Equilibrium problems techniques in the qualitative analysis of quasi-hemivariational inequalities

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Abstract

In this paper, we investigate the problem of existence of solutions of quasi-hemivariational inequalities. Some concepts of semicontinuity and hemicontinuity on subsets for functions as well as for set-valued mappings are developed and applied for solving quasi-hemivariational inequalities. Generalizations of some old results on the existence of solutions of equilibrium problems are obtained and applications to quasi-hemivariational inequalities are derived.

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Banach space $(E, \|, \|_E)$ which is continuously embedded in $L^p(\Omega; \mathbb{R}^n)$, for some $1 < p < +\infty$ and $n \geq 1$, where Ω is a bounded domain in $\mathbb{R}^m$, $m \geq 1$. Let i be the canonical injection of E into $L^p(\Omega; \mathbb{R}^n)$.

The aim of this paper is to study the existence of solutions for the following quasi-hemivariational inequality:

Find $u \in C$ and $u^* \in A(u)$ such that

$$\langle u^*, v \rangle_E + h(u) J^0(iu; iv) \geq \langle Fu, v \rangle_E \quad \forall v \in C, \tag{1.1}$$

where $A : E \rightrightarrows E^*$ is a nonlinear set-valued mapping, $F : E \to E^*$ is a nonlinear operator, $J : L^p(\Omega; \mathbb{R}^n) \to \mathbb{R}$ is a locally Lipschitzian functional and $h : E \to \mathbb{R}$ is a given nonnegative functional.

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Quasi-hemivariational inequalities are a generalization of hemivariational inequalities introduced by Panagiotopoulos in [17], [18] and describe several problems arising in mechanics and engineering, see also Costea & Rădulescu [9], Wangkeeree & Preechasilp [23], Rădulescu [22].

The quasi-hemivariational inequality problem (1.1) has been recently studied in Costea & Rădulescu [9] when C is the whole space E. For technical reasons, the authors considered the following quasi-hemivariational inequality:

Find $u \in C$ and $u^* \in A(u)$ such that

$$\langle u^*, v - u \rangle_E + h(u) J^0(iu; iv - iu) \geq \langle Fu, v - u \rangle_E \quad \forall v \in C. \qquad (1.2)$$

In Costea & Rădulescu [9], several results on the existence of solutions of the quasi-hemivariational problem (1.2) have been obtained in two cases: (i) when C is a nonempty, convex and compact subset of E; (ii) if C is a nonempty, convex, closed and bounded (then weakly compact) subset of a reflexive Banach space. Characterizations and applications for solving the quasi-hemivariational problem (1.1) when C is the whole space E are derived.

Clearly, if C is a linear subspace and in particular, if C is the whole space E, then the quasi-hemivariational problem (1.1) is equivalent to the quasi-hemivariational problem (1.2).

In this paper, we follow a direct approach by studying the existence of solutions of the quasi-hemivariational inequality problem (1.2) when C is a nonempty, closed and convex subset of E. All the results obtained can be then applied to the quasi-hemivariational inequality problem (1.1) when C is a linear subspace and in particular, when C is the whole space E.

In the first section of this paper, we introduce some concepts of continuity of functions and set-valued mappings and obtain some results and characterizations.

In the second section, we introduce a coercivity condition on a compact or weakly compact subset and use the concept of continuity on a subset for solving the quasi-hemivariational inequality problem (1.2) when C is a nonempty, closed and convex subset of E.

In the last part of the paper, we obtain some results on the existence of solutions of equilibrium problems by using the concept of continuity on a subset of equilibrium bifunctions in their first or second variable. Applications for solving quasi-hemivariational problems are given.

2. Notations and preliminary results

For a given Banach space $(X, \|.\|_X)$, we denote by X^* its dual space and by $\langle ., . \rangle_X$ (or simply by $\langle ., . \rangle$ if no confusion may arise), the duality pairing between X^* and X.

Recall that a function $\phi : X \to \mathbb{R}$ is called *locally Lipschitzian* if for every $u \in X$, there exists a neighborhood U of u and a constant $L_u > 0$ such that

$$|\phi(w) - \phi(v)| \leq L_u \|w - v\|_X \quad \forall u \in U, \forall v \in U.$$

If $\phi : X \to \mathbb{R}$ is locally Lipschitzian near $u \in X$, then the *generalized directional derivative* of ϕ at u in the direction of $v \in X$, denoted by $\phi^0(u, v)$, is defined by

$$\phi^0(u, v) = \limsup_{\substack{w \to u \\ \lambda \downarrow 0}} \frac{\phi(w + \lambda u) - \phi(w)}{\lambda}.$$

Among several important properties of the generalized directional derivative of locally Lipschitzian functions, we will make use in the present paper of the following properties (for proofs and related properties, we refer to Clarke [8, Proposition 2.1.1]).

