NONLOCAL p-LAPLACE EQUATIONS DEPENDING ON THE L^p NORM OF THE GRADIENT

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Abstract. We are studying a class of nonlinear nonlocal diffusion problems associated with a p-Laplace-type operator, where a nonlocal quantity is present in the diffusion coefficient. We address the issues of existence and uniqueness for the parabolic setting. Then we study the asymptotic behaviour of the solution for large time. For this purpose we introduce and investigate in details the associated stationary problem. Moreover, since the solutions of the stationary problem are also critical points of some energy functional, we make a classification of its critical points.

1. Introduction

We consider the problem of finding $u = u(x,t)$ weak solution to

$$\begin{cases} u_t - \nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 1$ with Lipschitz boundary Γ. We assume

$$a \text{ is continuous, } a(\xi) > 0, \ \forall \xi \in \mathbb{R}. \tag{1.2}$$

By $|\cdot|_p$ we denote the $L^p(\Omega)$-norm, $1 < p < +\infty$ and we assume

$$f = f(x) \in W^{-1,q}(\Omega) := \Big(W_0^{1,p}(\Omega)\Big)^*, \ u_0 \in W_0^{1,p}(\Omega) \cap L^2(\Omega), \ \frac{1}{p} + \frac{1}{q} = 1. \tag{1.3}$$

For notions on Sobolev spaces we refer to [5], [15], [16].

During the last decades many mathematicians have been studying problems associated with the p-Laplace operator, which appears in a variety of physical fields (see for instance [1], [2]). In particular a lot of attention has been devoted to nonlocal problems. One of the justification of such models lies in the fact that in reality the measurements are not made pointwise – but through some local average. Some interesting features of nonlocal problems and more motivation are described in [4], [5], [7], [8], [10] and in the references therein.

The elliptic problems with our type of nonlocality have been studied in [13], [14] and the stability issues for a local case were considered in [3]. Furthermore, the problem (1.1) was examined for $p = 2$ in [11] and [12].

We now describe the results obtained in this paper. In sections 2 and 3 we study the existence and uniqueness of a weak solution of problem (1.1). Next in section 4 we investigate the corresponding stationary problem and show that depending on the function a it can have from a unique up to a continuum of solutions. In particular, since stationary solutions are also critical points of some energy functional (see (2.8)) we prove the existence of its global minimizer.

The main results of this paper are contained in section 5, where we give the classification of the critical points of the energy functional define by (2.8), assuming that the function a satisfies just (1.2) and $1 < p < +\infty$. We also present an algorithm for finding a global minimizer or global minimizers (it may be not unique) of the energy (2.8).

Finally in the last section we study the asymptotic behaviour of the solution of problem (1.1) as time goes to infinity. We prove that the solution of problem (1.1) converges to a stationary solution, which is a global minimizer of (2.8), in case of uniqueness of such a stationary point. Moreover, we also present some local stability result for the case of uniqueness of a global minimizer.

2. Existence

Theorem 2.1. *Let the assumptions above hold and assume that there exist two constants* λ, Λ *such that*

$$0 < \lambda \leq a(\xi) \leq \Lambda, \quad \forall \xi \in \mathbb{R} \tag{2.1}$$

and that

$$f \in L^q(\Omega). \tag{2.2}$$

Then, for any $T > 0$ *there exists* u *solution to*

$$\begin{cases} u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap C([0,T]; L^2(\Omega)), \\ u_t \in L^q(0,T; W^{-1,q}(\Omega)), \\ u(\cdot, 0) = u_0, \\ \langle u_t, v\rangle + \displaystyle\int_\Omega a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle \\ \forall v \in W_0^{1,p}(\Omega) \text{ in } \mathcal{D}'(0,T), \end{cases} \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ *denotes the pairing between* $L^q(\Omega)$ *and* $L^p(\Omega)$, $u(t) = u(\cdot, t)$, $\mathcal{D}'(0,T)$ *is the space of distributions on* $(0,T)$.

Proof. Consider $\lambda_1, \dots, \lambda_n, \dots$ a basis in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$ smooth and that without loss of generality, we will suppose orthonormal in $L^2(\Omega)$. If $u_0 = \sum_i \beta_i \lambda_i$ consider

$$u_n(t) = \sum_{i=1}^n \gamma_i(t)\lambda_i$$

solution to

$$(2.4)\qquad \begin{cases} \displaystyle\int_\Omega u_n' v dx + a(\|\nabla u_n\|_p^p) \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla v dx = \langle f, v\rangle \\ \forall v \in [\lambda_1, \dots, \lambda_n], \\ \displaystyle u_n(0) = \sum_{i=1}^n \beta_i \lambda_i, \end{cases}$$

where $[\lambda_1, \dots, \lambda_n]$ is the space spanned by $\lambda_1, \dots, \lambda_n$. Taking $v = \lambda_j$ and using the fact that the λ_i's are orthonormal we see that (2.4) is equivalent to the Cauchy problem

$$(2.5)\qquad \begin{cases} \displaystyle\gamma_j'(t) = -a\Big(\Big\| \sum_{i=1}^n \gamma_i(t) \nabla\lambda_i \Big\|_p^p\Big) \int_\Omega \Big| \sum_{i=1}^n \gamma_i(t)\nabla\lambda_i \Big|^{p-2} \sum_{i=1}^n \gamma_i(t) \nabla\lambda_i \nabla\lambda_j dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \langle f, \lambda_j\rangle, \quad \forall j = 1, \dots n, \\ \gamma_j(0) = \beta_j, \quad \forall j = 1, \dots n. \end{cases}$$

Since the right hand side of the first equation above is continuous in γ_i this Cauchy problem possesses a solution. Moreover, using the formulation (2.4) and taking $v = u_n$ we see that

$$\int_\Omega u_n' u_n dx + a(\|\nabla u_n\|_p^p) \int_\Omega |\nabla u_n|^p dx = \langle f, u_n\rangle,$$

which implies using (2.1), Poincaré's and Young's inequalities

$$\frac{1}{2}\frac{d}{dt}|u_n|_2^2 + \lambda \int_\Omega |\nabla u_n|^p dx \le C|f|_q \|\nabla u_n\|_p \le \varepsilon \|\nabla u_n\|_p^p + C_\varepsilon |f|_q^q.$$

Choosing for instance $\varepsilon = \dfrac{\lambda}{2}$ we arrive to

$$\frac{1}{2}\frac{d}{dt}|u_n|_2^2 + \frac{\lambda}{2} \int_\Omega |\nabla u_n|^p dx \le C_\varepsilon |f|_q^q.$$

After an integration in t this leads to

$$(2.6)\qquad \frac{1}{2}|u_n(t)|_2^2 + \frac{\lambda}{2}\int_0^t \int_\Omega |\nabla u_n|^p dx dt \le C_\varepsilon \int_0^t |f|_q^q dt + \frac{1}{2}|u_n(0)|_2^2.$$

In particular we see that $|u_n(t)|_2$ remains bounded in time and thus the solution to (2.4) or (2.5) is global in time ($|\cdot|_2$ is just a norm in $[\lambda_1, \dots, \lambda_n]$, where all the norms are equivalent).

