IMPROVED ACCURACY OF INCOMPRESSIBLE APPROXIMATION OF COMPRESSIBLE EULER EQUATIONS

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Abstract. This article addresses a fundamental concern regarding the incompressible approximation of fluid motions, one of the most widely used approximations in fluid mechanics. Common belief is that its accuracy is $O(\varepsilon)$ where ε denotes the Mach number. In this article, however, we prove an $O(\varepsilon^2)$ accuracy for the incompressible approximation of the isentropic, compressible Euler equations thanks to several decoupling properties. At the initial time, the velocity field and its first time derivative are of $O(1)$ size, but the boundary conditions can be as stringent as the solid-wall type. The fast acoustic waves are still $O(\varepsilon)$ in magnitude, since the $O(\varepsilon^2)$ error is measured in the sense of Leray projection and more physically, in time-averages. We also show when a passive scalar is transported by the flow, it is $O(\varepsilon^2)$ accurate *point-wise in time* to use incompressible approximation for the velocity field in the transport equation.

1. Introduction and Statement of Main Theorem

All fluids are compressible, which generates acoustic waves. The restoring force is the pressure gradient which results from the fluid being compressed and decompressed. The Mach number, denoted by ε in our article, is defined as the typical value of the ratio of fluid speed over sound speed. In the very subsonic regime $\varepsilon \ll 1$, incompressible (vortical) fluid motions evolve in a slower time scale than acoustic wave propagation; then, incompressible approximation is often adopted so that effectively acoustic waves are filtered out. Numerous applications and theoretical studies rely on the validity of such approximation that indeed offers more convenience and simplicity than the compressible models.

Common belief is that the incompressible approximation introduces $O(\varepsilon)$ errors. In this article, however, we prove an improved $O(\varepsilon^2)$ error estimate between the isentropic, compressible Euler equations and its incompressible counterpart, thanks to several decoupling properties. The initial data is well-prepared in the sense that its first time derivative has $O(1)$ spatial norms, independent of the smallness of ε. In a loosely equivalent way, the velocity divergence is only $O(\varepsilon)$ in spatial norms and acoustic waves have only $O(\varepsilon)$ amplitudes as well. Higher time derivatives can still grow as $\varepsilon \to 0$. The central idea of time-averaging is repeatedly used to suppress the amplitude of acoustic waves by a factor of ε. Intuitively, acoustic waves oscillate fast at temporal frequencies of $O(\varepsilon^{-1})$, and therefore averaging them in time effectively cancels out the majority of oscillations.

We ought to point out that the nonlinear nature of fluid motions is bound to couple fast acoustic waves with the slower incompressible motions. Even when all acoustic waves are completely

1991 *Mathematics Subject Classification.* 35Q31 (Primary) 35L50, 76N15 (Secondary).

Key words and phrases. Incompressible approximation; compressible Euler equations; convergence rates; time averages; initial-boundary value problem; singular limits.

filtered out at the initial time, they are instantaneously generated from slow incompressible motions. In the atmosphere for example, the ubiquitous acoustic waves are emitted all the time, although most are inaudible to human ears[1].

To this end, time-averaging plays a crucial role to further suppress the "unwanted" contribution from acoustic waves to the incompressible dynamics. The physical relevance of time-averaging is evident from the popularity of its generalized version, time filtering. In fact, time filtering is necessary in dealing with observational and computational data when the resolution of fast acoustic waves suffers from a wide range of factors. To make even closer connection to applications, we will use time-averaging technique to show that, if a passive scalar is advected by flow generated by the compressible Eule equations, then it is $O(\varepsilon^2)$ accurate *point-wise in time* to replace the velocity with its incompressible counterpart(s).

Our techniques are applicable to general bounded domains subject to the solid-wall boundary condition. Several issues arise here: 1. nonlinear coupling of fast and slow dynamics does not decay or disperse in any strong sense; 2. Fourier analysis is not applicable here; 3. it is inconvenient to use straightforward energy estimates to obtain ε-independent energy estimates (in H^m norms) for the solution *and* its first time derivative over an $O(1)$ long time interval. The latter is because integrating by parts is not valid for estimating spatial derivatives subject to solid-wall boundary condition. These issues will be resolved by relying on time-averaging as well as vorticity formulations.

1.1. **Main results.** Upon rescaling and nondimensionalization, the isentropic, compressible Euler equations are expressed in terms of *total* density ρ^{tot} and velocity $\mathbf{v}$,

$$\begin{cases} \partial_t \rho^{\mathsf{tot}} + \nabla\cdot(\rho^{\mathsf{tot}}\mathbf{v}) = 0, \\ \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \dfrac{1}{\varepsilon^2}\,\dfrac{\nabla\pi(\rho^{\mathsf{tot}})}{\rho^{\mathsf{tot}}} = 0, \end{cases} \tag{1.1}$$

with Mach number $\varepsilon \ll 1$ bringing in acoustic waves oscillating on fast time scales. The pressure law $\pi \in C^\infty(\mathbb{R}^+)$ went through rescaling and affine transformation to satisfy

$$\pi(1) = \pi'(1) = 1. \tag{1.2}$$

It is then understood that $\rho^{\mathsf{tot}} - 1 \sim O(\varepsilon)$, so that the pressure gradient is approximated by $\nabla\rho^{\mathsf{tot}}$ and the linearized acoustic waves have both phase and group velocities at order $1/\varepsilon$, namely the rescaled sound speed.

Without loss of generality, we only consider a connected (but not necessarily simply connected) compact spatial domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ with the "solid-wall" boundary condition

$$\mathbf{v}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$$

where $\vec{\mathsf{n}} = \vec{\mathsf{n}}(x)$ is the outward normal to the static, smooth boundary $\partial\Omega$ which can be empty. The topology of Ω will occasionally be a concern, e.g. in Remark 2.2.

[1]The sound speed is around 330 meters per second in the lower atmosphere; human's hearing range starts from 20 Hertz. Therefore, we can not hear wave lengths longer than 17 meters in our everyday life.

The main goal of this article is to estimate, in terms of ε and initial data, the size of $(\mathbf{v} - \tilde{\mathbf{v}})$ where $\tilde{\mathbf{v}}$ solves the incompressible Euler equations

$$\partial_t \tilde{\mathbf{v}} + \tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}} + \nabla q = 0, \tag{1.3a}$$

$$\nabla\cdot\tilde{\mathbf{v}} = 0, \tag{1.3b}$$

$$\text{subject to} \qquad \tilde{\mathbf{v}}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0. \tag{1.3c}$$

$$\text{Initial data } \tilde{\mathbf{v}}_0 \text{ satisfies (1.3b), (1.3c).} \tag{1.3d}$$

Here, the scalar q is an auxiliary variable, also called pressure, that enforces the incompressible condition (1.3b). Without such a term, the above system would be overdetermined.

The spatial H^m norm is defined as usual,

$$\|f(x)\|_{\mathsf{H}^m} := \Big(\sum_{|\beta|\leqslant m}\int_\Omega |\partial_x^\beta f(x)|^2\,dx\Big)^{1/2}$$

where multi-index β indicates orders of derivatives taken on each spatial dimension. Let $\mathsf{H}^m(\Omega)$ denote the closure space of smooth functions with finite H^m norms. Of course, $\mathsf{L}^2(\Omega) = \mathsf{H}^0(\Omega)$.

Before stating the Main Theorem, we clarify one technical point. The time derivates at $t = 0$, denoted by $\partial_t^k(\rho_0^{\mathsf{tot}}, \mathbf{v}_0)$, can be calculated without knowing the solution for $t > 0$. This is because, by repeatedly taking time derivatives on (1.1), one can inductively express $\partial_t(\rho^{\mathsf{tot}}, \mathbf{v}), \partial_t^2(\rho^{\mathsf{tot}}, \mathbf{v})$, $\ldots \partial_t^k(\rho^{\mathsf{tot}}, \mathbf{v})$ solely in terms of $(\rho^{\mathsf{tot}}(t,x), \mathbf{v}(t,x))$ and their spatial derivatives up to the k-th order evaluated at each fixed time t.

Theorem 1.1. (Main Theorem) *Let integer $m \geqslant 4$ and parameter $\varepsilon \in [0, 1/2]$. Consider the compressible system (1.1) subject to initial data $(\rho_0^{\mathsf{tot}} - 1, \mathbf{v}_0) \in \mathsf{H}^m(\Omega)$. Assume $(\rho_0^{\mathsf{tot}}, \mathbf{v}_0)$ is compatible with the boundary condition, namely $(\partial_t^k \mathbf{v}_0)\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$ for $k < m$.*

Let $\tilde{\mathbf{v}}$ solve the incompressible system (1.3) subject to initial data $\tilde{\mathbf{v}}_0 = \mathcal{P}\mathbf{v}_0$. Here, $\mathcal{P}$, defined in (2.1) below, denotes the Leray projection into the incompressible velocity subspace.

Define

$$\mathrm{E}_0 := \big\|(\frac{\rho^{\mathsf{tot}} - 1}{\varepsilon}, \mathbf{v}_0)\big\|_{\mathsf{H}^m}, \qquad \mathrm{E}_{t,0} := \big\|\partial_t(\frac{\rho^{\mathsf{tot}}}{\varepsilon}, \mathbf{v}_0)\big\|_{\mathsf{H}^{m-1}}$$

Then, there exist constants E^*, T^*, C^* that only depend on m, Ω and pressure law $\pi(\cdot)$, so that with $\mathrm{E}_0 \leqslant E^*/\varepsilon$,

$$\sup_{t\in[0,T^*/\mathrm{E}_0]} \|\mathcal{P}\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathsf{H}^{m-3}} \leqslant C^*\,\varepsilon^2\,(\mathrm{E}_{t,0} + \mathrm{E}_0^2)\|\mathcal{P}\mathbf{v}_0\|_{\mathsf{H}^m}. \tag{1.4}$$

The proof is given in the last Section 5.

Remark 1.2. There are two bounding factors in (1.4) that depend on initial data. Regarding the $(\mathrm{E}_{t,0} + \mathrm{E}_0^2)$ factor, we note the compressible system (1.1) automatically enforces $\mathrm{E}_{t,0} \leqslant O(\mathrm{E}_0/\varepsilon)$. Then, using ill-prepared data that allow large time derivatives $\mathrm{E}_{t,0} \sim O(1/\varepsilon)$ and thus admit acoustic waves of $O(1)$ amplitudes, we would recover the $O(\varepsilon\mathrm{E}_0)$ error estimate for ill-prepared data previously proved by B. Cheng in [6]. Regarding the other factor $\|\mathcal{P}\mathbf{v}_0\|_{\mathsf{H}^m}$, in the extreme case with purely acoustic wave or potential flow initial data, $\mathcal{P}\mathbf{v} = 0$ is invariantly sustained by the compressible system whereas the incompressible system simply yields $\tilde{\mathbf{v}} = 0$. Then, both sides of (1.4) vanish, consistent with such well known invariance.

Remark 1.3. The local-in-time existence and uniqueness of $\mathscr{C}([0,T],\mathsf{H}^m(\Omega))$ solution to (1.1) has been established in [18]. The compatibility condition, $(\partial_t^k\mathbf{v}_0)\cdot\vec{\mathsf{n}}\big|_{\partial\Omega}=0$ for $k<m$, is both necessary and sufficient for H^m well-posedness, although this is not the main focus of this article. On the other hand, it is crucial in our study to obtain ε-independent upper bounds on $\|(\rho,\mathbf{v})\|_{\mathsf{H}^m}$ and $\|\partial_t(\rho,\mathbf{v})\|_{\mathsf{H}^{m-1}}$. This will be achieved in Section 4.

We used the clumsy notation of $\frac{\rho^{\mathsf{tot}}-1}{\varepsilon}$ to state the Main Theorem, as it will be replaced throughout the rest of this article with the density perturbation

$$\rho := \frac{\rho^{\mathsf{tot}}-1}{\varepsilon}. \tag{1.5}$$

With this notation,

$$\mathrm{E}_0=\|(\rho_0,\mathbf{v}_0)\|_{\mathsf{H}^m},\qquad \mathrm{E}_{t,0}=\|\partial_t(\rho_0,\mathbf{v}_0)\|_{\mathsf{H}^{m-1}}, \tag{1.6}$$

so that the Main Theorem is invariant under the hyperbolic scaling $\rho\to c\rho,\mathbf{v}\to c\mathbf{v},t\to t/c$.

Also, one can easily use (1.9) from below and Sobolev inequalities to show

$$\Big|\varepsilon\big\|\partial_t(\rho,\mathbf{v})\big\|_{\mathsf{H}^k}-\big\|(\nabla\cdot\mathbf{v},\nabla\rho)\big\|_{\mathsf{H}^k}\Big|\leqslant C\varepsilon\big\|(\rho,\mathbf{v})\big\|_{\mathsf{H}^{k+1}}\big\|(h_\varepsilon(\rho),\,\mathbf{v})\big\|_{\mathsf{H}^{k+1}}\quad\text{for } k\geqslant 2, \tag{1.7}$$

where $h_\varepsilon(\rho)=\rho+O(\varepsilon\rho^2)$ a la Taylor expansion. Thus, bounding the first time derivative $\partial_t(\rho,\mathbf{v})\sim O(1)$ is loosely equivalent to having $(\nabla\cdot\mathbf{v},\nabla\rho)\sim O(\varepsilon)$. If E_0 is already of $O(1)$, then preparing $\mathrm{E}_{t,0}\sim O(1)$ is equivalent to enforcing ρ_0 to be $O(\varepsilon)$ close to constant and $\mathbf{v}_0$ to be $O(\varepsilon)$ close to incompressibility.

In the final Section 5, we will also prove the following corollary without relying on Leray projection. Instead, the estimate is in terms of the physically relevant time-averages.

Corollary 1.4. *Under the same hypotheses as in the Main Theorem 1.1, there exists constants C^{**},C^{***} that only depends on m, Ω and pressure law $\pi(\cdot)$, so that for all times $T\in[0,T^*/\mathrm{E}_0]$,*

$$\varepsilon\big\|\mathbf{v}-\tilde{\mathbf{v}}\big\|_{\mathsf{H}^{m-3}}(T)+\Big\|\int_0^T\mathbf{v}-\tilde{\mathbf{v}}\Big\|_{\mathsf{H}^{m-3}}\leqslant C^{**}\,\varepsilon^2\,(\mathrm{E}_{t,0}+\mathrm{E}_0^2) \tag{1.8}$$

Moreover, if two scalars $\theta(t,x)$, $\tilde\theta(t,x)$ are transported by $\mathbf{v},\tilde{\mathbf{v}}$ respectively

$$\partial_t\theta+\mathbf{v}\cdot\nabla\theta=0,$$
$$\partial_t\tilde\theta+\tilde{\mathbf{v}}\cdot\nabla\tilde\theta=0,$$

subject to the same initial data $\theta_0=\tilde\theta_0\in\mathsf{H}^m(\Omega)$, then

$$\sup_{[0,T^*/\mathrm{E}_0]}\|\theta-\tilde\theta\|_{\mathsf{H}^{m-3}}\leqslant C^{***}\,\varepsilon^2\,(\mathrm{E}_{t,0}+\mathrm{E}_0^2)\,\|\mathcal{P}\mathbf{v}_0\|_{\mathsf{H}^m}\|\theta_0\|_{\mathsf{H}^m}$$

Apparently, by (1.4), the same $O(\varepsilon^2)$ accuracy holds true if we approximate the velocity by $\mathcal{P}\mathbf{v}$.