Suppose that $\phi : X \to \mathbb{R}$ is locally Lipschitzian near $u \in X$. Then,

1. the function $v \longmapsto \phi^0(u, v)$ is finite, positively homogeneous and subadditive;
2. the function $(u, v) \longmapsto \phi^0(u, v)$ is upper semicontinuous.

Before introducing some concepts of continuity we need in the paper, we recall here some general results on convergence of sequences.

Let X be a Hausdorff topological space. Recall that a subset B of X is said to be *sequentially closed* if whenever $(x_n)_n$ is a sequence in B converging to x, then $x \in B$. As well-known, a space is called *sequential* if every sequentially closed subset is closed. Every metric space and more generally, every Fréchet-Urysohn space is a sequential space. A space X is called *Fréchet-Urysohn space* if whenever x is in the closure of a subset B of X, there exists a sequence in B converging to x, see Engelking [12], Alleche & Calbrix [3] for further details.

The weak topology of Banach spaces is not sequential in general. However, bounded subsets of reflexive Banach spaces endowed with the weak topology have the following property: if a point x is in the weak closure of a bounded subset B of a reflexive Banach space, then there exists a sequence in B weakly converging to x (see Denkowski, Migórski & Papageorgiou [10, Proposition [3.6.23]]). Thus, every bounded and weakly sequentially closed subset of a reflexive Banach space is closed.

We say that a subset B has *the Fréchet-Urysohn property* if whenever x is in the closure of B, there exists a sequence in B converging to x. Every subset of a Fréchet-Urysohn space has the Fréchet-Urysohn property. Also there are some other interesting unbounded subsets of Banach spaces which have the Fréchet-Urysohn property, see Dilworth [11].

In the sequel, for a subset B of X, we denote by

$$\mathrm{Exp}(B) = \{x \in X \mid \exists (x_n)_n, x_n \in B, \forall n, x_n \longrightarrow x\},$$

the sequential explosion of B. Of course, $\mathrm{Exp}(B)$ is neither closed nor sequentially closed in general.

Let $x \in X$. A function $f : X \to \mathbb{R}$ is called

1. *sequentially upper semicontinuous at x* if for every sequence $(x_n)_n$ in X converging to x, we have

$$f(x) \geq \limsup_{n \to +\infty} f(x_n)$$

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where $\limsup\limits_{n\to+\infty} f(x_n) = \inf\limits_n \sup\limits_{k\geq n} f(x_k)$.

2. *sequentially lower semicontinuous at x* if $-f$ is sequentially upper semi-continuous at x, that is, for every sequence $(x_n)_n$ of X converging to x, we have

$$f(x) \leq \liminf_{n\to+\infty} f(x_n)$$

where $\liminf\limits_{n\to+\infty} f(x_n) = \sup\limits_n \inf\limits_{k\geq n} f(x_k)$.

The function f is said to be *sequentially upper* (resp. *sequentially lower*) *semicontinuous* on a subset S of X if it is sequentially upper (resp. sequentially lower) semicontinuous at every point of S.

If sequences are replaced by generalized sequences (nets) in the above definition of sequentially upper (resp. sequentially lower) semicontinuous function, we obtain the notion of upper (resp. lower) semicontinuous function.

The following result shows how easy is to construct sequentially upper (resp. sequentially lower) semicontinuous functions on a subset which are not sequentially upper (resp. sequentially lower) semicontinuous on the whole space.

Proposition 2.1. *Let $f : X \longrightarrow \mathbb{R}$ be a function and let S be a subset of X. If the restriction $f_{|U}$ of f on an open subset U containing S is sequentially upper (resp. sequentially lower) semicontinuous, then any extension of $f_{|U}$ to the space X is a sequentially upper (resp. sequentially lower) semicontinuous function on S.*

The following lemma provides us some properties of sequentially upper and sequentially lower semicontinuous functions on a subset.

