# Optimization problems involving the first Dirichlet eigenvalue and the torsional rigidity

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**Abstract**

We present some open problems and obtain some partial results for spectral optimization problems involving measure, torsional rigidity and first Dirichlet eigenvalue.

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## 1 Introduction

A shape optimization problem can be written in the very general form

$$\min\big\{F(\Omega)\ :\ \Omega\in\mathcal{A}\big\},$$

where $\mathcal{A}$ is a class of admissible domains and $F$ is a cost functional defined on $\mathcal{A}$. We consider in the present paper the case where the cost functional $F$ is related to the solution of an elliptic equation and involves the spectrum of the related elliptic operator. We speak in this case of *spectral optimization problems*. Shape optimization problems of spectral type have been widely considered in the literature; we mention for instance the papers [14], [**?**], [17], [19], [20], [21], [22], [29], and we refer to the books [16], [26], [27], and to the survey papers [2], [18], [25], where the reader can find a complete list of references and details.

In the present paper we restrict ourselves for simplicity to the Laplace operator $-\Delta$ with Dirichlet boundary conditions. Furthermore we shall assume that the admissible domains $\Omega$ are a priori contained in a given *bounded* domain $D\subset\mathbb{R}^d$. This assumption greatly simplifies several existence results that otherwise would require additional considerations in terms of concentration-compactness arguments [14], [31].

The most natural constraint to consider on the class of admissible domains is an inequality on their Lebesgue measure. Our admissible class $\mathcal{A}$ is then

$$\mathcal{A}=\big\{\Omega\subset D\ :\ |\Omega|\le 1\big\}.$$

Other kinds of constraints are also possible, but we concentrate here to the one above, referring the reader interested in possible variants to the books and papers quoted above.

The following two classes of cost functionals are the main ones considered in the literature.

*Integral functionals.* Given a right-hand side $f\in L^2(D)$, for every $\Omega\in\mathcal{A}$ let $u_\Omega$ be the unique solution of the elliptic PDE

$$-\Delta u=f \text{ in } \Omega,\qquad u\in H^1_0(\Omega).$$

The integral cost functionals are of the form

$$F(\Omega)=\int_\Omega j\big(x,u_\Omega(x),\nabla u_\Omega(x)\big)\,dx,$$

where $j$ is a suitable integrand that we assume convex in the gradient variable. We also assume that $j$ is bounded from below by

$$j(x,s,z)\ge -a(x)-c|s|^2,$$

with $a \in L^1(D)$ and $c$ smaller than the first Dirichlet eigenvalue of the Laplace operator $-\Delta$ in $D$. For instance, the energy $\mathcal{E}_f(\Omega)$ defined by

$$\mathcal{E}_f(\Omega) = \inf\left\{\int_D \Big(\frac{1}{2}|\nabla u|^2 - f(x)u\Big)\,dx \ : \ u \in H^1_0(\Omega)\right\},$$

belongs to this class since, integrating by parts its Euler-Lagrange equation, we have that

$$\mathcal{E}_f(\Omega) = -\frac{1}{2}\int_D f(x)u_\Omega\,dx,$$

which corresponds to the integral functional above with

$$j(x,s,z) = -\frac{1}{2}f(x)s.$$

The case $f = 1$ is particularly interesting for our purposes. We denote by $w_\Omega$ the *torsion function*, that is the solution of the PDE

$$-\Delta u = 1 \text{ in } \Omega, \qquad u \in H^1_0(\Omega),$$

and by the *torsional rigidity* $T(\Omega)$ the $L_1$ norm of $w_\Omega$,

$$T(\Omega) = \int_\Omega w_\Omega\,dx = -2\mathcal{E}_1(\Omega).$$

*Spectral functionals.* For every admissible domain $\Omega \in \mathcal{A}$ we consider the spectrum $\Lambda(\Omega)$ of the Laplace operator $-\Delta$ on $H^1_0(\Omega)$. Since $\Omega$ has a finite measure, the operator $-\Delta$ has a compact resolvent and so its spectrum $\Lambda(\Omega)$ is discrete:

$$\Lambda(\Omega) = \big(\lambda_1(\Omega), \lambda_2(\Omega), \dots\big),$$

where $\lambda_k(\Omega)$ are the eigenvalues counted with their multiplicity. The spectral cost functionals we may consider are of the form

$$F(\Omega) = \Phi\big(\Lambda(\Omega)\big),$$

for a suitable function $\Phi : \mathbb{R}^{\mathbb{N}} \to \overline{\mathbb{R}}$. For instance, taking $\Phi(\Lambda) = \lambda_k(\Omega)$ we obtain

$$F(\Omega) = \lambda_k(\Omega).$$

We take the torsional rigidity $T(\Omega)$ and the first eigenvalue $\lambda_1(\Omega)$ as prototypes of the two classes above and we concentrate our attention on cost functionals that depend on both of them. We note that, by the maximum principle, when $\Omega$ increases $T(\Omega)$ increases, while $\lambda_1(\Omega)$ decreases.

## 2 Statement of the problem

The optimization problems we want to consider are of the form

$$\min\big\{\Phi\big(\lambda_1(\Omega), T(\Omega)\big) \ : \ \Omega \subset D,\ |\Omega| \le 1\big\}, \tag{2.1}$$

where we have normalized the constraint on the Lebesgue measure of $\Omega$, and where $\Phi$ is a given continuous (or lower semi-continuous) and non-negative function. In the rest of this paper we often take for simplicity $D = \mathbb{R}^d$, even if most of the results are valid in the general case. For instance, taking $\Phi(a,b) = ka + b$ with $k$ a fixed positive constant, the quantity we aim to minimize becomes

$$k\lambda_1(\Omega) + T(\Omega) \qquad \text{with } \Omega \subset D, \text{ and } |\Omega| \le 1.$$

*Remark* 2.1. If the function $\Phi(a,b)$ is increasing with respect to $a$ and decreasing with respect to $b$, then the cost functional

$$F(\Omega) = \Phi\big(\lambda_1(\Omega), T(\Omega)\big)$$

turns out to be decreasing with respect to the set inclusion. Since both the torsional rigidity and the first eigenvalue are $\gamma$-continuous functionals and the function $\Phi$ is assumed lower semi-continuous, we can apply the existence result of [20], which provides the existence of an optimal domain.

In general, if the function $\Phi$ does not verify the monotonicity property of Remark 2.1, then the existence of an optimal domain is an open problem, and the aim of this paper is to discuss this issue. For simplicity of the presentation we limit ourselves to the two-dimensional case $d = 2$. The case of general $d$ does not present particular difficulties but requires the use of several $d-$ dependent exponents.

