ERROR ANALYSIS FOR AN ALE EVOLVING SURFACE FINITE ELEMENT METHOD

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ABSTRACT. We consider an arbitrary-Lagrangian-Eulerian evolving surface finite element method for the numerical approximation of advection and diffusion of a conserved scalar quantity on a moving surface. We describe the method, prove optimal order error bounds and present numerical simulations that agree with the theoretical results.

1. INTRODUCTION

For each $t \in [0,T], T > 0$, let $\Gamma(t)$ be a smooth connected hypersurface in $\mathbb{R}^{m+1}, m = 1,2,3$, oriented by the normal vector field $\boldsymbol{\nu}(\cdot,t)$, with $\Gamma_0 = \Gamma(0)$. We assume that there exists a diffeomorphism $\boldsymbol{G}(\cdot,t) : \Gamma_0 \to \Gamma(t)$, satisfying $\boldsymbol{G} \in \mathrm{C}^2([0,T], \mathrm{C}^2(\Gamma_0))$. We set $\boldsymbol{v}(\boldsymbol{G}(\cdot,t),t) = \partial_t \boldsymbol{G}(\cdot,t)$ with $\boldsymbol{G}(\cdot,0) = \boldsymbol{I}$ (the identity). Furthermore we assume that $\boldsymbol{v}(\cdot,t) \in \mathrm{C}^2(\Gamma(t))$. The given velocity field $\boldsymbol{v} = \boldsymbol{v}_{\boldsymbol{\nu}} + \boldsymbol{v}_{\boldsymbol{\mathcal{T}}}$ may contain both normal $\boldsymbol{v}_{\boldsymbol{\nu}}$ and tangential $\boldsymbol{v}_{\boldsymbol{\mathcal{T}}}$ components, i.e., $\boldsymbol{v}_{\boldsymbol{\nu}} = \boldsymbol{v} \cdot \boldsymbol{\nu}\boldsymbol{\nu}$ and $\boldsymbol{v}_{\boldsymbol{\mathcal{T}}} \cdot \boldsymbol{\nu} = 0$.

We focus on the following linear parabolic partial differential equation on $\Gamma(t)$;

$$\partial^\bullet_{\boldsymbol{v}} u + u \nabla_{\Gamma(t)} \cdot \boldsymbol{v} - \Delta_{\Gamma(t)} u = 0 \quad \text{on } \Gamma(t), \tag{1.1}$$

where, $\nabla_{\Gamma(t)} = \nabla - \nabla \cdot \boldsymbol{\nu}\boldsymbol{\nu}$ denotes the surface gradient, $\Delta_{\Gamma(t)} = \nabla_{\Gamma(t)} \cdot \nabla_{\Gamma(t)}$ the Laplace Beltrami operator and

$$\partial^\bullet_{\boldsymbol{v}} u = \partial_t u + \boldsymbol{v}_{\boldsymbol{\nu}} \cdot \nabla u + \boldsymbol{v}_{\boldsymbol{\mathcal{T}}} \cdot \nabla_{\Gamma(t)} u$$

is the material derivative with respect to the velocity field $\boldsymbol{v}$. In the case that the surface has a boundary we assume homogeneous Neumann boundary conditions, i.e.,

$$\left(u\boldsymbol{v} + \nabla_{\Gamma(t)} u\right) \cdot \boldsymbol{\mu} = 0 \quad \text{on } \partial\Gamma(t), \tag{1.2}$$

where $\boldsymbol{\mu}$ is the conormal to the boundary of the surface. If the boundary is empty no boundary conditions are needed. The upshot is that the total mass is conserved, i.e., $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} u = 0$.

We expect that our results apply also to the case of Dirichlet boundary conditions however in this setting one must also estimate the error due to boundary approximation which we neglect in this work.

