HOPF'S LEMMA FOR A CLASS OF SINGULAR/DEGENERATE PDE-S

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Abstract. This paper concerns Hopf's boundary point lemma, in certain $C^{1,Dini}$-type domains, for a class of singular/degenerate PDE-s, including p-Laplacian. Using geometric properties of levels sets for harmonic functions in convex rings, we construct subsolutions to our equations that play the role of a barrier from below. By comparison principle we then conclude Hopf's lemma.

1. Introduction

In this paper we consider Hopf's lemma, in certain $C^{1,Dini}$-type domains, for the following type of operators

$$\Delta_H u := \operatorname{div}(H(|\nabla u|)\nabla u), \tag{1}$$

where $H(t) = t^{-1}h(t)$, $h(0) = 0$ and $h(t)$ is a monotone increasing continuous function. We will call the weak solutions of (1) H-harmonic. The additional condition we impose on the Dini modulus of continuity $\epsilon(t)$ is that the function $t\epsilon(t)$ is convex (more discussion in Section 1.2).

Equation (1) is the Euler-Lagrangian of the functional

$$I(v) = \int_D F(|\nabla v|)dx, \tag{2}$$

where $F(t) = \int_0^t h(\tau)d\tau$, $F \in C^1([0,\infty))$, $F(0) = F'(0) = 0$ and F is strictly convex. Here and in the sequel $D \subset \mathbb{R}^n$ ($n \geq 2$) is a domain.

This type of operators arise in applications dealing with flows where the flow-rate is proportional to

$$H(|\nabla u|)\nabla u = h(|\nabla u|)\frac{\nabla u}{|\nabla u|}.$$

1.1. **Conditions on F.**

Let us now list all the assumptions we impose on h, that can formally be divided into three groups:

- **Physical** (see equations (3)),
- **Coercive** (see equation (4), or (5)),
- **Technical** (see equation (6)).

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The so-called physical conditions have been already presented in the introduction.

(3)
- $h(0) = 0$: with vanishing gradient flow vanishes;
- monotonicity of $h(t)$: larger gradient $\Longrightarrow$ more flow;
- H depends only on $|\nabla u|$: isotropy with respect to the position and direction.

The coercivity condition is needed to assure the existence. It is well known that this problem is, in general, ill-posed if h is bounded. The best illustration is the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

in the annulus $B_2\backslash B_1$, with boundary data 0 on ∂B_2 and M on ∂B_1. For large enough M the catenoid cannot reach the level M without leaving the domain $B_2\backslash B_1$ and the equation has no solution in $W^{1,1}$.

To avoid this we can impose the strong condition

$$ct^{p-1} < h(t) < Ct^{p-1} \tag{4}$$

for some $p > 1$, which makes the application of the direct methods of the calculus of variations in the Sobolev space $W^{1,p}(D)$, $p > 1$, possible (see [D]). In the special case $H(t) = t^{p-2}$ we obtain the p-Laplacian.

Alternatively we can impose a weaker coercivity condition and work in Sobolev-Orlicz spaces $W^{1,F}(D)$. Let us shortly introduce these spaces following [RR] (see also [RR1]). The Orlicz norm is defined as follows

$$\|u\|_F = \min\left\{M \;\Big|\; \int_D F\left(\frac{|u|}{M}\right)dx \le F(1)\right\}.$$

This norm defines the Banach space $L^F(D)$.

If we now denote by g the inverse function of h and define the Legendre transform of F by

$$F^*(t) = \int_0^t g(\tau)d\tau,$$

then assuming $h(1) = 1$ one can easily prove using Young's inequality the generalization of Hölder's inequality

$$\int_D uvdx \le \|u\|_F\|v\|_{F^*}.$$

If both F and its Legendre transform F^* satisfy the so-called Δ_2 condition, i.e., there exists $t_0 > 0$, $C_0 > 0$ such that

$$F(2t) \le C_0F(t) \text{ and } F^*(2t) \le C_0F^*(t) \text{ for } t > t_0, \tag{5}$$

then

$$(L^F(D))^* = L^{F^*}(D)$$

and, in particular, $L^F(D)$ is reflexive, since $(F^*)^* = F$ (Theorem 10, page 112, [RR]).