*Remark* 2.2. The following facts are well known.

i) If $B$ is a disk in $\mathbb{R}^2$ we have
$$T(B) = \frac{1}{8\pi}|B|^2.$$

ii) If $j_{0,1} \approx 2.405$ is the first positive zero of the Bessel functions $J_0(x)$ and $B$ is a disk of $\mathbb{R}^2$ we have
$$\lambda_1(B) = \frac{\pi}{|B|} j_{0,1}^2.$$

iii) The torsional rigidity $T(\Omega)$ scales as
$$T(t\Omega) = t^4 T(\Omega), \qquad \forall t > 0.$$

iv) The first eigenvalue $\lambda_1(\Omega)$ scales as
$$\lambda_1(t\Omega) = t^{-2}\lambda_1(\Omega), \qquad \forall t > 0.$$

v) For every domain $\Omega$ of $\mathbb{R}^2$ and any disk $B$ we have
$$|\Omega|^{-2} T(\Omega) \le |B|^{-2} T(B) = \frac{1}{8\pi}.$$

vi) For every domain $\Omega$ of $\mathbb{R}^2$ and any disk $B$ we have (Faber-Krahn inequality)
$$|\Omega|\lambda_1(\Omega) \ge |B|\lambda_1(B) = \pi j_{0,1}^2.$$

vii) A more delicate inequality is the so-called Kohler-Jobin inequality (see [28], [11]): for any domain $\Omega$ of $\mathbb{R}^2$ and any disk $B$ we have
$$\lambda_1^2(\Omega) T(\Omega) \ge \lambda_1^2(B) T(B) = \frac{\pi}{8} j_{0,1}^4.$$

We recall the following inequality, well known for planar regions (Section 5.4 in [30]), between torsional rigidity and first eigenvalue.

**Proposition 2.3.** *For every domain $\Omega \subset \mathbb{R}^d$ we have*

$$\lambda_1(\Omega) T(\Omega) \le |\Omega|.$$

*Proof.* By definition, $\lambda_1(\Omega)$ is the infimum of the Rayleigh quotient

$$\int_\Omega |\nabla u|^2 \, dx \Big/ \int_\Omega u^2 \, dx \qquad \text{over all } u \in H_0^1(\Omega),\ u \neq 0.$$

Taking as $u$ the torsion function $w_\Omega$, we have

$$\lambda_1(\Omega) \le \int_\Omega |\nabla w_\Omega|^2 \, dx \Big/ \int_\Omega w_\Omega^2 \, dx.$$

Since $-\Delta w_\Omega = 1$, an integration by parts gives

$$\int_\Omega |\nabla w_\Omega|^2 \, dx = \int_\Omega w_\Omega \, dx = T(\Omega),$$

while the Hölder inequality gives

$$\int_\Omega w_\Omega^2 \, dx \ge \frac{1}{|\Omega|} \left( \int_\Omega w_\Omega \, dx \right)^2 = \frac{1}{|\Omega|} \big(T(\Omega)\big)^2.$$

Summarizing, we have

$$\lambda_1(\Omega) \le \frac{|\Omega|}{T(\Omega)}$$

as required. □

*Remark* 2.4. The infimum of $\lambda_1(\Omega)T(\Omega)$ over open sets $\Omega$ of prescribed measure is zero. To see this, let $\Omega_n$ be the disjoint union of one ball of volume $1/n$ and $n(n-1)$ balls of volume $1/n^2$. Then the radius $R_n$ of the ball of volume $1/n$ is $(n\omega_d)^{-1/d}$ while the radius $r_n$ of the balls of volume $1/n^2$ is $(n^2\omega_d)^{-1/d}$, so that $|\Omega_n| = 1$,

$$\lambda_1(\Omega_n) = \lambda_1(B_{R_n}) = \frac{1}{R_n^2}\lambda_1(B_1) = (n\omega_d)^{2/d}\lambda_1(B_1),$$

and

$$\begin{aligned} T(\Omega_n) &= T(B_{R_n}) + n(n-1)T(B_{r_n}) = T(B_1)\big(R_n^{d+2} + n(n-1)r_n^{d+2}\big) \\ &= T(B_1)\omega_d^{-1-2/d}\big(n^{-1-2/d} + (n-1)n^{-1-4/d}\big). \end{aligned}$$

Therefore

$$\lambda_1(\Omega_n)T(\Omega_n) = \frac{\lambda_1(B_1)T(B_1)}{\omega_d} \frac{n^{2/d} + n - 1}{n^{1+2/d}},$$

which vanishes as $n \to \infty$.

In the next section we investigate the inequality of Proposition 2.3.

## 3 A sharp inequality between torsion and first eigenvalue

We define the constant

$$\mathcal{K}_d = \sup \left\{ \frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \ : \ \Omega \text{ open in } \mathbb{R}^d, \ |\Omega| < \infty \right\}.$$

We have seen in Proposition 2.3 that $\mathcal{K}_d \le 1$. The question is if the constant 1 can be improved.

Consider a ball $B$; performing the shape derivative as in [27], keeping the volume of the perturbed shapes constant, we obtain that for every field $V(x)$

$$\partial[\lambda_1(B)T(B)](V) = T(B)\partial[\lambda_1(B)](V) + \lambda_1(B)\partial[T(B)](V) = C_B \int_{\partial B} V \cdot n \, d\mathcal{H}^{d-1}$$

for a suitable constant $C_B$. Since the volume of the perturbed shapes is constant, we have

$$\int_{\partial B} V \cdot n \, d\mathcal{H}^{d-1} = 0,$$

where $\mathcal{H}^{d-1}$ denotes $(d-1)$-dimensional Hausdorff measure. This shows that balls are stationary for the functional

$$F(\Omega) = \frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|}.$$

Below we will show, by considering rectangles, that balls are not optimal. To do so we shall obtain a lower bound for the torsional rigidity of a rectangle.

**Proposition 3.1.** *In a rectangle $R_{a,b} = (-b/2, b/2) \times (-a/2, a/2)$ with $a \le b$ we have*

$$T(R_{a,b}) \ge \frac{a^3 b}{12} - \frac{11a^4}{180}.$$

*Proof.* Let us estimate the energy

$$\mathcal{E}_1(R_{a,b}) = \inf\left\{ \int_{R_{a,b}} \left( \frac{1}{2}|\nabla u|^2 - u \right) dx\, dy \ : \ u \in H_0^1(R_{a,b}) \right\}$$

by taking the function

$$u(x,y) = \frac{a^2 - 4y^2}{8}\theta(x),$$

where $\theta(x)$ is defined by

$$\theta(x) = \begin{cases} 1 & \text{,if } |x| \le (b-a)/2 \\ (b - 2|x|)/a & \text{,otherwise.} \end{cases}$$

We have

$$|\nabla u|^2 = \left( \frac{a^2 - 4y^2}{8} \right)^2 |\theta'(x)|^2 + y^2 |\theta(x)|^2,$$

so that

$$\begin{aligned} \mathcal{E}_1(R_{a,b}) &\le 2\int_0^{a/2} \left( \frac{a^2 - 4y^2}{8} \right)^2 dy \int_0^{b/2} |\theta'(x)|^2\, dx + 2\int_0^{a/2} y^2\, dy \int_0^{b/2} |\theta(x)|^2\, dx \\ &\quad - 4\int_0^{a/2} \frac{a^2 - 4y^2}{8}\, dy \int_0^{b/2} \theta(x)\, dx \\ &= \frac{a^4}{60} + \frac{a^3}{12}\left( \frac{b-a}{2} + \frac{a}{6} \right) - \frac{a^3}{6}\left( \frac{b-a}{2} + \frac{a}{4} \right) \\ &= -\frac{a^3 b}{24} + \frac{11a^4}{360}. \end{aligned}$$

The desired inequality follows since $T(R_{a,b}) = -2\mathcal{E}_1(R_{a,b})$. □

In $d$ dimensions we have the following.

