Strong converse for the classical capacity of all phase-insensitive bosonic Gaussian channels

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Abstract

We prove that a strong converse theorem holds for the classical capacity of all phase-insensitive bosonic Gaussian channels, when imposing a maximum photon number constraint on the inputs of the channel. The pure-loss, thermal, additive noise, and amplifier channels are all in this class of channels. The statement of the strong converse theorem is that the probability of correctly decoding a classical message rapidly converges to zero in the limit of many channel uses if the communication rate exceeds the classical capacity. We prove this theorem by relating the success probability of any code with its rate of data transmission, the effective dimension of the channel output space, and the purity of the channel as quantified by the minimum output entropy. Our result bolsters the understanding of the classical capacity of these channels by establishing it as a sharp dividing line between possible and impossible communication rates over them.

1 Introduction

One of the most fundamental tasks in quantum information theory is to determine the ultimate limits on achievable data transmission rates for a noisy communication channel. The classical capacity is defined as the maximum rate at which it is possible to send classical data over a quantum channel such that the error probability decreases to zero in the limit of many independent uses of the channel [14, 25]. As such, the classical capacity serves as a distinctive bound on our ability to faithfully recover classical information sent over the channel.

The above definition of the classical capacity states that (a) for any rate below capacity, one can communicate error free in the limit of many channels uses and (b) there cannot exist an error-free communication scheme in the limit of many channel uses whenever the rate exceeds the capacity. However, strictly speaking, for any rate R above capacity, the above definition leaves open the

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possibility for one to increase the communication rate R by allowing for some error $\varepsilon > 0$. Leaving room for the possibility of such a trade-off between the rate R and the error ε is the characteristic of a "weak converse," and the corresponding capacity is sometimes called the weak capacity. A *strong converse*, on the contrary, establishes the capacity as a very sharp threshold, so that there is no such room for a trade-off between rate and error in the limit of many independent uses of the channel. That is, it guarantees that the error probability of any communication scheme asymptotically converges to one if its rate exceeds the classical capacity.

Despite their significance in understanding the ultimate information-carrying capacity of noisy communication channels, strong converse theorems are known to hold only for a handful of quantum channels: for those with classical inputs and quantum outputs [20, 31] (see earlier results for all classical channels [32, 1]), for all covariant channels with additive minimum output Rényi entropy [16], for all entanglement-breaking and Hadamard channels [30], as well as for pure-loss bosonic channels [29].

In this paper, we consider the encoding of classical messages into optical quantum states and the transmission of these codewords over *phase-insensitive Gaussian channels*. In particular, we prove that a strong converse theorem holds for the classical capacity of these channels, when imposing a maximum photon number constraint on the inputs of the channel. Phase-insensitive Gaussian channels are invariant with respect to phase-space rotations [7], and they are considered to be one of the most practically relevant models to describe free space or optical fiber transmission, or transmission of classical messages through dielectric media, etc. In fact, phase-insensitive Gaussian channels constitute a broad class of noisy bosonic channels, encompassing all of the following: thermal noise channels (in which the signal photon states are mixed with a thermal state), additive noise channels (in which the input states are randomly displaced in phase space), and noisy amplifier channels [7, 11, 12].

In some very recent studies [11, 8, 18], a solution to the long-standing minimum output entropy conjecture [15, 9] has been established for all phase-insensitive Gaussian channels, demonstrating that the minimum output entropy for such channels is indeed achieved by the vacuum input state. The major implication of this work is that we now know the classical capacity of any phase-insensitive Gaussian channel whenever there is a mean photon-number constraint on the channel inputs (the capacity otherwise being infinite if there is no photon number constraint). For instance, consider the thermal noise channel represented by a beamsplitter with transmissivity $\eta \in [0, 1]$ mixing signaling photons (with average photon number N_S) with a thermal state of average photon number N_B. The results in [11] imply that the classical capacity of this channel is given by

$$g(\eta N_S + (1 - \eta)N_B) - g((1 - \eta)N_B),$$

where $g(x) \equiv (x + 1)\log_2(x + 1) - x\log_2 x$ is the entropy of a bosonic thermal state with mean photon number x. However, the corresponding converse theorem, which can be inferred as a further implication of their work, is only a weak converse, in the sense that the upper bound on the communication rate R of any coding scheme for the thermal noise channel can be written in the following form:

$$R \leq \frac{1}{1 - \varepsilon}[g(\eta N_S + (1 - \eta)N_B) - g((1 - \eta)N_B) + h_2(\varepsilon)],$$

where ε is the error probability, and $h_2(\varepsilon)$ is the binary entropy with the property that $\lim_{\varepsilon \to 0} h_2(\varepsilon) = 0$. That is, only in the limit $\varepsilon \to 0$ does the above expression serve as the classical capacity of the

channel, leaving room for a possible trade-off between rate and error probability. This observation also applies to the classical capacities of all other phase-insensitive Gaussian channels mentioned above.

In the present work, we prove that a strong converse theorem holds for the classical capacities of all phase-insensitive Gaussian channels when imposing a maximum photon-number constraint. This means that if we demand that the average code density operator for the codewords, which are used for transmission of classical messages, is constrained to have a large shadow onto a subspace with photon number no larger than some fixed amount, then the probability of successfully decoding the message converges to zero in the limit of many channel uses if the rate R of communication exceeds the classical capacity of these channels.

This paper is structured as follows. In Section 2, we review several preliminary ideas, including some definitions and notation for phase-insensitive Gaussian channels, and we recall structural decompositions of them that we exploit in our proof of the strong converse. We also recall the definition of the quantum Rényi entropy and an inequality that relates it to the smooth min-entropy [23]. In Section 3, we then prove our main result, i.e., that the strong converse holds for the classical capacity of all phase-insensitive Gaussian channels when imposing a maximum photon-number constraint. We conclude with a brief summary in Section 4 along with suggestions for future research.