Remark that $\|\nabla u_n\|_p$ remains bounded in time uniformly. To see that taking $v = u_n'$ in (2.4) we get

$$(2.7)\qquad \int_\Omega u_n'^2 dx + a(\|\nabla u_n\|_p^p) \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla u_n' dx = \langle f, u_n'\rangle.$$

Introducing

$$(2.8)\qquad E(u) = \frac{1}{p} A\left(\int_\Omega |\nabla u|^p dx\right) - \langle f, u\rangle$$

with

$$A(z) = \int_0^z a(s)ds, \tag{2.9}$$

we see that (2.7) can be written

$$\partial_t E(u_n) = -\int_\Omega u_n'^2 dx \le 0. \tag{2.10}$$

Thus $E(u_n)$ decreases in time and is bounded from above for every t. The bound for $\|\nabla u_n\|_p$ follows then from the estimate

$$E(u_n) \ge \frac{\lambda}{p}\|\nabla u_n\|_p^p - C|f|_q\|\nabla u_n\|_p. \tag{2.11}$$

From (2.6), (2.11) we deduce that

$$u_n \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)\cap W_0^{1,p}(\Omega)).$$

Furthermore, from the first equation in (2.4) and Hölder's inequality we derive easily

$$\int_\Omega u_n' v dx \le \Lambda \left(\int_\Omega |\nabla u_n|^{(p-1)q}dx\right)^{\frac{1}{q}} \left(\int_\Omega |\nabla v|^p dx\right)^{\frac{1}{p}} + C|f|_q \left(\int_\Omega |\nabla v|^p dx\right)^{\frac{1}{p}},$$

i.e.

$$|u_n'|_{-1,q} \le \Lambda \|\nabla u_n\|_p^{p-1} + C|f|_q,$$

and

$$u_n' \text{ is bounded in } L^q(0,T;W^{-1,q}(\Omega)) \subset L^q(0,T;W^{-1,q}(\Omega) + L^p(\Omega))$$

independently of n, $W^{-1,q}(\Omega) + L^p(\Omega)$ denotes a dual space to $W_0^{1,p}(\Omega)\cap L^q(\Omega)$. We have that

$$W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,q}(\Omega) + L^p(\Omega),$$

where the first embedding is compact (see [15]). Hence by Aubin-Lions lemma the embedding of

$$W := \{v \in L^p(0,T;W_0^{1,p}(\Omega)),\ v' \in L^q(0,T;W^{-1,q}(\Omega))\}$$

in $L^p(0,T;L^p(\Omega))$ is compact. Thus we can find a subsequence of n such that

$$u_n \rightharpoonup u \ \text{ in } L^p(0,T;W_0^{1,p}(\Omega)),$$

$$u_n \to u \ \text{ in } L^p(0,T;L^p(\Omega)),$$

$$\frac{1}{a(\|\nabla u_n\|_p^p)} \rightharpoonup a_\infty \text{ in } L^\infty(0,T) \ \text{ - weak*},$$

$$u_n(T) \rightharpoonup u(T) \ \text{ in } L^2(\Omega),$$

$$\nabla\cdot|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \chi \ \text{ in } L^q(0,T;W^{-1,q}(\Omega)).$$

In fact,

$$u_n' \in L^2(0,T;L^2(\Omega)) = L^2(Q_T), \quad Q_T = (0,T)\times\Omega. \tag{2.12}$$

Indeed, integrating (2.10) from 0 to T we derive

$$\int_0^T\int_\Omega |u_n'|^2 dx = E(u_n(0)) - E(u_n(T)). \tag{2.13}$$

Using the Young inequality in (2.11) we get

$$E(u) \ge \frac{\lambda}{p}\|\nabla u\|_p^p - \frac{C|f|_q^q}{\lambda^{\frac{q}{p}}q} - \frac{\lambda}{p}\|\nabla u\|_p^p = -\frac{1}{q}\left(\frac{C|f|_q}{\lambda^{\frac{1}{p}}}\right)^q,$$

hence $E(u_n)$ is bounded from below independently of n. Thus from (2.13) we obtain (2.12).

The fact that $u \in C([0,T], L^2(\Omega))$ follows by the standard argument (see [16]).

By rescaling the time in the following way, setting

$$\alpha(t) = \int_0^t a(\|\nabla u(\cdot,s)\|_p^p)ds, \tag{2.14}$$

we reduce solving the problem (1.1) to solving the problem (see [11]):

$$\begin{cases} w_t - \nabla \cdot |\nabla w|^{p-2}\nabla w = \dfrac{f}{a(\|\nabla w\|_p^p)} & \text{in } \Omega \times (0, \alpha(T)), \\ w = 0 & \text{on } \Gamma \times (0, \alpha(T)), \\ w(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{2.15}$$

where

$$w(x, \alpha(t)) = u(x,t).$$

Replacing in (2.14) u by u_n, we can also write the first equation of (2.4) as

$$\int_\Omega u_n' v dx + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx = \frac{\langle f, v\rangle}{a(\|\nabla u_n\|_p^p)}. \tag{2.16}$$

Now passing to the limit in (2.16) one has in the distributional sense in Q_T

$$u_t - \chi = a_\infty f \tag{2.17}$$

(therefore $u_t \in L^q(0,T; W^{-1,q}(\Omega))$).

Taking $v = u_n$ in (2.16) we obtain

$$\frac{1}{2}\frac{d}{dt}|u_n|_2^2 + \int_\Omega |\nabla u_n|^p dx = \frac{\langle f, u_n\rangle}{a(\|\nabla u_n\|_p^p)}$$

and by integration on $(0,T)$ we get

$$\int_{Q_T} |\nabla u_n|^p dxdt = \int_0^T \frac{\langle f, u_n\rangle}{a(\|\nabla u_n\|_p^p)} dt + \frac{|u_n(0)|_2^2}{2} - \frac{|u_n(T)|_2^2}{2}. \tag{2.18}$$

Since $u_n \to u$ in $L^p(Q_T)$, $\dfrac{f}{a(\|\nabla u_n\|_p^p)} \rightharpoonup a_\infty f$ in $L^q(Q_T)$ and using the fact that $\underline{\lim}_{n\to\infty} |u_n(T)|_2^2 \geq |u(T)|_2^2$ from (2.18) we get

$$\overline{\lim_{n\to\infty}} \int_{Q_T} |\nabla u_n|^p dxdt \leq \int_0^T a_\infty \langle f, u\rangle dt + \frac{|u_0|_2^2}{2} - \frac{|u(T)|_2^2}{2}. \tag{2.19}$$

Thus from the inequality

$$\int_{Q_T} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v)\cdot \nabla(u_n - v) dxdt \geq 0$$

we derive by taking the $\overline{\lim}$ for any $v \in L^p(0,T; W_0^{1,p}(\Omega))$

$$\int_0^T a_\infty \langle f, u\rangle dt + \frac{|u_0|_2^2}{2} - \frac{|u(T)|_2^2}{2} + \int_0^T \langle \chi, v\rangle dt \\ - \int_{Q_T} |\nabla v|^{p-2}\nabla v \cdot \nabla(u - v) dxdt \geq 0. \tag{2.20}$$

By integrating (2.17) after having multiplied by u we get

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 - \langle \chi, u\rangle = a_\infty \langle f, u\rangle$$

and integrating over $(0, T)$ we obtain

$$-\int_0^T \langle \chi, u\rangle dt = \int_0^T a_\infty \langle f, u\rangle dt + \frac{|u_0|_2^2}{2} - \frac{|u(T)|_2^2}{2}. \tag{2.21}$$

Thus combining (2.20), (2.21) we have

$$\int_0^T \langle -\chi + \nabla\cdot|\nabla v|^{p-2}\nabla v, u - v\rangle dt \geq 0 \quad \forall v \in L^p(0,T; W_0^{1,p}(\Omega)).$$

Taking $v = u - \delta w,\ \delta > 0$, we see

$$\int_0^T \langle -\chi + \nabla\cdot|\nabla(u-\delta w)|^{p-2}\nabla(u-\delta w), w\rangle dt \geq 0\ \forall w \in L^p(0,T; W_0^{1,p}(\Omega)).$$

