0 = \lim_{n\to+\infty}\langle I'(u_n),\varphi\rangle = \langle I'(u),\varphi\rangle, \quad \forall\,\varphi \in D(\mathbb{R}^N).$$

Then

$$\langle I'(u), u\rangle = 0. \tag{15}$$

Since $u_n \rightharpoonup u$ in E, we have $\|u\| \le \liminf\limits_{n\to+\infty}\|u_n\| = \lim\limits_{n\to+\infty}\|u_n\|$. We distinguish two cases:

A) **Compactness**: $\|u\| = \lim\limits_{n\to+\infty}\|u_n\|$, then

$$\limsup_{n\to+\infty}\|u_n\|_{q+1} \le \|u\|_{q+1} + \|\nabla u\|_2 - \liminf_{n\to+\infty}\|\nabla u_n\|_2.$$

Since

$$\|\nabla u\|_2 \le \liminf_{n\to+\infty}\|\nabla u_n\|_2, \quad \|u\|_{q+1} \le \liminf_{n\to+\infty}\|u_n\|_{q+1},$$

we get

$$\|u\|_{q+1} \le \liminf_{n\to+\infty}\|u_n\|_{q+1} \le \limsup_{n\to+\infty}\|u_n\|_{q+1} \le \|u\|_{q+1},$$

thus

$$\begin{cases} u_n & \to u \ \text{ a.e. } \ \text{in } \ \mathbb{R}^N \\ \|u_n\|_{q+1} & \to \|u\|_{q+1}. \end{cases}$$

By Brezis-Lieb lemma [14], we infer that

$$u_n \to u \ \text{ in } \ L^{q+1}(\mathbb{R}^N). \tag{16}$$

Therefore $\|\nabla u_n\|_2 \to \|\nabla u\|_2$. On the other hand

$$\int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^2 \, dx = \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - 2 \int_{\mathbb{R}^N} \nabla u_n \nabla u \, dx,$$

hence

$$\int_{\mathbb{R}^N} \nabla u_n \nabla u dx \to \int_{\mathbb{R}^N} |\nabla u|^2 \; dx.$$

Therefore

$$\|\nabla u_n - \nabla u\|_2 \to 0. \tag{17}$$

Combining relations (16) and (17), we deduce that $u_n \to u$ in E and the (PS) condition for I is satisfied.

B) **Dichotomy**: $\|u\| < \lim\limits_{n\to+\infty} \|u_n\|$. We prove that this case cannot occur. Set $v_n = u_n - u$.

Step 1: there exists $(y_n) \subset \mathbb{R}^N$ such that $v_n\,(.+y_n) \rightharpoonup v \neq 0$ in E. If not, for all $(y_n) \subset \mathbb{R}^N$, $v_n\,(.+y_n) \rightharpoonup 0$ in E. Then

$$\forall R > 0 \quad \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |v_n|^{q+1}(x) dx \to 0.$$

By [31, Lemma I.1, p. 231],

$$v_n \to 0 \;\; in \;\; L^p\left(\mathbb{R}^N\right), \quad \forall\, q+1 < p < 2^*. \tag{18}$$

On the other hand,

$$\begin{aligned} \langle I'(u_n), u_n\rangle &= \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x)u_n^2\right) dx - \int_{\mathbb{R}^N} a(x)\,|u_n|^{q+1}\, dx \\ &= \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)v_n^2\right) dx + \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{\mathbb{R}^N} \left(V(x)u^2 + 2\nabla v_n \nabla u\right) dx \\ &+ \int_{\mathbb{R}^N} 2V(x)v_n u dx - \int_{\mathbb{R}^N} a(x)\left(|u_n|^{q+1} - |u|^{q+1}\right) dx - \int_{\mathbb{R}^N} a(x)\,|u|^{q+1}\, dx. \end{aligned} \tag{19}$$

By (12) in Lemma 4.2, we obtain

$$\begin{aligned} |a(x)| \left| |u_n|^{q+1} - |u|^{q+1} - |v_n|^{q+1} \right| &= |a(x)| \left| |v_n + u|^{q+1} - |u|^{q+1} - |v_n|^{q+1} \right| \\ &\leq c\,|a(x)|\,|u|^q\, v_n. \end{aligned}$$