2 Preliminaries

2.1 Phase-insensitive Gaussian channels

Bosonic Gaussian channels play a key role in modeling optical communication channels, such as optical fibers or free space transmission. They are represented by completely positive and trace preserving (CPTP) maps evolving Gaussian input states into Gaussian output states [28, 6, 4]. (A Gaussian state is completely characterized by a mean vector and a covariance matrix [28].) Single-mode Gaussian channels are characterized by two matrices X and Y acting on the covariance matrix Γ of a single-mode Gaussian state in the following way:

$$\Gamma \longrightarrow \Gamma' = X\Gamma X^T + Y, \tag{1}$$

where X^T is the transpose of the matrix X. Here X and Y are both 2×2 real matrices, satisfying

$$Y \geq 0, \qquad \det Y \geq (\det X - 1)^2,$$

in order for the map to be a legitimate completely positive trace preserving map (see [28] for more details). A bosonic Gaussian quantum channel is said to be "quantum-limited' if the inequality above (involving $\det X$ and $\det Y$) is saturated [8, 11]. For instance, an amplifier channel $\mathcal{A}_G^N$ (characterized by its gain G and the number of noise photons N) is quantum-limited when the environment is in the vacuum state (we will denote such a quantum-limited amplifier by $\mathcal{A}_G^0$).

A whole subclass of these bosonic Gaussian channels consists of the phase-insensitive Gaussian channels [12, 28, 11, 18, 7, 24]. Phase insensitive channels correspond to the choice

$$X = \mathrm{diag}\left(\sqrt{\tau}, \sqrt{\tau}\right), \tag{2}$$
$$Y = \mathrm{diag}\left(\nu, \nu\right),$$

with $\tau, \nu \geq 0$ obeying the constraint above. The action of such phase-insensitive channels on an input signal mode can be uniquely described by their transformation of the symmetrically ordered characteristic function, defined as

$$\chi(\mu) \equiv \mathrm{Tr}[\rho D(\mu)], \tag{3}$$

where $D(\mu) \equiv \exp(\mu \hat{a}^\dagger - \mu^* \hat{a})$ is the displacement operator for the input signal mode $\hat{a}$ [28]. For the Gaussian channels, the transformed characteristic function at the output is given by $\chi'(\mu) = \chi(\sqrt{\tau}\mu) \exp(-\nu |\mu|^2 /2)$ [6, 11].

2.1.1 Examples

The canonical phase-insensitive Gaussian channels are the thermal noise channel, the additive noise channel, and the amplifier channel.

The thermal channel $\mathcal{E}_{\eta, N_B}$ can be represented by a beamsplitter of transmissivity $\eta \in [0, 1]$ that couples the input signal of mean photon number N_S with a thermal state of mean photon number N_B. The special case $N_B = 0$ corresponds to the pure-loss bosonic channel $\mathcal{E}_\eta$, where the state injected by the environment is the vacuum state.

In the additive noise channel $\mathcal{N}_{\bar{n}}$, each signal mode is randomly displaced in phase space according to a Gaussian distribution. The additive noise channel $\mathcal{N}_{\bar{n}}$ is completely characterized by the variance $\bar{n}$ of the noise introduced by the channel.

The quantum amplifier channel $\mathcal{A}_G^N$ is characterized by its gain $G \geq 1$ and the mean number of photons N in the associated ancilla input mode (which is in a thermal state).

The transformed characteristic functions for these Gaussian channels are given by [18, 9, 12]

$$\chi'(\mu) = \begin{cases} \chi(\sqrt{\eta}\mu) e^{-(1-\eta)(N_B+1/2)|\mu|^2} & \text{for } \mathcal{E}_{\eta, N_B} \\ \chi(\mu) e^{-\bar{n}|\mu|^2} & \text{for } \mathcal{N}_{\bar{n}} \\ \chi(\sqrt{G}\mu) e^{-(G-1)(N+1/2)|\mu|^2} & \text{for } \mathcal{A}_G^N. \end{cases} \tag{4}$$

2.1.2 Structural decompositions

Using the composition rule of Gaussian bosonic channels [3], any phase-insensitive Gaussian bosonic channel (let us denote it by $\mathcal{P}$) can be written as a concatenation of a pure-loss channel followed by a quantum-limited amplifier [7, 11]

$$\mathcal{P} = \mathcal{A}_G^0 \circ \mathcal{E}_T, \tag{5}$$

where $\mathcal{E}_T$ is a pure-loss channel with parameter $T \in [0, 1]$ and $\mathcal{A}_G^0$ is a quantum-limited amplifier with gain $G \geq 1$, these parameters chosen so that $\tau = TG$ and $\nu = G(1-T) + G - 1$ (with τ and ν defined in (2)).

For instance, the additive noise channel $\mathcal{N}_{\bar{n}}$ can be realized as a pure-loss channel with transmissivity $T = 1/(\bar{n}+1)$ followed by a quantum-limited amplifier channel with gain $G = \bar{n}+1$. Also, we can consider the thermal noise channel $\mathcal{E}_{\eta, N_B}$ as a cascade of a pure-loss channel with transmissivity $T = \eta/G$ followed by a quantum-limited amplifier channel with gain $G = (1 - \eta)N_B + 1$. These two observations are equivalent to

$$\mathcal{N}_{\bar{n}}(\rho) = (\mathcal{A}_{\bar{n}+1}^0 \circ \mathcal{E}_{\frac{1}{\bar{n}+1}})(\rho), \tag{6}$$

$$\mathcal{E}_{\eta, N_B}(\rho) = (\mathcal{A}_{(1-\eta)N_B+1}^0 \circ \mathcal{E}_T)(\rho). \tag{7}$$

The above structural decompositions are useful in establishing the classical capacity as well as the minimum output entropy for all phase-insensitive channels [11, 18, 8].

2.1.3 Classical capacitites of phase-insensitive channels

Holevo, Schumacher, and Westmoreland (HSW) characterized the classical capacity of a quantum channel $\mathcal{N}$ in terms of a quantity now known as the Holevo information [14, 25]

$$\chi(\mathcal{N}) \equiv \max_{\{p_X(x),\rho_x\}} I(X;B)_\rho, \tag{8}$$

where $\{p_X(x),\rho_x\}$ represents an ensemble of quantum states, and the quantum mutual information $I(X;B)_\rho \equiv H(X)_\rho + H(B)_\rho - H(XB)_\rho$, is defined with respect to a classical-quantum state $\rho_{XB} \equiv \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}(\rho_x)_B$. The above formula given by HSW for certain quantum channels is additive, in the sense that

$$\chi(\mathcal{N}^{\otimes n}) = n\chi(\mathcal{N}), \tag{9}$$

for any positive integer n. For such quantum channels, the HSW formula is indeed equal to the classical capacity of those channels. However, a regularization is thought to be required in order to characterize the classical capacity of quantum channels for which the HSW formula cannot be shown to be additive. The classical capacity in general is then characterized by the following regularized formula:

$$\chi_{\mathrm{reg}}(\mathcal{N}) \equiv \lim_{n\to\infty} \frac{1}{n}\chi(\mathcal{N}^{\otimes n}). \tag{10}$$