Proposition 2.2. *Let $f : X \longrightarrow \mathbb{R}$ be a function, S a subset of X and $a \in \mathbb{R}$.*

1. *If f is sequentially upper semicontinuous on S, then*

$$Exp\left(\{x \in X \mid f(x) \geq a\}\right) \cap S = \{x \in S \mid f(x) \geq a\}.$$

   Moreover, the trace on S of upper level sets of f are sequentially closed in S.

2. *If f is sequentially lower semicontinuous at S with respect to C, then*

$$Exp\left(\{x \in X \mid f(x) \leq a\}\right) \cap S = \{x \in S \mid f(x) \leq a\}.$$

   Moreover, the trace on S of lower level sets of f are sequentially closed in S.

Proof. The second statement being similar to the first, we prove only the case of the sequential upper semicontinuity. Let

$$x^* \in \mathrm{Exp}\left(\{x \in X \mid f(x) \geq a\}\right) \cap S.$$

Let $(x_n)_n$ be a sequence in $\mathrm{Exp}\left(\{x \in X \mid f(x) \geq a\}\right)$ converging to x^*. Since $x^* \in S$, then by the sequential upper semicontinuity of f on S, we have

$$f(x^*) \geq \limsup_{n\to+\infty} f(x_n) \geq a.$$

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Thus, $x^* \in \{x \in S \mid f(x) \geq a\}$. The converse holds from the fact that

$$\{x \in S \mid f(x) \geq a\} = \{x \in X \mid f(x) \geq a\} \cap S,$$

which is obvious as well as the sequential closeness in S of the trace on S of upper level sets of f. $\square$

The notions of upper and lower hemicontinuity are generalizations of the notions of sequential lower and sequential upper semicontinuity respectively when the space X is a real topological Hausdorff vector space. Recall that $f : X \to \mathbb{R}$ is called *upper hemicontinuous* at $x \in X$ if the restriction of f on any segment containing x is sequentially upper semicontinuous at x. It is called *lower hemicontinuous* at $x \in X$ if $-f$ is upper hemicontinuous at x.

The notions of upper and lower semicontinuity of set-valued mappings are the most known generalizations of the notion of continuity of functions to set-valued mappings.

Let X and Y be Hausdorff topological spaces. Recall that a set-valued mapping $T : X \rightrightarrows Y$ is said to be *lower semicontinuous* at $x \in X$, if for every open subset V of Y such that $V \cap T(x) \neq \emptyset$, there exists an open neighborhood U of x such that $V \cap T(x') \neq \emptyset$ for all $x' \in U$. Equivalently, $T : X \rightrightarrows Y$ is lower semicontinuous at $x \in X$ provided that T is continuous at x as a function from X to the hyperspace of subsets of Y endowed with the lower Vietoris topology. If the lower Vietoris topology is replaced by the upper Vietoris topology, then we obtain the definition of *the upper semicontinuity* of T at x, see Papageorgiou & Kyritsi-Yiallourou [19].

T is said to be lower semicontinuous on a subset S of X if T is lower semicontinuous at every point of S.

Here we introduce a generalization of lower semicontinuity of set valued functions when the space X is a real topological Hausdorff vector space. We say that a set-valued mapping $T : X \rightrightarrows Y$ is *lower quasi-hemicontinuous* at $x \in X$, if whenever $z \in X$ and $(\lambda_n)_n$ a sequence in $]0, 1[$ such that $\lim_{n \to +\infty} \lambda_n = 0$, there exists a sequence $(z_n^*)_n$ converging to some element x^* of $T(x)$ such that $z_n^* \in T(x + \lambda_n(z - x))$, for every n.

The set valued function T will be said lower quasi-hemicontinuous on a subset S of X if T is lower quasi-hemicontinuous at every point of S.

The following result shows that the notion of quasi-hemicontinuity of set-valued mappings is also a generalization of different other notions.

Proposition 2.3. *Let $T : X \rightrightarrows Y$ be a set-valued mapping and suppose that one of the following assumption hold:*

  1. *T is lower semicontinuous at $x \in X$;*
  2. *T has a continuous selection.*

Then T is lower quasi-hemicontinuous at x.