Since $v_n \rightharpoonup 0$ in E, we deduce that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} a(x)\left(|u_n|^{q+1} - |u|^{q+1}\right) dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} a(x)\,|v_n|^{q+1}(x) dx. \tag{20}$$

Using Hölder inequality in combination with relations (9) and (18), we obtain

$$\int_{\mathbb{R}^N} \left(a^+(x) + \chi_{B_R}(x)\right)|v_n|^{q+1}\, dx \leq \left\|a^+ + \chi_{B_R}\right\|_{\frac{2}{1-q}} \|v_n\|^{q+1}_{L^2(B(0,R))} \to 0, \tag{21}$$

Passing to the limit in (19) and using (9), (18), (20) and (21), we get

$$\begin{aligned}0=\lim_{n\to+\infty}\langle I'(u_n),u_n\rangle &= \langle I'(u),u\rangle+\lim_{n\to+\infty}\left(\int_{\mathbb{R}^N}\left(|\nabla v_n|^2-a(x)|v_n|^{q+1}\right)dx\right)\\ &= \langle I'(u),u\rangle+\lim_{n\to+\infty}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2+\left(a^-(x)+\chi_{B_R}\right)|v_n|^{q+1}\right)\\ &-\lim_{n\to+\infty}\int_{\mathbb{R}^N}\left(a^+(x)+\chi_{B_R}\right)|v_n|^{q+1}dx\\ &=\lim_{n\to+\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2+\left(a^-(x)+\chi_{B_R}\right)|v_n|^{q+1}dx.\\ &\geq\lim_{n\to+\infty}\int_{\mathbb{R}^N}\left(|\nabla v_n|^2+\min\left(\frac{-a_\infty}{2},1\right)|v_n|^{q+1}\right)dx\\ &\geq\lim_{n\to+\infty}\min\left(1,\min\left(\frac{-a_\infty}{2},1\right)\right)\int_{\mathbb{R}^N}\left(|\nabla v_n|^2+|v_n|^{q+1}\right)dx.\end{aligned}$$

Then $v_n\to 0$ in E, which yields a contradiction.

Step 2: (y_n) is not bounded. Indeed, suppose that (y_n) is bounded, there exists a subsequence of (y_n), also denoted by (y_n), such that $y_n\to y_0$. Then for all $\varphi\in D\left(\mathbb{R}^N\right)$

$$0=\lim_{n\to+\infty}\int_{\mathbb{R}^N}\varphi(x-y_n)v_n dx=\lim_{n\to+\infty}\int_{\mathbb{R}^N}\varphi(x)v_n(x+y_n)dx=\int_{\mathbb{R}^N}\varphi(x)v(x)dx.$$

Hence $v=0$ a.e. in $\mathbb{R}^N$, a contradiction.

Step 3: We show that v is a solution of the following problem:

$$(P_\infty)\begin{cases}-\Delta u+v_\infty u=a_\infty|u|^{q-1}u & \text{in }\mathbb{R}^N\\ u\in E\end{cases}$$

We first prove that problem (P_∞) admits only the trivial solution. Thus, since v solves (P_∞), we will obtain a contradiction.

Since (y_n) is not bounded, then $u_n(.+y_n)\rightharpoonup v$ in E. In fact, $u(.+y_n)\rightharpoonup\psi\in E$, hence

$$0=\lim_{n\to+\infty}\int_{\mathbb{R}^{\mathbb{N}}}u(x+y_n)\varphi(x)dx=\int_{\mathbb{R}^{\mathbb{N}}}\psi(x)\varphi(x)dx,\quad\forall\,\varphi\in D\left(\mathbb{R}^N\right).$$

It follows that $\psi=0$ a.e. Therefore

$$u_n(.+y_n)\rightharpoonup v\ \text{ in }\ E. \tag{22}$$

Let $\varphi\in D\left(\mathbb{R}^N\right)$. We have

$$\begin{aligned}\langle I'(u_n),\varphi(.-y_n)\rangle &=\int_{\mathbb{R}^N}\left(\nabla u_n\nabla\varphi(x-y_n)+V(x)u_n\varphi(x-y_n)\right)dx\\ &-\int_{\mathbb{R}^N}a(x)|u_n|^{q-1}u_n\varphi(x-y_n)dx\\ &=\int_{\mathbb{R}^N}\nabla u_n(x+y_n)\nabla\varphi(x)+V(x+y_n)u_n(x+y_n)\varphi(x)dx\\ &-\int_{\mathbb{R}N}a(x+y_n)|u_n|^{q-1}(x+y_n)u_n(x+y_n)\varphi(x)dx.\end{aligned}$$