**Proposition 3.2.** *If $\Omega_\varepsilon = \omega \times (-\varepsilon/2, \varepsilon, 2)$, where $\omega$ is a convex set in $\mathbb{R}^{d-1}$ with $|\omega| < \infty$, then*

$$T(\Omega_\varepsilon) = \frac{\varepsilon^3}{12}|\omega| + O(\varepsilon^4), \qquad \epsilon \downarrow 0.$$

We defer the proof to Section 5.

For a ball of radius $R$ we have

$$\lambda_1(B) = \frac{j_{d/2-1,1}^2}{R^2}, \qquad T(B) = \frac{\omega_d R^{d+2}}{d(d+2)}, \qquad |B| = \omega_d R^d, \tag{3.1}$$

so that

$$F(B) = \frac{\lambda_1(B) T(B)}{|B|} = \frac{j_{d/2-1,1}^2}{d(d+2)} := \alpha_d$$

For instance, we have

$$\alpha_2 \approx 0.723, \qquad \alpha_3 \approx 0.658, \qquad \alpha_4 \approx 0.612.$$

Moreover, since $j_{\nu,1} = \nu + O(\nu^{1/3})$, $\nu \to \infty$, we have that $\lim_{d\to\infty} \alpha_d = \frac{1}{4}$. A plot of $\alpha_d$ is given in Figure 1.

We now consider a slab $\Omega_\varepsilon = \omega \times (0, \varepsilon)$ of thickness $\varepsilon \to 0$. We have by separation of variables and Proposition 3.2 that

$$\lambda_1(\Omega_\varepsilon) = \frac{\pi^2}{\varepsilon^2} + \lambda_1(\omega) \approx \frac{\pi^2}{\varepsilon^2}, \qquad T(\Omega_\varepsilon) \approx \frac{\varepsilon^3 |\omega|}{12}, \qquad |\Omega_\varepsilon| = \varepsilon|\omega|,$$

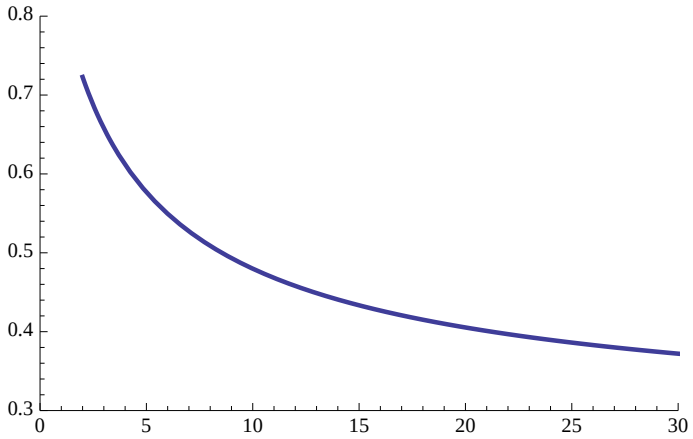


Figure 1: The plot of $\alpha_d$ for $2 \le d \le 30$.

so that

$$F(\Omega_\varepsilon) \approx \frac{\pi^2}{12} \approx 0.822.$$

This shows that in any dimension the slab is better than the ball. Using domains in $\mathbb{R}^d$ with $k$ small dimensions and $d-k$ large dimensions does not improve the value of the cost functional $F$. In fact, if $\omega$ is a convex domain in $\mathbb{R}^{d-k}$ and $B_k(\varepsilon)$ a ball in $\mathbb{R}^k$, then by Theorem 5.1 with $\Omega_\varepsilon = \omega \times B_k(\varepsilon)$ we have that

$$\lambda_1(\Omega_\varepsilon) \approx \frac{1}{\varepsilon^2}\lambda_1\big(B_k(1)\big), \qquad T(\Omega_\varepsilon) \approx \varepsilon^{k+2}|\omega|T(B_k(1)), \qquad |\Omega_\varepsilon| = \varepsilon^k|\omega||B_k(1)|,$$

so that

$$F(\Omega_\varepsilon) \approx \frac{j^2_{k/2-1,1}}{k(k+2)} \le \frac{\pi^2}{12}.$$

This supports the following.

**Conjecture 3.3.** *For every dimension $d$ we have $\mathcal{K}_d = \pi^2/12$, and no domain in $\mathbb{R}^d$ maximizes the functional $F$ for $d > 1$. The maximal value $\mathcal{K}_d$ is asymptotically reached by a thin slab $\Omega_\varepsilon = \omega \times (0, \varepsilon)$, with $\omega \subset \mathbb{R}^{d-1}$, as $\varepsilon \to 0$.*

# 4 The attainable set

In this section we bound the measure by $|\Omega| \le 1$. Our goal is to plot the subset of $\mathbb{R}^2$ whose coordinates are the eigenvalue $\lambda_1(\Omega)$ and the torsion $T(\Omega)$. It is convenient to change coordinates and to set for a given admissible domain $\Omega$,

$$x = \lambda_1(\Omega), \qquad y = \big(\lambda_1(\Omega)T(\Omega)\big)^{-1}.$$

In addition, define

$$E = \Big\{(x,y) \in \mathbb{R}^2 \ : \ x = \lambda_1(\Omega),\ y = \big(\lambda_1(\Omega)T(\Omega)\big)^{-1} \text{ for some } \Omega \text{ with } |\Omega| \le 1\Big\}.$$

Therefore, the optimization problem (2.1) can be rewritten as

$$\min\big\{\Phi\big(x, 1/(xy)\big) \ : \ (x,y) \in E\big\}.$$

**Conjecture 4.1.** *The set $E$ is closed.*

We remark that the conjecture above, if true, would imply the existence of a solution of the optimization problem (2.1) for many functions $\Phi$. Below we will analyze the variational problem in case $\Phi(x,y) = kx + \frac{1}{xy}$, where $k > 0$.

**Theorem 4.2.** *Let $d = 2, 3, \cdots$, and let*

$$k_d^* = \frac{1}{2d\omega_d^{4/d} j_{d/2-1,1}^2}.$$

*Consider the optimization problem*

$$\min\left\{k\lambda_1(\Omega) + T(\Omega) \ : \ |\Omega| \le 1\right\}. \tag{4.1}$$

*If $0 < k \le k_d^*$ then the ball with radius*

$$R_k = \left(\frac{2kdj_{d/2-1,1}^2}{\omega_d}\right)^{1/(d+4)} \tag{4.2}$$

*is the unique minimizer (modulo translations and sets of capacity 0).*
*If $k > k_d^*$ then the ball $B$ with measure 1 is the unique minimizer.*

*Proof.* Consider the problem (4.1) without the measure constraint

$$\min\left\{k\lambda_1(\Omega) + T(\Omega) \ : \ \Omega \subset \mathbb{R}^d\right\}. \tag{4.3}$$

Taking $t\Omega$ instead of $\Omega$ gives that

$$k\lambda_1(t\Omega) + T(t\Omega) = kt^{-2}\lambda_1(\Omega) + t^{d+2}T(\Omega).$$

The optimal $t$ which minimizes this expression is given by

$$t = \left(\frac{2k\lambda_1(\Omega)}{(d+2)T(\Omega)}\right)^{1/(d+4)}.$$

Hence (4.3) equals

$$\min\left\{(d+4)\left(\frac{k^{d+2}}{4(d+2)^{d+2}}T^2(\Omega)\lambda_1^{d+2}(\Omega)\right)^{1/(d+4)} \ : \ \Omega \subset \mathbb{R}^d\right\}. \tag{4.4}$$

By the Kohler-Jobin inequality in $\mathbb{R}^d$, the minimum in (4.4) is attained by any ball. Therefore the minimum in (4.3) is given by a ball $B_R$ such that

$$\left(\frac{2k\lambda_1(B_R)}{(d+2)T(B_R)}\right)^{1/(d+4)} = 1.$$

This gives (4.2). We conclude that the measure constrained problem (4.1) admits the ball $B_{R_k}$ as a solution whenever $\omega_d R_k^d \le 1$. That is $k \le k_d^*$.