Now analogously we can define the Sobolev-Orlicz space $W^{1,F}(D)$ by the Sobolev-Orlicz norm as sum of Orlicz norms of u and $|\nabla u|$. Under Δ_2 condition $W^{1,F}(D)$ will be reflexive and we can apply the direct methods of the calculus of variations.

The condition (5) is somewhat weaker since it requires polynomial of $F(t)$ growth only for large t and leaves more freedom for the behavior for small t.

In our proof the function

$$R(t) = \frac{F''(t)}{F'(t)}$$

plays an important role and we need the following technical condition on R. We assume that for every positive, monotone increasing, bounded function $c(s)$ in $\mathbb{R}^+$, $0 < c < c(s) < C < \infty$, there exist constants $\alpha > 0$ and $\beta > 0$, depending on function R, and constants c and C, such that

$$\int_t^T R(c(s)s)ds \geq \alpha \int_t^T R(\beta s)ds. \tag{6}$$

Remark 1. *Condition (6) is satisfied for more or less any "reasonable" function $R = F''/F'$. For monotone decreasing R one can take $\alpha = 1$, $\beta = C$ (this covers the case $F(t) = t^p$, $p > 1$). The authors think that it is easier to check the condition (6) for a given function F, than to try to introduce a broad class of functions satisfying it.*

Definition 2 (H-potential)**.** *For two convex domains $K_1 \Subset K_2$ we call the minimizer u of*

$$J(v) = \int_{K_2 \setminus K_1} F(|\nabla v|)dx \tag{7}$$

in the class of functions $\{v \in W_0^{1,F}(K_2) | \, v \equiv 1 \text{ on } K_1\}$ an H-potential (see [RR]*).*

1.2. **Liapunov-Dini boundary.** In the case of harmonic functions ($F(t) = t^2$) K.O. Widman ([W]) using the Green representation was able to prove a Hopf-type result for domains with Liapunov-Dini boundary (see below). Moreover, the following estimates for the second derivatives of the solution v of uniformly elliptic equation with Hölder continuous coefficients have been proved as well (see equation (2.4.1) in [W])

$$|D^2 v(x)| \leq C_D \frac{\epsilon(\delta(x))}{\delta(x)}, \tag{8}$$

where $\epsilon(t)$ is a Dini modulus of continuity (see (3) below) and $\delta(x)$ is the distance of the point x from the boundary. It is also shown that $C^{1,Dini}$ regularity is necessary for Hopf lemma in axially symmetric domains (see Remark 1 in [W]).

Since there is no Green representation for p-harmonic functions it is not possible to repeat Widman's direct estimates of the function's growth. Our proof is based on barrier construction and works for the general operator Δ_H under some regularity assumptions on the boundary.

Let us present the definition of Liapunov-Dini surface following [W].

Definition 3. *A modulus of continuity $\epsilon(r) \searrow 0$ as $r \to 0$ is called Dini modulus of continuity if $\int t^{-1}\epsilon(t)dt < \infty$.*

Definition 4. *A Liapunov-Dini surface S is a closed, bounded $(n-1)-$dimensional surface satisfying the following conditions:*
(a) At every point of S there is a uniquely defined tangent hyper-plane, and thus also a normal.
(b) There exits a Dini modulus of continuity $\epsilon(t)$ such that if β is the angle between two normals, and r is the distance between their foot points, then the inequality $\beta \le \epsilon(r)$ holds.
(c) There is a constant $\rho_S > 0$ such that for any point $x \in S$, any line parallel to the normal at x meets $S \cap B_{\rho_s}(x)$ at most once.

In simple words the above definitions says, that the surface S is locally the graph of a $C^{1,Dini}$ function in a ball of fixed radius.