Relation (22) yields

$$\int_{\mathbb{R}^N} \nabla u_n\left(x+y_n\right) \nabla \varphi(x) d x \rightarrow \int_{\mathbb{R}^N} \nabla v(x) \nabla \varphi(x) d x. \tag{23}$$

Since $(u_n(.+y_n))$ is bounded in E, $u_n(.+y_n) \rightarrow v$ in $L_{loc}^p\left(\mathbb{R}^N\right)$, for all $1 \leq p \leq 2^*$ (up a subsequence), $u_n(x+y_n) \rightarrow v$ a.e. in $\mathbb{R}^N$ and there exists $K \in L^p\left(\mathbb{R}^N\right)$ such that $\varphi\left|u_n(.+y_n)\right| \leq |K|$ in $\mathbb{R}^N$, $1 \leq p \leq 2^*$. Then, by (V_1), we obtain

$$\begin{cases} V\left(x+y_n\right) u_n\left(x+y_n\right) \varphi & \rightarrow v_{\infty} v \varphi \text{ a.e. in } \mathbb{R}^N \\ \left|V\left(x+y_n\right) u_n\left(x+y_n\right) \varphi\right| & \leq\|V\|_{\infty}|K||\varphi| \in L^1\left(\mathbb{R}^N\right). \end{cases}$$

Applying Lebesgue's dominated convergence theorem, we deduce that

$$\int_{\mathbb{R}^N} V\left(x+y_n\right) u_n\left(x+y_n\right) \varphi(x) d x \rightarrow v_{\infty} \int_{\mathbb{R}^N} v(x) \varphi(x) d x. \tag{24}$$

From hypothesis (A_1), we find

$$\begin{cases} a(x+y_n)\left|u_n\left(x+y_n\right)\right|^{q-1} u_n(x+y_n) \varphi & \rightarrow a_{\infty}|v|^{q-1} v \varphi \text{ a.e. in } \mathbb{R}^N \\ \left|a(x+y_n)\right|\left|u_n(x+y_n)\right|^q|\varphi| & \leq\|a\|_{\infty}|K|^q|\varphi| \in L^1\left(\mathbb{R}^N\right). \end{cases}$$

Next, by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n \rightarrow+\infty} \int_{\mathbb{R}^N} a(x+y_n)\left|u_n\right|^{q-1}(x+y_n) u_n(x+y_n) d x=a_{\infty} \int_{\mathbb{R}^N}|v|^{q-1} v \varphi(x) d x. \tag{25}$$

Combining relations (23), (24) and (25), we deduce that for all $\varphi \in D\left(\mathbb{R}^N\right)$,

$$\begin{aligned} 0 & =\lim_{n \rightarrow+\infty}\left\langle I'\left(u_n\right), \varphi\left(.-y_n\right)\right\rangle \\ & =\int_{\mathbb{R}^N}\left(\nabla v(x) \nabla \varphi(x)+v_{\infty} v \varphi\right) d x-a_{\infty} \int_{\mathbb{R}^N}|v|^{q-1} v \varphi(x) d x. \end{aligned}$$

Thus, v is a weak solution of problem (P_{∞}), hence $v=0$, which yields a contradiction. From steps 1, 2, and 3, we conclude that the dichotomy does not occur. The proof is complete. □

Lemma 4.4. *Assume (A_1) and (V_1) are fulfilled. Then for each $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that* $\sup_{u \in A_k} I(u)<0$.

Proof. We use some ideas developed in [27].