Next consider the case $k > k_d^*$. Let $B$ be the open ball with measure 1. It is clear that

$$\min\{k\lambda_1(\Omega) + T(\Omega) \ : \ |\Omega| \le 1\} \le k\lambda_1(B) + T(B).$$

To prove the converse we note that for $k > k_d^*$,

$$\begin{aligned}\min\{k\lambda_1(\Omega) + T(\Omega) : |\Omega| \le 1\}& \\ \ge \min\{(k-k_d^*)\lambda_1(\Omega) : |\Omega| \le 1\} &+ \min\{k_d^*\lambda_1(\Omega) + T(\Omega) : |\Omega| \le 1\}.\end{aligned} \tag{4.5}$$

The minimum in the first term in the right hand side of (4.5) is attained for $B$ by Faber-Krahn, whereas the minimum in second term is attained for $B_{R_{k_d^*}}$ by our previous unconstrained calculation. Since $|B_{R_{k_d^*}}| = |B| = 1$ we have by (4.5) that

$$\begin{aligned}\min\{k\lambda_1(\Omega) + T(\Omega) : |\Omega| \le 1\}& \\ &\ge (k-k_d^*)\lambda_1(B) + k_d^*\lambda_1(B) + T(B) \\ &= k\lambda_1(B) + T(B).\end{aligned}$$

Uniqueness of the above minimizers follows by uniqueness of Faber-Krahn and Kohler-Jobin. □

It is interesting to replace the first eigenvalue in (4.1) be a higher eigenvalue. We have the following for the second eigenvalue.

**Theorem 4.3.** *Let $d = 2, 3, \cdots$, and let*

$$l_d^* = \frac{1}{2d(2\omega_d)^{4/d} j_{d/2-1,1}^2}.$$

*Consider the optimization problem*

$$\min \{l\lambda_2(\Omega) + T(\Omega) \ : \ |\Omega| \leq 1\}. \tag{4.6}$$

*If $0 < l \leq l_d^*$ then the union of two disjoint balls with radii*

$$R_l = \left( \frac{l d j_{d/2-1,1}^2}{\omega_d} \right)^{1/(d+4)} \tag{4.7}$$

*is the unique minimizer (modulo translations and sets of capacity 0).*
*If $l > l_d^*$ then union of two disjoint balls with measure $1/2$ each is the unique minimizer.*

*Proof.* First consider the unconstrained problem

$$\min \left\{ l\lambda_1(\Omega) + T(\Omega) \ : \ \Omega \subset \mathbb{R}^d \right\}. \tag{4.8}$$

Taking $t\Omega$ instead of $\Omega$ gives that

$$l\lambda_2(t\Omega) + T(t\Omega) = lt^{-2}\lambda_2(\Omega) + t^{d+2}T(\Omega).$$

The optimal $t$ which minimizes this expression is given by

$$t = \left( \frac{2l\lambda_2(\Omega)}{(d+2)T(\Omega)} \right)^{1/(d+4)}.$$

Hence (4.8) equals

$$\min \left\{ (d+4) \left( \frac{l^{d+2}}{4(d+2)^{d+2}} T^2(\Omega) \lambda_2^{d+2}(\Omega) \right)^{1/(d+4)} \ : \ \Omega \subset \mathbb{R}^d \right\}. \tag{4.9}$$

It follows by the Kohler-Jobin inequality, see for example Lemma 6 in [9], that the minimizer of (4.9) is attained by the union of two disjoint balls $B_R$ and $B_R'$ with the same radius. Since $\lambda_2(B_R \cup B_R') = \lambda_1(B_R)$ and $T(B_R \cup B_R') = 2T(B_R)$ we have, using (3.1), that the radii of these balls are given by (4.7). We conclude that the measure constrained problem (4.6) admits the union of two disjoint balls with equal radius $R_l$ as a solution whenever $2\omega_d R_l^d \leq 1$. That is $l \leq l_d^*$.

Next consider the case $l > l_d^*$. Let $\Omega$ be the union of two disjoint balls $B$ and $B'$ with measure $1/2$ each. Then

$$\min\{l\lambda_2(\Omega) + T(\Omega) \ : \ |\Omega| \leq 1\} \leq l\lambda_1(B) + 2T(B).$$

To prove the converse we note that for $l > l_d^*$,

$$\begin{aligned} &\min\{l\lambda_2(\Omega) + T(\Omega) \ : \ |\Omega| \leq 1\} \\ &\qquad \geq \min\{(l - l_d^*)\lambda_2(\Omega) \ : \ |\Omega| \leq 1\} + \min\{l_d^*\lambda_2(\Omega) + T(\Omega) \ : \ |\Omega| \leq 1\}. \end{aligned} \tag{4.10}$$

The minimum in the first term in the right hand side of (4.10) is attained for $B \cup B'$ by the Krahn-Szegö inequality, whereas the minimum in second term is attained for the union of two disjoint balls with radius $R_{l_d^*}$ by our previous unconstrained calculation. Since $|B_{R_{l_d^*}}| = 1/2 = |B| = |B'|$ we have by (4.10) that

$$\begin{aligned} \min\{l\lambda_2(\Omega) + T(\Omega) \ : \ |\Omega| \leq 1\} &\geq (l - l_d^*)\lambda_1(B) + l_d^*\lambda_1(B) + 2T(B) \\ &= l\lambda_1(B) + 2T(B). \end{aligned}$$

Uniqueness of the above minimizers follows by uniqueness of Krahn-Szegö and Kohler-Jobin for the second eigenvalue. □

To replace the first eigenvalue in (4.1) be the $j$'th eigenvalue ($j > 2$) is a very difficult problem since we do not know the minimizers of the $j$'th Dirichlet eigenvalue with a measure constraint nor the minimizer of the $j$'th Dirichlet eigenvalue a torsional rigidity constraint. However, if these two problems have a common minimizer then information similar to the above can be obtained.

Putting together the facts listed in Remark 2.2 we obtain the following inequalities.

(i) By Faber-Krahn inequality we have $x \geq \pi j_{0,1}^2 \approx 18.168$.

(ii) By Conjecture 3.3 (if true) we have $y \geq 12/\pi^2 \approx 1.216$.

(iii) By the bound on the torsion of Remark 2.2 v) we have $xy \geq 8\pi \approx 25.133$.

(iv) By the Kohler-Jobin inequality we have $y/x \leq 8/(\pi j_{0,1}^4) \approx 0.076$.

(v) The set $E$ is *conical*, that is if a point $(x_0, y_0)$ belongs to $E$, then all the half-line $\big\{(tx_0, ty_0) \ : \ t \geq 1\big\}$ in contained in $E$. This follows by taking $\Omega_t = \Omega/t$ and by the scaling properties iii) and iv) of Remark 2.2.