Proof. The second statement is obvious. The first comes from the fact that T is lower semicontinuous at $x \in X$ if and only if for every generalized sequence

$(x_\lambda)_{\lambda \in \Lambda}$ converging to x, and for every $x^* \in T(x)$, there exists a generalized sequence $(x^*_\lambda)_{\lambda \in \Lambda}$ converging to x^* such that $x^*_\lambda \in T(x_\lambda)$, for every $\lambda \in \Lambda$, see Papageorgiou & Kyritsi-Yiallourou [19, Proposition 6.1.4]. $\square$

Although the notion of semicontinuity of set-valued mappings is important for the existence of continuous selections (Michael's selection theorem), it is not essential. This means that under additional conditions, different continuous set-valued mappings with respect to other hyperspace topology may have continuous selections and then, they are lower quasi-hemicontinuous. For further details on selection theory of set-valued mappings, we refer to Papageorgiou & Kyritsi-Yiallourou [19], Aubin & Frankowska [4], Repovš & Semenov [21].

As in Proposition 2.1, the following result shows how easy we construct lower quasi-hemicontinuous set-valued mapping on a subset without being lower quasi-hemicontinuous on the whole space.

Proposition 2.4. *Let $T : X \rightrightarrows Y$ be a set-valued mapping and let S be a subset of X. If the restriction $T_{|U}$ of T on an open and convex subset U containing S is lower quasi-hemicontinuous, then any extension of $T_{|U}$ to the space X is a lower quasi-hemicontinuous set-valued mapping on S.*

A set-valued mapping $T : E \rightrightarrows 2^{E^*}$ is said to be *relaxed α-monotone* if there exists a functional $\alpha : E \to \mathbb{R}$ such that for every $u, v \in E$, we have

$$\langle v^* - u^*, v - u \rangle \geq \alpha(v - u) \quad \forall u^* \in T(u), \forall v^* \in T(v).$$

3. Existence results for quasi-hemivariational inequalities

For any $v \in C$, we define the following set:

$$\Theta(v) = \left\{ u \in C \mid \inf_{v^* \in A(v)} \langle v^*, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle \geq \alpha(v - u) \right\}.$$

The following result should be compared with Theorem 3.1 and Theorem 3.2 in Costea & Rădulescu [9]. It provides us a result on the existence of solutions of quasi-hemivariational inequalities.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of the real Banach space E which is continuously imbedded in $L^p(\Omega; \mathbb{R}^n)$. Suppose that the following assumptions hold:*

1. *there exists a compact subset K of C and $v_0 \in K$ such that the following condition holds: for every $u \in C \setminus K$, there exists $v^* \in A(v_0)$ such that*

$$\langle v^*, v_0 - u \rangle + h(u) J^0(iu; iv_0 - iu) - \langle Fu, v_0 - u \rangle - \alpha(v_0 - u) < 0;$$

2. *$\alpha : E \to \mathbb{R}$ is a functional such that for every $u \in C$, $\lim_{n \to +\infty} \frac{\alpha(\lambda_n u)}{\lambda_n} = 0$ whenever $(\lambda_n)_n$ is a sequence in $]0, 1[$ such that $\lim_{n \to +\infty} \lambda_n = 0$ and $\limsup_{n \to +\infty} \alpha(u_n) \geq \alpha(u)$ whenever $(u_n)_n$ is a sequence in C converging to u;*

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3. A is relaxed α-monotone and lower quasi-hemicontinuous on K with respect to the weak* topology of E^*;

4. h is a nonnegative sequentially lower semicontinuous functional on K;

5. F is an operator such that for every $v \in C$, $u \mapsto \langle Fu, v - u \rangle$ is sequentially lower semicontinuous on K.

Then, the quasi-hemivariational inequality problem (1.2) has at least one solution.

Proof. By using the relaxed α-monotonicity of A and the subadditivity of the function $v \mapsto J^0(iu; iv)$, we obtain that the set-valued mapping $v \mapsto \Theta(v)$ is a KKM mapping. To do this, let $\{v_1, \ldots, v_n\} \subset C$ and put $u_0 = \sum_{k=1}^n \lambda_k v_k$ where $\lambda_k \in {]0, 1[}$ for every $k = 1, \ldots n$ and $\sum_{k=1}^n \lambda_k = 1$. Assume that $u_0 \notin \bigcup_{k=1}^n \Theta(v_k)$, then for every $k = 1, \ldots n$, we have

$$\inf_{v^* \in A(v_k)} \langle v^*, v_k - u_0 \rangle + h(u_0) J^0(iu_0; iv_k - iu_0) - \langle Fu, v_k - u_0 \rangle < \alpha(v_k - u_0).$$