Let R_0 and y_0 fixed by assumption (A_1) and consider the cube

$$D\left(R_0\right)=\left\{\left(x_1, \cdots, x_N\right) \in \mathbb{R}^N: \quad\left|x_i-y_i\right|<R_0, \quad 1 \leq i \leq N\right\}.$$

Fix $k \in \mathbb{N}$ arbitrarily. Let $n \in \mathbb{N}$ be the smallest integer such that $n^N \geq k$. We divide $D\left(R_0\right)$ equally into n^N small cubes, denote them by D_i with $1 \leq i \leq n^N$, by planes parallel to each face of $D\left(R_0\right)$. The edge of D_i has the length of $a=\frac{R_0}{n}$. We construct new cubes E_i in D_i such that E_i has the same center as that of D_i. The faces of E_i and D_i are parallel and the edge of E_i has the length of $\frac{a}{2}$. Thus, we can construct a function ψ_i, $1 \leq i \leq k$, such that

$$\operatorname{supp}\left(\psi_i\right) \subset D_i, \quad \operatorname{supp}\left(\psi_i\right) \cap \operatorname{supp}\left(\psi_j\right)=\emptyset \quad(i \neq j),$$

$$\psi_i(x)=1 \text{ for } x \in E_i, \quad 0 \leq \psi_i(x) \leq 1, \quad \forall x \in \mathbb{R}^N.$$

We denote

$$S^{k-1}=\left\{\left(t_1, \cdots, t_k\right) \in \mathbb{R}^k: \max_{1 \leq i \leq k}\left|t_i\right|=1\right\} \tag{26}$$

$$W_k = \left\{ \sum_{i=1}^{k} t_i \psi_i(x) : \quad (t_1, \cdots, t_k) \in S^{k-1} \right\} \subset E. \tag{27}$$

Since the mapping $(t_1, \cdots, t_k) \to \sum_{i=1}^{k} t_i \psi_i$ from S^{k-1} to W_k is odd and homeomorphic, we have $\gamma(W_k) = \gamma\left(S^{k-1}\right) = k$. But W_k is compact in E, thus there is a constant $\alpha_k > 0$ such that

$$\|u\|^2 \leq \alpha_k \quad \text{for all} \quad u \in W_k.$$

We recall the following inequality:

$$\|u\|_2 \leq c \|\nabla u\|_2^r \|u\|_{q+1}^{1-r} \leq c \|u\| , \tag{28}$$

with $r = \frac{2^*(q-1)}{2(2^*-q-1)}$. Then there is a constant $c_k > 0$ such that

$$\|u\|_2^2 \leq c_k \quad \text{for all} \quad u \in W_k.$$

Let $z > 0$ and $u = \sum_{i=1}^{k} t_i \psi_i(x) \in W_k$. We have

$$I(zu) \leq \frac{z^2}{2} \alpha_k + z^2 \frac{\|V\|_\infty}{2} c_k - \frac{1}{q+1} \sum_{i=1}^{k} \int_{D_i} a(x) |z t_i \psi_i|^{q+1} \, dx. \tag{29}$$

By (26), there exists $j \in [1, k]$ such that $|t_j| = 1$ and $|t_i| \leq 1$ for $i \neq j$. Then

$$\begin{aligned} \sum_{i=1}^{k} \int_{D_i} a(x) |z t_i \psi_i|^{q+1} \, dx = \int_{E_j} a(x) |z t_j \psi_j|^{q+1} \, dx + \\ \int_{D_j \setminus E_j} a(x) |z t_j \psi_j(x)|^{q+1} \, dx + \sum_{i \neq j} \int_{D_i} a(x) |z t_i \psi_i|^{q+1} \, dx. \end{aligned} \tag{30}$$

Since $\psi_j(x) = 1$ for $x \in E_j$ and $|t_j| = 1$, we have

$$\int_{E_j} a(x) |z t_j \psi_j|^{q+1} \, dx = |z|^{q+1} \int_{E_j} a(x) dx. \tag{31}$$

On the other hand by (A_1),

$$\int_{D_j \setminus E_j} a(x) |z t_j \psi_j|^{q+1} \, dx + \sum_{i \neq j} \int_{D_i} a(x) |z t_i \psi_i|^{q+1} \, dx \geq 0. \tag{32}$$

Relations (29), (30), (31) and (32) yield

$$\frac{I(zu)}{z^2} \leq \frac{\alpha_k}{2} + \frac{\|V\|_\infty}{2} c_k - \frac{|z|^{q+1}}{z^2} \inf_{1 \leq i \leq k} \left(\int_{E_i} a(x) dx \right), \tag{33}$$