(vi) The set $E$ is *vertically convex*, that is if a point $(x_0, y_0)$ belongs to $E$, then all points $(x_0, ty_0)$ with $1 \leq t \leq 8/(\pi j_{0,1}^4)$ belong to $E$. To see this fact, let $\Omega$ be a domain corresponding to the point $(x_0, y_0) \in E$. The *continuous Steiner symmetrization* path $\Omega_t$ (with $t \in [0, 1]$) then continuously deforms the domain $\Omega = \Omega_0$ into a ball $B = \Omega_1$, preserving the Lebesgue measure and decreasing $\lambda_1(\Omega_t)$ (see [13] where this tool has been developed, and Section 6.3 of [16] for a short survey). The curve

$$x(t) = \lambda_1(\Omega_t), \qquad y(t) = \big(\lambda_1(\Omega_t) T(\Omega_t)\big)^{-1}$$

then connects the point $(x_0, y_0)$ to the Kohler-Jobin line $\big\{y = 8x/(\pi j_{0,1}^4)\big\}$, having $x(t)$ decreasing. Since $\big(x(t), y(t)\big) \in E$, the conicity of $E$ then implies vertical convexity.

A plot of the constraints above is presented in Figure 2. Some particular cases can be computed

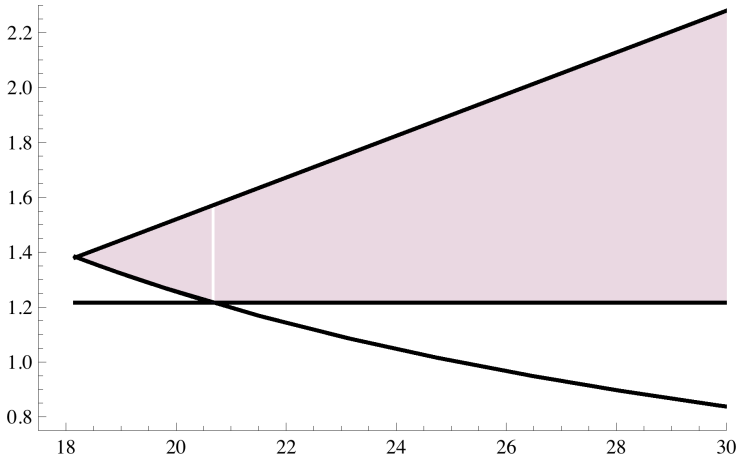


Figure 2: The admissible region $E$ is contained in the dark area.

explicitly. Consider $d = 2$, and let

$$\Omega = B_R \cup B_r, \text{with } B_R \cap B_r = \emptyset,\ r \leq R, \text{ and } \pi(R^2 + r^2) = 1.$$

An easy computation gives that

$$\lambda_1(\Omega) = \frac{j_{0,1}^2}{R^2}, \qquad T(\Omega) = \frac{2\pi^2 R^4 - 2\pi R^2 + 1}{8\pi},$$

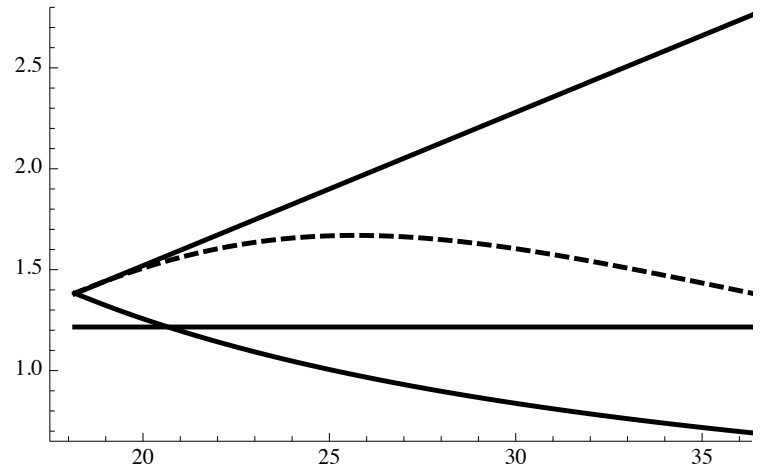


Figure 3: The dashed line corresponds to two disks of variable radii.

so that the curve

$$y = \frac{8\pi x}{x^2 - 2\pi j_{0,1}^2 x + 2\pi^2 j_{0,1}^4}, \qquad \pi j_{0,1}^2 \le x \le 2\pi j_{0,1}^2$$

is contained in $E$ (see Figure 3).
If we consider the rectangle

$$\Omega = (0,b) \times (0,a), \text{ with } a \le b, \text{ and } ab = 1,$$

we have by Proposition 3.1

$$\lambda_1(\Omega) = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \pi^2 \left(\frac{1}{a^2} + a^2\right), \qquad T(\Omega) \ge \frac{a^3 b}{12} - \frac{11a^4}{180} = \frac{a^2}{12} - \frac{11a^4}{180}.$$

Therefore

$$y \le h\big(x/(2\pi^2)\big), \qquad \text{where } h(t) = \frac{90}{\pi^2 t \left(11 + 15t - 22t^2 - (15+2t)\sqrt{t^2-1}\right)}, \quad t \ge 1.$$

By $E$ being conical the curve

$$y = h\big(x/(2\pi^2)\big) \qquad \pi^2 \le x < +\infty$$

is contained in $E$ (see Figure 4).

Besides the existence of optimal domains for the problem (2.1), the regularity of optimal shapes is another very delicate and important issue. Very little is known about the regularity of optimal domains for spectral optimization problems (see for instance [12], [15], [24], [31]); the cases where only the first eigenvalue $\lambda_1(\Omega)$ and the torsion $T(\Omega)$ are involved could be simpler and perhaps allow to use the free boundary methods developed in [1].

## 5 Torsional rigidity and the heat equation

It is well known that the rich interplay between elliptic and parabolic partial differential equations provide tools for obtaining results in one field using tools from the other. See for example the monograph by E. B. Davies [23], and [3, 5, 6, 7, 8, 10] for some more recent results. In this section we use some heat equation tools to obtain new estimates for the torsional rigidity. Before we do so we recall some basic facts relating the torsional rigidity to the heat equation. For an open set $\Omega$ in $\mathbb{R}^d$ with boundary $\partial\Omega$ we denote the Dirichlet heat kernel by $p_\Omega(x,y;t),\ x \in \Omega,\ y \in \Omega,\ t > 0$. So

$$u_\Omega(x;t) := \int_\Omega p_\Omega(x,y;t)\, dy,$$

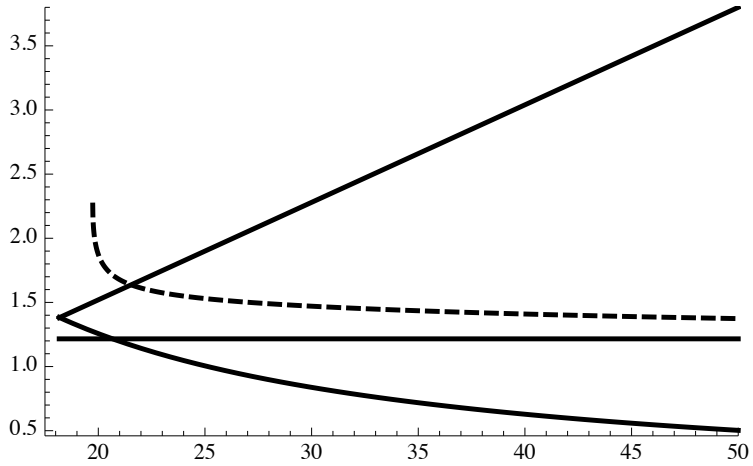


Figure 4: The dashed line is an upper bound to the line corresponding to rectangles.