For every $k = 1, \ldots n$, choose $v_k^* \in A(v_k)$ such that

$$\langle v_k^*, v_k - u_0 \rangle + h(u_0) J^0(iu_0; iv_k - iu_0) - \langle Fu, v_k - u_0 \rangle < \alpha(v_k - u_0).$$

Since A is relaxed α-monotone, then for every $u_0^* \in A(u_0)$, we have

$$\langle v_k^*, v_k - u_0 \rangle + h(u_0) J^0(iu_0; iv_k - iu_0) - \langle Fu, v_k - u_0 \rangle < \alpha(v_k - u_0)$$
$$\leq \langle v_k^* - u_0^*, v_k - u_0 \rangle.$$

Thus,

$$\langle u_0^*, v_k - u_0 \rangle + h(u_0) J^0(iu_0; iv_k - iu_0) - \langle Fu, v_k - u_0 \rangle < 0 \quad \forall u_0^* \in A(u_0).$$

Since the function $v \mapsto J^0(iu; iv)$ is subadditive, then for any $u_0^* \in A(u_0)$, we have

$$0 = \langle u_0^*, u_0 - u_0 \rangle + h(u_0) J^0(iu_0; iu_0 - iu_0) - \langle Fu, u_0 - u_0 \rangle$$
$$= \langle u_0^*, \sum_{k=1}^n \lambda_k(v_k - u_0) \rangle + h(u_0) J^0\left(iu_0; \sum_{k=1}^n \lambda_k(iv_k - iu_0)\right)$$
$$- \langle Fu, \sum_{k=1}^n \lambda_k(v_k - u_0) \rangle$$
$$\leq \sum_{k=1}^n \lambda_k \left(\langle u_0^*, v_k - u_0 \rangle + h(u_0) J^0(iu_0; iv_k - iu_0) - \langle Fu, v_k - u_0 \rangle \right) < 0.$$

This is a contradiction and then the set-valued mapping $v \mapsto \Theta(v)$ is a KKM mapping. Since $\Theta(v_0)$ is contained in K which is compact, then by Ky Fan's lemma [13], we have

$$\bigcap_{v \in C} \overline{\Theta(v)} \neq \emptyset.$$

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Now, we will prove that for every $v \in C$, we have

$$\overline{\Theta(v)} \cap K = \Theta(v) \cap K.$$

To do this, let $v \in C$ and $u \in \overline{\Theta(v)} \cap K$. Let $(u_n)_n$ be a sequence in $\Theta(v)$ converging to u. Let $v^* \in A(v)$ be arbitrary. We have for all $n \geq 1$

$$\alpha(v - u_n) \leq \langle v^*, v - u_n \rangle + h(u_n) J^0(iu_n; iv - iu_n) - \langle Fu_n, v - u_n \rangle \quad .$$

Since $u \in K$, then

$$\alpha(v - u) \leq \limsup_{n \to +\infty} \alpha(v - u_n)$$
$$\leq \limsup_{n \to +\infty} \left(\langle v^*, v - u_n \rangle + h(u_n) J^0(iu_n; iv - iu_n) - \langle Fu_n, v - u_n \rangle \right)$$
$$\leq \langle v^*, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle.$$

Thus, $u \in \Theta(v) \cap K$.

Now, by using the fact that $\Theta(v_0)$ is contained in K, we conclude that

$$\bigcap_{v \in C} \overline{\Theta(v)} = \bigcap_{v \in C} \Theta(v),$$

and then,

$$\bigcap_{v \in C} \Theta(v) \neq \emptyset.$$

Finally, let $u_0 \in \bigcap_{v \in C} \Theta(v)$. This means that $u_0 \in K$ and for every $w \in C$, we have

$$\inf_{w^* \in A(w)} \langle w^*, w - u \rangle + h(u) J^0(iu; iw - iu) - \langle Fu, w - u \rangle \geq \alpha(w - u).$$

Let $v \in C$ be arbitrary and define $w_n = u_0 + \lambda_n(v - u_0)$ where $(\lambda_n)_n$ is a sequence in $]0, 1[$ such that $\lim_{n \to +\infty} \lambda_n = 0$. By lower quasi-hemicontinuity of A on K, let $w_n^* \in A(w_n)$ be such that $w_n^* \xrightarrow{w^*} u_0^* \in A(u_0)$. Since the function $v \mapsto J^0(iu; iv)$ is positively homogeneous, we obtain

$$\langle w_n^*, v - u_0 \rangle + h(u_0) J^0(iu_0; iv - iu_0) - \langle Fu_0, v - u_0 \rangle \geq \frac{\alpha(\lambda_n(v - u))}{\lambda_n}.$$