By (33), we conclude that

$$\lim_{z \to 0} \sup_{u \in W_k} \frac{I(zu)}{z^2} = -\infty.$$

We fix z so small such that

$$\sup \{ I(u), u \in A_k \} < 0, \quad \text{where } A_k = z W_k \in \ \Gamma_k.$$

This concludes the proof. □

Lemma 4.5. *Assume (A_1) and (V_1) hold. Then I is bounded from below.*

Proof. By (A_1), we get

$$a^+ \in L^p\left(\mathbb{R}^N\right) \quad \text{for all} \quad 1 \leq p \leq +\infty. \tag{34}$$

Then

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V(x)u^2\right)dx - \frac{1}{q+1}\int_{\mathbb{R}^N} a(x)\,|u|^{q+1}\,dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 - V^-(x)u^2\right)dx - \frac{1}{q+1}\int_{\mathbb{R}^N} a^+(x)\,|u|^{q+1}\,dx \\ &\geq \left(\frac{1}{2} - \frac{\|V^-\|_{\frac{N}{2}}}{2s}\right)\|\nabla u\|_2^2 - \frac{\|a^+\|_{\frac{2^*}{2^*-q-1}}}{s^{\frac{q+1}{2}}}\|\nabla u\|_2^{q+1}. \end{aligned}$$

In view of (V_1) we conclude the proof. □

Proof of Theorem 2.1 concluded. We have $I(0) = 0$ and I is even. Combining Lemmas 4.3, 4.4 and 4.5, we deduce that the conditions (1) and (2) of Theorem 3.2 are satisfied. Thus, there exists a sequence $(u_n) \subset E$ such that $I(u_n) < 0$, $I'(u_n) = 0$ and $I(u_n) \to 0$, for all $n \geq 0$, hence u_n is a weak solution of problem (5).

By (V_1), we deduce that

$$\begin{aligned} \frac{1}{q+1}\langle I'(u_n), u_n\rangle - I(u_n) &= \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + V(x)u_n^2\right)dx \\ &\geq \left(\frac{1}{q+1} - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{2s}\left\|V^-\right\|_{\frac{N}{2}}\right)\|\nabla u_n\|_2^2. \end{aligned}$$

It follows that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx = 0. \tag{35}$$

On the other hand, by Hölder inequality, (9) and (35), we have

$$\begin{aligned} 0 &= \lim_{n\to+\infty}\left(I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\right) = \lim_{n\to+\infty}\left(\frac{1}{2} - \frac{1}{q+1}\right)\int_{\mathbb{R}^N} a(x)\,|u_n|^{q+1} \\ &= \left(\frac{1}{q+1} - \frac{1}{2}\right)\lim_{n\to+\infty}\left(\int_{\mathbb{R}^N}\left(a^-(x) + \chi_{B_R}\right)|u_n|^{q+1}\,dx - \int_{\mathbb{R}^N}\left(a^+(x) + \chi_{B_R}\right)|u_n|^{q+1}\right) \\ &\geq \left(\frac{1}{q+1} - \frac{1}{2}\right)\lim_{n\to+\infty}\left(\int_{\mathbb{R}^N}\left(a^-(x) + \chi_{B_R}\right)|u_n|^{q+1}\,dx - \frac{\|a^+ + \chi_{B_R}\|_{\frac{2^*}{2^*-q-1}}}{s^{\frac{q+1}{2}}}\|\nabla u_n\|_2^{q+1}\right) \\ &= \left(\frac{1}{q+1} - \frac{1}{2}\right)\lim_{n\to+\infty}\int_{\mathbb{R}^N}\left(a^-(x) + \chi_{B_R}\right)|u_n|^{q+1}\,dx \\ &\geq \left(\frac{1}{q+1} - \frac{1}{2}\right)\min\left(\frac{-a_\infty}{2}, 1\right)\lim_{n\to+\infty}\|u_n\|_{q+1}^{q+1}. \end{aligned}$$

This shows that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|u_n|^{q+1}\,dx = 0, \tag{36}$$

hence $\lim_{n\to+\infty} u_n = 0$ in E. This concludes the proof. □

5. **Proof of Theorem 2.2**

In this section, we define I and φ on X. We use standard arguments based on the fact that I' is a sequentially compact operator in order to prove that I satisfies the Palais-Smale condition. Then we deduce that problem (5) admits infinitely many nontrivial solutions in X.