is the unique weak solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u & x \in \Omega,\ t > 0, \\ \lim_{t \downarrow 0} u(x;t) = 1 & \text{in } L^2(\Omega), \\ u(x;t) = 0 & x \in \partial\Omega,\ t > 0. \end{cases}$$

The latter boundary condition holds at all regular points of $\partial\Omega$. We denote the heat content of $\Omega$ at time $t$ by

$$Q_\Omega(t) = \int_\Omega u_\Omega(x;t)\, dx.$$

Physically the heat content represents the amount of heat in $\Omega$ at time $t$ if $\Omega$ has initial temperature 1, while $\partial\Omega$ is kept at temperature 0 for all $t > 0$. Since the Dirichlet heat kernel is non-negative, and monotone in $\Omega$ we have that

$$0 \le p_\Omega(x,y;t) \le p_{\mathbb{R}^d}(x,y;t) = (4\pi t)^{-d/2} e^{-|x-y|^2/(4t)}. \tag{5.1}$$

It follows by either (5.1) or by the maximum principle that

$$0 \le u_\Omega(x;t) \le 1,$$

and that if $|\Omega| < \infty$ then

$$0 \le Q_\Omega(t) \le |\Omega|. \tag{5.2}$$

In the latter situation we also have an eigenfunction expansion for the Dirichlet heat kernel in terms of the Dirichlet eigenvalues $\lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots$, and a corresponding orthonormal set of eigenfunctions $\{\varphi_1, \varphi_2, \cdots\}$,

$$p_\Omega(x,y;t) = \sum_{j=1}^{\infty} e^{-t\lambda_j(\Omega)} \varphi_j(x)\varphi_j(y).$$

We note that the eigenfunctions are in $L^p(\Omega)$ for all $1 \le p \le \infty$. It follows by Parseval's formula that

$$Q_\Omega(t) = \sum_{j=1}^{\infty} e^{-t\lambda_j(\Omega)} \left( \int_\Omega \varphi_j\, dx \right)^2 \le e^{-t\lambda_1(\Omega)} \sum_{j=1}^{\infty} \left( \int_\Omega \varphi_j\, dx \right)^2 = e^{-t\lambda_1(\Omega)} |\Omega|. \tag{5.3}$$

Since the torsion function is given by

$$w_\Omega(x) = \int_0^\infty u_\Omega(x;t)\, dt,$$

we have that

$$T(\Omega) = \sum_{j=1}^{\infty} \lambda_j(\Omega)^{-1} \left( \int_\Omega \varphi_j \, dx \right)^2.$$

We recover Proposition 2.3. by integrating (5.3) with respect to $t$ over $[0, \infty)$:

$$T(\Omega) \leq \lambda_1(\Omega)^{-1} \sum_{j=1}^{\infty} \left( \int_\Omega \varphi_j \, dx \right)^2 = \lambda_1(\Omega)^{-1} |\Omega|.$$

Let $M_1$ and $M_2$ be two open sets in Euclidean space with finite Lebesgue measures $|M_1|$ and $|M_2|$ respectively. Let $M = M_1 \times M_2$. We have that

$$p_{M_1 \times M_2}(x, y; t) = p_{M_1}(x_1, y_1; t) p_{M_2}(x_2, y_2; t),$$

where $x = (x_1, x_2), y = (y_1, y_2)$. It follows that

$$Q_M(t) = Q_{M_1}(t) Q_{M_2}(t), \tag{5.4}$$

and

$$T(M) = \int_0^\infty Q_{M_1}(t) Q_{M_2}(t) \, dt. \tag{5.5}$$

Integrating (5.4) with respect to $t$, and using (5.2) for $M_2$ we obtain that

$$T(M) \leq T(M_1)|M_2|. \tag{5.6}$$

This upper bound should be "sharp" if the decay of $Q_{M_2}(t)$ with respect to $t$ is much slower than the decay of $Q_{M_1}(t)$. The result below makes this assertion precise in the case where $M_2$ is a convex set with $\mathcal{H}^{d_2-1}(\partial M_2) < \infty$. Here $\mathcal{H}^{d_2-1}$ denotes the $(d_2 - 1)$- dimensional Hausdorff measure.

**Theorem 5.1.** *Let $M = M_1 \times M_2$, where $M_1$ is an arbitrary open set with $|M_1| < \infty$, $d_1 = \dim M_1$, and $M_2$ is a convex open set with $\mathcal{H}^{d_2-1}(\partial M_2) < \infty$ and $d_2 = \dim M_2$. There exists a constant $\mathcal{C}_{d_2}$ depending on $d_2$ only such that*

$$T(M) \geq T(M_1)|M_2| - \mathcal{C}_{d_2} \lambda_1(M_1)^{-3/2} |M_1| \mathcal{H}^{d_2-1}(\partial M_2). \tag{5.7}$$

*Proof of Proposition* 3.2. Let $M_1 = (0, \epsilon) \subset \mathbb{R}$, $M_2 = \omega \subset \mathbb{R}^{d-1}$. Since the torsion function for $M_1$ is given by $x(\epsilon - x)/2$, $0 \leq x \leq \epsilon$ we have that $T(M_1) = \epsilon^3/12$. Then (5.6) proves the upper bound. The lower bound follows from (5.7) since $\lambda_1(M_1) = \pi^2/\epsilon^2$, $|M_1| = \epsilon$. □

*Proof.* With the notation above we have that

$$\begin{aligned} T(M) &= T(M_1)|M_2| - \int_0^\infty Q_{M_1}(t)(|M_2| - Q_{M_2}(t)) \, dt \\ &= T(M_1)|M_2| - \int_0^\infty Q_{M_1}(t) \int_{M_2} (1 - u_{M_2}(x_2; t)) \, dx_2 \, dt. \end{aligned}$$

We recall the following (Lemma 6.3 in [4]).