Letting $\delta \to 0$ we get easily

$$\int_0^T \langle -\chi + \nabla\cdot|\nabla u|^{p-2}\nabla u, w\rangle dt = 0 \quad \forall w \in L^p(0,T; W_0^{1,p}(\Omega))$$

and the equation (2.17) reads

$$u_t - \nabla\cdot|\nabla u|^{p-2}\nabla u = a_\infty f.$$

Going back to (2.19), (2.21) we derive

$$\overline{\lim_{n\to\infty}} \int_{Q_T} |\nabla u_n|^p dxdt \leq \int_{Q_T} |\nabla u|^p dxdt \ \left(\leq \underline{\lim_{n\to\infty}} \int_{Q_T} |\nabla u_n|^p dxdt\right)$$

and $\nabla u_n \to \nabla u$ in $L^p(Q_T)$ strongly. In other words

$$\int_0^T \int_\Omega |\nabla(u_n - u)|^p dxdt \to 0.$$

Up to a subsequence we have

$$\int_\Omega |\nabla(u_n - u)|_p^p dxdt \to 0 \ \text{ a.e. } t,$$

i.e. this implies

$$\|\nabla u_n\|_p^p \to \|\nabla u\|_p^p \ \text{ a.e. } t$$

and then $\dfrac{1}{a(\|\nabla u_n\|_p^p)} \to \dfrac{1}{a(\|\nabla u\|_p^p)}$ a.e. t, since the sequence is bounded this convergence take also place in any $L^p(0,T)$ and $a_\infty = \dfrac{1}{a(\|\nabla u\|_p^p)}$, which completes the proof. □

3. Uniqueness

For the reader convenience we start this section by formulating some auxiliary lemmas, used throughout the paper.

Lemma 3.1. *(see [6]) Let $1 < p < +\infty$. There exist positive constants c_p, C_p such that for every $\xi, \eta \in \mathbb{R}^n$*

$$c_p N_p(\xi,\eta) \le (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) \le C_p N_p(\xi,\eta),$$

where

$$N_p(\xi,\eta) = \{|\xi| + |\eta|\}^{p-2}|\xi-\eta|^2,$$

a dot denotes the Euclidean product in R^n.

Lemma 3.2. *Let a, b be non negative numbers. Then*

$$|a^p - b^p| \le p|a-b|\{a+b\}^{p-1}.$$

Proof. We can suppose $a > b$. Then

$$\begin{aligned} a^p - b^p &= \int_0^1 \frac{d}{dt}|b+t(a-b)|^p dt = p\int_0^1 |b+t(a-b)|^{p-1}\frac{b+t(a-b)}{|b+t(a-b)|}(a-b)dt \\ &\le p(a-b)\int_0^1 |ta+(1-t)b|^{p-1}dt \le p(a-b)\int_0^1 \{|a|+|b|\}^{p-1}dt \\ &= p|a-b|\{a+b\}^{p-1}. \end{aligned}$$

□

Theorem 3.1. *If in addition to the assumptions of Theorem 2.1 for some L it holds that*

$$|a(\xi) - a(\xi')| \le L|\xi - \xi'| \quad \forall \xi, \xi' \in \mathbb{R} \tag{3.1}$$

and

$$f \in L^2(\Omega),$$

then the solution to (2.3) is unique.

Proof. Let u_1, u_2 be two weak solutions to

$$\begin{cases} u \in L^p(0,T;W_0^{1,p}(\Omega)), \quad u_t \in L^q(0,T;W_0^{-1,q}(\Omega)), \\ u_t - \nabla\cdot|\nabla u|^{p-2}\nabla u = \dfrac{f}{a(\|\nabla u\|_p^p)}. \end{cases} \tag{3.2}$$

By subtraction we obtain

$$(u_1-u_2)_t - \nabla\cdot\left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right) = \left(\frac{1}{a(\|\nabla u_1\|_p^p)} - \frac{1}{a(\|\nabla u_2\|_p^p)}\right)f.$$

Multiplying by $u_1 - u_2$, integrating over Ω and using (2.1), (3.1) we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 + \int_\Omega \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)\cdot\nabla(u_1-u_2)dx \\ \le \frac{L}{\lambda^2}\left|\|\nabla u_1\|_p^p - \|\nabla u_2\|_p^p\right|\left|\int_\Omega f(u_1-u_2)dx\right| \end{aligned} \tag{3.3}$$

From Lemma 3.2 and the Hölder inequality we derive

$$\begin{aligned}
(3.4)\quad \big|\|\nabla u_1\|_p^p - \|\nabla u_2\|_p^p\big| &= \Big|\int_\Omega (|\nabla u_1|^p - |\nabla u_2|^p)dx\Big| \le \int_\Omega \Big||\nabla u_1|^p - |\nabla u_2|^p\Big|dx \\
&\le p\int_\Omega \big(|\nabla u_1| + |\nabla u_2|\big)^{p-1}|\nabla(u_1-u_2)|dx \\
&= p\int_\Omega \big(|\nabla u_1| + |\nabla u_2|\big)^{\frac{p}{2}}\big(|\nabla u_1| + |\nabla u_2|\big)^{\frac{p}{2}-1}|\nabla(u_1-u_2)|dx \\
&\le p\left(\int_\Omega \big(|\nabla u_1| + |\nabla u_2|\big)^p dx\right)^{\frac{1}{2}}\left(\int_\Omega \big(|\nabla u_1| + |\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2 dx\right)^{\frac{1}{2}}.
\end{aligned}$$

From Lemma 3.1 we obtain

$$\begin{aligned}
\int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\cdot\nabla(u_1-u_2)dx \\
\ge c_p\int_\Omega \big(|\nabla u_1| + |\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2dx.
\end{aligned}$$

Combining (3.3) and the two inequalities above leads to

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 &+ c_p\int_\Omega \big(|\nabla u_1| + |\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2dx \\
&\le \frac{Lp}{\lambda^2}\left|\int_\Omega f(u_1-u_2)dx\right|\left(\int_\Omega \big(|\nabla u_1| + |\nabla u_2|\big)^p dx\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_\Omega \big(|\nabla u_1| + |\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2dx\right)^{\frac{1}{2}} \\
&\le \frac{c_p}{2}\int_\Omega \big(|\nabla u_1| + |\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2dx + C(t)\int_\Omega |u_1-u_2|^2dx.
\end{aligned}$$

(In the last inequality above we use Young's inequality. Note that $C \in L^1(0,T)$). Therefore, we have

$$\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 \le C(t)\int_\Omega |u_1-u_2|^2dx.$$

The uniqueness follows then from Gronwall's inequality.

□

Theorem 3.2. *Let the assumptions* (1.2), (1.3) *hold and if in addition the function a is such that*

$$(3.5)\qquad s \mapsto a(s^p)s^{p-1} \textit{ is nondeacreasing,}$$

then the solution to (2.3) *is unique.*

Proof. Let u_1, u_2 be two solutions to (2.3), then taking $v = u_1 - u_2$ and by subtraction one has

$$\begin{aligned}
(3.6)\quad \frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 + \int_\Omega \Big(a(\|\nabla u_1\|_p^p)|\nabla u_1|^{p-2}\nabla u_1 \\
- a(\|\nabla u_2\|_p^p)|\nabla u_2|^{p-2}\nabla u_2\Big)\nabla(u_1-u_2)dx = 0.
\end{aligned}$$

By expanding the integral term I one gets

$$I = \int_\Omega \Big(a(\|\nabla u_1\|_p^p)|\nabla u_1|^p - a(\|\nabla u_1\|)|\nabla u_1|^{p-2}\nabla u_1 \nabla u_2 + a(\|\nabla u_2\|_p^p)|\nabla u_2|^p - a(\|\nabla u_2\|)|\nabla u_2|^{p-2}\nabla u_2 \nabla u_1 \Big) dx.$$