The recent breakthrough work in [11] (along with earlier results in [15]) has established the following expressions for the classical capacities of various phase-insensitive channels:

$$C(\mathcal{E}_{\eta,N_B}) = g(\eta N_S + (1-\eta)N_B) - g((1-\eta)N_B), \tag{11}$$

$$C(\mathcal{N}_{\bar{n}}) = g(N_S + \bar{n}) - g(\bar{n}), \tag{12}$$

$$C(\mathcal{A}_G^N) = g(GN_S + (G-1)(N+1)) - g((G-1)(N+1)), \tag{13}$$

where N_S is the mean input photon number. In general, the classical capacity of any phase-insensitive Gaussian channel is equal to [11]

$$g(N_S') - g(N_B'), \tag{14}$$

where $N_S' = \tau N_S + (\tau + \nu - 1)/2$ and $N_B' = (\tau + \nu - 1)/2$, with τ and ν defined in (2). In the above, N_S' is equal to the mean number of photons at the output when a thermal state of mean photon number N_S is input, and N_B' is equal to the mean number of noise photons when the vacuum state is sent in. Note that the capacities in (11), (12), and (13) all have this particular form (difference in the corresponding mean number of photons). The classical capacities specified above are attainable by using coherent state encoding schemes for the respective channels [15]. We will show in Section 3 that these expressions can also be interpreted as strong converse rates.

2.2 Quantum Rényi entropy and smooth min-entropy

The quantum Rényi entropy $H_\alpha(\rho)$ of a density operator ρ is defined for $0 < \alpha < \infty$, $\alpha \neq 1$ as

$$H_\alpha(\rho) \equiv \frac{1}{1-\alpha} \log_2 \mathrm{Tr}[\rho^\alpha]. \tag{15}$$

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It is a monotonic function of the "α-purity" $\text{Tr}[\rho^\alpha]$, and the von Neumann entropy $H(\rho)$ is recovered from it in the limit $\alpha \to 1$:

$$\lim_{\alpha \to 1} H_\alpha(\rho) = H(\rho) \equiv -\text{Tr}[\rho \log_2 \rho].$$

The min-entropy is recovered from it in the limit as $\alpha \to \infty$:

$$\lim_{\alpha \to \infty} H_\alpha(\rho) = H_{\min}(\rho) \equiv -\log \|\rho\|_\infty,$$

where $\|\rho\|_\infty$ is the infinity norm of ρ.

The quantum Rényi entropy of order $\alpha > 1$ of a thermal state with mean photon number N_B can be written as [13]

$$\frac{\log[(N_B + 1)^\alpha - N_B^\alpha]}{\alpha - 1}$$

For an additive noise channel $\mathcal{N}_{\bar{n}}$, the Rényi entropy $H_\alpha(\mathcal{N}_{\bar{n}}(\rho))$ for $\alpha > 1$ achieves its minimum value when the input ρ is the vacuum state $|0\rangle\langle 0|$ [18]:

$$\min_\rho H_\alpha(\mathcal{N}_{\bar{n}}(\rho)) = H_\alpha(\mathcal{N}_{\bar{n}}(|0\rangle\langle 0|)) = \frac{\log_2[(\bar{n} + 1)^\alpha - \bar{n}^\alpha]}{\alpha - 1} \text{ for } \alpha > 1. \tag{16}$$

Similarly, for the thermal noise channel $\mathcal{E}_{\eta, N_B}$, the Rényi entropy $H_\alpha(\mathcal{E}_{\eta, N_B}(\rho))$ for $\alpha > 1$ achieves its minimum value when the input ρ is the vacuum state $|0\rangle\langle 0|$ [18]:

$$\min_\rho H_\alpha(\mathcal{E}_{\eta, N_B}(\rho)) = H_\alpha(\mathcal{E}_{\eta, N_B}(|0\rangle\langle 0|)) = \frac{\log_2[((1 - \eta)N_B + 1)^\alpha - ((1 - \eta)N_B)^\alpha]}{\alpha - 1} \text{ for } \alpha > 1. \tag{17}$$

In general, the main result of [18] shows that the minimum output Rényi entropy of any phase-insensitive Gaussian channel $\mathcal{P}$ is achieved by the vacuum state:

$$\min_{\rho^{(n)}} H_\alpha(\mathcal{P}^{\otimes n}(\rho^{(n)})) = n H_\alpha(\mathcal{P}(|0\rangle\langle 0|)). \tag{18}$$

The above definition of the Rényi entropy can be generalized to the smooth Rényi entropy. This notion was first introduced by Renner and Wolf for classical probability distributions [23] and was later generalized to the quantum case (density operators). For a given density operator ρ, one can consider the set $\mathcal{B}^\varepsilon(\rho)$ of density operators $\tilde{\rho}$ that are ε-close to ρ in trace distance for $\varepsilon \geq 0$ [22]. The ε-smooth quantum Rényi entropy of order α of a density operator ρ is defined as [22]

$$H_\alpha^\varepsilon(\rho) \equiv \begin{cases} \inf_{\tilde{\rho} \in \mathcal{B}^\varepsilon(\rho)} H_\alpha(\tilde{\rho}) & 0 \leq \alpha < 1 \\ \sup_{\tilde{\rho} \in \mathcal{B}^\varepsilon(\rho)} H_\alpha(\tilde{\rho}) & 1 < \alpha < \infty \end{cases}. \tag{19}$$

In the limit as $\alpha \to \infty$, we recover the smooth min-entropy of ρ [22, 27]:

$$H_{\min}^\varepsilon(\rho) \equiv \sup_{\tilde{\rho} \in \mathcal{B}^\varepsilon(\rho)} \left[-\log_2 \|\tilde{\rho}\|_\infty\right]. \tag{20}$$

From the above, we see that the following relation holds

$$\inf_{\tilde{\rho} \in \mathcal{B}^\varepsilon(\rho)} \|\tilde{\rho}\|_\infty = 2^{-H_{\min}^\varepsilon(\rho)}. \tag{21}$$

A relation between the smooth min-entropy and the Rényi entropy of order $\alpha > 1$ is given by the following inequality [23]

$$H_{\min}^\varepsilon(\rho) \geq H_\alpha(\rho) - \frac{1}{\alpha - 1} \log\left(\frac{1}{\varepsilon}\right). \tag{22}$$

We will use this relation, along with the minimum output entropy results from [18], to prove the strong converse theorem for the classical capacity of all phase-insensitive Gaussian channels.