Letting n go to $+\infty$, we obtain that

$$\langle u_0^*, v - u_0 \rangle + h(u_0) J^0(iu_0; iv - iu_0) - \langle Fu_0, v - u_0 \rangle \geq 0$$

which completes the proof. $\square$

The following result is a generalization of Theorem 3.2 in Costea & Rădulescu [9]. It provides us with a second result on the existence of solutions of quasi-hemivariational inequalities.

Theorem 3.2. *Let C be a nonempty, closed and convex subset of the real reflexive Banach space E which is compactly imbedded in $L^p(\Omega; \mathbb{R}^n)$. Suppose that the following hypotheses are fulfilled:*

1. *there exist a weakly compact subset K of C and $v_0 \in K$ such that the following condition holds: for every $u \in C \setminus K$, there exists $v^* \in A(v_0)$ such that*

$$\langle v^*, v_0 - u \rangle + h(u) J^0(iu; iv_0 - iu) - \langle Fu, v_0 - u \rangle - \alpha(v_0 - u) < 0;$$

2. *$\alpha : E \to \mathbb{R}$ is a functional such that for every $u \in C$, $\lim\limits_{n \to +\infty} \frac{\alpha(\lambda_n u)}{\lambda_n} = 0$ whenever $(\lambda_n)_n$ is a sequence in $]0, 1[$ such that $\lim\limits_{n \to +\infty} \lambda_n = 0$ and $\limsup\limits_{n \to +\infty} \alpha(u_n) \geq \alpha(u)$ whenever $(u_n)_n$ is a sequence in C weakly converging to u;*

3. *A is relaxed α-monotone and lower quasi-hemicontinuous on K with respect to the weak* topology of E^*;*

4. *h is a nonnegative weakly sequentially lower semicontinuous functional on K;*

5. *F is an operator such that for every $v \in C$, $u \mapsto \langle Fu, v - u \rangle$ is weakly sequentially lower semicontinuous on K.*

Then the function $v \mapsto \Theta(v)$ is a KKM mapping and

$$Exp(\Theta(v)) \cap K = \Theta(v) \cap K \quad \forall v \in C.$$

If, in addition, $\Theta(v)$ has the Fréchet-Urysohn property, for every $v \in C$, then the quasi-hemivariational inequality problem (1.2) has at least one solution.

Proof. By the same proof as in Theorem 3.1, we obtain that the set-valued mapping $v \mapsto \Theta(v)$ is a KKM mapping.

Now, let $v \in C$ and $u \in Exp(\Theta(v)) \cap K$. Let $(u_n)_n$ be a sequence in $\Theta(v)$ weakly converging to u. Since the compact embedding i is compact, it maps weakly convergent sequences into strongly convergent sequences (see for example, Renardy & Rogers [20, Theorem 8.84]). Let $v^* \in A(v)$ be arbitrary. We have

$$\alpha(v - u_n) \leq \langle v^*, v - u_n \rangle + h(u_n) J^0(iu_n; iv - iu_n) - \langle Fu_n, v - u_n \rangle \quad \forall n.$$

Since $u \in K$, then

$$\begin{aligned}
\alpha(v - u) &\leq \limsup_{n \to +\infty} \alpha(v - u_n) \\
&\leq \limsup_{n \to +\infty} \left(\langle v^*, v - u_n \rangle + h(u_n) J^0(iu_n; iv - iu_n) - \langle Fu_n, v - u_n \rangle \right) \\
&\leq \langle v^*, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle.
\end{aligned}$$

Thus, $u \in \Theta(v) \cap K$.