To prove Theorem 2.2, we need the following auxiliary results.

Lemma 5.1. *Assume* (A_2)*,* (V_2) *and* (G_1) *are satisfied. Then* φ' *is a sequentially compact operator on* X*.*

Proof. By standard arguments, the functionals I and φ are well defined and of class C^1 on X.

Let $(u_n) \subset X$ be a bounded sequence. Then for all $h \in X$,

$$\langle \varphi'(u_n) - \varphi'(u), h\rangle = \int_{\mathbb{R}^N} \left[V(x)(u_n - u) - a(x)(g(u_n) - g(u))\right] h(x)dx.$$

Let $R > 0$ and $h \in X$ be such that $\|h\| = 1$. We have

$$\langle \varphi'(u_n) - \varphi'(u), h\rangle = J_1(n,h,R) + J_2(n,h,R),$$

where

$$J_1(n,h,R) = \int_{B_R} \left[V(x)(u_n - u) - a(x)(g(u_n) - g(u))\right] h(x)dx$$

$$J_2(n,h,R) = \int_{B_R^c} \left[V(x)(u_n - u) - a(x)(g(u_n) - g(u))\right] h(x)dx.$$

By Hölder's inequality, $(V_2), (A_2)$ and (G_1) we obtain

$$|J_2(n,h,R)| \le \int_{B_R^c} |V(x)(u_n - u)h(x) - a(x)(g(u_n) - g(u))h(x)|\, dx \le$$

$$\left(\int_{B_R^c} |V(x)|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} \left(\int_{B_R^c} |u_n - u|^{2^*} dx\right)^{\frac{1}{2^*}} \left(\int_{B_R^c} |h(x)|^{2^*} dx\right)^{\frac{1}{2^*}} +$$

$$c\left(\int_{B_R^c} |a(x)|^{\frac{2^*}{2^*-(q+1)}} dx\right)^{\frac{2^*-(q+1)}{2^*}} \left(\int_{B_R^c} (|u_n(x) + u(x)|)^{2^*} dx\right)^{\frac{q}{2^*}} \left(\int_{B_R^c} |h(x)|^{2^*} dx\right)^{\frac{1}{2^*}} \le$$

$$\left[\left(\int_{B_R^c} |V(x)|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} + \left(\int_{B_R^c} |a(x)|^{\frac{2^*}{2^*-(q+1)}} dx\right)^{\frac{2^*-(q+1)}{2^*}}\right].$$

The last expression can be made arbitrarily small by taking $R > 0$ large enough.

For J_1, since $V \in L^{\frac{N}{2}}(\mathbb{R}^N)$ and $a \in L^{\frac{2^*}{2^*-(q+1)}}(\mathbb{R}^N)$, we deduce that for all $\epsilon > 0$, there exists $\eta > 0$ such that

$$\left(\int_K |a(x)|^{\frac{2^*}{2^*-(q+1)}} dx\right)^{\frac{2^*-(q+1)}{2^*}} + \left(\int_K |V(x)|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} < \epsilon,$$

for all $K \subset B_R$ with $m(K) < \eta$ (see Dunford-Pettis [21]). Moreover,

$$\int_K |V(x)(u_n - u) - a(x)(g(u_n) - g(u))|\,|h(x)|\,dx \le$$
$$c\left(\int_K |V(x)|^{\frac{N}{2}}\,dx\right)^{\frac{2}{N}} + c\left(\int_K |a(x)|^{\frac{2^*}{2^*-(q+1)}}\,dx\right)^{\frac{2^*-(q+1)}{2^*}} \le c\epsilon,$$

where c is independent of n and h. By using the Vitali convergence theorem, we deduce that $J_1(n,h,R) \to 0$ as $n \to +\infty$ uniformly for $\|h\| = 1$. We conclude that $\varphi'(u_n) \to \varphi'(u)$ strongly in X'. The proof is complete. □

Lemma 5.2. *Assume that* (V_2), (A_2) *and* (G_1) *are satisfied. Then any* (PS)-*sequence of* I *is bounded in* X.