3 Strong converse for all phase-insensitive Gaussian channels

In this section, we consider the transmission of classical messages through phase-insensitive channels and show that a strong converse theorem holds for the classical capacity of these channels. Before doing so, we first make the following two observations:

- If the input signal states are allowed to have an arbitrarily large number of photons, then the classical capacity of the corresponding channel is infinite [15]. Thus, in order to have a sensible notion of the classical capacity for any quantum channel, one must impose some kind of constraint on the photon number of the states being fed into the channel. The most common kind of constraint is to demand that the mean number of photons in any signal transmitted through the channel can be at most $N_S \in [0, \infty)$. This is known as the mean photon number constraint and is commonly used in establishing the information-carrying capacity of a given channel [15, 10, 11]. However, following the same arguments as in [29] (and later in [2]), we can show that the strong converse need not hold under such a mean photon number constraint. So instead, we prove that the strong converse theorem holds under a maximum photon number constraint on the number of photons in the input states.

- Our proof of the strong converse theorem for the phase-insensitive channels can be regarded as a generalization of the arguments used in establishing the strong converse theorem for the classical capacity of the noiseless qubit channel [19, 16]. However, a comparison of our proof here and that for the noiseless qubit channel reveals that it is a significant generalization. Furthermore, our proof here also invites comparison with the proof of the strong converse for covariant channels with additive minimum output Rényi-entropy [16], especially since additivity of minimum output Rényi entropies plays a critical role in the present paper.

Let ρ_m denote a codeword of any code for communication over the phase-insensitive Gaussian channel $\mathcal{P}$. The *maximum photon number constaint* that we impose on the codebook is to require that the average code density operator $\frac{1}{M} \sum_m \rho_m$ (M is the total number of messages) has a large shadow onto a subspace with photon number no larger than some fixed amount nN_S. Mathematically, this constraint is given by

$$\frac{1}{M} \sum_m \text{Tr} \left\{ \Pi_{\lceil nN_S \rceil} \rho_m \right\} \geq 1 - \delta_1(n), \tag{23}$$

where $\delta_1(n)$ is a function that decreases to zero as n increases. In fact, the coherent-state encodings that attain the known capacities of the phase-insensitive channels do indeed satisfy the maximum photon number constraint, with an exponentially decreasing $\delta_1(n)$, if coherent states with average photon number per mode $< N_S - \delta$ are used, with δ being a small positive number (see Ref. [29] for an argument along these lines).

We can define a photon number cutoff projector Π_L projecting onto a subspace of n bosonic modes such that the total photon number is no larger than L:

$$\Pi_L \equiv \sum_{a_1, \ldots, a_n : \sum_i a_i \leq L} |a_1\rangle\langle a_1| \otimes \ldots \otimes |a_n\rangle\langle a_n|, \tag{24}$$

where $|a_i\rangle$ is a photon number state of photon number a_i. The rank of the projector $\Pi_{\lceil nN_S \rceil}$ is never larger than $2^{n[g(N_S)+\delta]}$ (Lemma 3 in [29]), i.e.,

$$\text{Tr} \left\{ \Pi_{\lceil nN_S \rceil} \right\} \leq 2^{n[g(N_S)+\delta]}, \tag{25}$$

where $\delta \geq \frac{1}{n}(\log_2 e + \log_2(1 + \frac{1}{N_S}))$, so that δ can be chosen arbitrarily small by taking n large enough.

The first important step in proving the strong converse is to show that if most of the probability mass of the input state of the phase-insensitive channel $\mathcal{P}$ is in a subspace with photon number no larger than nN_S, then most of the probability mass of the channel output is in a subspace with photon number no larger than nN_S', where N_S' is the mean energy of the output state. We state this as the following lemma:

Lemma 1 *Let $\rho^{(n)}$ denote a density operator on n modes that satisfies*

$$\text{Tr}\{\Pi_{\lceil nN_S \rceil}\rho^{(n)}\} \geq 1 - \delta_1(n),$$

where $\delta_1(n)$ is defined in (23). Let $\mathcal{P}$ be a phase-insensitive Gaussian channel with parameters τ and ν as defined in (2). Then

$$\text{Tr}\{\Pi_{\lceil nN_S' + \delta_2 \rceil}\mathcal{P}^{\otimes n}(\rho^{(n)})\} \geq 1 - \delta_1(n) - 2\sqrt{\delta_1(n)} - \delta_3(n),$$

where $N_S' = \tau N_S + (\tau + \nu - 1)/2$, $\mathcal{P}^{\otimes n}$ represents n instances of $\mathcal{P}$ that act on the density operator $\rho^{(n)}$, δ_2 is an arbitrarily small positive constant, and $\delta_3(n)$ is a function of n decreasing to zero as $n \to \infty$.

Proof. The proof of this lemma is essentially the same as the proof of Lemma 1 of [2], with some minor modifications. We include the details of it for completeness. We first recall the structural decomposition in (5) for any phase-insensitive channel:

$$\mathcal{P}(\rho) = \left(\mathcal{A}_G^0 \circ \mathcal{E}_T\right)(\rho),$$

i.e., that any phase-insensitive Gaussian channel can be realized as a concatenation of a pure-loss channel $\mathcal{E}_T$ of transmissivity T followed by a quantum-limited amplifier channel $\mathcal{A}_G$ with gain G, with $\tau = TG$ and $\nu = G(1 - T) + G - 1$. Thus, a photon number state $|k\rangle \langle k|$ input to the phase-insensitive noise channel leads to an output of the following form:

$$\mathcal{P}(|k\rangle \langle k|) = \sum_{m=0}^{k} p(m) \mathcal{A}_G^0(|m\rangle \langle m|), \tag{26}$$

where

$$p(m) = \binom{k}{m} T^m (1 - T)^{k-m}.$$

The quantum-limited amplifier channel has the following action on a photon number state $|m\rangle$ [7]:

$$\mathcal{A}_G^0(|m\rangle \langle m|) = \sum_{l=0}^{\infty} q(l|m) |l\rangle \langle l|,$$

where the conditional probabilities $q(l|m)$ are given by:

$$q(l|m) = \begin{cases} 0 & l < m \\ (1 - \mu^2)^{m+1} \mu^{2(l-m)} \binom{l}{l-m} & l \geq m \end{cases},$$

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where $\mu = \tanh r \in [0, 1]$, with r chosen such that $G = \cosh^2(r)$.