**Lemma 5.2.** *For any open set $\Omega$ in $\mathbb{R}^d$,*

$$u_\Omega(x; t) \geq 1 - 2 \int_{\{y \in \mathbb{R}^d : |y-x| > d(x)\}} p_{\mathbb{R}^d}(x, y; t) \, dy, \tag{5.8}$$

*where*

$$d(x) = \min\{|x - z| \ : \ z \in \partial\Omega\}.$$

Define for $r > 0$,

$$\partial M_2(r) = \{x \in M_2 : d(x) = r\}.$$

It is well known that (Proposition 2.4.3 in [16]) if $M_2$ is convex then

$$\mathcal{H}^{d_2-1}(\partial M_2(r)) \leq \mathcal{H}^{d_2-1}(\partial M_2). \tag{5.9}$$

By (5.3), (5.8) and (5.9) we obtain that

$$\begin{aligned}
&\int_0^\infty Q_{M_1}(t) \int_{M_2} (1 - u_{M_2}(x_2; t))\, dx_2\, dt \\
&\quad \leq 2|M_1|\mathcal{H}^{d_2-1}(\partial M_2) \int_0^\infty dt\, e^{-t\lambda_1(M_1)} \int_0^\infty dr \int_{\{z\in\mathbb{R}^{d_2}:|z-x|>r\}} p_{\mathbb{R}^{d_2}}(x, z; t)\, dz \\
&\quad = 2d_2\omega_{d_2}|M_1|\mathcal{H}^{d_2-1}(\partial M_2) \int_0^\infty dt\, e^{-t\lambda_1(M_1)} (4\pi t)^{-d_2/2} \int_0^\infty dr\, r^{d_2} e^{-r^2/(4t)} \\
&\quad = \mathcal{C}_{d_2}\lambda_1(M_1)^{-3/2}|M_1|\mathcal{H}^{d_2-1}(\partial M_2),
\end{aligned} \tag{5.10}$$

where

$$\mathcal{C}_{d_2} = \frac{\pi^{1/2} d_2 \Gamma((d_2+1)/2)}{\Gamma((d_2+2)/2)}.$$

□

It is of course possible, using the Faber-Krahn inequality for $\lambda_1(M_1)$, to obtain a bound for the right-hand side of (5.10) in terms of $|M_1|^{(d_1+3)/d_1}\mathcal{H}^{d_2-1}(\partial M_2)$.

Our next result is an improvement of Proposition 3.1. The torsional rigidity for a rectangle follows by substituting the formulae for $Q_{(0,a)}(t)$ and $Q_{(0,b)}(t)$ given in (5.12) below into (5.5). We recover the expression given on p.108 in [30]:

$$T(R_{a,b}) = \frac{64ab}{\pi^6} \sum_{k=1,3,\cdots} \sum_{l=1,3,\cdots} k^{-2} l^{-2} \left(\frac{k^2}{a^2} + \frac{l^2}{b^2}\right)^{-1}.$$

Nevertheless the following result is not immediately obvious.

**Theorem 5.3.**

$$\left| T(R_{a,b}) - \frac{a^3 b}{12} + \frac{31\zeta(5)a^4}{2\pi^5} \right| \leq \frac{a^5}{15b}, \tag{5.11}$$

*where*

$$\zeta(5) = \sum_{k=1}^{\infty} \frac{1}{k^5}.$$

*Proof.* A straightforward computation using the eigenvalues and eigenfunctions of the Dirichlet Laplacian on the interval together with the first identity in (5.3) shows that

$$Q_{(0,a)}(t) = \frac{8a}{\pi^2} \sum_{k=1,3,\ldots} k^{-2} e^{-t\pi^2 k^2/a^2}. \tag{5.12}$$

We write

$$Q_{(0,b)}(t) = b - \frac{4t^{1/2}}{\pi^{1/2}} + \left( Q_{(0,b)}(t) + \frac{4t^{1/2}}{\pi^{1/2}} - b \right). \tag{5.13}$$

The constant term $b$ in the right-hand side of (5.13) gives, using (5.12), a contribution

$$\begin{aligned}
\frac{8ab}{\pi^2} \int_{[0,\infty)} dt \sum_{k=1,3,\ldots} k^{-2} e^{-t\pi^2 k^2/a^2} &= \frac{8a^3 b}{\pi^4} \sum_{k=1,3,\ldots} k^{-4} \\
&= \frac{8a^3 b}{\pi^4} \left( \sum_{k=1}^{\infty} k^{-4} - \sum_{k=2,4,\ldots} k^{-4} \right) = \frac{15a^3 b}{2\pi^4}\zeta(4) \\
&= \frac{a^3 b}{12},
\end{aligned}$$

which jibes with the corresponding term in (5.11). In a very similar calculation we have that the $-\frac{4t^{1/2}}{\pi^{1/2}}$ term in the right-hand side of (5.13) contributes

$$-\frac{32a}{\pi^{5/2}}\int_{[0,\infty)} dt\, t^{1/2} \sum_{k=1,3,\ldots} k^{-2} e^{-t\pi^2 k^2/a^2} = -\frac{31\zeta(5)a^4}{2\pi^5},$$

which jibes with the corresponding term in (5.11). It remains to bound the contribution from the expression in the large round brackets in (5.11). Applying formula (5.12) to the interval $(0,b)$ instead and using the fact that $\sum_{k=1,3,\cdots} k^{-2} = \pi^2/8$ gives that

$$\begin{aligned} Q_{(0,b)}(t) - b + \frac{4t^{1/2}}{\pi^{1/2}} &= \frac{8b}{\pi^2} \sum_{k=1,3,\ldots} k^{-2}\left(e^{-t\pi^2 k^2/b^2} - 1\right) + \frac{4t^{1/2}}{\pi^{1/2}} \\ &= -\frac{8}{b} \sum_{k=1,3,\ldots} \int_{[0,t]} d\tau e^{-\tau\pi^2 k^2/b^2} + \frac{4t^{1/2}}{\pi^{1/2}} \\ &= -\frac{8}{b}\int_{[0,t]} d\tau \left(\sum_{k=1}^{\infty} e^{-\tau\pi^2 k^2/b^2} - \sum_{k=1}^{\infty} e^{-4\tau\pi^2 k^2/b^2}\right) + \frac{4t^{1/2}}{\pi^{1/2}}. \end{aligned} \tag{5.14}$$

In order to bound the right-hand side of (5.14) we use the following instance of the Poisson summation formula.

$$\sum_{k\in\mathbb{Z}} e^{-t\pi k^2} = t^{-1/2} \sum_{k\in\mathbb{Z}} e^{-\pi k^2/t},\ t>0.$$

We obtain that

$$\sum_{k=1}^{\infty} e^{-t\pi k^2} = \frac{1}{(4t)^{1/2}} - \frac{1}{2} + t^{-1/2} \sum_{k=1}^{\infty} e^{-\pi k^2/t},\ t>0.$$

Applying this identity twice (with $t = \pi\tau/b^2$ and $t = 4\pi\tau/b^2$ respectively) gives that the right-hand side of (5.14) equals

$$-\frac{8}{\pi^{1/2}}\int_{[0,t]} d\tau \left(\tau^{-1/2}\sum_{k=1}^{\infty} e^{-k^2b^2/\tau} - (4\tau)^{-1/2}\sum_{k=1}^{\infty} e^{-k^2b^2/(4\tau)}\right).$$

Since $k \mapsto e^{-k^2b^2/\tau}$ is non-negative and decreasing,

$$\sum_{k=1}^{\infty} \tau^{-1/2} e^{-k^2b^2/\tau} \le \tau^{-1/2}\int_{[0,\infty)} dk e^{-k^2b^2/\tau} = \pi^{1/2}(2b)^{-1}$$

. It follows that

$$\left| Q_{(0,b)}(t) - b + \frac{4t^{1/2}}{\pi^{1/2}} \right| \le \frac{8t}{b},\ t>0.$$

So the contribution of the third term in (5.13) to $T(R_{a,b})$ is bounded in absolute value by

$$\begin{aligned} \frac{64a}{\pi^2 b}\int_{[0,\infty)} dt\, t \sum_{k=1,3,\ldots} k^{-2} e^{-t\pi^2 k^2/a^2} &= \frac{64a^5}{\pi^6 b} \sum_{k=1,3,\ldots} k^{-6} \\ &= \frac{63a^5}{\pi^6 b}\zeta(6) \\ &= \frac{a^5}{15b}. \end{aligned}$$

This completes the proof of Theorem 5.3. □

The Kohler-Jobin theorem mentioned in Section 2 generalizes to $d$- dimensions. For any open set $\Omega$ with finite measure the ball minimizes $T(\Omega)\lambda_1^{(d+2)/2}$. We refer to [9] for an elementary heat equation proof of the following.