Proof. Let $(u_n) \subset X$ be a (PS)-sequence. Then there exists $\alpha > 0$ such that $I(u_n) \le \alpha$. By Hölder inequality and conditions $(A_2), (V_2), (G_1)$, we have

$$\begin{aligned}\alpha \ge I(u_n) &\ge \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2(x) - V^-(x)u_n(x)^2\right)dx - \int_{\mathbb{R}^N} a(x)G(u_n(x))dx\\ &\ge \left(\frac{1}{2} - \frac{1}{2s}\left\|V^-\right\|_{\frac{N}{2}}\right)\|u_n\|_X^2 - s^{\frac{-q-1}{2}}\|a\|_{\frac{2^*}{2^*-(q+1)}}\|u_n\|_X^{q+1}.\end{aligned}$$

Since $0 < q < 1$, the last inequality shows that (u_n) is bounded in X. The proof is complete. □

As a consequence, we obtain the following result.

Lemma 5.3. *Assume that* (V_2), (A_2) *and* (G_1) *are satisfied. Then* I *satisfies the* (PS) *condition in* X.

Proof. Set

$$\begin{aligned}F:\ & D^{1,2}(\mathbb{R}^N) && \to \left(D^{1,2}(\mathbb{R}^N)\right)'\\ & u && \longmapsto F(u),\quad \langle F(u), v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v\ \ dx,\quad \forall v \in D^{1,2}(\mathbb{R}^N).\end{aligned}$$

Then F is an isomorphism. Let (u_n) be a (PS) sequence of I, hence

$$u_n = F^{-1}(\varphi'(u_n)) + o(1). \tag{37}$$

By Lemma 5.2, (u_n) is bounded in X. Sine φ' is a compact operator and using (37), we deduce that (u_n) is strongly convergent in X (up a subsequence). □

Lemma 5.4. *Assume* $(G_1), (V_2)$ *and* (A_2) *are satisfied. Then* I *is bounded from below.*

Proof. By $(G_1), (V_2)$ and (A_2), we have

$$\begin{aligned}I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u(x)|^2 + V(x)u^2(x)\right)dx - \int_{\mathbb{R}^N} a(x)G(u(x))\,dx\\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u(x)|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)u^2(x)dx - \int_{\mathbb{R}^N} a(x)G(u(x))\,dx\\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u(x)|^2\,dx - \frac{1}{2s}\left\|V^-\right\|_{\frac{N}{2}}\|u\|_X^2 - s^{\frac{-q-1}{2}}\|a\|_{\frac{2^*}{2^*-(q+1)})}\|u\|_X^{q+1}\\ &\ge \left(\frac{1}{2} - \frac{1}{2s}\left\|V^-\right\|_{\frac{N}{2}}\right)\|u\|_X^2 - s^{\frac{-q-1}{2}}\|a\|_{\frac{2^*}{2^*-(q+1)}}\|u\|_X^{q+1}.\end{aligned}$$

Since $1 < q+1 < 2$, we deduce that I is bounded from below. The proof is complete. $\square$

Next, we prove the geometric condition required by Theorem 3.2.

Lemma 5.5. *Assume (A_2), (V_2), (G_1), (G_2), and (G_3) are satisfied. Then for each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that* $\sup_{u \in A_k} I(u) < 0$.

Proof. By using conditions (G_2) and (G_3), the proof is similar to that Lemma 4.4. $\square$

Proof of theorem 2.2 concluded. The energy functional I is even and $I(0) = 0$. By Lemmas 5.3 and 5.4, condition (1) of Theorem 3.2 is satisfied. In view of Lemma 5.5, condition (2) of theorem 3.2 is also satisfied. Thus, there exists a sequence (u_k) such that $c_k = I(u_k)$ is a critical value of I, $c_k < 0$, $c_k \to 0$, for all $k \geq 0$. This means that (u_k) are weak solutions of problem (5) and (u_k) is a (PS) sequence of I. Then, by Lemma 5.2, (u_k) is bounded. $\square$

Remark 5.6. *If $g(x) = |x|^{q-1} x$, $\;0 < q < 1$, then $u_n \to 0$ in X. In fact, by (V_2), we have*

$$\begin{aligned} 0 = \frac{1}{q+1} \langle I'(u_n), u_n \rangle - I(u_n) &= \left(\frac{1}{q+1} - \frac{1}{2} \right) \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x) |u_n|^2 \right) dx \\ &\geq \left(\frac{1}{q+1} - \frac{1}{2} \right) \left(1 - \frac{\|V^-\|_{\frac{N}{2}}}{s} \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 (x) dx. \end{aligned}$$

Since $I'(u_n) = 0$ and $\lim\limits_{n \to +\infty} I(u_n) = 0$, we deduce that $u_n \to 0$ in X.