The following variational formulation of (1.1) was derived in [1];

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)} \varphi = \int_{\Gamma(t)} u \partial^\bullet_{\boldsymbol{v}} \varphi, \tag{1.3}$$

where φ is a sufficiently smooth test function defined on the space-time surface

$$\mathcal{G}_T := \bigcup_{t \in [0,T]} \Gamma(t) \times \{t\}.$$

In [1], a finite element approximation was proposed for (1.3) using piecewise linear finite elements on a triangulated surface interpolating (at the nodes) $\Gamma(t)$, the vertices of the triangulated surface were moved with the material velocity (of points on $\Gamma(t)$) $\boldsymbol{v}$. In this work we adopt a similar setup, in that we propose a finite element approximation using piecewise linear finite elements on a triangulated surface interpolating (at the nodes) $\Gamma(t)$, however we move the vertices of the triangulated surface with the velocity $\boldsymbol{v}_a =$

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$\boldsymbol{v} + \boldsymbol{a}_{\mathcal{T}}$, where $\boldsymbol{a}_{\mathcal{T}}$ is an *arbitrary tangential velocity field* ($\boldsymbol{a}_{\mathcal{T}} \cdot \boldsymbol{\nu} = 0$) that if $\Gamma(t)$ has a boundary has zero conormal component on the boundary, i.e., for $t \in [0, T]$,

$$(\boldsymbol{v} - \boldsymbol{v}_a) \cdot \boldsymbol{\mu} = 0 \quad \text{on } \partial\Gamma(t). \tag{1.4}$$

Furthermore we assume that $\boldsymbol{v}_a$ satisfies the same smoothness assumptions as the material velocity $\boldsymbol{v}$, i.e., there exits a diffeomorphism $\tilde{\boldsymbol{G}}(\cdot, t) : \Gamma_0 \to \Gamma(t)$, satisfying $\tilde{\boldsymbol{G}} \in \mathrm{C}^2([0, T], \mathrm{C}^2(\Gamma_0))$ with $\boldsymbol{v}_a(\tilde{\boldsymbol{G}}(\cdot, t), t) = \partial_t \tilde{\boldsymbol{G}}(\cdot, t)$ and with $\tilde{\boldsymbol{G}}(\cdot, 0) = \boldsymbol{I}$ (the identity) and $\boldsymbol{v}_a(\cdot, t) \in \mathrm{C}^2(\Gamma(t))$.

For a sufficiently smooth function f, we have that

$$\partial^\bullet_{\boldsymbol{v}_a} f = \partial_t f + \boldsymbol{v}_a \cdot \nabla f = \partial_t f + \boldsymbol{v}_{\boldsymbol{\nu}} \cdot \nabla f + (\boldsymbol{a}_{\mathcal{T}} + \boldsymbol{v}_{\mathcal{T}}) \cdot \nabla_{\Gamma(t)} f = \partial^\bullet_{\boldsymbol{v}} f + \boldsymbol{a}_{\mathcal{T}} \cdot \nabla_{\Gamma(t)} f.$$

Thus we may write the following equivalent variational formulation to (1.1), which will form the basis for our finite element approximation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)} \varphi = \int_{\Gamma(t)} u \partial^\bullet_{\boldsymbol{v}_a} \varphi - u \boldsymbol{a}_{\mathcal{T}} \cdot \nabla_{\Gamma(t)} \varphi, \tag{1.5}$$

where φ is a sufficiently smooth test function defined on $\mathcal{G}_T$. We note that (1.5) may be thought of as a weak formulation of an advection diffusion-equation on a surface with material velocity $\boldsymbol{v}_a$, in which the advection $\boldsymbol{a}_{\mathcal{T}}$ is governed by some external process other than material transport. Hence the results we present are also an analysis of a numerical scheme for an advection diffusion equation on an evolving surface with another source of advective transport other than that due to the material velocity.