Recall that $a(\|\nabla u_i\|_p^p)$, $i = 1, 2$ are independent of x and can be pulled out of the integrals. Using Hölder's inequality we see

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla u_2 dx \leq \|\nabla u_2\|_p \|\nabla u_1\|_p^{p-1},$$

$$\int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla u_1 dx \leq \|\nabla u_1\|_p \|\nabla u_2\|_p^{p-1}.$$

Then using (3.5) it comes

$$\begin{aligned} I &\geq a(\|\nabla u_1\|_p^p)\Big(\|\nabla u_1\|_p^p - \|\nabla u_1\|_p^{p-1}\|\nabla u_2\|_p\Big) \\ &\quad + a(\|\nabla u_2\|_p^p)\Big(\|\nabla u_2\|_p^p - \|\nabla u_2\|_p^{p-1}\|\nabla u_1\|_p\Big) \\ &= \Big(a(\|\nabla u_1\|_p^p)\|\nabla u_1\|_p^{p-1} - a(\|\nabla u_2\|_p^p)\|\nabla u_2\|_p^{p-1}\Big)(\|\nabla u_1\|_p - \|\nabla u_2\|_p) \geq 0. \end{aligned}$$

Hence, (3.6) implies

$$\frac{d}{dt}|u_1 - u_2|_2^2 \leq 0,$$

therefore the result follows. □

Remark 3.1. Note that (3.5) holds in particular for a nondecreasing.

4. The stationary problem

In this section we consider the associated stationary problem to the problem (1.1), that is the following problem

$$\begin{cases} -\nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{4.1}$$

We will assume here that $f \in W^{-1,q}(\Omega)$. In a weak form u is a weak solution to

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \int_\Omega a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{4.2}$$

Here and after $\langle \cdot, \cdot \rangle$ denotes the pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$. In order to solve the stationary problem we introduce φ the solution to

$$\begin{cases} \varphi \in W_0^{1,p}(\Omega), \\ \int_\Omega |\nabla \varphi|^{p-2}\nabla \varphi \nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{4.3}$$

It is known that for $f \in W^{-1,q}(\Omega)$ (4.3) admits a unique solution [6].

Theorem 4.1. *Suppose that* (1.2) *holds,* $1 < p < +\infty$. *Then for* $f \in W^{-1,q}(\Omega)$, *the mapping* $u \mapsto \|\nabla u\|_p^p$ *is one-to-one mapping from the set of solutions to* (4.2) *onto the set of solutions in* $\mathbb{R}$ *of the equation*

$$a(\mu)^{\frac{p}{p-1}}\mu = \|\nabla\varphi\|_p^p. \tag{4.4}$$

Proof. Let u be a solution to the stationary problem, then

$$\int_\Omega a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u\nabla v dx = \langle f, v\rangle = \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla v dx \quad \forall v \in W_0^{1,p}(\Omega), \tag{4.5}$$

which implies

$$a(\|\nabla u\|_p^p)^{\frac{1}{p-1}}u = \varphi, \tag{4.6}$$

from where follows

$$a(\|\nabla u\|_p^p)^{\frac{p}{p-1}}\|\nabla u\|_p^p = \|\nabla\varphi\|_p^p. \tag{4.7}$$

Hence $\|\nabla u\|_p^p$ is a solution to (4.4).

Let now μ be a solution to (4.4), u denotes the solution to

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(\mu)|\nabla u|^{p-2}\nabla u\nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega), \end{cases} \tag{4.8}$$

then $a(\mu)^{\frac{1}{p-1}}u = \varphi$. Therefore, we get

$$a(\mu)^{\frac{p}{p-1}}\|\nabla u\|_p^p = \|\nabla\varphi\|_p^p = a(\mu)^{\frac{p}{p-1}}\mu \;\Rightarrow\; \|\nabla u\|_p^p = \mu$$

and u is a solution to (4.2). Now to show the injectivity we have

$$\|\nabla u_1\|_p^p = \|\nabla u_2\|_p^p \;\Rightarrow\; a(\|\nabla u_1\|_p^p) = a(\|\nabla u_2\|_p^p) \;\Rightarrow\; u_1 = u_2,$$

due to the uniqueness of the solution of (4.8).

□

Remark 4.1. The stationary points are determined by the solutions to

$$a(\mu) = \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1}. \tag{4.9}$$

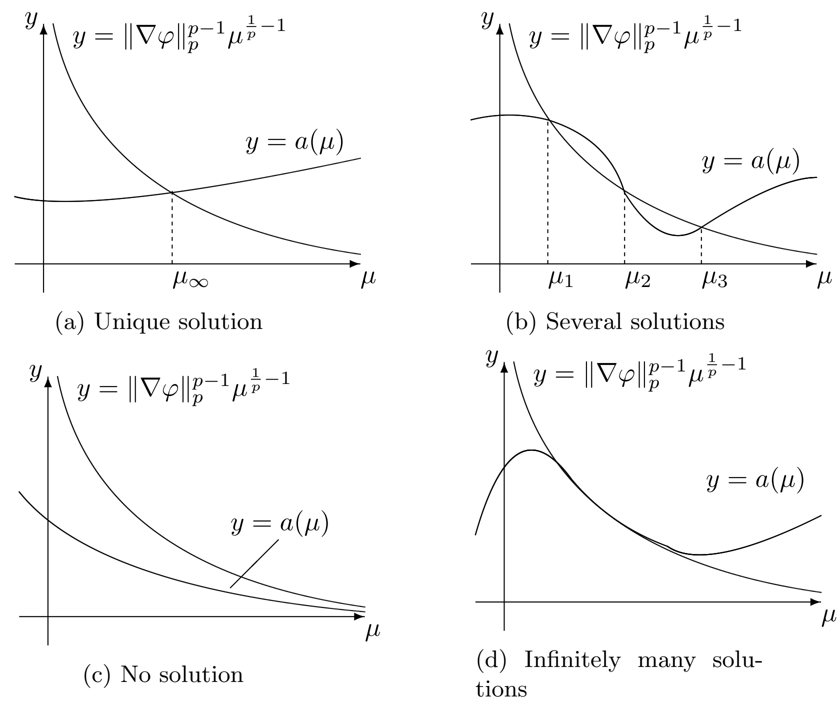
Thus it can happen that there is one solution, several, infinitely many solutions or no solution (just in case where a is not bounded away from 0). It depends on the function a, see Figure 4.1. In the case where (3.5) holds the set of stationary points is an interval which is reduced to a point when $a(s^p)s^{p-1}$ is increasing.

The solutions of the problem (4.2) can be also found as critical points of the energy $E(u)$, defined by

$$E(u) = \frac{1}{p}A\left(\int_\Omega |\nabla u|^p dx\right) - \langle f, v\rangle, \tag{4.10}$$

where

$$A(z) = \int_0^z a(s)ds \tag{4.11}$$



(a) Unique solution (b) Several solutions

(c) No solution (d) Infinitely many solutions

Figure 4.1

and

$$E'(u) = -\nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u - f. \tag{4.12}$$

If u_∞ is a critical point of E on $W_0^{1,p}(\Omega)$ then u_∞ is a solution to (4.2). Indeed, if u_∞ is a critical point then for arbitrary $v \in W_0^{1,p}(\Omega)$ it holds

$$\frac{d}{d\delta}E(u_\infty + \delta v)\Big|_{\delta=0}$$
$$= \left(a(\|\nabla(u_\infty + \delta v)\|_p^p) \int_\Omega |\nabla(u_\infty + \delta v)|^{p-2}\nabla(u_\infty + \delta v)\nabla v - \langle f, v\rangle \right)\Big|_{\delta=0} = 0.$$

Thus

$$a(\|\nabla u_\infty\|_p^p) \int_\Omega |\nabla u_\infty|^{p-2}\nabla u_\infty \nabla v - \langle f, v\rangle = 0, \quad \forall v \in W_0^{1,p}(\Omega),$$

namely u_∞ is a solution to (4.2) and a stationary point.