Since in general the function $t\epsilon(t)$ is not convex we introduce a subclass of Dini modules of continuity as follows.

Definition 5. *A Dini modulus of continuity $\epsilon(r)$ is called convex-Dini if the function $t\epsilon(t)$ is convex.*

Remark 6. *Note that domains with $C^{1,\alpha}$ boundary are convex-Dini. The introduction of such a sub-class is necessary because our proof relies on the construction of barriers in convex rings with $C^{1,Dini}$ boundary. It it in general not true that for any Dini modulus of continuity $\epsilon(t)$ there is another Dini modulus of continuity $\tilde{\epsilon}(t) \ge \epsilon(t)$ such that $t\tilde{\epsilon}(t)$ is convex.*

Remark 7. *Note that if the Dini modulus of continuity is convex-Dini then a domain D with Liapunov-Dini boundary satisfies a kind of inner (outer) convex $C^{1,Dini}$ condition in the following sense: There exists a convex Liapunov-Dini domain K such that for any point $x_0 \in \partial D$ there exists a translation and rotation K_{x_0} of the domain K satisfying*

$$K_{x_0} \subset D, \ (K_{x_0} \subset \mathbb{R}^n \backslash D) \ \text{and} \ \partial K_{x_0} \cap \partial D = \{x_0\}.$$

Moreover, we can take

$$K = K_{r_D} = B_{r_D}((0, \dots, 0, r_D)) \cap \{x \mid x_n > 2|x'|\epsilon(|x'|)\},$$

where $x = (x', x_n)$*,* $r_D < \rho_{\partial D}/2$ *and* ϵ *is the convex-Dini modulus of continuity. Without loss of generality (if necessary by modifying* ϵ *at "corners" with* $\partial B_{r_D}((0,\dots,0,r_D))$*) we can assume that* K *has smooth boundary. Let us also observe that for any* $a > 0$ *by taking* r_D *small enough we can have*

$$B_{(1-a)r_D}((0,\dots,0,r_D)) \Subset K.$$

Let us assume that for r_D

(9) $$B_{\frac{3}{4}r_D}((0,\dots,0,r_D)) \Subset K.$$

In the sequel we will use as barriers the H-potentials in the convex rings

(10) $$K \backslash B_{r_D/2}((0,\dots,0,r_D)) \text{ and } B_{3r_D}((0,\dots,0,-r_D)) \backslash (-K),$$

where $-K$ is obtained from K by symmetry with respect to the origin. We will refer to convex rings (10) as inner and out convex rings.

2. The Main Result and its proof

The main result of this paper is the following extension of the Kjel-Ove Widman's result to a wider class of operators (1), for which p-Laplacian is a particular case, in Liapunov-Dini domains with convex-Dini modulus of continuity. As we will see later (Remark 9) our result yields the boundary Harnack principle for H-harmonic functions in domains with convex-Dini boundary.

Theorem 1. *Assume* u *is an* H*-harmonic function in the domain* D*. Further assume* $0 \in \partial D$*,* ∂D *satisfies the inner convex Dini condition at* 0 *and*

$$u(x) > u(0) \text{ for all } x \in D.$$

Then there exist positive constants r_0 *and* c *such that*

$$\max_{B_r \cap D} u(x) - u(0) > cr,$$

for $0 < r < r_0$*.*

Proof. Without loss of generality we can assume that the outer normal of ∂D a the origin is $(0,\dots,0,-1)$. Since for the solutions of (1) we have the maximum principle (see Appendix II) we need to construct a barrier in the inner convex ring from (10).

Actually we will construct barriers in arbitrary convex ring $K_2 \backslash K_1$, where $K_1 \Subset K_2$ are two convex domains with Liapunov-Dini boundary.