The original (Lagrangian) evolving surface finite element method (ESFEM) was proposed and analysed in [1], where optimal error bounds were shown for the error in the energy norm in the semidiscrete (space discrete) case. Optimal L_2 error bounds for the semidiscrete case were shown in [2] and an optimal error bound for the full discretisation was shown in [3]. High order Runge-Kutta time discretisations and BDF timestepping schemes for the ESFEM were analysed in [4] and [5] respectively. There has also been recent work on the analysis of ESFEM approximations of the Cahn-Hilliard equation on an evolving surface [6], scalar conservation laws on evolving surfaces [7] and the wave equation on an evolving surface [8]. For an overview of finite element methods for PDEs on fixed and evolving surfaces see [9]. Although the analytical results have thus far focussed on the case where the discrete velocity is an interpolation of the continuous material velocity, the Lagrangian setting, in many applications it proves computationally efficient to consider a mesh velocity which is different to the interpolated material velocity. In particular it appears that the arbitrary tangential velocity, that we consider in this study can be chosen such that beneficial mesh properties are observed in practice. This provides the motivation for this work in which we analyse an ESFEM where the material velocity of the mesh is different to (the interpolant of) the material velocity of the surface, i.e., an arbitrary Lagrangian-Eulerian ESFEM (ALE-ESFEM). We refer to [10] for extensive computational investigations of the ALE-ESFEM that we analyse in this study. For examples in the numerical simulation of mathematical models for cell motility and biomembranes, where the ALE approach proves computationally more robust than the Lagrangian approach, we refer to [11, 12, 13, 14].

Our main results are Theorems 4.3 and 5.4 where we show optimal order error bounds for the semidiscrete (space discrete, time continuous) and fully discrete numerical schemes. The fully discrete bound is proved for a second order backward difference time discretisation. An optimal error bound is also stated for an implicit Euler time discretisation. While the fully discrete bound is proved independently of the bound on the semidiscretisation, we believe that the analysis of the semidiscrete scheme may prove a useful starting point for the analysis of other time discretisations. We also observe that analysis holds for smooth flat surfaces, i.e, bulk domains with smooth boundary. Thus the analysis we present is also an analysis of ALE schemes for PDEs in evolving bulk domains. We report on numerical simulations of the fully discrete scheme that support our theoretical results and illustrate that the arbitrary tangential velocity may be chosen such that the meshes generated during the evolution are more suitable than in the

Lagrangian case. We also investigate numerically the long time behaviour of solutions to (1.1) with different initial data when the evolution of the surface is a periodic function of time. Our numerical results indicate that in the example we consider the solution converges to the same periodic solution for different initial data.

The original ESFEM was formulated for a surface with a smooth material velocity that had both normal and tangential components [1]. Hence many of the results from the literature are applicable in the present setting of a smooth arbitrary tangential velocity.

2. Setup

We start by introducing an abstract notation in which we formulate the problem.

2.1. **Definition** (Bilinear forms)**.** For $\varphi, \psi \in \mathrm{H}^1(\Gamma(t)), \boldsymbol{w} \in \mathrm{C}^2(\Gamma(t))$ we define the following bilinear forms

$$a(\varphi(\cdot,t),\psi(\cdot,t)) = \int_{\Gamma(t)} \nabla_{\Gamma(t)}\varphi(\cdot,t)\cdot\nabla_{\Gamma(t)}\psi(\cdot,t) \tag{2.1}$$

$$m(\varphi(\cdot,t),\psi(\cdot,t)) = \int_{\Gamma(t)} \varphi(\cdot,t)\psi(\cdot,t) \tag{2.2}$$

$$g(\varphi(\cdot,t),\psi(\cdot,t);\boldsymbol{w}(\cdot,t)) = \int_{\Gamma(t)} \varphi(\cdot,t)\psi(\cdot,t)\nabla_{\Gamma(t)}\cdot\boldsymbol{w}(\cdot,t) \tag{2.3}$$

$$b(\varphi(\cdot,t),\psi(\cdot,t);\boldsymbol{w}(\cdot,t)) = \int_{\Gamma(t)} \varphi(\cdot,t)\nabla_{\Gamma(t)}\psi(\cdot,t)\cdot\boldsymbol{w}(\cdot,t). \tag{2.4}$$