Theorem 4.2. *Let* (2.1) *holds,* $f \in W^{-1,q}(\Omega)$*, then* $E(u)$ *admits a global minimizer on* $W_0^{1,p}(\Omega)$.

Proof. To prove this theorem we will use the direct method of calculus of variations. We claim that E is coercive and bounded from below. Indeed, Hölder's and Poincaré's inequalities imply

$$|\langle f, u\rangle| \le |f|_{-1,q}\|\nabla u\|_p,$$

therefore

$$E(u) = \frac{1}{p}A(\|\nabla u\|_p^p) - \langle f, v\rangle \ge \frac{\lambda}{p}\|\nabla u\|_p^p - |f|_{-1,q}\|\nabla u\|_p. \tag{4.13}$$

Since $p > 1$ the coerciveness follows. Now coming back to (4.13) and using Young's inequality we obtain

$$E(u) \geq \frac{\lambda}{p}\|\nabla u\|_p^p - \frac{(|f|_{-1,q})^q}{\lambda^{\frac{q}{p}} q} - \frac{\lambda}{p}\|\nabla u\|_p^p = -\frac{1}{q}\left(\frac{|f|_{-1,q}}{\lambda^{\frac{1}{p}}}\right)^q. \tag{4.14}$$

Thus E is also bounded from below.

Let $u_n \in W_0^{1,p}(\Omega)$ be a minimizing sequence of E. From (4.13) it follows that u_n is bounded in $W_0^{1,p}(\Omega)$. Hence for some $u_\infty \in W_0^{1,p}(\Omega)$ we have

$$u_n \rightharpoonup u_\infty \quad \text{in} \quad W_0^{1,p}(\Omega).$$

Next we show that E is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. In fact, it holds that

$$\varliminf_{n\to\infty} \|\nabla u_n\|_p^p \geq \|\nabla u_\infty\|_p^p$$

(the norm is weakly lower semicontinuous). Considering a subsequence u_{n_k} such that

$$\varliminf_{n\to\infty} \|\nabla u_n\|_p^p = \lim_{k\to\infty} \|\nabla u_{n_k}\|_p^p$$

and due to the fact that u_{n_k} is a minimizing sequence we see

$$\begin{aligned}\inf_{W_0^{1,p}(\Omega)} E(u) = \lim_k E(u_{n_k}) &= \frac{1}{p}\int_0^{\lim \|\nabla u_{n_k}\|_p^p} a(s)ds - \langle f, u_\infty\rangle \\ &\geq \frac{1}{p}\int_0^{\|\nabla u_\infty\|_p^p} a(s)ds - \langle f, u_\infty\rangle = E(u_\infty),\end{aligned}$$

which implies u_∞ is a minimizer of E on $W_0^{1,p}(\Omega)$. Therefore, the result follows. □

Note that the minimizer might be not unique.

5. Remarks on the stationary points

Suppose first we are in case of Figure 5.2, then we have:

Theorem 5.1. *Let u_1 be the stationary point corresponding to μ_1 such that*

$$a(\mu) < \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall \mu \in (\underline{\mu}, \mu_1), \tag{5.1}$$

$$a(\mu) > \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall \mu \in (\mu_1, \overline{\mu}). \tag{5.2}$$

Then u_1 is a local minimizer for E. More precisely one has:

$$E(u_1) < E(u) \quad \forall u \neq u_1,\ \|\nabla u\|_p^p \in (\underline{\mu}, \overline{\mu}).$$

Proof. Recall that by Theorem 4.1 we have that

$$\mu_1 = \|\nabla u_1\|_p^p, \quad u_1 = \frac{\varphi}{a(\mu_1)^{\frac{1}{p-1}}}. \tag{5.3}$$

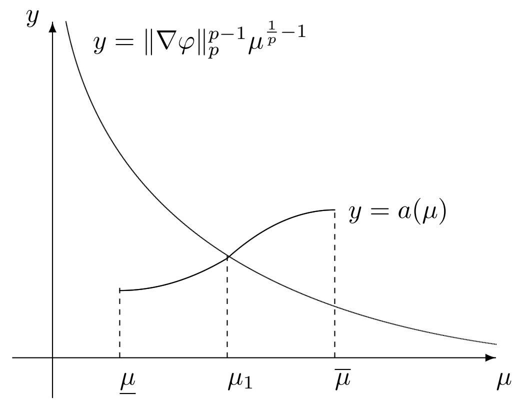


Figure 5.2

(i) Suppose $\|\nabla u\|_p^p > \mu_1$. Then from (2.8), (5.2) we have

$$
\begin{aligned}
E(u) - E(u_1) &= \frac{1}{p}\int_{\|\nabla u_1\|_p^p}^{\|\nabla u\|_p^p} a(s)ds - \langle f, u\rangle + \langle f, u_1\rangle \\
&> \frac{1}{p}\|\nabla\varphi\|_p^{p-1}\int_{\|\nabla u_1\|_p^p}^{\|\nabla u\|_p^p} s^{\frac{1}{p}-1}ds - \langle f, u\rangle + \langle f, u_1\rangle \\
&= \|\nabla\varphi\|_p^{p-1}\|\nabla u\|_p - \|\nabla\varphi\|_p^{p-1}\|\nabla u_1\|_p - \langle f, u\rangle + \langle f, u_1\rangle.
\end{aligned} \tag{5.4}
$$

From (4.3) and using Hölder's inequality we see

$$
\begin{aligned}
|\langle f, u\rangle| &= \left|\int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla u dx\right| \\
&\le \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}\left(\int_\Omega |\nabla\varphi|^{q(p-1)}dx\right)^{\frac{1}{q}} = \|\nabla u\|_p\|\nabla\varphi\|_p^{p-1},
\end{aligned} \tag{5.5}
$$

where $q = \frac{p}{p-1}$. Now by (4.4) and (5.3) we obtain

$$
\begin{aligned}
\langle f, u_1\rangle &= \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla u_1 dx \\
&= \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\frac{\nabla\varphi}{a(\mu_1)^{\frac{1}{p-1}}}dx = \|\nabla\varphi\|_p^p\frac{\|\nabla u_1\|_p}{\|\nabla\varphi\|_p} = \|\nabla\varphi\|_p^{p-1}\|\nabla u_1\|_p.
\end{aligned} \tag{5.6}
$$

Hence, combining (5.4) – (5.6) we derive

$$E(u) > E(u_1) \quad \text{for} \quad \|\nabla u\|_p^p > \mu_1.$$

(ii) Suppose now $\|\nabla u\|_p^p < \mu_1$. Then as above we get

$$E(u) - E(u_1) = -\frac{1}{p}\int_{\|\nabla u\|_p^p}^{\|\nabla u_1\|_p^p} a(s)ds - \langle f, u\rangle + \langle f, u_1\rangle,$$

and by (5.1), (5.5), (5.6) we can conclude

$$E(u) - E(u_1) > -\frac{1}{p}\|\nabla\varphi\|_p^{p-1}\int_{\|\nabla u\|_p^p}^{\|\nabla u_1\|_p^p} s^{\frac{1}{p}-1}ds - \langle f,u\rangle + \langle f,u_1\rangle \tag{5.7}$$

$$= -\|\nabla\varphi\|_p^{p-1}\|\nabla u_1\|_p + \|\nabla\varphi\|_p^{p-1}\|\nabla u\|_p - \langle f,u\rangle + \langle f,u_1\rangle \geq 0.$$

Thus we have

$$E(u) > E(u_1) \quad \text{for} \quad \|\nabla u\|_p^p \in (\underline{\mu},\overline{\mu}),\ u \neq u_1.$$

□

Remark 5.1. If $u \neq u_1$ one does not have necessarily $\|\nabla u\|_p^p \neq \|\nabla u_1\|_p^p = \mu_1$ and the proof of the theorem is incomplete. But if $\|\nabla u\|_p^p = \|\nabla u_1\|_p^p$ one has (see above) $0 \leq E(u) - E(u_1) = \langle f, u - u_1\rangle$. If this last quantity is vanishing we will show in Lemma 5.2 that $u = u_1$.