Suppose now that $\Theta(v)$ has the Fréchet-Urysohn property, for every $v \in C$. Then

$$Exp(\Theta(v)) = \overline{\Theta(v)} \quad \forall v \in C$$

where the closure is taken with respect to the weak topology. Since the set-valued mapping $v \mapsto \Theta(v)$ is a KKM mapping and since $\Theta(v_0)$ is contained in K which is weakly compact, then by Ky Fan's lemma, we have

$$\bigcap_{v \in C} Exp(\Theta(v)) \neq \emptyset.$$

By the same arguments as in the proof of Theorem 3.1, we conclude that

$$\bigcap_{v \in C} Exp(\Theta(v)) = \bigcap_{v \in C} \Theta(v),$$

and then,

$$\bigcap_{v \in C} \Theta(v) \neq \emptyset.$$

Also, by a similar proof as in Theorem 3.1, we conclude that the quasi-hemivariational problem (1.2) has at least one solution. $\square$

4. Equilibrium problems versus quasi-hemivariational inequality problems and applications

Equilibrium problems are very general and they include as particular cases, Nash equilibrium problems and convex minimization problems. Relevant applications in physics, optimization and economics are described by models based on equilibrium problems.

Let C be a nonempty, closed and convex subset of a real topological Hausdorff vector space X. An *equilibrium problem* in the sense of Blum and Oettli [7] (see also Alleche [1, 2], Bianchi & Schaible [6] and the references therein) is a problem of the form:

$$\text{Find} \quad u \in C \quad \text{such that} \quad \Phi(u, v) \geq 0 \quad \forall v \in C \qquad (4.1)$$

where $\Phi : C \times C \to \mathbb{R}$ is a bifunction such that $\Phi(u, u) \geq 0$, for every $u \in C$. Such a bifunction is called an *equilibrium bifunction*.

We present in this section some results about the existence of solutions of equilibrium problems and apply these results for solving quasi-hemivariational inequalities.

In the sequel, we define the following sets: for every $v \in C$, we put

$$\Phi^+(v) = \{u \in C \mid \Phi(u, v) \geq 0\}$$

and

$$\Phi^-(v) = \{u \in C \mid \Phi(v, u) \leq 0\}.$$

Recall that a function $f : C \to \mathbb{R}$ is said to be

1. *semistrictly quasiconvex* on C if, for every $u_1, u_2 \in C$ such that $f(u_1) \neq f(u_2)$, we have

$$f(\lambda u_1 + (1 - \lambda) u_2) < \max\{f(u_1), f(u_2)\} \quad \forall \lambda \in \,]0, 1[\,.$$

2. *explicitly quasiconvex* on C if it is quasiconvex and semistrictly quasiconvex (see for example Avriel, Diewert, Schaible & Zang [5]).

The following result extends the well-known Ky Fan's minimax inequality theorem (see Fan [14], Kassay [15], Konnov [16]) for sequentially upper semicontinuous bifunctions on their first variable on a subset of a real Banach space.

Theorem 4.1. *Let C be a nonempty, closed and convex subset of the real Banach space E. Let $\Phi : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction and suppose that the following assumptions hold:*

1. *Φ is quasiconvex in its second variable on C;*
2. *there exists a compact subset K of C and $v_0 \in K$ such that*

$$\Phi(u, v_0) < 0 \quad \forall u \in C \setminus K;$$

3. *Φ is sequentially upper semicontinuous in its first variable on K.*

Then the equilibrium problem (4.1) has a solution.

Proof. Since Φ is an equilibrium bifunction, then $\overline{\Phi^+(v)}$ is nonempty and closed, for every $v \in C$.

By quasiconvexity of Φ in its second variable, the mapping $v \mapsto \Phi^+(v)$ is a KKM mapping (see for example, Alleche [2], Bianchi & Schaible [6], Fan [13, 14], Kassay [15]), and since $\Phi^+(v_0)$ is contained in the compact subset K, then by Ky Fan's lemma, we have

$$\bigcap_{v \in C} \overline{\Phi^+(v)} \neq \emptyset.$$

On the other hand, we have

$$\bigcap_{v \in C} \overline{\Phi^+(v)} = \bigcap_{y \in C} \left(\overline{\Phi^+(v)} \cap K \right).$$

Since

$$\mathrm{Exp}\left(\Phi^+(v)\right) = \overline{\Phi^+(v)} \quad \forall v \in C,$$

then by Proposition 2.2, we have

$$\overline{\Phi^+(v)} \cap K = \Phi^+(v) \cap K \quad \forall v \in C.$$

Thus,

$$\bigcap_{v \in C} \Phi^+(v) = \bigcap_{v \in C} \overline{\Phi^+(v)} \neq \emptyset$$

which completes the proof. $\square$

As well-known in the literature, the equilibrium problem (4.1) can be also solved when the bifunction Φ is not upper semicontinuous on its first variable. In this case some additional conditions are needed.