We may now write the equation (1.5) as

$$\frac{\mathrm{d}}{\mathrm{d}t}m(u,\varphi) + a(u,\varphi) = m\left(u,\partial^\bullet_{\boldsymbol{v}_a}\varphi\right) - b(u,\varphi;\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}). \tag{2.5}$$

In [1] the authors showed existence of a weak solution to (1.3) and hence a weak solution exists to the (reformulated) problem (1.5), furthermore for sufficiently smooth initial data the authors proved the following estimate for the solution of (1.3) and hence of (1.5)

$$\sup_{(0,T)}\|u\|^2_{\mathrm{L}_2(\Gamma)} + \int_0^T \left\|\nabla_{\Gamma(t)}u\right\|^2_{\mathrm{L}_2(\Gamma(t))} \leq c\|u_0\|^2_{\mathrm{L}_2(\Gamma(t))}, \tag{2.6}$$

$$\int_0^T \|\partial^\bullet_{\boldsymbol{v}}u\|^2_{\mathrm{L}_2(\Gamma(t))} + \sup_{(0,T)}\|\nabla_\Gamma u\|_{\mathrm{L}_2(\Gamma)} \leq c\|u_0\|^2_{\mathrm{H}^1(\Gamma(t))}. \tag{2.7}$$

We immediately conclude that as $\partial^\bullet_{\boldsymbol{v}_a}u - \partial^\bullet_{\boldsymbol{v}}u = \boldsymbol{a}_{\boldsymbol{\mathcal{T}}}\cdot\nabla_\Gamma u$ the bound (2.7) holds with the material derivative with respect to the material velocity replaced with the material derivative with respect to the ALE-velocity. See [15, 16, 17] for further discussion on the well-posedness of the weak formulation of the continuous problem.

3. Surface finite element discretisation

3.1. **Surface discretisation.** The smooth surface $\Gamma(t)$ is interpolated at nodes $\boldsymbol{X}_j(t) \in \Gamma(t)$ ($j = 1,\ldots,J$) by a discrete evolving surface $\Gamma_h(t)$. For each t, $\Gamma_h(t)$ is linear for $m=1$, polygonal for $m=2$ or polyhedral for $m=3$. These nodes move with velocity $\mathrm{d}\boldsymbol{X}_j(t)/\mathrm{d}t = \boldsymbol{v}_a(\boldsymbol{X}_j(t),t)$ and hence the nodes of the discrete surface $\Gamma_h(t)$ remain on the surface $\Gamma(t)$ for all $t\in[0,T]$. The discrete surface,

$$\Gamma_h(t) = \bigcup_{K(t)\in\mathcal{T}_h(t)} K(t)$$

is the union of m-dimensional simplices $K(t)$ that is assumed to form an admissible triangulation $\mathcal{T}_h(t)$; see [1] for details. We assume that the maximum diameter of the simplices is bounded uniformly in time and we denote this bound by h which we refer to as the mesh size.

We assume that for each point x on $\Gamma_h(t)$ the exists a unique point $\boldsymbol{p}(\boldsymbol{x},t)$ on $\Gamma(t)$ such that for $t \in [0,T]$ (see [9] for sufficient conditions such that this assumption holds)

$$\boldsymbol{x} = \boldsymbol{p}(\boldsymbol{x},t) + d(\boldsymbol{x},t)\boldsymbol{\nu}(\boldsymbol{x},t). \tag{3.1}$$

For a continuous function η_h defined on $\Gamma_h(t)$ we define its lift η_h^l onto $\Gamma(t)$ by extending constantly in the normal direction $\boldsymbol{\nu}$ (to the continuous surface) as follows

$$\eta_h^l(\boldsymbol{p},t) = \eta_h(\boldsymbol{x}(\boldsymbol{p},t),t) \quad \text{for } \boldsymbol{p} \in \Gamma(t), \tag{3.2}$$

where $\boldsymbol{x}(\boldsymbol{p},t)$ is defined by (3.1).