Remark 5.2. If one assumes

$$a(\mu) \leq \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall \mu \leq \mu_1,$$

$$a(\mu) \geq \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall \mu \geq \mu_1.$$

Then one gets only

$$E(u) \geq E(u_1).$$

Thus E can posses infinitely many global minimizers (see Figure 4.1d).

Lemma 5.1. *Let u_2 be the stationary point corresponding to μ_2 such that*

$$a(\mu) > \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall \mu \in (\underline{\mu}, \mu_2), \tag{5.8}$$

$$a(\mu) < \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall \mu \in (\mu_2, \overline{\mu}) \tag{5.9}$$

(*see Figure 5.3*). *Then u_2 is a point of local maximum for E in the direction of φ, where φ is the solution of the problem* (4.3). *More precisely one has:*

$$E(u_2) > E(u_2 + \delta\varphi),$$

for every $\delta \neq 0$ such that

$$\delta \geq -\frac{1}{a(\mu_2)^{\frac{1}{p-1}}}\ ,\ \|\nabla(u_2 + \delta\varphi)\|_p^p \in (\underline{\mu},\overline{\mu}).$$

Proof. As above by Theorem 4.1 we have that

$$\mu_2 = \|\nabla u_2\|_p^p, \quad u_2 = \frac{\varphi}{a(\mu_2)^{\frac{1}{p-1}}}. \tag{5.10}$$

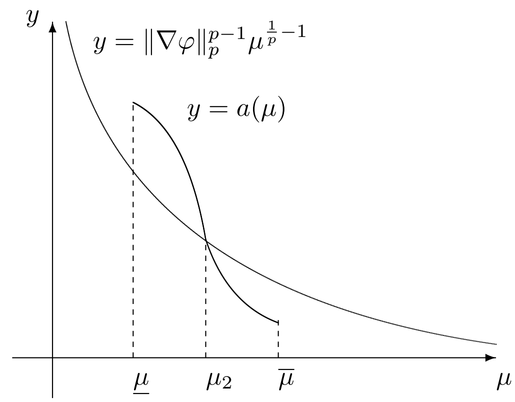


Figure 5.3

(i) Let us first assume that $\|\nabla(u_2+\delta\varphi)\|_p^p > \mu_2$. Then from (2.8), (4.3), (5.9) we have

$$
\begin{aligned}
E(u_2+\delta\varphi) - E(u_2) &= \frac{1}{p}\int_{\|\nabla u_2\|_p^p}^{\|\nabla(u_2+\delta\varphi)\|_p^p} a(s)ds - \delta\langle f,\varphi\rangle \\
&< \frac{1}{p}\|\nabla\varphi\|_p^{p-1}\int_{\|\nabla u_2\|_p^p}^{\|\nabla(u_2+\delta\varphi)\|_p^p} s^{\frac{1}{p}-1}ds - \delta\|\nabla\varphi\|_p^p \\
&= \|\nabla\varphi\|_p^{p-1}\big(\|\nabla(u_2+\delta\varphi)\|_p - \|\nabla u_2\|_p\big) - \delta\|\nabla\varphi\|_p^p \\
&= \|\nabla\varphi\|_p^{p-1}\left(\left|\frac{1}{a(\mu_2)^{\frac{1}{p-1}}}+\delta\right|\|\nabla\varphi\|_p - \frac{\|\nabla\varphi\|_p}{a(\mu_2)^{\frac{1}{p-1}}}\right) - \delta\|\nabla\varphi\|_p^p = 0,
\end{aligned}
\tag{5.11}
$$

if $\dfrac{1}{a(\mu_2)^{\frac{1}{p-1}}} + \delta \geq 0$. Thus it holds that

$$E(u_2+\delta\varphi) < E(u_2) \quad \text{for} \quad \|\nabla(u_2+\delta\varphi)\|_p^p > \mu_2.$$

(ii) Suppose now $\|\nabla(u_2+\delta\varphi)\|_p^p < \mu_2$. Then similarly, from (2.8), (4.3), (5.8) we get

$$
\begin{aligned}
E(u_2+\delta\varphi) - E(u_2) &= -\frac{1}{p}\int_{\|\nabla(u_2+\delta\varphi)\|_p^p}^{\|\nabla u_2\|_p^p} a(s)ds - \delta\langle f,\varphi\rangle \\
&< -\|\nabla\varphi\|_p^{p-1}\big(\|\nabla u_2\|_p - \|\nabla(u_2+\delta\varphi)\|_p\big) - \delta\|\nabla\varphi\|_p^p = 0
\end{aligned}
\tag{5.12}
$$

as in part (i).

Hence,

$$E(u_2+\delta\varphi) < E(u_2) \quad \text{for} \quad \|\nabla(u_2+\delta\varphi)\|_p^p < \mu_2.$$

□

Lemma 5.2. *Let u be a solution to the problem* (4.2). *Suppose that* (1.2) *holds and that $\psi \in W_0^{1,p}(\Omega)$, $\psi \neq 0$ is such that*

$$\langle f, \psi \rangle = 0. \tag{5.13}$$

Then

$$E(u+\psi) > E(u), \tag{5.14}$$

i.e. u is a point of minimum for E in any direction of the hyperplane defined by (5.13).

Proof. Let us consider ψ which satisfies (5.13). Then for $\|\nabla(u+\psi)\|_p > \|\nabla u\|_p$ from (2.1) we have

$$E(u+\psi) - E(u) = \frac{1}{p}\int_{\|\nabla u\|_p^p}^{\|\nabla(u+\psi)\|_p^p} a(s)ds > 0.$$

Hence it remains to prove that $\|\nabla(u+\psi)\|_p > \|\nabla u\|_p$. Due to (5.13) and since $a > 0$ we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \psi dx = 0.$$

Then we see

$$\begin{aligned}\|\nabla(u+\psi)\|_p^p - \|\nabla u\|_p^p &= \int_0^1 \frac{d}{ds}\int_\Omega |\nabla(u+s\psi)|^p dxds \\ &= p\int_0^1\int_\Omega |\nabla(u+s\psi)|^{p-2}\nabla(u+s\psi)\nabla\psi dxds \\ &= p\int_0^1\int_\Omega \big(|\nabla(u+s\psi)|^{p-2}\nabla(u+s\psi) - |\nabla u|^{p-2}\nabla u\big)\nabla\psi dxds.\end{aligned}$$

From Lemma 3.1 we have

$$\begin{aligned}&\big(|\nabla(u+s\psi)|^{p-2}\nabla(u+s\psi) - |\nabla u|^{p-2}\nabla u\big)\nabla(s\psi) \\ &\qquad\qquad \geq c_p\big(|\nabla(u+s\psi)| + |\nabla u|\big)^{p-2}|\nabla(s\psi)|^2.\end{aligned}$$

This shows that $\|\nabla(u+\psi)\|_p - \|\nabla u\|_p \geq 0$. If the equality holds then