Note in (1.8) the gained ε factor thanks to time averaging. Also, it is $O(\varepsilon^2)$ accurate *point-wise in time* to use both $\tilde{\mathbf{v}}$ and $\mathcal{P}\mathbf{v}$ approximations for $\mathbf{v}$ in the transport equation.

1.2. **Literature.** There have been numerous results regarding the singular limits of compressible Euler equations and other fluid equations in various settings. None of them has shown the $O(\varepsilon^2)$ accuracy. There have been studies regarding $O(\varepsilon^2)$ corrections to the incompressible approximation (e.g. [10] for periodic domains) – our result here confirms that such correction is in fact zero for Euler equations in general spatial domains. We point to two survey papers for some comprehensive lists of references: [21] with emphases on hyperbolic PDEs and homogenization in mixed; [15] with emphases on viscous fluids and weak solutions. To mention only a few earliest works in terms of well-prepared data, we refer to [8, 9, 14, 3, 13, 22]. In a closely related paper [5], the bounded derivative method is applied to numerical schemes from geophysical applications. Well-prepared conditions on initial data were later removed for problems in the whole space ([25]), in an exterior domain ([11, 12]) and in a torus ([20]). These arguments more or less rely on use of Fourier analysis and/or dispersive nature of the underlying wave equations.

Singular limit problems in a bounded spatial domain, on the other hand, remain much less studied. In [18], Schochet proved the same low-Mach-number limit with solid-wall boundary condition and well-prepared initial data. A main challenge in this setting is the presence of characteristic boundary. Rauch elaborated in [16] that, near the boundary, only estimates along tangential directions are available. In [17], Secchi proved the strong convergence of $\mathcal{P}\mathbf{v}$ for 3D Euler equations with ill-prepared initial data, without obtaining convergence rates. Very recently, B. Cheng proved $O(\varepsilon)$ convergence rate for ill-prepared data in [6]. The time-averaging technique used there inspired this current study; also see Cheng & Mahalov [7] for time-averaging applied to geophysical models on a sphere.

1.3. **Formulations.** Further to (1.5), we rewrite (1.1) in terms of unknown pair $(\rho,\, \mathbf{v})$,

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = -\varepsilon^{-1}\nabla\cdot\mathbf{v} \tag{1.9a}$$

$$\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + h_\varepsilon(\rho)\nabla\rho = -\varepsilon^{-1}\nabla\rho \tag{1.9b}$$

$$\text{subject to} \quad \mathbf{v}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0. \tag{1.10}$$

$$\text{Here,} \quad h_\varepsilon(\rho) := \left(\frac{\pi'(1+\varepsilon\rho)}{1+\varepsilon\rho} - 1\right)\frac{1}{\varepsilon}. \tag{1.11}$$

By (1.2) and Taylor expansion, $h_\varepsilon(\rho) = (\pi''(1) - \pi'(1))\rho + O(\varepsilon\rho^2)$. In a more compact form,

$$\partial_t\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix} + \mathcal{N}\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix} = -\varepsilon^{-1}\mathcal{L}\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix}$$

with anti-symmetric operator

$$\mathcal{L}\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix} := \begin{pmatrix}\nabla\cdot\mathbf{v}\\ \nabla\rho\end{pmatrix}. \tag{1.12}$$

For purely aesthetic reasons, we will use notations $\mathcal{L}\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix}$ and $\mathcal{L}(\rho,\mathbf{v})$ interchangeably.

In light of (1.7), the boundedness of first time derivative $\partial_t(\rho,\mathbf{v}) \sim O(1)$ and the boundedness of $\mathcal{L}(\rho,\mathbf{v}) \sim O(\varepsilon)$ are closely related.

Later in the article, we will apply the Leray projection to the compressible system, which effectively annihilates $\varepsilon^{-1}\mathcal{L}$. This gives a decomposition of the solution space into slow and fast subspaces, and correspondingly a decomposition of the compressible system into a slow one governing the incompressible motions and a fast one governing the rapidly oscillating acoustic waves.

The slow dynamics is very closely related to the vorticity equations. Apply $\nabla\times$ to (1.9b) so that the cancellation property $\nabla\times\nabla = 0$ yields the equation for vorticity $\omega := \nabla \times \mathbf{v}$

$$\partial_t\omega + \mathbf{v}\cdot\nabla\omega + (\nabla\cdot\mathbf{v})\omega = 0 \qquad \text{in 2D,} \tag{1.13a}$$

$$\text{and} \quad \partial_t\omega - \omega\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\omega + (\nabla\cdot\mathbf{v})\omega = 0 \qquad \text{in 3D} \tag{1.13b}$$

without $O(\varepsilon^{-1})$ terms contributing to $\partial_t\omega$. Thus, vorticity ω evolves on a slow time scale. We have to alert that the vortical dynamics may not contain all the information of the slow dynamics if the spatial domain is not contractible. Take Remark 2.2 for example. The point vortex restricted in a ring-shaped domain forms a steady solution of incompressible Euler equations; together with an axisymmetric profile properly chosen for ρ, it also solves the compressible system. However, both vorticity and divergence are identically zero; therefore the vorticity equation, even combined with the divergence equation, does not retain all dynamical information required to solve for $\mathbf{v}$.

Nevertheless, the vorticity equation has the simple structure of a transport equation and does not require solving the extra pressure variable, so it is very widely used in practice. We will also take advantage of its nice structure to simplify estimate proofs in Section 3, 4.

The rest of this article is organized as follows. In Section 2, we define the Leray projection, prove its properties using elliptic PDE theory and use it to extract the slow dynamics from the compressible system. Section 3 contains probably the most novelty. It explains how to use the time-averaging technique to obtain pointwise-in-time error estimates. A decoupling property of the compressible system allows us gain an extra ε factor as long as the data is well-prepared. Without concerns for boundary, the reader does not have to rely on the next Section 4. Here, we use mixed norms to obtain ε-independent bounds on the sizes of the solution and its first time derivative. The methods used here are partially similar to those of [19], but we work with both ill-prepared and well-prepared data. The final Section 5 completes proofs of the Main Theorem 1.1, Corollary 1.4 and makes further comments.

Remark 1.5. The Sobolev inequalities used throughout this article can be summarized as follows. Given functions $f_1(x), f_2(x), \ldots, f_j(x)$ and a product $(\partial_x^{\beta_1} f_1)\,(\partial_x^{\beta_2} f_2)\ldots(\partial_x^{\beta_j} f_j)$ with multi-indices $\beta_1, ..., \beta_j$ satisfying

$$|\beta_1 + ... + \beta_j| \leqslant 2m - 3 \qquad \text{and} \quad m \geqslant |\beta_1| \geqslant |\beta_2| \geqslant \ldots \geqslant |\beta_j|, \tag{1.14}$$

we have $\big\|\prod_{i=1}^j \partial_x^{\beta_i} f_i\big\|_{\mathsf{L}^2} \leqslant \big\|\partial_x^{\beta_1} f_1\big\|_{\mathsf{L}^2} \prod_{i=2}^j \big|\partial_x^{\beta_i} f_i\big|_{\mathsf{L}^\infty(\Omega)}$ and therefore, by Sobolev inequalities and Ω being two or three dimensional,

$$\Big\|\prod_{i=1}^{j} \partial_x^{\beta_i} f_i\Big\|_{\mathsf{L}^2} \lesssim \prod_{i=1}^{j}\Big\|f_i\Big\|_{\mathsf{H}^m} \qquad \text{if (1.14) holds.} \tag{1.15}$$

Here and below, the "similarly less than" notation $a \lesssim b$ is understood as

$$a \leqslant Cb \text{ for a constant } C \text{ solely depending on } m, \Omega \text{ and pressure law } \pi(\cdot)$$

2. Projections onto Slow and Fast Subspaces

Define X to be the space of incompressible velocity fields subject to solid-wall boundary condition,

$$\mathsf{X} := \mathsf{L}^2 \text{ closure of } \left\{ \mathbf{v}^{\rm inc} \in \mathscr{C}^1(\overline{\Omega}) \,\middle|\, \nabla\cdot\mathbf{v}^{\rm inc} = 0,\ \mathbf{v}^{\rm inc}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0 \right\}$$

Define $\mathcal{P}$ as the L^2-orthogonal projection onto X so that, for any $\mathbf{v}, \mathbf{v}' \in \mathsf{L}^2(\Omega)$,

$$\mathcal{P}^2\mathbf{v} = \mathcal{P}\mathbf{v} \in \mathsf{X}, \tag{2.1a}$$

$$\int_\Omega (\mathbf{v} - \mathcal{P}\mathbf{v})\cdot(\mathcal{P}\mathbf{v}') = 0. \tag{2.1b}$$

In fact, $\mathcal{P}$ is the classical Leray projection subject to solid-wall boundary condition. Then, define

$$\mathcal{Q} := 1 - \mathcal{P}.$$

These projections can be characterized conveniently by elliptic PDE as follows.

Proposition 2.1. For any $\mathbf{v} \in \mathsf{H}^k(\Omega)$ with $k \geqslant 1$, we have

$$\mathcal{Q}\mathbf{v} = \nabla\phi \in \mathsf{H}^k(\Omega) \quad \text{with} \quad \phi \text{ solving} \begin{cases} \Delta\phi = \nabla\cdot\mathbf{v} & \text{in } \Omega \\ \nabla\phi\cdot\vec{\mathsf{n}} = \mathbf{v}\cdot\vec{\mathsf{n}} & \text{in } \partial\Omega \end{cases} \tag{2.2}$$

ϕ is unique up to an added constant and thus $\mathcal{Q}\mathbf{v}$ is unique.

Here and below, we always assume $k \geqslant 1$ whenever the trace of an $\mathsf{H}^k(\Omega)$ function is involved.

Proof. The solvability of (2.2) follow from standard elliptic PDE theory (e.g. [23], Ch. 5, Prop. 7.7). It suffices to verify $(1 - \mathcal{Q})$ with $\mathcal{Q}$ given in (2.2) equals $\mathcal{P}$ which is uniquely defined in (2.1a), (2.1b).

Obviously, (2.2) implies $(1 - \mathcal{Q})\mathbf{v} \in \mathsf{X}$ for any $\mathbf{v} \in \mathsf{H}^k(\Omega)$. Then, for any $\mathbf{v}^{\rm inc} \in \mathsf{X} \bigcap \mathsf{H}^k(\Omega)$, the uniqueness (up to a harmless constant) of ϕ in (2.2) implies that $\mathcal{Q}\mathbf{v}^{\rm inc} = 0$. More precisely, $(1 - \mathcal{Q})$ is a projection so that, for any $\mathbf{v} \in \mathsf{H}^k$, $(1 - \mathcal{Q})^2\mathbf{v} = (1 - \mathcal{Q})\mathbf{v}$.

It remains to show $\mathcal{Q}\mathbf{v}$ is L^2-orthogonal to any $\mathbf{v}^{\rm inc} \in \mathsf{X}$. Since $\mathsf{X} \bigcap \mathscr{C}^\infty(\overline{\Omega})$ is a dense subset of X in the L^2 topology, it suffices to only consider smooth $\mathbf{v}^{\rm inc}$ as a typical element of X. Then,

$$\int_\Omega \mathbf{v}^{\rm inc}\cdot(\mathcal{Q}\mathbf{v})\,dx = \int_\Omega \mathbf{v}^{\rm inc}\cdot\nabla\phi\,dx \overset{(a)}{=} \int_\Omega \nabla\cdot(\mathbf{v}^{\rm inc}\phi)\,dx \overset{(b)}{=} \int_{\partial\Omega} \vec{\mathsf{n}}\cdot\mathbf{v}^{\rm inc}\phi\,ds \overset{(c)}{=} 0,$$

where $(a), (c)$ are due to the definition of $\mathbf{v}^{\rm inc}$ and (b) due to the divergence theorem. □

This proposition shows that $\mathcal{Q}\mathbf{v}$ is always a perfect gradient and therefore its curl vanishes.

$$\nabla\times(\mathcal{Q}\mathbf{v}) = 0, \qquad \nabla\times(\mathcal{P}\mathbf{v}) = \nabla\times\mathbf{v}. \tag{2.3}$$

Compare them to the definitional facts,

$$\nabla\cdot(\mathcal{P}\mathbf{v}) = 0, \qquad \nabla\cdot(\mathcal{Q}\mathbf{v}) = \nabla\cdot\mathbf{v}.$$

This is to say, $\mathcal{P}\mathbf{v}$ contains all the information of the vorticity (but not necessarily vice versa!) and $\mathcal{Q}\mathbf{v}$ contains all the information of the divergence.

From here on, we will interchangeably use $\mathbf{v}^P$ for $\mathcal{P}\mathbf{v}$ and $\mathbf{v}^Q$ for $\mathcal{Q}\mathbf{v}$.

2.1. **Boundedness of Projections and Elliptic Estimates.** Operators $\mathcal{L}, \nabla\times, \mathcal{P}, \mathcal{Q}$ are all elliptic operators with nontrivial kernels, and we will employ elliptic estimates with boundary conditions to estimate them. The papers of Agmon, Douglis and Nirenberg [1], [2] establish a "Complementing Boundary Condition" that is necessary and sufficient for the solution operator of a s-th order elliptic PDE system to be $\mathscr{C}^k \to \mathscr{C}^{k+s}$ and $\mathsf{H}^k \to \mathsf{H}^{k+s}$. To treat the Euler equations (e.g. [4]), only a particular case is used: for any velocity field $\mathbf{v}$ with enough regularity,

$$\|\mathbf{v}\|_{\mathsf{H}^k} \lesssim \|\nabla\cdot\mathbf{v}\|_{\mathsf{H}^{k-1}} + \|\nabla\times\mathbf{v}\|_{\mathsf{H}^{k-1}} + \|\mathbf{v}\|_{\mathsf{L}^2}, \quad \text{if } \mathbf{v}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0 \text{ and } k \geqslant 1. \tag{2.4}$$

Remark 2.2. We alert that the $\|\mathbf{v}\|_{\mathsf{L}^2}$ term above may not be dropped if, for example, Ω is not contractible. Consider a ring-shaped domain $\Omega = \{(x,y)\,|\,1 < x^2+y^2 < 2\}$ and a point vortex $\mathbf{v} = \nabla^\perp \ln|x^2+y^2|$. Then, $\nabla\cdot\mathbf{v} = \nabla\times\mathbf{v} = 0$ and $\mathbf{v}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$, but $\mathbf{v} \neq 0$.