We assume that the triangulated and continuous surfaces are such that for each simplex $K(t) \in \Gamma_h(t)$ there is a unique $k(t) \subset \Gamma(t)$, whose edges are the unique projections of the edges of $K(t)$ onto $\Gamma(t)$. The union of the $k(t)$ induces an exact triangulation of $\Gamma(t)$ with curved edges. We refer for example to [9] for necessary conditions for this assumption to be satisfied.

3.2. **Definition** (Surface finite element spaces)**.** For each $t \in [0,T]$ we define the finite element spaces together with their associated lifted finite element spaces

$$\begin{aligned} \mathcal{S}_h(t) &= \left\{ \Phi \in C^0\left(\Gamma_h(t)\right) | \Phi|_K \text{ is linear affine for each } K \in \mathcal{T}_h(t) \right\}, \\ \mathcal{S}_h^l(t) &= \left\{ \varphi = \Phi^l | \Phi \in \mathcal{S}_h(t) \right\}. \end{aligned}$$

Let $\chi_j(\cdot,t)$ $(j = 1,\ldots,N)$ be the nodal basis of $\mathcal{S}_h(t)$, so that, denoting by $\{\boldsymbol{X}_j\}_{j=1}^J$ the vertices of $\Gamma_h(t)$, $\chi_j(\boldsymbol{X}_i(t),t) = \delta_{ji}$. The discrete surface moves with the piecewise linear velocity $\boldsymbol{V}_h^a$ and by $\boldsymbol{T}_h^a$ we denote the interpolant of the arbitrary tangential velocity $\boldsymbol{a}_{\mathcal{T}}$

$$\boldsymbol{V}_h^a(x,t) = \sum_{j=1}^J \boldsymbol{v}_a(\boldsymbol{X}_j(t),t)\chi_j(x,t), \tag{3.3}$$

$$\boldsymbol{T}_h(x,t) = \sum_{j=1}^J \boldsymbol{a}_{\mathcal{T}}(\boldsymbol{X}_j(t),t)\chi_j(x,t). \tag{3.4}$$

The discrete surface gradient is defined piecewise on each surface simplex $K(t) \in \mathcal{T}_h(t)$ as

$$\nabla_{\Gamma_h} g = \nabla g - \nabla g \cdot \boldsymbol{\nu}_h \boldsymbol{\nu}_h,$$

where $\boldsymbol{\nu}_h$ denotes the normal to the discrete surface defined element wise.

3.3. **Material derivatives.** We now relate the material velocity $\boldsymbol{V}_h^a$ of the triangulated surface Γ_h to the material velocity $\boldsymbol{v}_h^a$ of the smooth triangulated surface. For each $\boldsymbol{X}(t)$ on $\Gamma_h(t)$ there is a unique $\boldsymbol{Y}(t) = \boldsymbol{p}(\boldsymbol{X}(t),t) \in \Gamma(t)$ with

$$(\dot{\boldsymbol{Y}})(t) = \partial_t \boldsymbol{p}(\boldsymbol{X}(t),t) + \boldsymbol{V}_h^a(\boldsymbol{X}(t),t) \cdot \nabla \boldsymbol{p}(\boldsymbol{X}(t),t) := \boldsymbol{v}_h^a(\boldsymbol{p}(\boldsymbol{X}(t),t),t), \tag{3.5}$$

where $\boldsymbol{p}$ is as in (3.1). We note that $\boldsymbol{v}_h^a$ is not the interpolant of the velocity $\boldsymbol{v}_a$ into the space $\mathcal{S}_h^l$ (c.f., [2]). We denote by $\boldsymbol{t}_h^a = (\boldsymbol{T}_h^a)^l$ the lift of the velocity $\boldsymbol{T}_h^a$ to the smooth surface.