The conditional distribution $q(l|m)$ has the two important properties of having finite second moment and exponential decay with increasing photon number. The property of exponential decay with increasing l follows from

$$(1 - \mu^2)^{m+1} \mu^{2(l-m)} \binom{l}{l-m} = (1 - \mu^2)^{m+1} \mu^{-2m} 2^{-2\log_2\left(\frac{1}{\mu}\right)l} \binom{l}{l-m}$$
$$\leq (1 - \mu^2)^{m+1} \mu^{-2m} 2^{-2\log\left(\frac{1}{\mu}\right)l} 2^{lh_2\left(\frac{l-m}{l}\right)}$$
$$= (1 - \mu^2)^{m+1} \mu^{-2m} 2^{-l\left[2\log\left(\frac{1}{\mu}\right) - h_2\left(\frac{l-m}{l}\right)\right]}$$

The inequality applies the bound $\binom{n}{k} \leq 2^{nh_2(k/n)}$ (see (11.40) of [5]), where $h_2(x)$ is the binary entropy with the property that $\lim_{x \to 1} h_2(x) = 0$. Thus, for large enough l, it will be the case that $2\log\left(\frac{1}{\mu}\right) - h_2\left(\frac{l-m}{l}\right) > 0$, so that the conditional distribution $q(l|m)$ has exponential decay with increasing l. We can also then conclude that this distribution has a finite second moment. It follows from (26) that

$$\mathcal{P}\left(|k\rangle\langle k|\right) = \sum_{l=0}^{\infty} \left[\sum_{m=0}^{k} p(m) q(l|m)\right] |l\rangle\langle l|. \tag{27}$$

The eigenvalues above (i.e., $\sum_{m=0}^{k} p(m) q(l|m)$) represent a distribution over photon number states at the output of the phase-insensitive channel $\mathcal{P}$, which we can write as a conditional probability distribution $p(l|k)$ over l given the input with definite photon number k. This probability distribution has its mean equal to $\tau k + (\tau + \nu - 1)/2$, since the mean energy of the input state is k. Furthermore, this distribution inherits the properties of having a finite second moment and an exponential decay to zero as $l \to \infty$.

For example, we can consider the thermal noise channel $\mathcal{E}_{\eta, N_B}$ with the structural decomposition given by (7)

$$\mathcal{E}_{\eta, N_B}(\rho) = (\mathcal{A}^0_{(1-\eta)N_B+1} \circ \mathcal{E}_{\eta/((1-\eta)N_B+1)})(\rho).$$

The mean of the corresponding distribution for this channel when a state of definite photon number k is input, following the above arguments, is equal to $\eta k + (1 - \eta) N_B$.

The argument from here is now exactly the same as the proof of Lemma 1 of [2] (starting from (40) of [2]). We include it here for completeness. We now suppose that the input state satisfies the maximum photon-number constraint in (23), and apply the Gentle Measurement Lemma [21, 31] to obtain the following inequality

$$\text{Tr}\left\{\Pi_{\lceil nN'_S+\delta_2\rceil} \mathcal{P}^{\otimes n}\left(\rho^{(n)}\right)\right\} \geq \text{Tr}\left\{\Pi_{\lceil nN'_S+\delta_2\rceil} \mathcal{P}^{\otimes n}\left(\Pi_{\lceil nN_S\rceil}\rho^{(n)}\Pi_{\lceil nN_S\rceil}\right)\right\} - 2\sqrt{\delta_1(n)}, \tag{28}$$

where $N'_S = \tau N_S + (\tau + \nu - 1)/2$. Since there is photodetection at the output (i.e., the projector $\Pi_{\lceil n\eta N'_S+\delta_2\rceil}$ is diagonal in the number basis), it suffices for us to consider the input $\Pi_{\lceil nN_S\rceil}\rho^{(n)}\Pi_{\lceil nN_S\rceil}$ to be diagonal in the photon-number basis, and we write this as

$$\rho^{(n)} = \sum_{a^n:\sum_i a_i \leq \lceil nN_S\rceil} p(a^n) |a^n\rangle\langle a^n|,$$

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where $|a^n\rangle$ represents strings of photon number states. We then find that (28) is equal to

$$\sum_{a^n:\sum_i a_i \leq \lceil nN_S \rceil} p\left(a^n\right) \mathrm{Tr}\left\{\left(\Pi_{\lceil nN_S'+\delta_2 \rceil}\right) \mathcal{P}^{\otimes n}\left(|a^n\rangle\langle a^n|\right)\right\} - 2\sqrt{\delta_1(n)}$$

$$= \sum_{a^n:\sum_i a_i \leq \lceil nN_S \rceil} p\left(a^n\right) \sum_{l^n:\sum_i l_i \leq \lceil nN_S'+\delta_2 \rceil} p\left(l^n|a^n\right) - 2\sqrt{\delta_1(n)}, \tag{29}$$

where the distribution $p\left(l^n|a^n\right) \equiv \prod_{i=1}^{n} p\left(l_i|a_i\right)$ with $p\left(l_i|a_i\right)$ coming from (27).

In order to obtain a lower bound on the expression in (29), we analyze the term

$$\sum_{l^n:\sum_i l_i \leq \lceil nN_S'+\delta_2 \rceil} p\left(l^n|a^n\right) \tag{30}$$

on its own under the assumption that $\sum_i a_i \leq \lceil nN_S \rceil$. Let $L_i|a_i$ denote a conditional random variable with distribution $p\left(l_i|a_i\right)$, and let $\overline{L^n}|a^n$ denote the sum random variable:

$$\overline{L^n}|a^n \equiv \sum_i L_i|a_i,$$

so that

$$\sum_{l^n:\sum_i l_i \leq \lceil nN_S'+\delta_2 \rceil} p\left(l^n|a^n\right) = \mathrm{Pr}\left\{\overline{L^n}|a^n \leq nN_S'+\delta_2\right\}$$

$$= \mathrm{Pr}\left\{\overline{L^n}|a^n \leq n\left(\tau N_S + \left(\tau+\nu-1\right)/2\right) + \delta_2\right\} \tag{31}$$

$$\geq \mathrm{Pr}\left\{\overline{L^n}|a^n \leq n\left(\tau\frac{1}{n}\sum_i a_i + \left(\tau+\nu-1\right)/2 + \delta_2\right)\right\}, \tag{32}$$