6. Proof of Theorem 2.3

In this section we change the condition (V_2) by (V_1) and we suppose that a satisfies (A_3). Under the last conditions, the functional I is not well defined both on X and on E, then we define it on the space Y. First, we start by showing that $(Y, \langle \; \rangle)$ is a Hilbert space and it is embedded into $L^p(\mathbb{R}^N)$ for $2 \leq p \leq 2^*$.

Lemma 6.1. *Assume that (V_1) holds. Then*

$$u \to \left(\int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + V^+(x) u^2(x) \right) dx \right)^{\frac{1}{2}},$$

defines a norm on Y, which is equivalent to the usual norm in $H^1(\mathbb{R}^N)$

$$\|u\|_{H^1} = \|\nabla u\|_2 + \|u\|_2 .$$

Proof. By (V_1), there exists $R > 0$ such that

$$\frac{v_\infty}{2} \leq V^+(x) \leq \|V\|_\infty \quad \text{for all} \quad x \in B_R^c.$$

Then

$$\int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + V^+(x)u^2(x)\right) dx =$$
$$\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx + \int_{B_R} V^+(x)u^2(x)dx + \int_{B_R^c} V^+(x)u^2(x) \leq$$
$$\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx + \left\|V^+\right\|_{L^{\frac{N}{2}}(B_R)} \|u\|_{2^*}^2 + \|V\|_\infty \int_{B_R^c} u^2(x)dx \leq$$
$$\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx + s^{-1}\left\|V^+\right\|_{L^{\frac{N}{2}}(B_R)} \|\nabla u\|_2^2 + \|V\|_\infty \int_{B_R^c} u^2(x)dx \leq$$
$$(1+c_1)\int_{\mathbb{R}^{\mathbb{N}}} |\nabla u(x)|^2\, dx + \|V\|_\infty \int_{B_R^c} u^2(x)dx \leq$$
$$\max\left((1+c_1), \|V\|_\infty\right) \int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + u^2(x)\right) dx,$$

with $r_1 = s^{-1}\left\|V^+\right\|_{L^{\frac{N}{2}}(B_R)}$.

On the other hand, we have

$$\int_{B_R} |u(x)|^2\, dx \leq (m(B_R))^{\frac{2}{N}} \left(\int_{B_R} |u(x)|^{2^*}\, dx\right)^{\frac{2}{2^*}} \leq \frac{(m(B_R))^{\frac{2}{N}}}{s} \int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx.$$

Then

$$\int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + |u(x)|^2\right) dx =$$
$$\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx + \int_{B_R} |u(x)|^2\, dx + \int_{B_R^c} |u(x)|^2\, dx \leq$$
$$\left(1 + \frac{(m(B_R))^{\frac{2}{N}}}{s}\right) \int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx + \frac{2}{v_\infty}\int_{B_R^c} V^+(x)\,|u(x)|^2\, dx \leq$$
$$\max\left(\frac{2}{v_\infty}, \left(1 + \frac{(m(B_R))^{\frac{2}{N}}}{s}\right)\right) \int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + V^+(x)\,|u(x)|^2\right) dx.$$

It follows that $\|\ \|_Y$ is equivalent to $\|\ \|_{H^1}$. We conclude that $(Y,\ \|\ \|_Y)$ is a Hilbert space. By the Sobolev embedding theorem, $Y \hookrightarrow L^p$, $2 \leq p \leq 2^*$, see [3]. □

By using Lemma 6.1, the proof of Theorem 2.3, with slight modification, is similar to that of Theorem 2.2.

Remark 6.2. *If* $g(x) = |x|^{q-1}x$, $0 < q < 1$, *then* $u_n \to 0$ *in* Y.

Remark 6.3. *In Theorems 2.1, 2.2, and 2.3, we can suppose that* u_0 *is a nonnegative solution of* (5)*, since*

$$I(u_0) = I(|u_0|) = c_0.$$

In such a case, u_0 *is called a ground state for* I.

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