We are now in a position to define material derivatives on the triangulated surfaces. Given the velocity field $\boldsymbol{V}_h^a \in (\mathcal{S}_h)^{m+1}$ and the associated velocity $\boldsymbol{v}_h^a$ on $\Gamma(t)$ we define discrete material derivatives on $\Gamma_h(t)$ and $\Gamma(t)$ element wise as follows

$$\partial^\bullet_{h,\boldsymbol{V}_h^a} \Phi_h|_{K(t)} = \left(\partial_t \Phi_h + \boldsymbol{V}_h^a \cdot \nabla \Phi_h\right)|_{K(t)}, \tag{3.6}$$

$$\partial^\bullet_{h,\boldsymbol{v}_h^a} \varphi_h|_{k(t)} = \left(\partial_t \varphi_h + \boldsymbol{v}_h^a \cdot \nabla \varphi_h\right)|_{k(t)}. \tag{3.7}$$

The following transport property of the finite element basis functions was shown in [1]

$$\partial^{\bullet}_{h,\boldsymbol{V}^a_h}\chi_j = 0, \quad \partial^{\bullet}_{h,\boldsymbol{v}^a_h}\chi^l_j = 0, \tag{3.8}$$

which implies that for $\Phi_h = \sum_j \Phi_j(t)\chi_j(\cdot,t) \in \mathcal{S}_h(t)$ with $\varphi_h = \Phi^l_h \in \mathcal{S}^l_h(t)$

$$\partial^{\bullet}_{h,\boldsymbol{V}^a_h}\Phi_h(\cdot,t) = \sum_{j=1}^{J}\dot{\Phi}_j(t)\chi_j(\cdot,t), \quad \partial^{\bullet}_{h,\boldsymbol{v}^a_h}\varphi_h(\cdot,t) = \sum_{j=1}^{J}\dot{\Phi}_j(t)\chi_j(\cdot,t)^l. \tag{3.9}$$

We now introduce the notation we need to formulate and analyse the fully discrete scheme. Let N be a positive integer, we define the uniform timestep $\tau = T/N$. For each $n \in \{0,\dots,N\}$ we set $t^n = n\tau$. For a discrete time sequence f^n, $n \in \{0,\dots,N\}$ we introduce the notation

$$\partial_\tau f^n = \frac{1}{\tau}\left(f^{n+1} - f^n\right). \tag{3.10}$$

For $n \in \{0,\dots,N\}$ we denote by $\mathcal{S}^n_h = \mathcal{S}_h(t^n)$ and by $\mathcal{S}^{n,l}_h = \mathcal{S}^l_h(t^n)$. For $j = \{1,\dots,J\}$, we set

$$\chi_j(\cdot,t^n) = \chi^n_j, \quad \chi^l_j(\cdot,t^n) = \chi^{n,l}_j \tag{3.11}$$

and employ the notation

$$\Phi^n_h = \sum_{j=1}^{J}\Phi^n_j\chi^n_j \in \mathcal{S}^n_h, \quad \varphi^n_h = \Phi^{n,l} \in \mathcal{S}^{n,l}_h. \tag{3.12}$$

Following [3] we find it convenient to define for $\alpha = -1, 0, 1$ and $t \in [t^{n-1}, t^{n+1}]$

$$\underline{\Phi}^{n+\alpha}_h(\cdot,t) = \sum_{j=1}^{J}\Phi^{n+\alpha}_j\chi_j(\cdot,t) \in \mathcal{S}_h(t), \tag{3.13}$$

$$\underline{\varphi}^{n+\alpha}_h(\cdot,t) = \left(\underline{\Phi}^{n+\alpha}_h(\cdot,t)\right)^l \in \mathcal{S}^l_h(t). \tag{3.14}$$