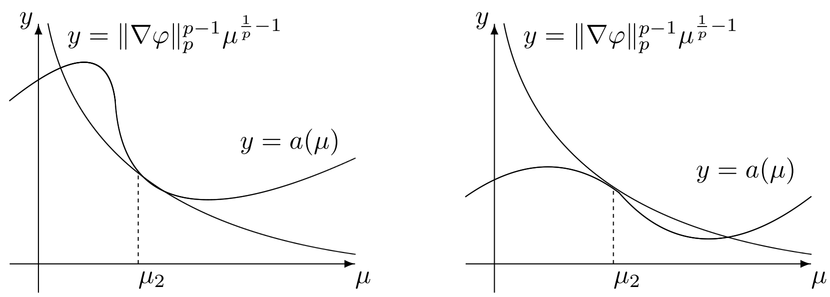
$$\big(|\nabla(u+s\psi)| + |\nabla u|\big)^{p-2}|\nabla\psi|^2 = 0 \ \text{ a.e. } x \in \Omega, \ s \in (0,1).$$

This implies that for $|\nabla u| = 0$ we have $|\nabla \psi| = 0$ and for $|\nabla u| \neq 0$ as well. Thus $\psi = 0$, which contradicts our assumptions. This completes the proof of the theorem. □

Theorem 5.2. *Let $f \not\equiv 0$,* (1.2) *holds, u_2 be a solution to* (4.2) *such that* (5.8), (5.9) *hold* (*see Figure 5.3, u_2 corresponds to μ_2*). *Then u_2 is a saddle point for the energy* (2.8).

Proof. The statement of the theorem is a consequence of Lemmas 5.1 and 5.2. □

Remark 5.3. The same situation occurs if the graph of a is not crossing the graph of y and touching it (see Figure 5.4).



(a) (b)

Figure 5.4

Theorem 5.3. *Let u^* be a solution of the problem* (4.2) *corresponding to the solutions μ^* of the equation* (4.4)*. Let*

$$y(s) = \|\nabla\varphi\|_p^{p-1} s^{\frac{1}{p}-1}, \tag{5.15}$$

then one has

$$E(u^*) = \frac{1}{p}\int_0^{\mu^*}(a(s) - y(s))ds.$$

Proof. From (2.8) one has

$$E(u^*) = \frac{1}{p}\int_0^{\|\nabla u^*\|_p^p} a(s)ds - \langle f, u^*\rangle.$$

Due to the definition of u^* (see (5.3)) we get

$$\frac{1}{p}\int_0^{\mu^*} y(s)ds = \frac{1}{p}\|\nabla\varphi\|_p^{p-1}\int_0^{\|\nabla u^*\|_p^p} s^{\frac{1}{p}-1}ds$$
$$= \|\nabla\varphi\|_p^{p-1}\|\nabla u^*\|_p = \langle f, u^*\rangle.$$

(see (5.6)). Hence, the result follows.

□

Corollary 5.1. *Let u_1, u_2 be two solutions of the problem* (4.2) *corresponding to the solutions $\mu_1 < \mu_2$ of the equation* (4.4) *respectively. Then one has*

$$E(u_1) - E(u_2) = -\frac{1}{p}\int_{\mu_1}^{\mu_2}(a(s) - y(s))ds =: -\frac{1}{p}A_{12} \tag{5.16}$$

and

$$A_{12} > 0 \ \Rightarrow\ E(u_1) < E(u_2);$$
$$A_{12} < 0 \ \Rightarrow\ E(u_2) < E(u_1);$$
$$A_{12} = 0 \ \Rightarrow\ E(u_1) = E(u_2),$$

Corollary 5.2. *Let u_1 and u_2 be two solutions of the problem* (4.2) *corresponding to the solutions $\mu_1 < \mu_2$ of the equation* (4.4)*. If we assume that*

$$a(\mu) > y(\mu) \quad \textit{for} \quad \mu_1 < \mu < \mu_2 \tag{5.17}$$

$$\left(\textit{resp.} \quad a(\mu) < y(\mu), \quad a(\mu) = y(\mu)\right), \tag{5.18}$$

then

$$E(u_1) < E(u_2) \quad (\textit{resp. } E(u_1) > E(u_2),\ E(u_1) = E(u_2)).$$

Corollary 5.3. *The absolute minimum of E corresponds to a point μ_∞ such that*

$$\int_{\mu_\infty}^{\mu} (a(s) - y(s))ds \geq 0, \quad \forall \mu > \mu_\infty, \quad \mu \textit{ corresponding to a stationary point,}$$

$$\int_{\mu}^{\mu_\infty} (a(s) - y(s))ds \leq 0, \quad \forall \mu < \mu_\infty, \quad \mu \textit{ corresponding to a stationary point.}$$

Therefore, due to Theorem 5.3 and its corollaries we can compare the energy at any two different stationary points and we can find a global minimizer of the energy $E(u)$.

Example 5.1. *Let u_i, $i = 1, 2, 3$ be solutions of the problem* (4.2) *corresponding to the solutions μ_i, $i = 1, 2, 3$ of the equation* (4.4) *such as on Figure 5.5. Then by*

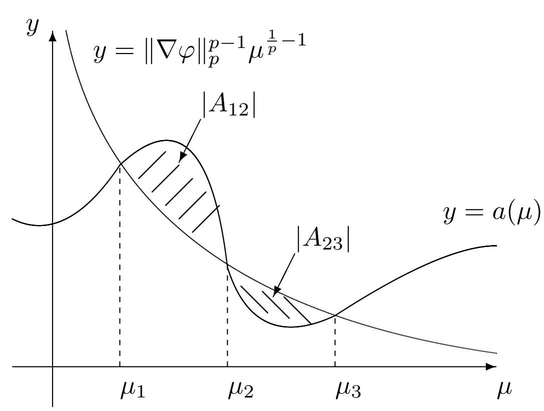


Figure 5.5. Several solutions

Corollary 5.2 we get that

$$E(u_1) < E(u_2), \quad E(u_3) < E(u_2).$$

It is left to compare the energy at the points u_1 and u_3. By Corollary 5.1 we see that

$$E(u_1) - E(u_3) = -\frac{1}{p}A_{13} = -\frac{1}{p}\left(|A_{12}| - |A_{23}|\right) < 0,$$

where

$$A_{ij} := \int_{\mu_i}^{\mu_j} (a(s) - y(s))ds, \quad i = 1, 2,\ j = 2, 3. \tag{5.19}$$

Hence, u_1 is a global minimizer of the energy $E(u)$ defined by (2.8).

Remark 5.4. We label the solutions to (4.4) as $\mu_1 < \mu_2 < \ldots < \mu_N$ with the convention that we choose only one point μ_i in the interval $(\underline{\mu}_i, \overline{\mu}_i)$ when the solutions consist of one interval $(\underline{\mu}_i, \overline{\mu}_i)$ (see Figure 5.6). We denote by $\{u\}_1, \{u\}_2, \ldots \{u\}_N$ the sets of solutions of (4.2), corresponding to $\mu_1 < \mu_2 < \ldots < \mu_N$ solutions of (4.4). Then due to our convention we see that $\{u\}_i$ can consist of one point or infinitely many points. By Corollary 5.2 for arbitrary $u \in \{u\}_i$, $i \in I := \{1, \ldots, N\}$

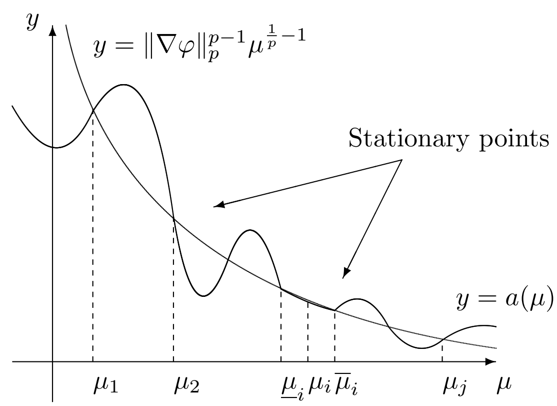


Figure 5.6. Infinitely many solutions

it holds that $E(u) = E_i$, $i \in I$. Therefore in the case when the stationary problem (4.2) is having infinitely many solutions, the energy (2.8) can have a unique, several or infinitely many global minimizers.