where $\left(\tau+\nu-1\right)/2$ represents the mean number of noise photons injected by the channel, and the inequality follows from the constraint $\sum_i a_i \leq \lceil nN_S \rceil$. Since

$$\mathbb{E}\left\{L_i|a_i\right\} = \tau a_i + \left(\tau+\nu-1\right)/2,$$

it follows that

$$\mathbb{E}\left\{\overline{L^n}|a^n\right\} = n\left(\tau\frac{1}{n}\sum_i a_i + \left(\tau+\nu-1\right)/2\right),$$

and so the expression in (32) is the probability that a sum of independent random variables deviates from its mean by no more than δ_2. To obtain a bound on the probability in (32) from below, we now follow the approach in [2] employing the truncation method (see Section 2.1 of [26] for more details), in which each random variable $L_i|a_i$ is split into two parts:

$$\left(L_i|a_i\right)_{>T_0} \equiv \left(L_i|a_i\right)\mathcal{I}\left(\left(L_i|a_i\right)>T_0\right),$$
$$\left(L_i|a_i\right)_{\leq T_0} \equiv \left(L_i|a_i\right)\mathcal{I}\left(\left(L_i|a_i\right)\leq T_0\right),$$

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where $\mathcal{I}(\cdot)$ is the indicator function and T_0 is a truncation parameter taken to be very large (much larger than $\max_i a_i$, for example). We can then split the sum random variable into two parts as well:

$$\overline{L^n}|a^n = \left(\overline{L^n}|a^n\right)_{>T_0} + \left(\overline{L^n}|a^n\right)_{\leq T_0}$$
$$\equiv \sum_i \left(L_i|a_i\right)_{>T_0} + \sum_i \left(L_i|a_i\right)_{\leq T_0}.$$

We can use the union bound to argue that

$$\Pr\left\{\overline{L^n}|a^n \geq \mathbb{E}\left\{\overline{L^n}|a^n\right\} + n\delta_2\right\} \leq \Pr\left\{\left(\overline{L^n}|a^n\right)_{>T_0} \geq \mathbb{E}\left\{\left(\overline{L^n}|a^n\right)_{>T_0}\right\} + n\delta_2/2\right\}$$
$$+ \Pr\left\{\left(\overline{L^n}|a^n\right)_{\leq T_0} \geq \mathbb{E}\left\{\left(\overline{L^n}|a^n\right)_{\leq T_0}\right\} + n\delta_2/2\right\}. \quad (33)$$

The idea from here is that for a random variable $L_i|a_i$ with sufficient decay for large values, we can bound the first probability for $\left(\overline{L^n}|a^n\right)_{>T_0}$ from above by ε/δ_2 for ε an arbitrarily small positive constant (made small by taking T_0 larger) by employing the Markov inequality. We then bound the second probability for $\left(\overline{L^n}|a^n\right)_{\leq T_0}$ using a Chernoff bound, since these random variables are bounded. This latter bound has an exponential decay with increasing n due to the ability to use a Chernoff bound. So, the argument is just to make ε arbitrarily small by increasing the truncation parameter T_0, and for n large enough, exponential convergence to zero kicks in. We point the reader to Section 2.1 of [26] for more details. By using either approach, we arrive at the following bound:

$$\sum_{l^n:\sum_i l_i \leq \lceil nN'_S + \delta_2 \rceil} p\left(l^n|a^n\right) \geq 1 - \delta_3(n),$$

where $\delta_3(n)$ is a function decreasing to zero as $n \to \infty$. Finally, we put this together with (29) to obtain

$$\mathrm{Tr}\left\{\Pi_{\lceil nN'_S + \delta_2 \rceil} \mathcal{P}^{\otimes n}\left(\rho^{(n)}\right)\right\}$$
$$\geq \sum_{a^n:\sum_i a_i \leq \lceil nN_S \rceil} p\left(a^n\right) \sum_{l^n:\sum_i l_i \leq \lceil nN'_S + \delta_2 \rceil} p\left(l^n|a^n\right) - 2\sqrt{\delta_1(n)}$$
$$\geq \left(1 - \delta_1(n)\right)\left(1 - \delta_3(n)\right) - 2\sqrt{\delta_1(n)}$$
$$\geq 1 - \delta_1(n) - \delta_3(n) - 2\sqrt{\delta_1(n)},$$

thereby completing the proof. ∎

Let Λ_m denote a decoding POVM acting on the output space of n instances of the phase-insensitive channel. In what follows, we prove the strong converse theorem for the classical capacity of all phase-insensitive Gaussian channels.

Theorem 1 *Let $\mathcal{P}$ be a phase-insensitive Gaussian channel with parameters τ and ν as defined in (2). The average success probability p_{succ} of any code for this channel satisfying (23) is bounded as*

$$p_{\mathrm{succ}} = \frac{1}{M}\sum_m \mathrm{Tr}\{\Lambda_m \mathcal{P}^{\otimes n}(\rho_m)\} \leq$$

$$2^{-nR}2^{n\left[g\left(N'_S\right) - H_\alpha(\mathcal{P}(|0\rangle\langle0|)) + \delta_2 + \frac{1}{n(\alpha-1)}\log(1/\varepsilon)\right]} + \varepsilon + 2\sqrt{\delta_1(n) + 2\sqrt{\delta_1(n)} + \delta_3(n)}, \quad (34)$$

where $\alpha > 1$, $\varepsilon \in (0,1)$, $N_S' = \tau N_S + (\tau + \nu - 1)/2$, $\mathcal{P}^{\otimes n}$ denotes n instances of $\mathcal{P}$, $\delta_1(n)$ is defined in (23), δ_2 is an arbitrarily small positive constant and $\delta_3(n)$ is a function decreasing with n (both defined in Lemma 1).