Now, set $\mathbf{v} = \mathbf{v}^P$ in (2.4) and use the facts that $\nabla\cdot\mathbf{v}^P = 0$, $\mathbf{v}^P\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$ to obtain, without boundary condition on $\mathbf{v}$,

$$\|\mathbf{v}^P\|_{\mathsf{H}^k} \lesssim \|\nabla\times\mathbf{v}^P\|_{\mathsf{H}^{k-1}} + \|\mathbf{v}^P\|_{\mathsf{L}^2} \quad \text{for } k \geqslant 1. \tag{2.5}$$

This gives a bound on the high norms of $\mathbf{v}^P$ using the high norms of $\nabla\times\mathbf{v}^P = \nabla\times\mathbf{v}$ (by (2.3)) and the L^2 norm of $\mathbf{v}^P$.

Therefore, $\mathcal{P}$, $\mathcal{Q}$ are bounded operators in $\mathsf{H}^k(\Omega)$ regardless of boundary condition,

$$\|\mathcal{P}\mathbf{v}\|_{\mathsf{H}^k} + \|\mathcal{Q}\mathbf{v}\|_{\mathsf{H}^k} \lesssim \|\mathbf{v}\|_{\mathsf{H}^k} \quad \text{for } k \geqslant 0 \tag{2.6}$$

The case of $k = 0$ is due to the definition of $\mathcal{P}$ and the Pythagorean theorem. Obvisouly, one can reverse the direction of this inequality due to $\mathcal{P} + \mathcal{Q} = 1$.

Similar to (2.5), we can bound the norms of $\mathbf{v}^Q$ using norms of $\nabla\cdot\mathbf{v}^Q = \nabla\cdot\mathbf{v}$. In fact, set $\mathbf{v} = \mathbf{v}^Q$ in (2.4), use (2.3) and the fact $\mathbf{v}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = \mathbf{v}^Q\cdot\vec{\mathsf{n}}\big|_{\partial\Omega}$ to obtain

$$\|\mathbf{v}^Q\|_{\mathsf{H}^k} \lesssim \|\nabla\cdot\mathbf{v}^Q\|_{\mathsf{H}^{k-1}} + \|\mathbf{v}^Q\|_{\mathsf{L}^2} \quad \text{if} \quad \mathbf{v}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0 \tag{2.7}$$

A notable feature of the above inequality is, unlike in Remark 2.2, the L^2 norm term above can be dropped regardless of topology of the spatial domain.

Proposition 2.3. Let $k \geqslant 1$. For any $\mathbf{v} \in \mathsf{H}^k(\Omega)$ subject to $\mathbf{v}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$,

$$\|\mathbf{v}^Q\|_{\mathsf{H}^k} \lesssim \|\nabla\cdot\mathbf{v}^Q\|_{\mathsf{H}^{k-1}} = \|\nabla\cdot\mathbf{v}\|_{\mathsf{H}^{k-1}}.$$

Proof. Having (2.7) established, it suffices to show $\|\mathbf{v}^Q\|_{\mathsf{L}^2} \leqslant C_\Omega \|\nabla\cdot\mathbf{v}\|_{\mathsf{L}^2}$.

Take any smooth test velocity field $\mathbf{v}'$. By Leray projection and Proposition 2.1, $\mathbf{v}' = \nabla\psi + \mathcal{P}\mathbf{v}'$. Then, we know $\int_\Omega \mathbf{v}^Q\cdot\mathbf{v}' = \int_\Omega \mathbf{v}\cdot\nabla\psi$ by the orthogonality of $\mathcal{P}$, $\mathcal{Q}$, and $\|\nabla\psi\|_{\mathsf{L}^2} \leqslant \|\mathbf{v}'\|_{\mathsf{L}^2}$ by the Pythagorean theorem. Thus, a sufficient condition for the desired inequality $\int_\Omega \mathbf{v}^Q\cdot\mathbf{v}' \leqslant C_\Omega \|\nabla\cdot\mathbf{v}\|_{\mathsf{L}^2}\|\mathbf{v}'\|_{\mathsf{L}^2}$ is,

$$\int_\Omega \mathbf{v}\cdot\nabla\psi \leqslant C_\Omega \|\nabla\cdot\mathbf{v}\|_{\mathsf{L}^2}\|\nabla\psi\|_{\mathsf{L}^2} \tag{2.8}$$

Since Ω is compact, this is done by choosing ψ with zero mean, applying the divergence theorem on the LHS and then applying the Hölder and Poincaré inequalities. □

2.2. **Slow and Fast dynamics.** Next, we want to extract the slow dynamics from (1.9) in the form of an evolutionary system that is free of $O(\varepsilon^{-1})$ time derivative. One could apply $\mathcal{K}$ to cancel $\varepsilon^{-1}\mathcal{L}$ and get the vorticity equations (1.13a) or (1.13b), but the comments thereafter suggests that the vortical dynamics does not necessary retain all the information of the slow dynamics. For such a reason, we will instead apply $\mathcal{P}$ on (1.9b). The intuition is, if we define $\mathcal{P}^\sharp\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix} := \begin{pmatrix}0\\ \mathcal{P}\mathbf{v}\end{pmatrix}$ so that $\mathcal{L}\mathcal{P}^\sharp = 0$ and $\mathcal{L}$, $\mathcal{P}^\sharp$ are (skew-)symmetric, then hopefully they commute $\mathcal{P}^\sharp\mathcal{L} = 0$ and therefore applying $\mathcal{P}^\sharp$ to (1.9) will eliminate the $\varepsilon^{-1}\mathcal{L}$ term. This is easily proved via duality argument if $\partial\Omega = \varnothing$, and can still be established in general provided the boundary conditions are taken care of.

Proposition 2.4. For any scalar $\rho \in \mathsf{H}^1(\Omega)$,

$$\mathcal{P}(\nabla\rho) = 0.$$

Proof. By definition, we have $\nabla\cdot(\mathcal{P}\mathbf{v}') = 0$ and $(\mathcal{P}\mathbf{v}')\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$ for any $\mathbf{v}' \in \mathsf{H}^1(\Omega)$. Therefore,

$$0 = \int_\Omega \rho\nabla\cdot(\mathcal{P}\mathbf{v}') = -\int_\Omega (\nabla\rho)\cdot(\mathcal{P}\mathbf{v}').$$

Then, apply (2.1b) twice,

$$0 = \int_\Omega (\nabla\rho)\cdot(\mathcal{P}\mathbf{v}') = \int_\Omega \mathcal{P}(\nabla\rho)\cdot(\mathcal{P}\mathbf{v}') = \int_\Omega \mathcal{P}(\nabla\rho)\cdot(\mathbf{v}')$$

Since $\mathsf{H}^1(\Omega)$ is dense in $\mathsf{L}^2(\Omega)$, we have $\int_\Omega \mathcal{P}(\nabla\rho)\cdot(\mathbf{v}') = 0$ for any $\mathbf{v}' \in \mathsf{L}^2$. Thus, $\mathcal{P}(\nabla\rho) = 0$. □

Now, we apply $\mathcal{P}$ on (1.9b) to obtain the slow dynamics,

$$-\partial_t\mathbf{v}^P = \mathcal{P}\big(\mathbf{v}\cdot\nabla\mathbf{v}\big) = \mathcal{P}\big(\mathbf{v}^P\cdot\nabla\mathbf{v}^P\big) + \mathcal{P}\big(\mathbf{v}^P\cdot\nabla\mathbf{v}^Q + \mathbf{v}^Q\cdot\nabla\mathbf{v}^P\big) + \mathcal{P}\big(\mathbf{v}^Q\cdot\nabla\mathbf{v}^Q\big) \tag{2.9}$$

Here, we used the fact that $h_\varepsilon(\rho)\nabla\rho$ is a perfect gradient, therefore by the above lemma, $\mathcal{P}(h_\varepsilon(r)\nabla r) = 0$. Now, apply $\mathcal{Q}$ to (1.9b) and keep (1.9a) as is to arrive at the fast dynamics,

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = -\varepsilon^{-1}\nabla\cdot\mathbf{v} \tag{2.10a}$$

$$\partial_t\mathbf{v}^Q + \mathcal{Q}(\mathbf{v}\cdot\nabla\mathbf{v}) + h_\varepsilon(\rho)\nabla\rho = -\varepsilon^{-1}\nabla\rho \tag{2.10b}$$

This way, the original system is decomposed into (2.10) governing the fast variables $\rho, \mathbf{v}^Q$ with $O(\varepsilon^{-1})$ coefficients, and (2.9) governing the slow variable $\mathbf{v}^P$ whose first time derivative is $O(1)$. Note density component is identically zero in the slow variable.

A key **decoupling** property is that the "fast-fast" product in (2.9) vanishes completely.

Lemma 2.5. *For any* $\mathbf{v}^Q \in \mathsf{H}^2(\Omega)$,

$$\mathcal{P}\big(\mathbf{v}^Q\cdot\nabla\mathbf{v}^Q\big) = 0.$$

Proof. By Proposition 2.1, there exists a scalar function ϕ so that $\mathbf{v}^Q = \nabla\phi$. Then, $\mathbf{v}^Q\cdot\nabla\mathbf{v} = (\nabla\phi)\cdot\nabla(\nabla\phi) = \frac{1}{2}\nabla|\nabla\phi|^2$. So, by Proposition 2.4, $\mathcal{P}\big(\mathbf{v}^Q\cdot\nabla\mathbf{v}^Q\big)$ vanishes. □

Therefore, by defining $\mathrm{B}\big(\mathbf{v}_1, \mathbf{v}_2\big) := \mathbf{v}_1\cdot\nabla\mathbf{v}_2 + \mathbf{v}_2\cdot\nabla\mathbf{v}_1$, we rewrite (2.9) as

$$-\partial_t\mathbf{v}^P = \mathcal{P}\big(\mathbf{v}^P\cdot\nabla\mathbf{v}^P\big) + \mathcal{P}\mathrm{B}\big(\mathbf{v}^P, \mathbf{v}^Q\big) \tag{2.11}$$

If we set $\mathbf{v}^Q \equiv 0$ by brutal force, this equation is reduced to the incompressible Euler equations.

Proposition 2.6. Consider a velocity field $\tilde{\mathbf{v}} \in \mathscr{C}^2([0,T^*] \times \Omega)$. Then, $\tilde{\mathbf{v}}$ solves the *actual* incompressible Euler equations (1.3a), (1.3b), (1.3c) if and only if it solves

$$- \partial_t \tilde{\mathbf{v}} = \mathcal{P}\big(\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\big) \tag{2.12}$$

with the same initial data $\tilde{\mathbf{v}}_0$ satisfying (1.3d).

Proof. " Only if ". Assume $\tilde{\mathbf{v}}$ solves (1.3a), (1.3b), (1.3c). Apply $\mathcal{P}$ on (1.3a). On the LHS, because $\partial_t\tilde{\mathbf{v}}$ also satisfies (1.3b), (1.3c), we have $\mathcal{P}(\partial_t\tilde{\mathbf{v}}) = \partial_t\tilde{\mathbf{v}}$ by the definition of $\mathcal{P}$. Also, we have $\mathcal{P}(\nabla q) = 0$ by Proposition 2.4. Therefore, applying $\mathcal{P}$ on (1.3a) gives us exactly (2.12).

" If ". Assume $\tilde{\mathbf{v}}$ solves (2.12), which can be recast as

$$\partial_t\tilde{\mathbf{v}} + \tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}} - \mathcal{Q}\big(\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\big) = 0$$

By Proposition 2.1, there exists a scalar ϕ so that $\mathcal{Q}\big(\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\big) = \nabla\phi$ and therefore the above equation is of the same form as (1.3a) with pressure $q = -\phi$. Also, since $\nabla\cdot\tilde{\mathbf{v}}_0 = 0$ and taking divergence on (2.12) gives $\partial_t(\nabla\cdot\tilde{\mathbf{v}}) = 0$, we have $\nabla\cdot\tilde{\mathbf{v}} = 0$, i.e. (1.3b) satisfied for all times in $[0,T^*]$. Finally, restrict (2.12) on $\partial\Omega$, take its dot product with $\vec{\mathsf{n}}$ and use the definition of $\mathcal{P}$ to obtain $\partial_t(\tilde{\mathbf{v}}\cdot\vec{\mathsf{n}}) = 0$ on $\partial\Omega$. Since (1.3d) ensures $\tilde{\mathbf{v}}_0\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$, we have $\tilde{\mathbf{v}}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$, i.e. (1.3c) validated for all times in $[0,T^*]$. □

Problem is, in order to estimate the difference of (2.11) and (2.12), how can we bound the "slow-fast" term $\mathcal{P}\mathrm{B}\big(\mathbf{v}^P,\mathbf{v}^Q\big)$ by $O(\varepsilon^2)$? Because of nonlinearity, the slow subspace $\ker\mathcal{L}$ is not invariant under the coupled slow-fast dynamics. This means, even with initial data $\mathbf{v}_0^Q = 0$ and $\mathbf{v}_0^P \sim O(1)$, nonlinear coupling can lead to $\mathbf{v}^Q \sim O(\varepsilon)$ in later times.

To this end, we bring to focus the key idea of this article: a generic compressible solution $\mathbf{v}$, without being $O(\varepsilon^2)$ pointwise in time, can still be $O(\varepsilon^2)$ in terms of its *time-averages* as long as $\mathbf{v}_0^Q \sim O(\varepsilon)$ initially. Such estimate in turn will suffice to make $(\mathbf{v}^P - \tilde{\mathbf{v}}) \sim O(\varepsilon^2)$ pointwise in time. This is the subject of next section.

3. Pointwise-in-time Error Estimates Using Time-averages

In this section, we demonstrate in Lemma 3.1 and Theorem 3.3 the crucial role of time-averages in estimating $(\mathbf{v}^P - \tilde{\mathbf{v}})$ **pointwise in time**. For brevity, throughout this section, we assume solutions $\rho,\mathbf{v},\tilde{\mathbf{v}} \in \mathscr{C}([0,T],\mathsf{H}^m(\Omega))$ for integer $m \geqslant 4$.