From the Hopf lemma for harmonic functions ([W]) we know that if $\Delta w = 0$ in $K_2 \backslash K_1$, with boundary values $w = 0$ on ∂K_2 and $w = 1$ on ∂K_1 then $\nabla w \neq 0$ on $\partial(K_2 \backslash K_1)$. Now we will prove the existence of a convex, smooth, monotone increasing function $f : [0,1] \to [0,1]$, $f(0) = 0$, $f(1) = 1$, $f'(0) > 0$, $f'(1) < \infty$ such that

(11) $$\Delta_H f(w) \geq 0$$

in $K_2 \setminus K_1$. This will mean that the function $f(w)$ is a subsolution for Δ_H, has non-vanishing gradient at any boundary point and thus can be used as a barrier.

We start by computing

$$\Delta_H f(w) = H(|\nabla f(w)|)\Delta f(w) + \frac{H'(|\nabla f(w)|)}{|\nabla f(w)|}\Delta_\infty f(w),$$

where $\Delta_\infty u = \nabla u D^2 u \nabla u$ is the ∞-Laplace operator. Using

$$\Delta f(w) = f'(w)\Delta w + f''(w)|\nabla w|^2 = f''(w)|\nabla w|^2,$$

and

$$\Delta_\infty f(w) = (f'(w))^3 \Delta_\infty w + (f'(w))^2 f''(w)|\nabla w|^4,$$

we arrive at

$$\Delta_H f(w) = H(f'(w)|\nabla w|) f''(w)|\nabla w|^2 + \frac{H'(f'(w)|\nabla w|)}{f'(w)|\nabla w|}\Big[(f'(w))^3\Delta_\infty w + (f'(w))^2 f''(w)|\nabla w|^4\Big]. \tag{12}$$

We thus need to find a function f, such that $f'(t) > 0$ for $t \in [0,1]$ and $\Delta_H f(w) \geq 0$. To comply with the latter we need (see (12))

$$f''(w)|\nabla w|^2\Big[H(f'(w)|\nabla w|) + f'(w)|\nabla w| H'(f'(w)|\nabla w|)\Big] \geq - \frac{H'(f'(w)|\nabla w|)}{f'(w)|\nabla w|}(f'(w))^3 \Delta_\infty w,$$

which after substitution $H'(t) = \frac{F''(t)}{t} - \frac{F'(t)}{t^2}$ and $H(t) + tH'(t) = F''(t)$ simplifies to

$$f''(w) \geq \left(\frac{F'(f'(w)|\nabla w|)}{F''(f'(w)|\nabla w|)} - f'(w)|\nabla w|\right)|\nabla w|^{-5}\Delta_\infty w. \tag{13}$$

Let us note that

$$0 = \Delta w = \partial_{\nu\nu} w - (n-1)\kappa \partial_\nu w, \tag{14}$$

where ν is the unit vector in the direction of ∇w and κ is the mean curvature of the level set; here we have used that the level sets of a positive harmonic potential are smooth. From this we conclude

$$\Delta_\infty w = (\partial_\nu w)^2 \partial_{\nu\nu} w = (n-1)\kappa |\nabla w|^3.$$

Since the level sets of a harmonic potential in a convex ring are convex (see [L]), the mean curvature and thus $\Delta_\infty w$ is positive near the boundary. This along with $f'(w) > 0$ and (13) (which is yet to be proven) implies that it is enough to find a function f such that

$$f''(w) R(f'(w)|\nabla w|) \geq |\nabla w|^{-5}\Delta_\infty w, \tag{15}$$

where $R(t) = \frac{F''(t)}{F'(t)}$.

The Hopf lemma proved in [W] for harmonic functions yields

$$0 < c < |\nabla w(x)| < C < \infty, \tag{16}$$

where the constants c and C depend only on the convex ring. Using this we easily obtain

$$c \min(w(x), 1 - w(x)) \le \delta(x) \le C \min(w(x), 1 - w(x))$$

and together with (8)

$$|\nabla w|^{-5} |\Delta_\infty w| \le c^{-3} |D^2 w| \le c^{-3} C_D \min \frac{\epsilon(\delta(x))}{\delta(x)} \le \\ c^{-3} C_D \frac{\epsilon(C \min(w, 1 - w))}{c \min(w, 1 - w)} =: \zeta(w),$$

where $\zeta(t) \in L^1([0, 1])$ depend only on the convex ring.