**Theorem 5.4.** *If $T(\Omega) < \infty$ then the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$ is discrete, and*

$$T(\Omega) \geq \left(\frac{2}{d+2}\right)\left(\frac{4\pi d}{d+2}\right)^{d/2} \sum_{k=1}^{\infty} \lambda_k^{-(d+2)/2}.$$

We obtain, using the Ashbaugh-Benguria theorem (p.86 in [26]) for $\lambda_1(\Omega)/\lambda_2(\Omega)$, that

$$T(\Omega)\lambda_1^{(d+2)/2} \geq \left(\frac{2}{d+2}\right)\left(\frac{4\pi d}{d+2}\right)^{d/2} \Gamma\left(1+\frac{d}{2}\right)\left(1+\left(\frac{\lambda_1(B)}{\lambda_2(B)}\right)^{(d+2)/2}\right). \tag{5.15}$$

The constant in the right-hand side of (5.15) is for $d = 2$ off by a factor $\frac{j_{0,1}^4 j_{1,1}^4}{8(j_{0,1}^4 + j_{1,1}^4)} \approx 3.62$ if compared with the sharp Kohler-Jobin constant. We also note the missing factor $m^{m/(m+2)}$ in the right-hand side of (57) in [9].

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# References

[1] H.W. Alt, L.A. Caffarelli: *Existence and regularity for a minimum problem with free boundary.* J. Reine Angew. Math., **325** (1981), 105–144.

[2] M.S. Ashbaugh: *Open problems on eigenvalues of the Laplacian.* In "Analytic and Geometric Inequalities and Applications", Math. Appl. **478**, Kluwer Acad. Publ., Dordrecht (1999), 13–28.

[3] M. van den Berg: *On the spectrum of the Dirichlet Laplacian for horn-shaped regions in $\mathbb{R}^n$ with infinite volume.* J. Funct. Anal., **58** (1984), 150–156.

[4] M. van den Berg, E.B. Davies: *Heat flow out of regions in $\mathbb{R}^m$.* Math. Z., **202** (1989), 463–482.

[5] M. van den Berg, R. Bañuelos: *Dirichlet eigenfunctions for horn-shaped regions and Laplacians on cross sections.* J. Lond. Math. Soc., **53** (1996), 503–511.

[6] M. van den Berg, E. Bolthausen: *Estimates for Dirichlet eigenfunctions.* J. Lond. Math. Soc., **59** (1999), 607–619.

[7] M. van den Berg, T. Carroll: *Hardy inequality and $L^p$ estimates for the torsion function.* Bull. Lond. Math. Soc., **41** (2009), 980–986.

[8] M. van den Berg: *Estimates for the torsion function and Sobolev constants.* Potential Anal., **36** (2012), 607–616.

[9] M. van den Berg, M. Iversen: *On the minimization of Dirichlet eigenvalues of the Laplace operator.* J. Geom. Anal., **23** (2013), 660–676.

[10] M. van den Berg, D. Bucur: *On the torsion function with Robin or Dirichlet boundary conditions.* J. Funct. Anal., **266** (2014), 1647–1666.

[11] L. Brasco: *On torsional rigidity and principal frequencies: an invitation to the Kohler-Jobin rearrangement technique.* ESAIM COCV, to appear.

[12] T. Briançon, J. Lamboley: *Regularity of the optimal shape for the first eigenvalue of the Laplacian with volume and inclusion constraints.* Ann. Inst. H. Poincaré Anal. Non Linéaire, **26** (2009), 1149–1163.

[13] F. Brock: *Continuous Steiner symmetrization.* Math. Nachr., **172** (1995), 25–48.

[14] D. Bucur: *Minimization of the k-th eigenvalue of the Dirichlet Laplacian.* Arch. Rational Mech. Anal., **206** (2012), 1073–1083.

[15] D. Bucur: *Regularity of optimal convex shapes.* J. Convex Anal., **10** (2003), 501–516.

[16] D. Bucur, G. Buttazzo: *Variational Methods in Shape Optimization Problems.* Progress in Nonlinear Differential Equations **65**, Birkhäuser Verlag, Basel (2005).

[17] D. Bucur, G. Buttazzo, I. Figueiredo: *On the attainable eigenvalues of the Laplace operator.* SIAM J. Math. Anal., **30** (1999), 527–536.

[18] G. Buttazzo: *Spectral optimization problems.* Rev. Mat. Complut., **24** (2011), 277–322.

[19] G. Buttazzo, G. Dal Maso: *Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions.* Appl. Math. Optim., **23** (1991), 17–49.

[20] G. Buttazzo, G. Dal Maso: *An existence result for a class of shape optimization problems..* Arch. Rational Mech. Anal., **122** (1993), 183–195.

[21] G. Buttazzo, B. Velichkov: *Shape optimization problems on metric measure spaces.* J. Funct. Anal., **264** (2013), 1–33.

[22] G. Buttazzo, B. Velichkov: *Some new problems in spectral optimization.* Banach Center Publications, (to appear), available on `http://cvgmt.sns.it` and on `http://arxiv.org`.

[23] E.B. Davies: *Heat kernels and spectral theory.* Cambridge University Press, Cambridge (1989).

[24] G. De Philippis, B. Velichkov: *Existence and regularity of minimizers for some spectral optimization problems with perimeter constraint.* Preprint available at: `http://cvgmt.sns.it/paper/2110/`.

[25] A. Henrot: *Minimization problems for eigenvalues of the Laplacian.* J. Evol. Equ., **3** (2003), 443–461.

[26] A. Henrot: *Extremum Problems for Eigenvalues of Elliptic Operators.* Frontiers in Mathematics, Birkhäuser Verlag, Basel (2006).

[27] A. Henrot, M. Pierre: *Variation et Optimisation de Formes. Une Analyse Géométrique.* Mathématiques & Applications **48**, Springer-Verlag, Berlin (2005).

[28] M.T. Kohler-Jobin: *Une méthode de comparaison isopérimétrique de fonctionnelles de domaines de la physique mathématique. I. Une démonstration de la conjecture isopérimétrique $P\lambda^2 \geq \pi j_0^4/2$ de Pólya et Szegö.* Z. Angew. Math. Phys., **29** (1978), 757–766.

[29] D. Mazzoleni, A. Pratelli: *Existence of minimizers for spectral problems.* J. Math. Pures Appl., **100** (2013), 433–453.

[30] G. Polya, G.Szegö: *Isoperimetric inequalities in mathematical physics.* Ann. of Math. Stud. **27**, Princeton University Press, Princeton (1951).

[31] B. Velichkov: *Existence and regularity results for some shape optimization problems.* PhD Thesis available at (`http://cvgmt.sns.it`).

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