Define the time-averaging (indeed, integrating) operator

$$\overline{\mathbf{v}}(T,x) := \int_0^T \mathbf{v}(t,x)\,dt$$

First, estimate the slow-fast product $\mathrm{B}\big(\mathbf{v}^P,\mathbf{v}^Q\big)$ of (2.11), which is the extra term compared to the incompressible system (2.12). By the product rule, $\mathrm{B}\big(\mathbf{v}^P,\mathbf{v}^Q\big) = \partial_t\mathrm{B}\big(\mathbf{v}^P,\overline{\mathbf{v}^Q}\big) - \mathrm{B}\big(\partial_t\mathbf{v}^P,\overline{\mathbf{v}^Q}\big)$, so we apply time averaging,

$$\begin{aligned}\overline{\mathrm{B}\big(\mathbf{v}^P,\mathbf{v}^Q\big)}(t) &= \mathrm{B}\big(\mathbf{v}^P,\overline{\mathbf{v}^Q}\big)\Big|_0^t - \int_0^t \mathrm{B}\big(\partial_t\mathbf{v}^P,\overline{\mathbf{v}^Q}\big)\\ &= \mathrm{B}\big(\mathbf{v}^P,\overline{\mathbf{v}^Q}\big)\Big|_0^t + \int_0^t \mathrm{B}\big(\mathcal{P}(\mathbf{v}^P\cdot\nabla\mathbf{v}^P) + \mathcal{P}\mathrm{B}\big(\mathbf{v}^P,\mathbf{v}^Q\big),\ \overline{\mathbf{v}^Q}\big)\end{aligned}$$

where $\partial_t \mathbf{v}^P$ was replaced via the slow dynamics (2.11). Apply the boundedness of $\mathcal{P}$ in (2.6) and the Sobolev inequalities (with $m \geqslant 4$),

$$\sup_{[0,T]} \|\overline{\mathrm{B}\big(\mathbf{v}^P, \mathbf{v}^Q\big)}\|_{\mathsf{H}^{m-2}} \lesssim \sup_{[0,T]} \Big\{ \|\overline{\mathbf{v}^Q}\|_{\mathsf{H}^{m-1}} \|\mathbf{v}^P\|_{\mathsf{H}^m} (1 + T\|\mathbf{v}\|_{\mathsf{H}^m}) \Big\} \tag{3.1}$$

Then, as the common factor in the above RHS, the spatial norms of $\overline{\mathbf{v}^Q}$ are expected to vanish at $O(\varepsilon^2)$. In fact, by Proposition 2.3,

$$\|\overline{\mathbf{v}^Q}\|_{\mathsf{H}^{m-1}} \lesssim \|\overline{\nabla\cdot\mathbf{v}}\|_{\mathsf{H}^{m-2}} \lesssim \|\nabla(\overline{\nabla\cdot\mathbf{v}})\|_{\mathsf{H}^{m-3}}$$

where the last estimate is due to the Poincaré inequality and the zero spatial mean of $\nabla\cdot\mathbf{v}$. Now, replace $\nabla(\overline{\nabla\cdot\mathbf{v}})$ on the RHS using the continuity equation (2.10a),

$$\|\overline{\mathbf{v}^Q}\|_{\mathsf{H}^{m-1}}(T) \lesssim \|\nabla(\overline{\nabla\cdot\mathbf{v}})\|_{\mathsf{H}^{m-3}} = \varepsilon \left\| \nabla(\rho(T,\cdot) - \rho_0) + \nabla \int_0^T \nabla\cdot(\rho\mathbf{v}) \right\|_{\mathsf{H}^{m-3}}. \tag{3.2}$$

Already, we have gained an ε factor in the bound of $\overline{\mathbf{v}^Q}$. But this is not enough for $O(\varepsilon^2)$. A key **decoupling property** here is that the quadratic terms in the RHS contain no "slow-slow" product which would've made $O(\|\mathbf{v}^P\|^2)$ contribution. Instead, everything in the RHS of (3.2) has a factor from the fast variables $\rho, \mathbf{v}^Q$, or more precisely $(\nabla\cdot\mathbf{v}^Q, \nabla\rho) = \mathcal{L}(\rho, \mathbf{v})$. Thus, combine (3.2) and Sobolev inequalities (with $m \geqslant 4$) to get

$$\|\overline{\mathbf{v}^Q}\|_{\mathsf{H}^{m-1}}(T) \lesssim \varepsilon \sup_{[0,T]} \Big\{ \big\|\mathcal{L}(\rho,\mathbf{v})\big\|_{\mathsf{H}^{m-2}} (1 + T\|(\rho,\mathbf{v})\|_{\mathsf{H}^{m-1}}) \Big\}. \tag{3.3}$$

Plug it into (3.1) to prove the following lemma.

Lemma 3.1. *Let integer $m \geqslant 4$. Suppose the compressible Euler equations (1.9) admit solution $(\rho, \mathbf{v}) \in \mathscr{C}([0,T], \mathsf{H}^m(\Omega))$. Then, there exists a constant C solely depending on m, Ω, so that*

$$\sup_{[0,T]} \|\overline{\mathrm{B}\big(\mathbf{v}^P, \mathbf{v}^Q\big)}\|_{\mathsf{H}^{m-2}} \leqslant C\,\varepsilon \sup_{[0,T]} \Big\{ \big\|\mathcal{L}(\rho,\mathbf{v})\big\|_{\mathsf{H}^{m-2}} (1 + T\|(\rho,\mathbf{v})\|_{\mathsf{H}^{m-1}}) \|\mathbf{v}^P\|_{\mathsf{H}^m} (1 + T\|\mathbf{v}\|_{\mathsf{H}^m}) \Big\}$$

Meanwhile, estimate (1.7) relates $\mathcal{L}(\rho,\mathbf{v}) \sim O(\varepsilon)$ to $\partial_t(\rho,\mathbf{v}) \sim O(1)$. Thus, we will show in Section 4, by preparing initial data so that $\big\|\partial_t(\rho,\mathbf{v})\big\|_{\mathsf{H}^{m-1}} \sim O(1)$ at $t = 0$, it will remain $O(1)$ for finite times, eventually making the RHS in the above lemma $O(\varepsilon^2)$.

Having bounded the time-average of the "extra term" $\mathrm{B}\big(\mathbf{v}^P, \mathbf{v}^Q\big)$ in the compressible system (2.11), we move on to show how it helps us to estimate $(\mathbf{v}^P - \widetilde{\mathbf{v}})$. Before the main result in Theorem 3.3, we prove following technical lemma showing that error propagation in a (bilinear) PDE system heavily relies on the time-averages of its coefficients.

Lemma 3.2. *Assume $v_1, v_2, u_1, u_2, \gamma_1, \gamma_2 \in \mathscr{C}([0,T], \mathsf{H}^k(\Omega))$ for $k \geqslant 1$ and satisfy PDE systems*

$$\partial_t u_i + \mathrm{b}(v_i,\, u_i) = \gamma_i, \qquad i = 1, 2, \qquad \textit{subject to the same initial data} \tag{3.4}$$

Assume also bilinear operator b *satisfies*

$$\langle u, \mathrm{b}(v,u)\rangle_{\mathsf{H}^{k-1}} \leqslant C_1 \|v\|_{\mathsf{H}^k} \|u\|^2_{\mathsf{H}^{k-1}} \tag{3.5a}$$

$$\|\mathrm{b}(v,u)\|_{\mathsf{H}^{k-1}} \leqslant C_2 \|v\|_{\mathsf{H}^k} \|u\|_{\mathsf{H}^k} \tag{3.5b}$$

for any v as linear combination of v_1, v_2 and any u as linear combination of u_1, u_2, $\overline{\gamma_1 - \gamma_2}$.

Then, with $M := \sup_{[0,T]}\big\{\|v_1\|_{\mathsf{H}^k} + \|u_2\|_{\mathsf{H}^k}\big\}$ *and* $C := \sup\{C_1, C_2\}$,

$$\sup_{[0,T]}\|u_1-u_2\|_{\mathsf{H}^{k-1}} \leqslant e^{CMT}\sup_{[0,T]}\big\|\overline{\gamma_1-\gamma_2}\big\|_{\mathsf{H}^k} + \big(e^{CMT}-1\big)\sup_{[0,T]}\|v_1-v_2\|_{\mathsf{H}^k}$$

Under the same hypotheses, one can also prove

$$\sup_{[0,T]}\|u_1-u_2\|_{\mathsf{H}^{k-2}} \leqslant e^{CMT}\sup_{[0,T]}\Big\{\big\|\overline{\gamma_1-\gamma_2}\big\|_{\mathsf{H}^{k-1}} + \big\|\overline{v_1-v_2}\big\|_{\mathsf{H}^{k-1}}(1+T\|\partial_t u_2\|_{H^{k-1}})\Big\}$$

but we will neither prove nor use it in this article, only proving a simplified version for the last part of Corollary 1.4.

Notice the solution error (u_1-u_2) measured pointwise in time is affected by the source terms γ_i only via the time average $\overline{(\gamma_1-\gamma_2)}$. On the other hand, $|\gamma_1-\gamma_2|$ can still be large. Similar effect comes from (v_1-v_2) as well.

Proof. (Lemma 3.2). Set $i=1,2$ in (3.4) and subtract them to get

$$\partial_t(u_1-u_2) + \mathrm{b}(v_1,\, u_1-u_2) + \mathrm{b}(v_1-v_2,\, u_2) = \gamma_1-\gamma_2$$

Then, define the time integral of the right hand side

$$\xi := \overline{(\gamma_1-\gamma_2)}.$$

and replace the RHS of the previous equation with $\partial_t\xi$, recasting it into

$$\partial_t(u_1-u_2-\xi) + \mathrm{b}(v_1, u_1-u_2-\xi) + \mathrm{b}(v_1,\xi) + \mathrm{b}(v_1-v_2,\, u_2) = 0.$$

It is easy to see that $\partial_t(u_1-u_2-\xi) \in \mathscr{C}([0,T], \mathsf{H}^{k-1}(\Omega))$ due to (3.5). Take its H^{k-1} inner product with $(u_1-u_2-\xi)$ and apply (3.5) to get

$$\frac{1}{2}\frac{d}{dt}\|u_1-u_2-\xi\|^2_{\mathsf{H}^{k-1}} \leqslant C\|u_1-u_2-\xi\|_{\mathsf{H}^{k-1}}(\|v_1\|_{\mathsf{H}^k}+\|u_2\|_{\mathsf{H}^k})$$
$$\Big(\|u_1-u_2-\xi\|_{\mathsf{H}^{k-1}} + \|\xi\|_{\mathsf{H}^k} + \|v_1-v_2\|_{\mathsf{H}^k}\Big)$$

Relax the last two terms $\|\xi\|_{\mathsf{H}^k} + \|v_1-v_2\|_{\mathsf{H}^k}$ to a t-independent $\sup_{[0,T]}(\|\xi\|_{\mathsf{H}^k} + \|v_1-v_2\|_{\mathsf{H}^k})$ and also relax $(\|v_1\|_{\mathsf{H}^k}+\|u_2\|_{\mathsf{H}^k})$ to a t-independent M. Then, integrate this Gronwall's inequality from 0 to T to obtain, noting $u_1(0,\cdot)-u_2(0,\cdot)=\xi(0,\cdot)=0$,

$$\|u_1-u_2-\xi\|_{\mathsf{H}^{k-1}}(T) \leqslant (e^{CMT}-1)\sup_{[0,T]}(\|\xi\|_{\mathsf{H}^k} + \|v_1-v_2\|_{\mathsf{H}^k})$$

By the triangle inequality, this concludes the proof. □

The above lemma can be applied to compare the *vorticity formulations* of (2.11) and (2.12). When H^k estimates are sought in the presence of boundary conditions, it is more convenient to deal with vorticities which are governed by transport equations. There will be no boundary terms containing the highest order spatial derivatives, thanks to the simple (bilinear) structure of vorticity equations. The downside, however, is that the vorticity does not necessarily retain all information of an incompressible velocity field – cf. Remark 2.2.

Apply $\nabla\times$ to the incompressible system (1.3a) and arrive at vorticity equation,

$$\partial_t\widetilde{\omega} + \widetilde{\mathbf{v}}\cdot\nabla\widetilde{\omega} = 0 \qquad \text{in 2D,} \tag{3.6a}$$

$$\partial_t\widetilde{\omega} - \widetilde{\omega}\cdot\nabla\widetilde{\mathbf{v}} + \widetilde{\mathbf{v}}\cdot\nabla\widetilde{\omega} = 0 \qquad \text{in 3D} \tag{3.6b}$$

Then, rewrite its compressible counterpart (1.13) into as similar as possible formulation

$$\partial_t\omega + \mathbf{v}^P\cdot\nabla\omega = -\Big(\mathbf{v}^Q\cdot\nabla\omega + (\nabla\cdot\mathbf{v}^Q)\omega\Big) \qquad \text{in 2D,} \tag{3.7a}$$

$$\partial_t\omega - \omega\cdot\nabla\mathbf{v}^P + \mathbf{v}^P\cdot\nabla\omega = -\Big(\mathbf{v}^Q\cdot\nabla\omega + (\nabla\cdot\mathbf{v}^Q)\omega - \omega\cdot\nabla\mathbf{v}^Q\Big) \qquad \text{in 3D} \tag{3.7b}$$

Here, $\mathbf{v}^P, \mathbf{v}^Q$ are separated to respect the fact that $\widetilde{\mathbf{v}}$ in (3.6) is only a slow variables. There is no separation of ω because, by (2.3), $\omega = \nabla\times\mathbf{v} = \nabla\times\mathbf{v}^P$ is purely slow. Thus, the bilinear terms in the LHS of the above 4 equations only contain "slow-slow" products, and the bilinear terms on the RHS only contain "slow-fast" products.

Theorem 3.3. (Time-averaging estimates) *Consider the slow dynamics (2.11) and the incompressible Euler equations (2.12), subject to the solid-wall boundary condition and the same initial data $\mathbf{v}_0^P = \widetilde{\mathbf{v}}_0$. Suppose $(\rho, \mathbf{v}, \widetilde{\mathbf{v}}) \in \mathscr{C}([0, T^\flat], \mathsf{H}^m(\Omega))$ for $m \geqslant 4$. Then, there exists constants D_1, D_2 only dependent on m, Ω so that,*

$$\sup_{[0,T]} \|\widetilde{\mathbf{v}} - \mathbf{v}^P\|_{\mathsf{H}^{m-3}} \leqslant \varepsilon\, D_1 \sup_{[0,T]}\Big\{\big\|\mathcal{L}(\rho,\mathbf{v})\big\|_{\mathsf{H}^{m-2}}\|\mathbf{v}^P\|_{\mathsf{H}^m}\Big\}$$

for $T \in [0, T^\flat] \bigcap \Big[0,\ D_2/\sup_{[0,T^\flat]}\|(\rho,\mathbf{v},\widetilde{\mathbf{v}})\|_{\mathsf{H}^m}\Big]$

Proof. By the virtue of (2.5), we estimate respectively the high norms of vorticity and the L^2 norm of velocity, in the hope that a factor of $\overline{\mathrm{B}\big(\mathbf{v}^P, \mathbf{v}^Q\big)}$ will appear in both estimates.

(i). Estimates on high norms of $\nabla\times(\widetilde{\mathbf{v}} - \mathbf{v}^P) = \widetilde{\omega} - \omega$. Fit (3.6) and (3.7) into the framework of Lemma 3.2 so that,

$$\begin{aligned} u_1 &= \widetilde{\omega}, & v_1 &= \widetilde{\mathbf{v}} \\ u_2 &= \omega, & v_2 &= \mathbf{v}^P \\ \gamma_1 &= 0, & \gamma_2 &= \text{RHS of (3.7)}, \end{aligned}$$

$$\mathrm{b}(v,u) = \begin{cases} v\cdot\nabla u & \text{in 2D (3.6a), (3.7a)} \\ v\cdot\nabla u - u\cdot\nabla v & \text{in 3D (3.6b), (3.7b)} \end{cases}$$

Note both forms of $\mathrm{b}(\cdot,\cdot)$ satisfy (3.5). In particular, for any v as linear combination of $\mathbf{v}^P, \widetilde{\mathbf{v}}$, we have $\mathrm{b}(\cdot,\cdot)$ satisfies (3.5a) with $k = m-3$, thanks to the solid-wall boundary condition.