In order to have (15) we need to construct a function f such that

$$f''(w) R(f'(w)|\nabla w|) \ge \zeta(w). \tag{17}$$

For any $x \in K_2 \backslash K_1$ let us denote by ℓ_x the gradient flow line of w which contains x. Let us parametrize the curve ℓ_x by $w \in [0, 1]$. We can now integrate (17) on any ℓ_x in parameter w

$$\int_{w_1}^{w_2} f''(w) R(f'(w)|\nabla w|) dw \ge \int_{w_1}^{w_2} \zeta(w) dw, \tag{18}$$

Observe that since the level sets of w are convex the function $|\nabla w|$ on ℓ_x as a function of w are monotone increasing (see equation (14)), but on the other hand we know that it is bounded by (16). Thus we can apply our technical condition (6)

$$\int_{w_1}^{w_2} f''(w) R(f'(w)|\nabla w|) dw = \int_{f'(w_1)}^{f'(w_2)} R(c(s)s) ds \ge \alpha \int_{f'(w_1)}^{f'(w_2)} R(\beta s) ds,$$

where $s = f'(w)$ and the function $c(s) = |\nabla w|(s) > 0$ is a monotone function such that $c < c(s) < C$.

If we now construct a function f such that

$$\alpha \int_{f'(w_1)}^{f'(w_2)} R(\beta s) ds \ge \int_{w_1}^{w_2} \zeta(w) dw \tag{19}$$

for all $0 < w_1 < w_2 < 1$, then for this function f the inequality (18) will be satisfied for all gradient flow lines ℓ_x and thus the inequality (17) will be satisfied everywhere in $K_2 \backslash K_1$, and we would be done.

Since $F'(0) = 0$ and $F'(\infty) = \infty$, the function $R(t) = \frac{F''(t)}{F'(t)}$ is not integrable near zero and at $+\infty$, due to

$$\int_t^T R(\tau) d\tau = \log \frac{F'(T)}{F'(t)}.$$

As $w_1 \to 0$ and $w_2 \to 1$ the right hand side of (19) remains bounded ($\zeta \in L^1(0,1)$) and we can write

$$\alpha \int_{f'(w_1)}^{f'(w_2)} R(\beta s) ds = \frac{\alpha}{\beta} \left(\log F'(\beta f'(w_2)) - \log F'(\beta f'(w_1))\right) = \int_{w_1}^{w_2} \zeta(w) dw. \tag{20}$$

Now we can take $f'(0) = m > 0$ and construct

$$f'(w) = \beta^{-1} g\left(F'(\beta m) e^{\frac{\beta}{\alpha}\int_0^w \zeta(\tau)d\tau}\right), \tag{21}$$

where g is the inverse function of $h = F'$ on $\mathbb{R}^+$. Thus we obtained that (11) is satistied for $f(w) = \int_0^w f'(\tau)d\tau$, where f' is given by (21).

By changing the parameter $m \in (0,\infty)$ we can construct f such that $f(1)$ is any positive number.

The proof of the theorem now will easily follow from applying the barrier $f(w)$ in the inner convex ring (10) with the parameter m to be chosen such that

$$f(1) = \min_{x \in B_{r_D/2}((0,\dots,0,r_D))} u(x) > 0.$$

□

Remark 8. *Observe that if $f(w)$ is a sub-solution of (1) then in general we cannot say anything about the function $\alpha f(w)$, and we should construct the appropriate sub-solution by changing the parameter m in (21).*

Remark 9. *If the boundary value of a non-negative H-harmonic function u vanishes in a neighborhood of y and the boundary ∂D satisfies the outer convex $C^{1,Dini}$ condition at y, then we can apply the super-solution barrier $f(1) - f(w)$ in the outer convex ring (see (10)), and obtain the Lipschitz bound*

$$u(x) \leq CM\, dist(x, K),$$

for $x \in B_{r_D}(y) \cap D$, where r, C depending only on D and $M = \max_{B_{r_D}(y)} u$.