6. Asymptotic behaviour

We start with a lemma:

Lemma 6.1. *Let u be a weak solution to* (1.1) *and suppose that* (2.1) *holds. There exists a sequence t_k such that*

$$u_k = u(\cdot, t_k) \to u_\infty \text{ in } W_0^{1,p}(\Omega) \text{ as } t_k \to +\infty,$$

where u_∞ is a stationary point.

Proof. Taking $v = u_t$ in (2.3) we obtain

$$a(||\nabla u||_p^p) \int_\Omega |\nabla u|^{p-2} \nabla u \nabla u_t dx - \langle f, u_t \rangle = -|\partial_t u|_2^2,$$

$$\partial_t E(u) = -|\partial_t u|_2^2 \leq 0.$$

Hence, $E(u(t)) \leq E(u_0)$ and $E(u(t))$ decreases with the time. Remark that from (4.13) we have that $||\nabla u||_p^p$ is uniformly bounded in t. Since E is also bounded from below (see (4.14)), then it follows

$$E(u(t)) \to E_\infty \tag{6.1}$$

(E_∞ is some constant). From above we get

$$E(u(t)) - E(u(s)) = -\int_s^t |\partial_t u|_2^2(\xi)d\xi,$$

$$\int_s^\infty |\partial_t u|_2^2(\xi)d\xi < +\infty,$$

which implies for a sequence t_k

$$\partial_t u(\cdot, t_k) \to 0 \text{ in } L^2(\Omega).$$

From the equation in (2.3) – with $u_k = u(\cdot, t_k)$ – we obtain

$$\int_\Omega \partial_t u(\cdot, t_k) u_k dx + a(||\nabla u_k||_p^p) \int_\Omega |\nabla u_k|^p dx = \langle f, u_k\rangle. \tag{6.2}$$

We can show as in (2.6) that

$$|u(t)|_2^2 \le C.$$

Up to a subsequence it holds that

$$u(\cdot, t_k) = u_k \rightharpoonup u_\infty \text{ in } W_0^{1,p}(\Omega),$$

$$u_k \rightharpoonup u_\infty \text{ in } L^2(\Omega),$$

$$||\nabla u_k||_p^p \to l_\infty,$$

$$g_k = |\nabla u_k|^{p-2}\nabla u_k \rightharpoonup g_\infty \text{ in } (L^q(\Omega))^n.$$

Passing to the limit in (6.2) we see

$$a(l_\infty) l_\infty = \langle f, u_\infty\rangle. \tag{6.3}$$

Next taking $v \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ we get

$$\int_\Omega \partial_t u(\cdot, t_k) v dx + a(||\nabla u_k||_p^p) \int_\Omega |\nabla u_k|^{p-2} \nabla u_k \nabla v dx = \langle f, v\rangle.$$

Passing to the limit we derive

$$a(l_\infty) \int_\Omega g_\infty \nabla v dx = \langle f, v\rangle. \tag{6.4}$$

Since $a > 0$ combining (6.3), (6.4) we can conclude that

$$l_\infty = \int_\Omega g_\infty \nabla u_\infty dx.$$

We claim that $u_k \to u_\infty$ strongly in $W_0^{1,p}(\Omega)$. Indeed, for $p \ge 2$ there exists a constant $C_p > 0$ such that

$$\chi_k = \int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2}\nabla u_\infty\right) \nabla(u_k - u_\infty) dx$$
$$\ge C_p \int_\Omega |\nabla(u_k - u_\infty)|^p dx.$$

Developing

$$\chi_k = \int_\Omega |\nabla u_k|^p dx - \int_\Omega g_k \nabla u_\infty dx - \int_\Omega |\nabla u_\infty|^{p-2}\nabla u_\infty \nabla u_k dx + \int_\Omega |\nabla u_\infty|^p dx$$
$$\to l_\infty - \int_\Omega g_\infty \nabla u_\infty dx - \int_\Omega |\nabla u_\infty|^p dx + \int_\Omega |\nabla u_\infty|^p dx = 0.$$

This implies

$$l_\infty = \lim_k ||\nabla u_k||_p^p = ||\nabla u_\infty||_p^p,\ g_\infty = |\nabla u_\infty|^{p-2}\nabla u_\infty.$$

Hence u_∞ is a stationary point.

To show that $u_k \to u_\infty$ in $W_0^{1,p}(\Omega)$ strongly in case $1 < p < 2$ it is enough to notice that by Lemma 3.1 one has

$$c_p \int_\Omega |\nabla(u_k - u_\infty)|^2 \big(|\nabla u_k| + |\nabla u_\infty|\big)^{p-2} dx \le \chi_k \to 0.$$

Writing

$$\int_\Omega |\nabla(u_k - u_\infty)|^p dx = \int_\Omega |\nabla(u_k - u_\infty)|^p \big(|\nabla u_k| + |\nabla u_\infty|\big)^{\frac{(p-2)p}{2}} \big(|\nabla u_k| + |\nabla u_\infty|\big)^{\frac{(2-p)p}{2}} dx$$

and using Hölder's inequality with $\dfrac{2}{p}, \dfrac{2}{2-p}$ it comes

$$\int_\Omega |\nabla(u_k - u_\infty)|^p dx \le \left(\int_\Omega |\nabla(u_k - u_\infty)|^2 \big(|\nabla u_k| + |\nabla u_\infty|\big)^{p-2} dx\right)^{\frac{p}{2}} \left(\int_\Omega \big(|\nabla u_k| + |\nabla u_\infty|\big)^p dx\right)^{\frac{2-p}{2}} \le C\chi_k \to 0.$$

This completes the proof of the lemma.

□

Corollary 6.1. *Suppose that E admits a unique global minimizer u_∞ (u_∞ is also a solution to the problem* (4.2)*) and that the initial value u_0 of* (2.3) *satisfies $E(u_0) < E(u_i)$ for any stationary point $u_i \neq u_\infty$. Then*

$$u(\cdot, t) \to u_\infty \text{ in } W_0^{1,p}(\Omega) \text{ as } t \to +\infty.$$

Proof. Recall that we have

$$E(u) \le E(u_0) < E(u_i),\ u_i \neq u_\infty.$$

Then by Lemma 6.1 and (6.1) we get

$$E(u(t)) \to E(u_\infty),$$

where u_∞ is the global minimizer of E and a solution of the problem (4.2). Due to the fact that $u(t)$ is uniformly bounded in $W_0^{1,p}(\Omega)$ for some subsequence we have

$$u(\cdot, t_k) \rightharpoonup v_\infty \text{ in } W_0^{1,p}(\Omega).$$

Then by the weak lower semicontinuity of E (see the proof of Theorem 4.2) we obtain

$$E(u_\infty) = \lim_{t_k \to \infty} E(u(t_k)) \ge E(v_\infty).$$

Since u_∞ is a unique global minimizer of E, then it holds that $E(u_\infty) < E(v_\infty)$ for $u_\infty \neq v_\infty$, hence $u_\infty = v_\infty$. This holds for every subsequence and the convergence is in fact strong (see Lemma 6.1), therefore the result follows.

□

Remark 6.1. In the case where $a(s^p)s^{p-1}$ is increasing (see (3.5) and Remark 4.1) the problem has a single stationary point and for any initial data $u(\cdot, t)$ converges to this stationary point. We refer to [9] for more asymptotic analysis.

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