For the source terms, γ_2 only consists of "slow-fast" products, so we should expect an estimate of γ_2 similar to that of $\mathrm{B}\big(\mathbf{v}^P, \mathbf{v}^Q\big)$. In fact, the correspondence between (2.11) and (3.7) implies

$$\nabla\times\mathcal{P}\mathrm{B}\big(\mathbf{v}^P, \mathbf{v}^Q\big) = -\gamma_2 \implies \|\overline{\gamma_1 - \gamma_2}\|_{\mathsf{H}^{m-3}} \leqslant \big\|\overline{\mathrm{B}\big(\mathbf{v}^P, \mathbf{v}^Q\big)}\big\|_{\mathsf{H}^{m-2}}$$

Combine it and Lemma 3.2 (with $k = m-3$) to obtain, for any $T \in [0, T^\flat]$,

$$\sup_{[0,T]}\|\widetilde{\omega} - \omega\|_{\mathsf{H}^{m-4}} \leqslant e^{CMT}\sup_{[0,T]}\big\|\overline{\mathrm{B}\big(\mathbf{v}^P, \mathbf{v}^Q\big)}\big\|_{\mathsf{H}^{m-2}} + \big(e^{CMT} - 1\big)\sup_{[0,T]}\big\|\widetilde{\mathbf{v}} - \mathbf{v}^P\big\|_{\mathsf{H}^{m-3}} \tag{3.8}$$

where we relaxed the value of M so that $M = \sup_{[0,T^\flat]}\|(\rho,\mathbf{v},\widetilde{\mathbf{v}})\|_{\mathsf{H}^m}$.

(ii). Estimates on L^2 norm of $(\tilde{\mathbf{v}}-\mathbf{v}^P)$. Fit the compressible Euler equations (2.11) and its incompressible counterpart (2.12) into the framework of Lemma 3.2 so that,

$$\begin{aligned} u_1 = v_1 = \tilde{\mathbf{v}}, &\qquad \gamma_1 = 0 \\ u_2 = v_1 = \mathbf{v}^P, &\qquad \gamma_2 = -\mathcal{P}\mathrm{B}\big(\mathbf{v}^P,\mathbf{v}^Q\big) \\ \mathrm{b}(v,u) &= \mathcal{P}(v\cdot\nabla u). \end{aligned}$$

Note only estimate (3.5a) with $k=1$ needs to be carefully verified with boundary condition. In fact, since $u_i=\mathcal{P}u_i$, $v_i=\mathcal{P}v_i$, $\gamma_i=\mathcal{P}\gamma_i$, we have that any u,v relevant to the conditions below (3.5) are in the image of $\mathcal{P}$, thus making

$$\begin{aligned} \int_\Omega u\cdot\mathcal{P}(v\cdot\nabla u) &= \int_\Omega u\cdot(v\cdot\nabla u) \qquad \ldots \text{by } \mathsf{L}^2\text{-orthogonality of } \mathcal{P},\mathcal{Q} \\ = \int_\Omega \frac{1}{2}v\cdot\nabla|u|^2 &= -\int_\Omega \frac{1}{2}(\nabla\cdot v)|u|^2 = 0 \end{aligned}$$

Therefore, applying Lemma 3.2 with $k=1$, we arrive at

$$\sup_{[0,T]}\|\tilde{\mathbf{v}}-\mathbf{v}^P\|_{\mathsf{L}^2} \leqslant \text{RHS of (3.8)}$$

Combine the above estimate with (3.8) and elliptic estimate (2.5) to arrive at

$$\sup_{[0,T]}\|\tilde{\mathbf{v}}-\mathbf{v}^P\|_{\mathsf{H}^{m-3}} \leqslant c_1\cdot\text{RHS of (3.8)}$$

Now we have "closed" the estimate of $\|\tilde{\mathbf{v}}-\mathbf{v}^P\|_{\mathsf{H}^{m-3}}$ by using itself to bound itself, and also its coefficient on the RHS is less than 1 for short enough time T. Then, rearrange it to get

$$\sup_{[0,T]}\|\tilde{\mathbf{v}}-\mathbf{v}^P\|_{\mathsf{H}^{m-3}} \leqslant \frac{e^{CMT}}{c_1+1-e^{CMT}}\sup_{[0,T]}\big\|\overline{\mathrm{B}\big(\mathbf{v}^P,\mathbf{v}^Q\big)}\big\|_{\mathsf{H}^{m-2}} \leqslant c_2\sup_{[0,T]}\big\|\overline{\mathrm{B}\big(\mathbf{v}^P,\mathbf{v}^Q\big)}\big\|_{\mathsf{H}^{m-2}}$$

as long as $T\leqslant T^\flat$ and $T\leqslant(\ln(c_1/2+1))/(CM)=:D_2/M$.

Combine it with Lemma 3.1, relaxing terms such as $T\|\mathbf{v}\|_{\mathsf{H}^m}$ to D_1 due to the range of T in the last line of the theorem, to conclude the proof. □

4. Estimates Independent of ε

What does it mean to have estimates independent of ε? Basically, we want calculations to be invariant under the hyperbolic scaling $\rho\to c\rho, \mathbf{v}\to c\mathbf{v}, t\to t/c$. In Theorem 4.6 for example, we will show H^m solutions exist at least for time interval of order $1/\|(\rho_0,\mathbf{v}_0)\|_{\mathsf{H}^m}$. During this time interval, the solution's H^m norm is at most inflated by a constant and very importantly, the H^{m-1} norm of its first time derivative is also only inflated by a constant. The latter estimate can be loosely stated as "what starts well-prepared, stays well-prepared". Recall that having $\partial_t(\rho,\mathbf{v})\sim O(1)$ is equivalent to having $(\nabla\cdot\mathbf{v},\nabla\rho)\sim O(\varepsilon)$ as detailed in (1.7)

They main difficulty is, even though $\mathcal{L}$ is skew-self-adjoint, namely $\int_\Omega\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix}\cdot\mathcal{L}\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix}dx=0$ for $\mathbf{v}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega}=0$, it is in general not the case for the spatial derivatives, namely

$$\int_\Omega \partial_x^\beta\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix}\cdot\partial_x^\beta\mathcal{L}\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix}dx\neq 0 \quad \text{if} \quad \beta\neq 0 \text{ and } \partial\Omega\neq\varnothing.$$

This would've introduced $O(\varepsilon^{-1})$ terms in the energy estimate. In addition, a very sophisticated mollification procedure would've been needed in estimating the highest spatial derivatives [16].

On the other hand, we can recruit higher time derivatives since $\partial\Omega$ is static and,

$$\int_\Omega \partial_t^k \begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix} \cdot \partial_t^k \mathcal{L}\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix} dx = 0 \qquad \text{if} \quad \partial_t^k \begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix} \in \mathsf{H}^1(\Omega) \text{ and } \mathbf{v}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0. \tag{4.1}$$

If one considers $[0,T] \times \partial\Omega$ as the lateral boundary of the time-space domain, then the ∂_t derivatives are precisely taken in the tangential directions, resonating with the argument in [16] that, near a characteristic boundary, tangential and normal derivatives are estimated differently.

We first rescale the original system into an equivalent one without explicit dependence on ε. At the end of this section, we scale it back to the original formulation for which the essential results will still be independent of ε, so long as hyperbolic scaling is respected in the estimates.

Recall the pressure law $\pi(\rho^{\mathsf{tot}}) = \pi(1+\varepsilon\rho)$ rescaled to satisfy $\pi(1) = \pi'(1) = 1$. Introduce new variable r satifying

$$\pi(1+\varepsilon\rho) = 1 + \varepsilon r \iff \pi^{inv}(1+\varepsilon r) = 1 + \varepsilon\rho = \rho^{\mathsf{tot}} \tag{4.2}$$

where π^{inv} denotes the functional inverse of π. Note by Taylor expansion $r = \rho + O(\varepsilon\rho^2)$. Then, the Euler equations (1.1) are reformulated as, term-by-term,

$$\begin{cases} \dfrac{\varepsilon}{\pi'(\pi^{inv}(1+\varepsilon r))}(\partial_t r + \mathbf{v}\cdot\nabla r) + \pi^{inv}(1+\varepsilon r)\nabla\cdot\mathbf{v} = 0, \\ \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \dfrac{1}{\pi^{inv}(1+\varepsilon r)}\dfrac{\nabla r}{\varepsilon} = 0. \end{cases}$$

Then, replace

$$\mathbf{v} = \check{\mathbf{v}}/\varepsilon, \quad r = \check{r}/\varepsilon, \quad t = \tau\varepsilon \tag{4.3}$$

and rewrite it as a symmetric hyperbolic PDE system for the rescaled variable $V := (\check{r}, \check{\mathbf{v}})$,

$$\partial_\tau V + \check{\mathbf{v}}\cdot\nabla V = -\sigma(\check{r})\mathcal{L}(V), \qquad \check{\mathbf{v}}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0 \tag{4.4}$$

where diagonal matrix

$$\sigma(\check{r}) := \operatorname{diag}\big\{\pi^{inv}(1+\check{r})\pi'(\pi^{inv}(1+\check{r})), \frac{1}{\pi^{inv}(1+\check{r})}, \ldots, \frac{1}{\pi^{inv}(1+\check{r})}\big\}.$$

This rescaled, ε-free system will be the main subject of this section.

Since $\sigma(0) = I$ and $\sigma(\check{r})$ is $\mathscr{C}^\infty$, there exists a constant R_π only depending on $\pi(\cdot)$ such that

$$\text{for } |\check{r}| \leqslant R_\pi, \quad \text{diagonal matrice } \sigma(\check{r}) \in [1/2, 2] \tag{4.5}$$

For example, if $\pi(\cdot)$ satisfies the γ-power law, then by the scaling assumption (1.2), it must be $\pi(\rho^{\mathsf{tot}}) = (\gamma - 1 + (\rho^{\mathsf{tot}})^\gamma)/\gamma$ so that $\pi^{inv}(1+\check{r}) = (1+\gamma\check{r})^{1/\gamma}$. Because physics suggests $\gamma \in [1,2]$, it is easy to calculate $R_\pi = (1-2^{-\gamma})/\gamma$.

Introduce a mixed norm for $V(\tau, x)$ at any fixed time τ

$$|\!|\!| V |\!|\!|_m(\tau) := \Big(\sum_{k=0}^m \|\partial_\tau^k V\|^2_{\mathsf{H}^{m-k}(\Omega)}(\tau)\Big)^{1/2}.$$

It is a somewhat lazy way to encapsulate all mixed space-time derivatives up to order m.

For brevity, we make a priori assumption throughout this section that

$$\text{integer } m \geqslant 4, \;\; V \in \mathscr{C}([0,\tau^*], \mathsf{H}^m(\Omega)), \;\; |\!|\!| V |\!|\!|_m(\tau) \leqslant 1 \quad \text{and} \quad |\breve{r}(\tau,x)| \leqslant R_\pi \tag{4.6}$$

for all $\tau \in [0,\tau^*]$, unless specified otherwise. The last two inequalities are not stringent at all because by the scaling of V in (4.3), (4.4), one expects $|\!|\!| V |\!|\!|_m$ and $|\breve{r}|$ to be $O(\varepsilon)$ as long as the original unknown satisfies scaling $\|(r,\mathbf{v})\|_{\mathsf{H}^m} \sim O(1)$.

4.1. **Mixed norms.** First, we establish some basic facts of $|\!|\!| \cdot |\!|\!|$ as direct consequences of Sobolev inequalities. Given function $f(\tau,x)$,

$$|f|_{\mathsf{L}^\infty(\Omega)} \lesssim \|f\|_{\mathsf{H}^2(\Omega)}, \quad \text{thus} \quad |\partial^\beta_{\tau,x} f|_{\mathsf{L}^\infty(\Omega)} \lesssim |\!|\!| f |\!|\!|_{|\beta|+2} \tag{4.7}$$

More generally, given functions $f_1(\tau,x), f_2(\tau,x), \dots, f_j(\tau,x)$ and a product of mixed mixed derivatives $(\partial^{\beta_1}_{\tau,x} f_1)\,(\partial^{\beta_2}_{\tau,x} f_2)\dots(\partial^{\beta_j}_{\tau,x} f_j)$ with multi-indices $\beta_1, ..., \beta_j$ satisfying

$$|\beta_1 + ... + \beta_j| \leqslant 2m-3 \quad \text{and} \quad m \geqslant |\beta_1| \geqslant |\beta_2| \geqslant \ldots \geqslant |\beta_j|, \tag{4.8}$$

we have $\big\|\prod_{i=1}^j \partial^{\beta_i}_{\tau,x} f_i\big\|_{\mathsf{L}^2(\Omega)} \leqslant \big\|\partial^{\beta_1}_{\tau,x} f_1\big\|_{\mathsf{L}^2(\Omega)} \prod_{i=2}^j \big|\partial^{\beta_i}_{\tau,x} f_i\big|_{\mathsf{L}^\infty(\Omega)}$ and therefore by (4.7),

$$\Big\|\prod_{i=1}^j \partial^{\beta_i}_{\tau,x} f_i\Big\|_{\mathsf{L}^2} \lesssim \prod_{i=1}^j |\!|\!| f_i |\!|\!|_m \quad \text{if (4.8) holds.} \tag{4.9}$$

Estimates on $\sigma(\breve{r})$ and its matrix inverse $\sigma^{-1}(\breve{r})$ will also be needed. By (4.5), $\breve{r}$-derivatives of σ and σ^{-1} can be bounded by constants only depending on pressure law $\pi(\cdot)$ and the order of derivatives, i.e.

$$\left|\frac{d^k}{d\breve{r}^k}\sigma(\breve{r})\right| + \left|\frac{d^k}{d\breve{r}^k}\sigma^{-1}(\breve{r})\right| \lesssim 1 \quad \text{for} \quad |\breve{r}| \leqslant R_\pi. \tag{4.10}$$

Also, by the mean value theorem and $\sigma(0) = I$,

$$|\sigma(\breve{r}) - I| + \big|\sigma^{-1}(\breve{r}) - I\big| \lesssim |\breve{r}| \quad \text{for} \quad |\breve{r}| \leqslant R_\pi. \tag{4.11}$$

Now, inductively apply the chain rule and product rule to obtain, for multi-index β,

$$\partial^\beta_{\tau,x}\sigma(\breve{r}) = \text{linear combination of } \frac{d^j\sigma(\breve{r})}{d\breve{r}^j}\,(\partial^{\beta_1}_{\tau,x}\breve{r})\,(\partial^{\beta_2}_{t,x}\breve{r})\dots(\partial^{\beta_j}_{\tau,x}\breve{r})$$
$$\text{over all integers } j \in [1,|\beta|] \text{ and multi-indices satisfying } \;\beta_1 + ... + \beta_j = \beta$$

Therefore, by (4.9), (4.10) and the assumptions $|\!|\!| V |\!|\!|_m \leqslant 1$, $m \geqslant 4$ set in (4.6),

$$\|\partial^\beta_{\tau,x}\sigma(\breve{r})\|_{\mathsf{L}^2} \lesssim |\!|\!| \breve{r} |\!|\!|_m, \qquad \text{for } 1 \leqslant |\beta| \leqslant m.$$