Proof. This proof is very similar to the proof of Theorem 2 of [2], with the exception that we can now invoke the main result of [18] (that the minimum output entropy for Rényi entropies of arbitrary order is attained by the vacuum state input). Consider the success probability of any code satisfying the maximum photon-number constraint (23):

$$
\begin{aligned}
\frac{1}{M} \sum_m \mathrm{Tr}\{\Lambda_m \mathcal{P}^{\otimes n}(\rho_m)\} &\leq \frac{1}{M} \sum_m \mathrm{Tr}\{\Lambda_m \Pi_{\lceil nN_S'\rceil} \mathcal{P}^{\otimes n}(\rho_m) \Pi_{\lceil nN_S'\rceil}\} \\
&\quad + \frac{1}{M} \sum_m \left\| \Pi_{\lceil nN_S'\rceil} \mathcal{P}^{\otimes n}(\rho_m) \Pi_{\lceil nN_S'\rceil} - \mathcal{P}^{\otimes n}(\rho_m) \right\|_1 \\
&\leq \frac{1}{M} \sum_m \mathrm{Tr}\{\Lambda_m \Pi_{\lceil nN_S'\rceil} \mathcal{P}^{\otimes n}(\rho_m) \Pi_{\lceil nN_S'\rceil}\} \\
&\quad + 2\sqrt{\delta_1(n) + 2\sqrt{\delta_1(n)}} + \delta_3(n).
\end{aligned}
$$

The first inequality is a special case of the inequality

$$
\mathrm{Tr}\{\Lambda\sigma\} \leq \mathrm{Tr}\{\Lambda\rho\} + \|\rho - \sigma\|_1, \tag{35}
$$

which holds for $0 \leq \Lambda \leq I$, $\rho, \sigma \geq 0$, and $\mathrm{Tr}\{\rho\}, \mathrm{Tr}\{\sigma\} \leq 1$. The second inequality is obtained by invoking Lemma 1 and the Gentle Measurement Lemma [21, 31] for ensembles.

Note that in the above, the second term vanishes as $n \to \infty$; hence it suffices to focus on the first term, which by cyclicity of trace yields

$$
\frac{1}{M} \sum_m \mathrm{Tr}\{\Lambda_m \Pi_{\lceil nN_S'\rceil} \mathcal{P}^{\otimes n}(\rho_m) \Pi_{\lceil nN_S'\rceil}\} = \frac{1}{M} \sum_m \mathrm{Tr}\{\Pi_{\lceil nN_S'\rceil} \Lambda_m \Pi_{\lceil nN_S'\rceil} \mathcal{P}^{\otimes n}(\rho_m)\}. \tag{36}
$$

At this point, we consider the set of all states $\widetilde{\sigma}_m$ that are ε-close in trace distance to each output of the phase-insensitive channel $\mathcal{P}^{\otimes n}(\rho_m)$ (let us denote this set by $\mathcal{B}^\varepsilon(\mathcal{P}^{\otimes n}(\rho_m))$). This consideration will allow us to relate the success probability to the smooth min-entropy. We find the following upper bound on (36):

$$
\begin{aligned}
\frac{1}{M} \sum_m \mathrm{Tr}\{\Pi_{\lceil nN_S'\rceil} \Lambda_m \Pi_{\lceil nN_S'\rceil} \mathcal{P}^{\otimes n}(\rho_m)\} &\leq \frac{1}{M} \sum_m \mathrm{Tr}\{\Pi_{\lceil nN_S'\rceil} \Lambda_m \Pi_{\lceil nN_S'\rceil} \widetilde{\sigma}_m\} + \varepsilon \\
&\leq \frac{1}{M} \sum_m \mathrm{Tr}\{\Pi_{\lceil nN_S'\rceil} \Lambda_m \Pi_{\lceil nN_S'\rceil}\} \|\widetilde{\sigma}_m\|_\infty + \varepsilon.
\end{aligned}
$$

We can now optimize over all of the states $\widetilde{\sigma}_m$ that are ε-close to $\mathcal{P}^{\otimes n}(\rho_m)$, leading to the tightest upper bound on the success probability

$$
\frac{1}{M} \sum_m \mathrm{Tr}\{\Pi_{\lceil nN_S'\rceil} \Lambda_m \Pi_{\lceil nN_S'\rceil} \mathcal{P}^{\otimes n}(\rho_m)\} \tag{37}
$$

$$
\leq \frac{1}{M} \sum_m \mathrm{Tr}\{\Pi_{\lceil nN_S'\rceil} \Lambda_m \Pi_{\lceil nN_S'\rceil}\} \inf_{\widetilde{\sigma}_m \in \mathcal{B}^\varepsilon(\mathcal{P}^{\otimes n}(\rho_m))} \|\widetilde{\sigma}_m\|_\infty + \varepsilon. \tag{38}
$$

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Since the quantity $\inf_{\widetilde{\sigma}_m \in \mathcal{B}^\varepsilon(\mathcal{P}^{\otimes n}(\rho_m))} \|\widetilde{\sigma}_m\|_\infty$ is related to the smooth min-entropy via

$$\inf_{\widetilde{\sigma}_m \in \mathcal{B}^\varepsilon(\mathcal{P}^{\otimes n}(\rho_m))} \|\widetilde{\sigma}_m\|_\infty = 2^{-H_{\min}^\varepsilon\left(\mathcal{P}^{\otimes n}(\rho_m)\right)},$$

the upper bound in (38) gives

$$\frac{1}{M}\sum_m \text{Tr}\{\Pi_{\lceil nN_S'\rceil}\Lambda_m\Pi_{\lceil nN_S'\rceil}\}2^{-H_{\min}^\varepsilon\left(\mathcal{P}^{\otimes n}(\rho_m)\right)} + \varepsilon$$

$$\leq \frac{1}{M}\sum_m \text{Tr}\{\Pi_{\lceil nN_S'\rceil}\Lambda_m\Pi_{\lceil nN_S'\rceil}\}\sup_\rho 2^{-H_{\min}^\varepsilon\left(\mathcal{P}^{\otimes n}(\rho)\right)} + \varepsilon$$

$$= \frac{1}{M}2^{-\inf_\rho H_{\min}^\varepsilon\left(\mathcal{P}^{\otimes n}(\rho)\right)}\text{Tr}\{\Pi_{\lceil nN_S'\rceil}\} + \varepsilon$$

$$\leq 2^{-nR}2^{-\inf_\rho H_{\min}^\varepsilon\left(\mathcal{P}^{\otimes n}(\rho)\right)}2^{n[g(N_S')+\delta]} + \varepsilon. \tag{39}$$

The first inequality follows by taking a supremum over all input states. The first equality follows because $\sum_m \Lambda_m = I$ for the set of decoding POVM measurements $\{\Lambda_m\}$, and the second inequality is a result of the upper bound on the rank of the photon number cutoff projector in (25). We have also used the fact that the rate of the channel is expressed as $R = (\log_2 M)/n$, where M is the number of messages.