Remark 10. *One can make the condition (6) even weaker: there exists a constant $\alpha > 0$ and a monotone increasing continuous function*

$$L : \mathbb{R}^+ \to \mathbb{R}^+, \ \ L(0) = 0, \ \ L(\infty) = \infty$$

such that

$$\int_t^T R(c(s)s)ds \geq \alpha \int_{L(t)}^{L(T)} R(s)ds. \tag{22}$$

For functions F satisfying (22) one will obtain

$$f'(t) = L^{-1} g\left(F'(L(m)) e^{\frac{1}{\alpha}\int_0^t \zeta(\tau)d\tau}\right),$$

where g is the same as in (21).

3. Appendix: A comparison principle

The comparison principle for p-harmonic functions is well known (see [HKM]), but for the general operator Δ_H we could not find a reference. Therefore we shall present a proof of this.

Theorem 2. *Let u be a weak solution of (1)and v be its weak sub-solution in the domain D with C^1 boundary. Further let $v \leq u$ on ∂D in the sense of trace operator. Then $v \leq u$ in D.*

Proof. Let us denote by F^* the Legandre transform of F. Observe that $g(t) = (F^*(t))'$ is the inverse function of the function $h(t) = F'(t)$. By Young's inequality

$$ab \leq F(a) + F^*(b)$$

and the equality holds if and only if $b = h(a)$.

If u is weak solution of (1) then from the convexity of F it follows that

$$\int_D F(|\nabla u|)dx \leq \int_D F(|\nabla w|)dx$$

for any w such that $u - w \in W_0^{1,p}(D)$. Otherwise

$$\int_D F(|\nabla(u + t(w - u))|) \leq (1 - t)\int_D F(|\nabla u|)dx + t\int_D F(|\nabla w|)dx < \int_D F(|\nabla u|)dx - \epsilon t$$

for some $\epsilon > 0$ and differentiating in t we obtain

$$\int_D H(|\nabla u|)\nabla u \nabla(u - w)dx < 0, \tag{23}$$

which gives a contradiction after approximating $u - w \in W_0^{1,p}(D)$ by a test function $\phi \in C_0^\infty(D)$.

Let us now assume that $v \not\leq u$ and take as a test function $\psi = (v - u)^+$. By the definition of the sub-solution

$$\int_D H(|\nabla v|)\nabla v \nabla \psi dx \leq 0.$$

Thus

$$\int_{D_1} H(|\nabla v|)\nabla v \nabla v dx \leq \int_{D_1} H(|\nabla v|)\nabla v \nabla u dx,$$

where $D_1 = \text{supp}\psi \subset D$. Using now Young's inequality we obtain

$$\int_{D_1} H(|\nabla v|)\nabla v \nabla u dx \leq \int_{D_1} h(|\nabla v|)|\nabla u| dx \leq \int_{D_1} F(|\nabla u|)dx + \int_{D_1} F^*(h(|\nabla v|))dx. \tag{24}$$

Since $h(t) = F'(t) = tH(t)$, and Young's inequality is an equality for $b = h(a)$, we deduce

$$\int_{D_1} H(|\nabla v|)\nabla v \nabla v dx = \int_{D_1} h(|\nabla v|)|\nabla v| dx = \int_{D_1} F(|\nabla v|)dx + \int_{D_1} F^*(h(|\nabla v|))dx. \tag{25}$$

By (23)–(25) we arrive at

$$\int_{D_1} F(|\nabla v|)dx \leq \int_{D_1} F(|\nabla u|)dx,$$

where the inequality is strict unless $\nabla u = \nabla v$ a.e. in D_1; a contradiction in since $u - v \in W_0^{1,p}(D_1)$. □

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