Obviously, this estimate works for matrix inverse σ^{-1} as well. Sum such estimates for all $|\beta|$ from 1 to m, and use (4.11) for the $|\beta| = 0$ case to arrive at

$$|\!|\!| \sigma(\breve{r}) - I |\!|\!|_m + |\!|\!| \sigma^{-1}(\breve{r}) - I |\!|\!|_m \lesssim |\!|\!| \breve{r} |\!|\!|_m. \tag{4.12}$$

Similar estimate works for $\partial_\tau\sigma$. In fact, applying (4.9) to $\partial_\tau\sigma = (\frac{d}{d\breve{r}}\sigma(\breve{r}) - \frac{d}{d\breve{r}}\sigma(0))\partial_\tau\breve{r} + \frac{d}{d\breve{r}}\sigma(0)\partial_\tau\breve{r}$, and noting that $\frac{d}{d\breve{r}}\sigma(\breve{r}) - \frac{d}{d\breve{r}}\sigma(0)$ can be bounded in a way similar to (4.12) , namely, $|\!|\!| \frac{d}{d\breve{r}}\sigma(\breve{r}) - \frac{d}{d\breve{r}}\sigma(0) |\!|\!|_{m-1} \lesssim |\!|\!| \breve{r} |\!|\!|_{m-1} \leqslant 1$, we have

$$|\!|\!| \partial_\tau\sigma(\breve{r}) |\!|\!|_{m-1} + |\!|\!| \partial_\tau\sigma^{-1}(\breve{r}) |\!|\!|_{m-1} \lesssim |\!|\!| \partial_\tau\breve{r} |\!|\!|_{m-1}. \tag{4.13}$$

Now, thanks to the ε-free formulation of (4.4), combine (4.12), (4.13) with Sobolev inequalities (1.15) to iteratively estimate $\partial_\tau V, \partial_\tau^2 V, \ldots, \partial_\tau^m V$ and obtain, under the a priori assumption (4.6)

$$\|\partial_\tau V\|_{m-1} \lesssim \|\partial_\tau V\|_{\mathsf{H}^{m-1}} \leqslant \|V\|_m \lesssim \|V\|_{\mathsf{H}^m} \tag{4.14}$$

4.2. **Vorticity estimates.** Define $\breve{\omega} := \nabla \times \check{\mathbf{v}}$. Take $\nabla\times$ of the momentum equation in (4.4),

$$\partial_\tau \breve{\omega} + \mathrm{b}^{vor}(\check{\mathbf{v}}, \breve{\omega}) = 0 \tag{4.15}$$

$$\text{where} \qquad \mathrm{b}^{vor}(\mathbf{v}, \omega) = \begin{cases} \mathbf{v}\cdot\nabla\omega + (\nabla\cdot\mathbf{v})\omega & \text{in 2D} \\ \mathbf{v}\cdot\nabla\omega + (\nabla\cdot\mathbf{v})\omega - \omega\cdot\nabla\mathbf{v} & \text{in 3D} \end{cases}$$

It is then an exercise of energy estimates to show, with any $\check{\mathbf{v}}$ satisfying (4.6) and $\mathbf{v}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$,

$$\|\breve{\omega}\|^2_{\mathsf{H}^{m-1}}\Big|_{\tau_1}^{\tau_2} \lesssim \int_{\tau_1}^{\tau_2} \|\check{\mathbf{v}}\|_{\mathsf{H}^m} \|\breve{\omega}\|^2_{\mathsf{H}^{m-1}} \qquad \text{for} \quad 0 \leqslant \tau_1 < \tau_2 \leqslant \tau^*. \tag{4.16}$$

We note by passing that, when estimating the $(m-1)$th spatial derivatives of $\breve{\omega}$, one should mollify $\breve{\omega}$ by first extending it outside $\partial\Omega$ (if there is one), then convolving it with a smooth kernel and restricting it back to Ω — cf. [24, pp. 557]. Velocity $\check{\mathbf{v}}$ is not mollified because it is assumed to be H^m, which allows us to utilize the $\check{\mathbf{v}}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$ condition and divergence theorem to show such identities as $\int_\Omega (\check{\mathbf{v}}\cdot\nabla f) f\, dx = -\frac{1}{2}\int_\Omega (\nabla\cdot\check{\mathbf{v}})|f|^2\, dx$ for $f \in \mathsf{H}^1(\Omega)$.

Next, recall (2.4) which gives a bound for V using $\mathcal{L}(V) = (\nabla\cdot\check{\mathbf{v}}, \nabla\check{r})$ and $\nabla\times\check{\mathbf{v}}$,

$$\|V\|_{\mathsf{H}^k} \lesssim \|\mathcal{L}(V)\|_{\mathsf{H}^{k-1}} + \|\nabla\times\check{\mathbf{v}}\|_{\mathsf{H}^{k-1}} + \|V\|_{\mathsf{L}^2}, \quad \text{if } \check{\mathbf{v}}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0 \text{ and } k \geqslant 1. \tag{4.17}$$

So it remains to estimate $\mathcal{L}(V) = (\nabla\cdot\check{\mathbf{v}}, \nabla\check{r})$. In light of (4.1) and the fact that $\mathcal{L}(V)$ and $\partial_\tau V$ are connected via (4.4), we now move on to estimate $\|\partial_t^k V\|_{\mathsf{L}^2}$ for $k = 0, 1, \ldots m$.

4.3. **Diagnostic estimates and recurrence.** We now connect mixed norms of $\mathcal{L}(V)$ and $\partial_\tau V$ via rescaled system (4.4) and its time derivatives. We call such estimates "diagnostic", as opposed to "prognostic" estimates such as (4.16) for $\breve{\omega}$ in the form of integral and differential inequalities of the Gronwall type. Diagnostic estimates do NOT rely on evolutionary properties of (4.4) and will not involve Gronwall type inequalities. They instead come from algebraic manipulation of (4.4) at a fixed time τ, typically using the product rule and Sobolev inequalities.

Elliptic estimate (4.17) and the fact that $\partial_\tau^k \check{\mathbf{v}}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$ for $k \in [0, m-1]$ imply,

$$\|\partial_\tau^k V\|_{\mathsf{H}^{m-k}} \lesssim \|\mathcal{L}(\partial_\tau^k V)\|_{\mathsf{H}^{m-k-1}} + \|\nabla\times\partial_\tau^k\check{\mathbf{v}}\|_{\mathsf{H}^{m-k-1}} + \|\partial_\tau^k V\|_{\mathsf{L}^2}.$$

On the other hand, take ∂_τ^k derivative on (4.4) to get, for $k \in [0, m-1]$,

$$-\mathcal{L}(\partial_\tau^k V) = \partial_\tau^{k+1} V + \partial_\tau^k(\check{\mathbf{v}}\cdot\nabla V) + \partial_\tau^k(\sigma - I)\mathcal{L}(V).$$

Combine these two to obtain a recursive inequality, for every $k \in [0, m-1]$,

$$\|\partial_\tau^k V\|_{\mathsf{H}^{m-k}} \lesssim \|\partial_\tau^{k+1} V\|_{\mathsf{H}^{m-k-1}} + \left\|\partial_\tau^k\Big(\check{\mathbf{v}}\cdot\nabla V,\, (\sigma - I)\mathcal{L}(V),\, \breve{\omega}\Big)\right\|_{\mathsf{H}^{m-k-1}} + \|\partial_\tau^k V\|_{\mathsf{L}^2}. \tag{4.18}$$

At one end of this recursive chain is $\|V\|_{\mathsf{H}^m}$, the desirable norm, and at the other end is $\|\partial_\tau^m V\|_{\mathsf{H}^0}$, which will be estimated prognostically using energy method in the next subsection.

Now, exclude $k=0$ and connect from $k=1$ to $m-1$, using the definition of $\|\!|\cdot|\!\|$ and replace $\partial_\tau\breve{\omega}$ with bilinear terms a la (4.15) to get

$$\|\partial_\tau V\|_{\mathsf{H}^{m-1}} \lesssim \|\!|\partial_\tau\Big(\breve{\mathbf{v}}\cdot\nabla V,\,(\sigma-I)\mathcal{L}(V)\Big)|\!\|_{m-2} + \|\!|\mathrm{b}^{vor}(\breve{\mathbf{v}},\breve{\omega})|\!\|_{m-2} + \sum_{k=1}^{m}\|\partial_\tau^k V\|_{\mathsf{L}^2}$$

Combine it with Sobolev inequalities (4.9), bounds on $(\sigma-I)$ in (4.12) and the equivalence of $\|\!|\cdot|\!\|_m$ and $\|\cdot\|_{\mathsf{H}^m}$ in (4.14) to reach the following lemma.

Lemma 4.1. (Diagnostic estimates on first time derivative) *Consider $V(\tau,x)$, a solution of (4.4) in the a priori setting of (4.6). Then,*

$$\|\partial_\tau V\|_{\mathsf{H}^{m-1}} \lesssim \|V\|_{\mathsf{H}^m}^2 + \sum_{k=1}^{m}\|\partial_\tau^k V\|_{\mathsf{L}^2}$$

The extra quadratic term will cause no trouble, due to the scaling argument below (4.6).

Combine it with the $k=0$ case of (4.18) and apply (4.12), (4.14) to similarly deduce

$$2d_1\|V\|_{\mathsf{H}^m} \leqslant \|\breve{\omega}\|_{\mathsf{H}^{m-1}} + \|V\|_{\mathsf{H}^m}^2 + \sum_{k=0}^{m}\|\partial_\tau^k V\|_{\mathsf{L}^2}$$

for some constant d_1. Notice the vorticity $\breve{\omega}$ stays as is.

Finally, relax one of the $\|V\|_{\mathsf{H}^m}$ factors on the RHS to d_1 and absorb the associated quadratic term into the LHS, reaching the following lemma.

Lemma 4.2. (Diagnostic estimates) *Consider $V(\tau,x)$, a solution of (4.4) in the a priori setting of (4.6). If furthermore $\|V\|_{\mathsf{H}^m}\leqslant d_1$ for some constant d_1 solely depending on m,Ω and pressure law $\pi(\cdot)$, then,*

$$d_1\|V\|_{\mathsf{H}^m} \leqslant \|\breve{\omega}\|_{\mathsf{H}^{m-1}} + \sum_{k=0}^{m}\|\partial_\tau^k V\|_{\mathsf{L}^2}.$$

4.4. **Prognostic estimates of $\|\partial_\tau^k V\|_{\mathsf{L}^2}$.** Take the ∂_τ^k derivative of (4.4) and single out the highest derivatives

$$\partial_\tau(\partial_\tau^k V) + \breve{\mathbf{v}}\cdot\nabla\partial_\tau^k V + \sigma\mathcal{L}(\partial_\tau^k V) = -[\partial_\tau^k,\,\sigma\mathcal{L}+(\breve{\mathbf{v}}\cdot\nabla)]V =: R^{(k)}$$

Here, the commutator term $[\partial_\tau^k,\,\sigma\mathcal{L}+(\breve{\mathbf{v}}\cdot\nabla)]V$ is understood in the sense of Leibniz product rule, so it contains τ derivatives up to the k-th order. It can be estimated diagnostically.

Proposition 4.3. (Commutator estimates) Given a solution $V(\tau,x)$ to (4.4) satisfying a priori assumption (4.6). Then, for all $k\in[0,m-1]$, (we are being cautious not to include $k=m$)

$$\left\|R^{(k)}\right\|_{\mathsf{L}^2} \lesssim \|\partial_\tau V\|_{\mathsf{H}^{m-1}}\|V\|_{\mathsf{H}^m}.$$

Proof. The case $k=0$ is trivial, so we consider $k\in[1,m-1]$. By Sobolev inequalities (4.9),

$$\left\|[\partial_\tau^k,\,\sigma\mathcal{L}+(\breve{\mathbf{v}}\cdot\nabla)]V\right\|_{\mathsf{L}^2} \lesssim \|\!|(\partial_\tau\sigma,\,\partial_\tau\breve{\mathbf{v}})|\!\|_{m-1}\|\!|\partial_x V|\!\|_{m-1}$$

Here, we used the fact that, in the expansion of $[\partial_\tau^k,\,\sigma\mathcal{L}+(\breve{\mathbf{v}}\cdot\nabla)]V$, every product contains a factor with at least one τ derivative taken on σ or $\breve{\mathbf{v}}$. Combine it with (4.13) and (4.14) to conclude the proof. □

Before obtaining estimates of $\|\partial_t^k V\|_{\mathsf{L}^2}$ in the next lemma, we carry out some calculation for any W with the same number of components as V and with regularity $W \in \mathsf{H}^1(\Omega)$, $\partial_\tau W \in \mathsf{L}^2(\Omega)$.

$$\begin{aligned}
&2\int_\Omega (\sigma^{-1}W)\cdot(\partial_\tau W + \check{\mathbf{v}}\cdot\nabla W)\\
&\overset{(a)}{=} \int_\Omega \partial_\tau(\sigma^{-1}W\cdot W) + \int_\Omega \check{\mathbf{v}}\cdot\nabla(\sigma^{-1}W\cdot W) - \Big(\int_\Omega (\partial_\tau\sigma^{-1})W\cdot W + \int_\Omega (\check{\mathbf{v}}\cdot\nabla\sigma^{-1})W\cdot W\Big)\\
&\overset{(b)}{=} \int_\Omega \partial_\tau(\sigma^{-1}W\cdot W) + \int_\Omega \check{\mathbf{v}}\cdot\nabla(\sigma^{-1}W\cdot W) - \int_\Omega (\partial_\tau\check{r} + \check{\mathbf{v}}\cdot\nabla\check{r})(\frac{d}{d\check{r}}\sigma^{-1}W\cdot W)\\
&\overset{(c)}{=} \int_\Omega \partial_\tau(\sigma^{-1}W\cdot W) - \int_\Omega (\nabla\cdot\check{\mathbf{v}})(\sigma^{-1}W\cdot W) - \int_\Omega (\partial_\tau\check{r} + \check{\mathbf{v}}\cdot\nabla\check{r})(\frac{d}{d\check{r}}\sigma^{-1}W\cdot W)\\
&\overset{(d)}{=} \int_\Omega \partial_\tau(\sigma^{-1}W\cdot W) - \int_\Omega \nabla\cdot\check{\mathbf{v}}(\sigma^{-1} - \sigma_{1,1}\frac{d}{d\check{r}}\sigma^{-1})W\cdot W.
\end{aligned}$$

Here, (a) is by the product rule and the fact that σ is diagonal, (b) is by the chain rule, (c) is by the divergence theorem and $\check{\mathbf{v}}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$, and (d) is a simple substitution via the mass equation of (4.4) with $\sigma_{1,1}$ denoting the first entry of matrix σ.