Observe that the success probability is now related to the smooth min-entropy, and we can now exploit the following relation between smooth min-entropy and the Rényi entropies for $\alpha > 1$ [23]:

$$H_{\min}^\varepsilon(\omega) \geq H_\alpha(\omega) - \frac{1}{\alpha - 1}\log\left(\frac{1}{\varepsilon}\right).$$

Using the above inequality and the fact that the "strong" Gaussian optimizer conjecture has been proven for the Rényi entropies of all orders [18] (recall (18)), we get that

$$\inf_\rho H_{\min}^\varepsilon\left(\mathcal{P}^{\otimes n}(\rho)\right) \geq n\left[H_\alpha\left(\mathcal{P}(|0\rangle\langle 0|)\right) - \frac{1}{n(\alpha - 1)}\log\left(\frac{1}{\varepsilon}\right)\right]. \tag{40}$$

The first term on the right hand side is a result of the fact that the vacuum state minimizes the Rényi entropy of all orders at the output of a phase-insensitive Gaussian channel. ∎

By tuning the parameters α and ε appropriately, we recover the strong converse theorem:

Corollary 1 (Strong converse) *Let $\mathcal{P}$ be a phase-insensitive Gaussian channel with parameters τ and ν as defined in (2). The average success probability p_{succ} of any code for this channel satisfying (23) is bounded as*

$$p_{\text{succ}} = \frac{1}{M}\sum_m \text{Tr}\{\Lambda_m\mathcal{P}^{\otimes n}(\rho_m)\} \leq$$

$$2^{-nR}2^{n[g(N_S')-g(N_B')+\delta_2+\delta_5/\delta_4+\delta_4 C(N_B')]}$$

$$+ 2^{-n\delta_5} + 2\sqrt{\delta_1(n) + 2\sqrt{\delta_1(n)} + \delta_3(n)}, \tag{41}$$

where $N_S' = \tau N_S + (\tau + \nu - 1)/2$, $N_B' \equiv (\tau + \nu - 1)/2$, $\mathcal{P}^{\otimes n}$ denotes n instances of $\mathcal{P}$, $\delta_1(n)$ is defined in (23), δ_2 is an arbitrarily small positive constant and $\delta_3(n)$ is a function decreasing with

n (both defined in Lemma 1), δ_4 and δ_5 are arbitrarily small positive constants such that δ_5/δ_4 is arbitrarily small, and $C\left(N'_B\right)$ is a function of N'_B only. Thus, for any rate $R > g\left(N'_S\right) - g\left(N'_B\right)$, it is possible to choose the parameters such that the success probability of any family of codes satisfying (23) decreases to zero in the limit of large n.

Proof. In Theorem 1, we can pick $\alpha = 1 + \delta_4$ and $\varepsilon = 2^{-n\delta_5}$, with $\delta_5 > 0$ much smaller than $\delta_4 > 0$ such that δ_5/δ_4 is arbitrarily small, and the terms on the right hand side in (40) simplify to

$$n\left[H_{1+\delta_4}\left(\mathcal{P}(|0\rangle\langle 0|)\right) - \frac{\delta_5}{\delta_4}\right].$$

The output state $\mathcal{P}(|0\rangle\langle 0|)$ for the phase-insensitive channel with the vacuum state as the input is a thermal state with mean photon number $N'_B \equiv (\tau + \nu - 1)/2$. The quantum Rényi entropy of order $\alpha > 1$ of a thermal state with mean photon number N'_B is given by [9]

$$\frac{\log\left[\left(N'_B + 1\right)^{\alpha} - N'^{\alpha}_B\right]}{\alpha - 1}. \tag{42}$$

Lemma 6.3 of Tomamichel's thesis gives us the following inequality for a general state (for α close enough to one):

$$H_{\alpha}\left(\rho\right) \geq H\left(\rho\right) - 4\left(\alpha - 1\right)\left(\log v\right)^2,$$

where

$$v \equiv 2^{-\frac{1}{2}H_{3/2}(\rho)} + 2^{\frac{1}{2}H_{1/2}(\rho)} + 1$$

For a thermal state, we find using (42) that

$$H_{3/2}\left(\rho\right) = 2\log\left[\left(N'_B + 1\right)^{3/2} - N'^{3/2}_B\right],$$

$$H_{1/2}\left(\rho\right) = -2\log\left[\left(N'_B + 1\right)^{1/2} - N'^{1/2}_B\right],$$

so that

$$v\left(N'_B\right) = \left[\left(N'_B + 1\right)^{3/2} - N'^{3/2}_B\right]^2 + \left[\left(N'_B + 1\right)^{1/2} - N'^{1/2}_B\right]^{-2} + 1.$$

We then find that

$$H_{1+\delta_4}\left(\mathcal{P}(|0\rangle\langle 0|)\right) \geq H\left(\mathcal{P}(|0\rangle\langle 0|)\right) - \delta_4 C\left(N'_B\right)$$
$$= g\left(N'_B\right) - \delta_4 C\left(N'_B\right),$$

where

$$C\left(N'_B\right) \equiv 4\left[\log v\left(N'_B\right)\right]^2.$$

We now recover the bound in the statement of the corollary. ∎

Finally, we recall the capacities of the phase-insensitive channels in (11), (12), and (13). Comparing them with the statement of Corollary 1, we can conclude that these expressions indeed represent strong converse rates for these respective channels, since the success probability when communicating above these rates decreases to zero in the limit $n \to \infty$.

4   Conclusion

Phase-insensitive Gaussian channels represent physical noise models which are relevant for optical communication, including attenuation, thermalization, or amplification of optical signals. In this paper, we combine the proofs in [2] with the recent results of [11, 18] to prove that a strong converse theorem holds for the classical capacity of all phase-insensitive Gaussian quantum channels. We showed that the success probability of correctly decoding classical information asymptotically converges to zero in the limit of many channel uses, if the communication rate exceeds the capacity. Our result thus establishes the capacity of these channels as a very sharp dividing line between possible and impossible communication rates through these channels. This result might find an immediate application in proving security of noisy quantum storage models of cryptography [17] for continuous-variable systems.

As an open question, one might attempt to establish a strong converse for the classical capacity of all *phase-sensitive* Gaussian channels. Another area of research where our result might be extended is in the setting of network information theory—for example, one might consider establishing a strong converse for the classical capacity of the multiple-access bosonic channels, in which two or more senders communicate to a common receiver over a shared communication channel [33].

5   Acknowledgements

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