The bifunction $\Phi : C \times C \longrightarrow \mathbb{R}$ is said to be *pseudomonotone* on C if

$$\Phi(u, v) \geq 0 \Longrightarrow \Phi(v, u) \leq 0, \quad \forall u, v \in C.$$

The following result extends (under the settings of the real Banach space E) some results of Alleche [2], Bianchi & Schaible [6] on the existence of solutions for pseudomontone equilibrium problems.

Theorem 4.2. *Let C be a nonempty, closed and convex subset of the real Banach space E. Let $\Phi : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction and suppose that the following assumptions hold:*

1. *Φ is pseudomonotone on C;*
2. *there exists a compact subset K of C and $v_0 \in K$ such that*

$$\Phi(u, v_0) < 0 \quad \forall u \in C \setminus K;$$

3. *Φ is upper hemicontinuous in its first variable on K;*
4. *Φ is explicitly quasiconvex in its second variable on C;*
5. *Φ is sequentially lower semicontinuous in its second variable on K.*

Then, the equilibrium problem (4.1) has a solution.

Proof. By the same proof as in Theorem 4.1, we obtain that

$$\bigcap_{v \in C} \left(\overline{\Phi^+(v)} \cap K \right) = \bigcap_{v \in C} \overline{\Phi^+(v)} \neq \emptyset.$$

Since Φ is sequentially lower semicontinuous in its second variable on K, then by applying Proposition 2.2, we have

$$\overline{\Phi^-(v)} \cap K = \Phi^-(v) \cap K \quad \forall v \in C.$$

From pseudo-monotonicity, we have $\Phi^+(v) \subset \Phi^-(v)$, for every $v \in C$. It follows that

$$\bigcap_{v \in C} \left(\overline{\Phi^+(v)} \cap K \right) \subset \bigcap_{v \in C} \left(\Phi^-(v) \cap K \right).$$

By using the the hemicontinuity of Φ in its first variable on K and the explicit quasi-convexity (see Alleche [2, Lemma 2.4]), we have

$$\bigcap_{v \in C} \left(\Phi^-(v) \cap K \right) \subset \bigcap_{v \in C} \Phi^+(v).$$

A combination of the above statements yields

$$\bigcap_{y \in C} \overline{\Phi^+(y)} = \bigcap_{y \in C} \Phi^+(y).$$

This completes the proof. □

Of course, Theorem 4.1 and Theorem 4.2 remain true if the real Banach space E is replaced by a real topological Hausdorff vector space such that the subset C is a Fréchet-Urysohn space.

Now we apply the above theorems to derive results on the existence of solution of quasi-hemivariational inequalities.

Define the equilibrium bifunction $\Psi : C \times C \to \mathbb{R}$ by

$$\Psi(u,v) = \inf_{v^* \in A(v)} \langle v^*, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle.$$

Although we are aware of the intrinsic properties of the generalized directional derivative, we do not know if Ψ satisfies any condition of continuity or of convexity in its first or second variable. In other words, even under assumptions of Theorem 3.1 and Theorem 3.2, it is not clear whether Ψ satisfies any condition of Theorem 4.1 or Theorem 4.1.

The following result provides us with a a sufficient condition for solving the quasi-hemivariational inequality problem (1.2). Note that the concept of relaxed α-monotonicity is no longer needed.

Theorem 4.3. *Let C be a nonempty, closed and convex subset of the real Banach space E. Suppose that A is lower quasi-hemicontinuous on K with respect to the weak* topology of E^*. If the equilibrium problem*

$$\text{find } u \in C \text{ such that } \Psi(u,v) \geq 0 \quad \forall v \in C$$

has a solution, then the quasi-hemivariational inequality problem (1.2) has a solution.

Let us point out that by a classical method, we can also define an equilibrium bifunction $\Psi : C \times C \to \mathbb{R}$ as follows:

$$\Psi(u,v) = \sup_{u^* \in A(u)} \langle u^*, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle.$$

Clearly, any solution of the quasi-hemivariational inequality problem (1.2) is a solution of the equilibrium problem

$$\text{Find} \quad u \in C \quad \text{such that} \quad \Psi(u,v) \geq 0 \quad \forall v \in C. \tag{4.2}$$

The converse do not hold easily as in Theorem 4.3 and it seems to need additional conditions on the values of the set-valued mapping A.

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