Add and subtract a $2\int_\Omega (\sigma^{-1}W)\cdot(\sigma\mathcal{L}(W)) = 2\int_\Omega W\cdot\mathcal{L}(W)$ term and rearrange it,

$$\begin{aligned}
\frac{d}{d\tau}\int_\Omega \sigma^{-1}W\cdot W = &-2\int_\Omega W\cdot\mathcal{L}(W) + \int_\Omega \nabla\cdot\check{\mathbf{v}}(\sigma^{-1} - \sigma_{1,1}\frac{d}{d\check{r}}\sigma^{-1})W\cdot W\\
&+2\int_\Omega (\sigma^{-1}W)\cdot\Big(\partial_\tau W + \check{\mathbf{v}}\cdot\nabla W + \sigma\mathcal{L}(W)\Big)
\end{aligned}$$

Use (4.10) to bound the maxima of $\sigma, \sigma^{-1}, \partial_{\check{r}}\sigma^{-1}$, and also use $|\nabla\cdot\check{\mathbf{v}}|_{\mathsf{L}^\infty} \lesssim \|V\|_{\mathsf{H}^m}$ to arrive at,

$$\begin{aligned}
\left|\frac{d}{d\tau}\|W\|^2_{\mathsf{L}^2_\sigma}\right| \lesssim &\left|\int_\Omega W\cdot\mathcal{L}(W)\right| + \|V\|_{\mathsf{H}^m}\|W\|^2_{\mathsf{L}^2_\sigma}\\
&+\|\partial_\tau W + \check{\mathbf{v}}\cdot\nabla W + \sigma\mathcal{L}(W)\|_{\mathsf{L}^2}\|W\|_{\mathsf{L}^2_\sigma}
\end{aligned} \tag{4.19}$$

where

$$\|W\|_{\mathsf{L}^2_\sigma} := \Big(\int_\Omega \sigma^{-1}W\cdot W dx\Big).$$

Lemma 4.4. (Prognostic estimates) *Consider $V(\tau,x)$ a solution to (4.4) satisfying a priori assumption (4.6). Then, for all $k \in [0,m]$ and $0 \leqslant \tau_1 < \tau_2 \leqslant \tau^*$,*

$$\|\partial_\tau^k V\|^2_{\mathsf{L}^2_\sigma}\Big|_{\tau_1}^{\tau_2} \lesssim \int_{\tau_1}^{\tau_2}\Big(\|\partial_\tau^k V\|_{\mathsf{L}^2_\sigma} + \|\partial_\tau V\|_{\mathsf{H}^{m-1}}\Big)\|\partial_\tau^k V\|_{\mathsf{L}^2_\sigma}\|V\|_{\mathsf{H}^m}.$$

Note $k = m$ is included here. Also, for $k \geqslant 1$, both sides are in some sense quadratic in $\partial_\tau V$, which will eventually yield desirable bound on the inflation of $\|\partial_\tau V\|_{\mathsf{H}^{m-1}}$.

Proof. First, restrict the value of $k \in [0, m-1]$ so that $\partial_\tau^k V \in \mathsf{H}^1(\Omega)$ and $\partial_\tau^k\check{\mathbf{v}}\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$ is well-defined, allowing us to apply the divergence theorem to have

$$\int_\Omega \partial_\tau^k V\cdot\mathcal{L}(\partial_\tau^k V) = 0, \qquad \text{if } k \in [0, m-1]. \tag{4.20}$$

Thus, set $W = \partial_\tau^k V$ in (4.19) and in the RHS, apply (4.20) to cancel out the first term and use Proposition 4.3 to estimate the last term to prove the lemma for $k \in [0, m-1]$.

The tricky part is the highest derivative $\partial_\tau^m V$ which is merely in $\mathsf{L}^2(\Omega)$. Thus, (4.19) and (4.20) are not directly applicable here, although a version of Proposition 4.3 is. One remedy is to apply mollification in time, namely a time filter, to increase time regularity. Here instead, we demonstrate the closely related time-averaging technique.

Let W in (4.19) be the time average of $\partial_\tau^m V$, i.e. with small $\delta > 0$, setting

$$W = W^\delta := \frac{1}{\delta}\int_\tau^{\tau+\delta} \partial_\tau^m V = \frac{1}{\delta}\partial_\tau^{m-1} V\Big|_\tau^{\tau+\delta}$$

Then, W^δ is still in $\mathsf{H}^1(\Omega)$ with its τ derivative in $\mathsf{L}^2(\Omega)$, validating (4.19) and making the first term in the RHS zero. Moreover, for the last term in (4.19),

$$\begin{aligned}
&\delta\cdot\Big(\partial_\tau W^\delta + \check{\mathbf{v}}\cdot\nabla W^\delta + \mathcal{L}(W^\delta)\Big)\\
&= \left(\partial_\tau\partial_\tau^{m-1}V + \check{\mathbf{v}}\cdot\nabla\partial_\tau^{m-1}V + \mathcal{L}(\partial_\tau^{m-1}V)\right)\Big|_\tau^{\tau+\delta} - \left(\check{\mathbf{v}}\Big|_\tau^{\tau+\delta}\right)\cdot\nabla\partial_\tau^{m-1}V(\tau+\delta,\cdot)\\
&= -\int_\tau^{\tau+\delta}\partial_\tau R^{(m-1)} - \left(\int_\tau^{\tau+\delta}\partial_\tau\check{\mathbf{v}}\right)\cdot\nabla\partial_\tau^{m-1}V(\tau+\delta,\cdot)
\end{aligned}$$

In the RHS, the $\partial_\tau R^{(m-1)}$ term, measured in L^2 norm, is $\lesssim \|\partial_\tau V\|_{\mathsf{H}^{m-1}}\|V\|_{\mathsf{H}^m}$, estimated using the same technique as Proposition 4.3. The $\partial_\tau\check{\mathbf{v}}$ term has a maximum bounded by $\|\partial_\tau V\|_{\mathsf{H}^{m-1}}$. Combining these estimates with Hölder's inequality $\left(\int_a^b f\,d\tau\right)^2 \leqslant |b-a|\int_a^b f^2\,d\tau$, we arrive at

$$\begin{aligned}
\|\partial_\tau W^\delta + \check{\mathbf{v}}\cdot\nabla W^\delta + \mathcal{L}(W^\delta)\|_{\mathsf{L}^2}^2 \lesssim &\left(\frac{1}{\delta}\int_\tau^{\tau+\delta}\|\partial_\tau V\|_{\mathsf{H}^{m-1}}^2\|V\|_{\mathsf{H}^m}^2\right) + \\
&\left(\frac{1}{\delta}\int_\tau^{\tau+\delta}\|\partial_\tau V\|_{\mathsf{H}^{m-1}}^2\right)\|V\|_{\mathsf{H}^m}^2(\tau+\delta)
\end{aligned}$$

Now integrate (4.19) over $[\tau_1, \tau_2-\delta]$, and apply the above estimate and the fact that $\int_\Omega W^\delta\cdot\mathcal{L}(W^\delta) = 0$. Then, pass the limit as $\delta \to 0+$ to prove the lemma for $k = m$. Note by $V \in \mathscr{C}([0,\tau^*], \mathsf{H}^m)$, we have $\partial_\tau^k V \in \mathscr{C}([0,\tau^*], \mathsf{H}^{m-k})$, and thus $\lim\limits_{\delta\to 0}\|W^\delta - \partial_\tau^m V\|_{\mathsf{L}^2}(\tau) = 0$. □

4.5. **Estimates of $\|(\rho,\mathbf{v})\|_{\mathsf{H}^m}$ and $\|\partial_t(\rho,\mathbf{v})\|_{\mathsf{H}^{m-1}}$.** We will still mostly work with $V = \varepsilon(r,\mathbf{v})$, only reconnecting with $\mathbf{v}$ and $r \approx \rho$ in Theorem 4.6 near the end of this section. The goal to keep in mind is the existence time in terms of τ at the same order of $1/\|V_0\|_{\mathsf{H}^m}$ with the $\|V\|_{\mathsf{H}^m}$ norm only inflated by a constant. We also need to see similar inflation of $\|\partial_\tau V\|_{\mathsf{H}^{m-1}}$ but will tolerate some additional term that is *quadratic* in $\|V\|_{\mathsf{H}^m}$.

Lemma 4.5. (Estimates on V and $\partial_\tau V$) *Consider a solution of (4.4), $V \in \mathscr{C}([0,\tau^*], \mathsf{H}^m(\Omega))$ with $m \geqslant 4$. Then, there exist positive constants $\tau^\sharp, C_v, C_1, C_2, C_3$ that solely depend on m, Ω and pressure law $\pi(\cdot)$ so that, if $\|V\|_{\mathscr{C}([0,\tau^*],\mathsf{H}^m)} \leqslant C_v$, then for times $\tau \in [0,\tau^*]\bigcap[0,\tau^\sharp/\|V_0\|_{\mathsf{H}^m}]$,*

$$\|V\|_{\mathsf{H}^m} \leqslant C_1\|V_0\|_{\mathsf{H}^m} \tag{4.21}$$

$$\|\partial_\tau V\|_{\mathsf{H}^{m-1}} \leqslant C_2\|\partial_\tau V_0\|_{\mathsf{H}^{m-1}} + C_3\|V_0\|_{\mathsf{H}^m}^2 \tag{4.22}$$

Notice, under the hyperbolic rescaling (4.3), one has $\|V\|_{\mathsf{H}^m} = \varepsilon\|(r,\check{\mathbf{v}})\|_{\mathsf{H}^m}$ and $\|\partial_\tau V\|_{\mathsf{H}^m} = \varepsilon^2\|\partial_t(r,\check{\mathbf{v}})\|_{\mathsf{H}^m}$ both of which are respected in this lemma.

Proof. We pick constant C_v so that $\|V\|_{\mathscr{C}([0,\tau*],\mathsf{H}^m)} \leqslant C_v$ implies a priori assumption (4.6) and $\|V\|_{\mathsf{H}^m} \leqslant d_1$ as required by Lemma 4.2.

Introduce the shorthand notations

$$f(\tau) := \|\partial_\tau V\|_{\mathsf{H}^{m-1}}(\tau), \qquad F(\tau) := \|V\|_{\mathsf{H}^m}(\tau),$$

$$\phi(\tau) := \Big(\sum_{k=1}^{m} \|\partial_\tau^k V\|^2_{\mathsf{L}^2_\sigma}\Big)^{1/2}, \qquad \Phi(\tau) := \Big(\|\breve{\omega}\|^2_{\mathsf{H}^{m-1}} + \|V\|^2_{\mathsf{L}^2_\sigma} + \phi^2(\tau)\Big)^{1/2}$$

where f, ϕ only involves 1st and higher ∂_τ derivatives.

By definition and $|\sigma| \in [1/2, 2]$, we have $\phi \leqslant \sqrt{2}|\!|\!|\partial_\tau V|\!|\!|_{m-1}$ and $\Phi \leqslant \sqrt{2}|\!|\!|V|\!|\!|_m$. Combine it with (4.14) to get

$$\phi \lesssim f, \qquad \Phi \lesssim F. \tag{4.23}$$

Meanwhile, we have been gathering diagnostic estimates in Lemmas 4.1, 4.2, i.e.

$$f \lesssim (F^2 + \phi), \qquad F \lesssim \Phi \tag{4.24}$$

and prognostic estimates in (4.16) and Lemma 4.4 which sum up to

$$\phi^2\Big|_{\tau_1}^{\tau_2} \lesssim \int_{\tau_1}^{\tau_2} (\phi + f)\phi F, \qquad \Big|_{\tau_1}^{\tau_2}\Phi^2 \lesssim \int_{\tau_1}^{\tau_2} (\Phi + f)\Phi F \tag{4.25}$$

(i). Estimate of $\Phi(\tau)$. Use the second part (4.25), relax f to F a la (4.14) and relax F to Φ a la (4.24) to obtain $\Phi^2\Big|_{\tau_1}^{\tau_2} \leqslant 2c_1 \displaystyle\int_{\tau_1}^{\tau_2} \Phi^3$ for some constant c_1. Thus, $\Phi^2(\tau) \leqslant \Phi^2(0) + 2c_1 \displaystyle\int_0^\tau \Phi^3$ and by the continuity of $\Phi(\tau)$ and the comparison principle,

$$\Phi \leqslant \tilde{\Phi} \quad \text{solving} \quad \frac{d}{d\tau}(\tilde{\Phi})^2 = 2c_1(\tilde{\Phi})^3, \quad \tilde{\Phi}(0) = \Phi(0)$$

$$\implies \Phi(\tau) \leqslant \tilde{\Phi}(\tau) = \frac{\Phi(0)}{1 - c_1\Phi(0)\tau}$$

as long as the RHS is bounded. Thus

$$1 - c_1\Phi(0)\tau \geqslant 1/2 \implies \Phi(\tau) \leqslant 2\Phi(0). \tag{4.26}$$

Then, by the equivalence of F, Φ as in (4.23), (4.24), we proved (4.21) as well as the τ interval prescribed above it.

(ii). Estimate of $\phi(\tau)$. Combine the first parts of (4.24), (4.25), and relax F to $F(0)$ a la (4.21),

$$\phi^2\Big|_{\tau_1}^{\tau_2} \leqslant 2c_2 \int_{\tau_1}^{\tau_2} (\phi + F^2(0))\cdot\phi\cdot F(0).$$

By the continuity of $\phi(\tau)$ and the comparison principle,

$$\phi \leqslant \tilde{\phi} \quad \text{solving} \quad \frac{d}{d\tau}(\tilde{\phi})^2 = 2c_2(\tilde{\phi} + F^2(0))\cdot\tilde{\phi}\cdot F(0), \quad \tilde{\phi}(0) = \phi(0)$$

$$\implies \phi(\tau) \leqslant \tilde{\phi}(\tau) = -F^2(0) + e^{c_2 F(0)\tau}(\phi(0) + F^2(0)).$$

Combine it with the τ interval above (4.21) and the first parts of (4.23), (4.24) to prove (4.22). □

This lemma leads to the final theorem of this section.

Theorem 4.6. (Uniform estimates) *Under the same hypotheses as Main Theorem 1.1 with* $\mathrm{E}_0, \mathrm{E}_{t,0}$ *equivalently given in (1.6), there exist constants* $E^*, T^\sharp, C^\sharp, C_t^\sharp, C_{\mathcal{L}}^\sharp$ *that solely depend on* m, Ω *and pressure law* $\pi(\cdot)$ *so that,*

$$\mathrm{E}_0 \leqslant E^*/\varepsilon \quad \textit{implies there exists a unique solution} \quad (\rho, \mathbf{v}) \in \mathscr{C}^1\left([0, T^\sharp/\mathrm{E}_0] \times \Omega\right).$$

More precisely,

$$\|(\rho, \mathbf{v})\|_{\mathscr{C}([0,T^\sharp/\mathrm{E}_0],\mathsf{H}^m)} \leqslant C^\sharp \mathrm{E}_0, \tag{4.27a}$$

$$\|\partial_t(\rho, \mathbf{v})\|_{\mathscr{C}([0,T^\sharp/\mathrm{E}_0],\mathsf{H}^{m-1})} \leqslant C_t^\sharp(\mathrm{E}_{t,0} + \mathrm{E}_0^2), \tag{4.27b}$$

$$\|\mathcal{L}(\rho, \mathbf{v})\|_{\mathscr{C}([0,T^\sharp/\mathrm{E}_0],\mathsf{H}^{m-1})} \leqslant C_{\mathcal{L}}^\sharp\, \varepsilon\, (\mathrm{E}_{t,0} + \mathrm{E}_0^2). \tag{4.27c}$$

Proof. The short time existence of classical solutions is established in [18], so we only prove the estimates here. The continuation method is always at our disposal, since the compatibility condition $(\partial_t^k \mathbf{v}_0)\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = (\partial_\tau^k \check{\mathbf{v}}_0)\cdot\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$ is invariant under hyperbolic rescaling (4.3).

First of all, by the close relation of r and ρ in (4.2) namely $\check{r} = \varepsilon r = \pi(1+\varepsilon\rho) - \pi(1)$ and $\varepsilon\rho = \pi^{inv}(1+\varepsilon r) - \pi^{inv}(1)$, we can use similar technique for proving (4.12), (4.13) to show that

$$\begin{aligned}
\|\varepsilon\rho\|_{\mathsf{H}^m} \leqslant 1,\ \ |\varepsilon\rho| \leqslant 1/2 &\implies \|r\|_{\mathsf{H}^m} \lesssim \|\rho\|_{\mathsf{H}^m},\ \ \|\partial_t r\|_{\mathsf{H}^{m-1}} \lesssim \|\partial_t\rho\|_{\mathsf{H}^{m-1}},\\
\|\varepsilon r\|_{\mathsf{H}^m} \leqslant 1,\ \ |\varepsilon r| \leqslant R_\pi &\implies \|\rho\|_{\mathsf{H}^m} \lesssim \|r\|_{\mathsf{H}^m},\ \ \|\partial_t\rho\|_{\mathsf{H}^{m-1}} \lesssim \|\partial_t r\|_{\mathsf{H}^{m-1}}.
\end{aligned}$$

Therefore, by choosing constants wisely, it suffices to show there exist similar universal constants $e^*, \tau^\sharp, c_0, c_1, c_2$, so that, $\|V_0\|_{\mathsf{H}^m} \leqslant e^*$ implies

$$\|V\|_{\mathscr{C}([0,\tau^\sharp/\|V_0\|_{\mathsf{H}^m}],\mathsf{H}^m)} \leqslant c_0\|V_0\|_{\mathsf{H}^m}, \tag{4.28a}$$

$$\|\partial_\tau V\|_{\mathscr{C}([0,\tau^\sharp/\|V_0\|_{\mathsf{H}^m}],\mathsf{H}^{m-1})} \leqslant c_1(\|\partial_\tau V_0\|_{\mathsf{H}^{m-1}} + \|V_0\|_{\mathsf{H}^m}^2), \tag{4.28b}$$

$$\|\mathcal{L}(V)\|_{\mathscr{C}([0,\tau^\sharp/\|V_0\|_{\mathsf{H}^m}],\mathsf{H}^{m-1})} \leqslant c_2(\|\partial_\tau V_0\|_{\mathsf{H}^{m-1}} + \|V_0\|_{\mathsf{H}^m}^2). \tag{4.28c}$$

Indeed, choose $e^* = C_v/C_1$ with $C_1 > 1$ and C_v used in Lemma 4.5. Then, by continuity argument and Lemma 4.5, the a priori assumption $\|V\|_{\mathsf{H}^m} \leqslant C_v$ as well as (4.28a), (4.28b) remain true in the time interval prescribed. Finally, (4.28c) is by a simple deduction from (4.28a), (4.28b), the ε-free formulation (4.4), Sobolev inequalities and bounds of σ in (4.12).

□

5. Proof of the Main Theorem and Concluding Remarks

Now we prove the Main Theorem 1.1 using the time-averaging estimates in Theorem 3.3 and the ε-independent estimates in Theorem 4.6.

Proof. (Main Theorem 1.1). First, regarding the time interval of validity, by (4.27a) of Theorem 4.6, replace $T^\flat$ with $T^\sharp/\mathrm{E}_0$ in the last line of Theorem 3.3 to get

$$T \in \left[0, T^\sharp/\mathrm{E}_0\right] \bigcap \left[0,\ D_2/\sup_{[0,T^\flat]} \|(\rho, \mathbf{v}, \tilde{\mathbf{v}})\|_{\mathsf{H}^{m-1}}\right]$$

By estimate (4.27a) again, and by a similar estimate well known to be true for $\tilde{\mathbf{v}}$ (e.g. [24, Ch. 17, Thm. 3.2]), we further shorten the second time interval to $[0, D_2/(C^\sharp\mathrm{E}_0)]$. Therefore, take $T^* := \min\{T^\sharp, D_2/C^\sharp\}$ to make both Theorem 3.3 and 4.6 valid for $T \in [0, T^*/\mathrm{E}_0]$.

On this time interval, Theorem 3.3 guarantees

$$\sup_{[0,T]} \|\widetilde{\mathbf{v}} - \mathbf{v}^P\|_{\mathsf{H}^{m-3}} \leqslant D_1\, \varepsilon \sup_{[0,T]} \big\|\mathcal{L}(\rho,\mathbf{v})\big\|_{\mathsf{H}^{m-2}} \sup_{[0,T]} \|\mathbf{v}^P\|_{\mathsf{H}^m}$$

One ε factor is in place, and (4.27c) of Theorem 4.6 guarantees another ε factor from $\mathcal{L}(\rho,\mathbf{v})$.

So, our last job is to bound $\|\mathbf{v}^P\|_{\mathsf{H}^m}$ in terms of $\|\mathbf{v}^P_0\|_{\mathsf{H}^m}$. Scale vorticity estimate (4.16) back to variables ω, t so that $\|\omega\|^2_{\mathsf{H}^{m-1}}\Big|_{t_1}^{t_2} \lesssim \displaystyle\int_{t_1}^{t_2} \|\mathbf{v}\|_{\mathsf{H}^m}\|\omega\|^2_{\mathsf{H}^{m-1}}$. Apply energy method to (2.11), noting $\mathbf{v}^P{\cdot}\vec{\mathsf{n}}\big|_{\partial\Omega} = \mathbf{v}^Q{\cdot}\vec{\mathsf{n}}\big|_{\partial\Omega} = 0$, to get $\|\mathbf{v}^P\|^2_{\mathsf{L}^2}\Big|_{t_1}^{t_2} \lesssim \displaystyle\int_{t_1}^{t_2} \|\mathbf{v}\|_{\mathsf{H}^m}\|\mathbf{v}^P\|^2_{\mathsf{L}^2}$. Combine these two Gronwall inequalities with elliptic estimate (2.5) to obtain $\|\mathbf{v}^P\|_{\mathsf{H}^m}(T) \lesssim \|\mathbf{v}^P_0\|_{\mathsf{H}^m} \exp(T \sup_{[0,T]} \|\mathbf{v}\|_{\mathsf{H}^m})$. Then, the exponent can be relaxed to $C^\sharp T^\sharp$ due to (4.27a). The proof is complete! □

The proof of Corollary 1.4 is as follows. First, regarding $\|\mathbf{v} - \widetilde{\mathbf{v}}\|_{\mathsf{H}^{m-3}}$, combine (4.27c) of Theorem 4.6 with elliptic estimates from Proposition 2.3 to obtain $\|\mathbf{v}^Q\|_{\mathsf{H}^m} \lesssim \varepsilon(\mathrm{E}_{t,0} + \mathrm{E}_0^2)$. By the $O(\varepsilon^2)$ estimate of $\|\mathbf{v}^P - \widetilde{\mathbf{v}}\|_{\mathsf{H}^{m-3}}$ from the Main Theorem, this is more than enough to prove the $O(\varepsilon)$ estimate of $\|\mathbf{v} - \widetilde{\mathbf{v}}\|_{\mathsf{H}^{m-3}}$.

Secondly, regarding $\displaystyle\int_0^T \mathbf{v} - \widetilde{\mathbf{v}}$, use estimate (3.3) of $\overline{\mathbf{v}^Q}$ and estimate (4.27c) of $\mathcal{L}(\rho,\mathbf{v})$ to obtain

$$\|\int_0^T \mathbf{v}^Q\|_{\mathsf{H}^{m-1}} \lesssim \varepsilon^2(\mathrm{E}_{t,0} + \mathrm{E}_0^2)(1 + T\sup_{[0,T]} \|(\rho,\mathbf{v})\|_{\mathsf{H}^{m-1}}) \lesssim \varepsilon^2(\mathrm{E}_{t,0} + \mathrm{E}_0^2) \tag{5.1}$$

where the $T\sup_{[0,T]} \|(\rho,\mathbf{v})\|_{\mathsf{H}^{m-1}}$ term is absorbed into a constant, due to (4.27a). Meanwhile, the $O(\varepsilon^2)$ estimate of $\|\mathbf{v}^P - \widetilde{\mathbf{v}}\|_{\mathsf{H}^{m-3}}$ from the Main Theorem apparently implies,

$$\|\int_0^T \mathbf{v}^P - \widetilde{\mathbf{v}}\|_{\mathsf{H}^{m-3}} \lesssim \varepsilon^2(\mathrm{E}_{t,0} + \mathrm{E}_0^2)\|\mathbf{v}^P_0\|_{\mathsf{H}^m} T \lesssim \varepsilon^2(\mathrm{E}_{t,0} + \mathrm{E}_0^2)$$

where the last estimate is due to $T \leqslant T^*/\mathrm{E}_0 \lesssim T^*/\|\mathbf{v}^P_0\|_{\mathsf{H}^m}$. Combine the above two estimates to complete the proof of (1.8).

Lastly, regarding the transport equations in Corollary 1.4, it is proven in the same fashion as Lemma 3.2. First, subtract them and rearrange

$$\partial_t(\theta - \widetilde{\theta}) + \widetilde{\mathbf{v}}{\cdot}\nabla(\theta - \widetilde{\theta}) = (\widetilde{\mathbf{v}} - \mathbf{v}^P){\cdot}\nabla\theta - \mathbf{v}^Q{\cdot}\nabla\theta = (\widetilde{\mathbf{v}} - \mathbf{v}^P){\cdot}\nabla\theta - \partial_t\xi$$

where $\xi := \overline{\mathbf{v}^Q{\cdot}\nabla\theta}$. Combine ∂_t terms together and write an equation in terms of $(\theta - \widetilde{\theta} + \xi)$,

$$\partial_t(\theta - \widetilde{\theta} + \xi) + \widetilde{\mathbf{v}}{\cdot}\nabla(\theta - \widetilde{\theta} + \xi) = (\widetilde{\mathbf{v}} - \mathbf{v}^P){\cdot}\nabla\theta + \widetilde{\mathbf{v}}{\cdot}\nabla\xi. \tag{5.2}$$

Now, perform integrating by parts on ξ and use the transport equation of θ itself,

$$\xi(T,\cdot) = \overline{\mathbf{v}^Q}{\cdot}\nabla\theta\Big|_0^T - \int_0^T \overline{\mathbf{v}^Q}{\cdot}\nabla\partial_t\theta = \overline{\mathbf{v}^Q}{\cdot}\nabla\theta\Big|_0^T + \int_0^T \overline{\mathbf{v}^Q}{\cdot}\nabla(\mathbf{v}{\cdot}\nabla\theta)$$

Take the H^{m-2} norm, apply (5.1) and Sobolev inequalities (1.15) to get

$$\|\xi\|_{\mathsf{H}^{m-2}} \lesssim \varepsilon^2(\mathrm{E}_{t,0} + \mathrm{E}_0^2)\sup_{[0,T]} \|\theta\|_{\mathsf{H}^m}\big(\|1 + T\sup_{[0,T]} \|\mathbf{v}\|_{\mathsf{H}^m}\big) \tag{5.3}$$

Apply the standard energy method to the transport equation of θ, using the same mollification for proving vorticity estimate (4.16), to get

$$\|\theta\|^2_{\mathsf{H}^m}\Big|_{t_1}^{t_2} \lesssim \int_{t_1}^{t_2} \|\mathbf{v}\|_{\mathsf{H}^m}\|\theta\|^2_{\mathsf{H}^m} \implies \sup_{[0,T]}\|\theta\|_{\mathsf{H}^m} \lesssim \|\theta_0\|_{\mathsf{H}^m}\exp(T\sup_{[0,T]}\|\mathbf{v}\|_{\mathsf{H}^m}).$$

Plug it into (5.3) and absorb all $T\sup_{[0,T]}\|\mathbf{v}\|_{\mathsf{H}^m}$ terms into a constant a la (4.27a),

$$\sup_{[0,T]}\|\xi\|_{\mathsf{H}^{m-2}} \lesssim \varepsilon^2(\mathrm{E}_{t,0}+\mathrm{E}_0^2)\,\|\theta_0\|_{\mathsf{H}^m} \tag{5.4}$$

Then, apply energy method on (5.2), bounding the RHS with the above estimate and (1.4),

$$\begin{aligned}\|\theta-\widetilde{\theta}+\xi\|^2_{\mathsf{H}^{m-3}}\Big|_{t_1}^{t_2} \lesssim \int_{t_1}^{t_2}\|\theta-\widetilde{\theta}+\xi\|_{\mathsf{H}^{m-3}}\Big\{&\|\widetilde{\mathbf{v}}\|_{\mathsf{H}^m}\|\theta-\widetilde{\theta}+\xi\|_{\mathsf{H}^{m-3}}\\ &+\varepsilon^2(\mathrm{E}_{t,0}+\mathrm{E}_0^2)\,(\|\theta_0\|_{\mathsf{H}^m}\cdot\|\widetilde{\mathbf{v}}\|_{\mathsf{H}^m}+\|\mathbf{v}_0^P\|_{\mathsf{H}^m}\cdot\|\theta\|_{\mathsf{H}^m})\Big\}.\end{aligned}$$

Since $\theta-\widetilde{\theta}+\xi=0$ at $t=0$, this Gronwall inequality implies a desirable $O(\varepsilon^2)$ estimate for $\|\theta-\widetilde{\theta}+\xi\|_{\mathsf{H}^{m-3}}$ and together with (5.4), it proves the last inequality of Corollary 1.4.

For future studies, we like to comment on the possibilities of sharpening the error estimates for practical use such as numerical analysis, for the incompressible approximation is so ubiquitously important. One aspect is to get some good bounds on the inequality constants C^*, T^* etc. This can benefit from using optimal constants in the Sobolev inequalities, making all $\lesssim$ relations explicitly $\leqslant$ relations. In addition, for the easier case $\partial\Omega=\varnothing$, one can drastically reduce the steps of the energy method in Section 4, potentially reducing constants as well. Another aspect is to utilize dispersive and/or dissipative mechanisms which the current article does not reply on. It will be very interesting to see what role they can play when combined with time-averaging.

Furthermore, we note that it is easy to extend our techniques to domains living in two and three dimensional Riemannian manifolds, in which case two major analytical tools remain valid: Stokes' theorem as generalization of divergence theorem and Sobolev inequalities. Also, calculations carried out in this article mostly rely on a handful of coordinate-independent operators , i.e $\nabla, \nabla\cdot, \nabla\times, \mathbf{v}\cdot\nabla, \Delta$. Then, our results and techniques can be applied to interesting areas such as geophysical fluid dynamics on a sphere and relativistic fluid dynamics.

Acknowledgement

B. Cheng would like to thank I. Gallagher for very useful discussion while both were participating the "Mathematics for the Fluid Earth" programme at the Isaac Newton Institute, Autumn 2013. B. Cheng thanks INI for its hospitality and is very grateful to the organizers of this programme for offering a Visiting Fellowship.

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