We will also make use of the following notation for $t \in [t^n, t^{n+1}]$

$$\Phi^L_h(\cdot,t) = \frac{t^{n+1}-t}{\tau}\Phi^n_h(\cdot) + \frac{t-t^n}{\tau}\Phi^{n+1}_h(\cdot), \tag{3.15}$$

$$\varphi^L_h(\cdot,t) = \frac{t^{n+1}-t}{\tau}\varphi^n_h(\cdot) + \frac{t-t^n}{\tau}\varphi^{n+1}_h(\cdot). \tag{3.16}$$

We now introduce a concept of material derivatives for time discrete functions as defined in [3]. Given $\Phi^n_h \in \mathcal{S}^n_h$ and $\Phi^n_h \in \mathcal{S}^{n+1}_h$ we define the time discrete material derivative as follows

$$\partial^{\bullet}_{h,\boldsymbol{V}^a_h}\Phi^n_h = \sum_{j=1}^{J}\partial_\tau\Phi^n_j\chi^n_j \in \mathcal{S}^n_h, \quad \partial^{\bullet}_{h,\boldsymbol{v}^a_h}\varphi^n_h = \sum_{j=1}^{J}\partial_\tau\Phi^n_j\chi^{n,l}_j \in \mathcal{S}^{n,l}_h. \tag{3.17}$$

The following observations are taken from [3], for $n \in 0,\dots,N-1$

$$\partial^{\bullet}_{h,\boldsymbol{V}^a_h}\chi^n_j = 0, \partial^{\bullet}_{h,\boldsymbol{v}^a_h}\chi^{n,l}_j = 0. \tag{3.18}$$

On $[t^{n-1}, t^{n+1}]$, for $\alpha = -1, 0, 1$

$$\partial^{\bullet}_{h,\boldsymbol{V}^a_h}\underline{\Phi}^{n+\alpha}_h = 0, \quad \partial^{\bullet}_{h,\boldsymbol{v}^a_h}\underline{\varphi}^{n+\alpha}_h = 0, \tag{3.19}$$

which implies

$$\underline{\Phi}^{n+1}_h(\cdot,t^n) = \Phi^n_h + \tau\partial^{\bullet}_{h,\boldsymbol{V}^a_h}\Phi^n_h, \quad \underline{\varphi}^{n+1}_h(\cdot,t^n) = \varphi^n_h + \tau\partial^{\bullet}_{h,\boldsymbol{v}^a_h}\varphi^n_h. \tag{3.20}$$

3.4. **Definition** (Discrete bilinear forms)**.** We define the analogous bilinear forms to those defined in Definition 2.1 as follows, for $\Phi_h \in \mathcal{S}_h(t)$, $\Psi_h \in \mathcal{S}_h(t)$ and $\boldsymbol{W}_h \in (\mathcal{S}_h(t))^{m+1}$

$$
\begin{aligned}
a_h\left(\Phi_h(\cdot,t),\Psi_h(\cdot,t)\right) &= \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)}\Phi_h(\cdot,t)\cdot\nabla_{\Gamma_h(t)}\Psi_h(\cdot,t)\\
m_h\left(\Phi_h(\cdot,t),\Psi_h(\cdot,t)\right) &= \int_{\Gamma_h(t)} \Phi_h(\cdot,t)\Psi_h(\cdot,t)\\
g_h\left(\Phi_h(\cdot,t),\Psi_h(\cdot,t);\boldsymbol{W}_h(\cdot,t)\right) &= \int_{\Gamma_h(t)} \Phi_h(\cdot,t)\Psi_h(\cdot,t)\nabla_{\Gamma_h(t)}\cdot\boldsymbol{W}_h(\cdot,t)\\
b_h\left(\Phi_h(\cdot,t),\Psi_h(\cdot,t);\boldsymbol{W}_h(\cdot,t)\right) &= \int_{\Gamma_h(t)} \Phi_h(\cdot,t)\boldsymbol{W}_h(\cdot,t)\cdot\nabla_{\Gamma_h(t)}\Psi_h(\cdot,t)\\
\tilde{a}_h\left(\Phi_h(\cdot,t),\Psi_h(\cdot,t);\boldsymbol{W}_h(\cdot,t)\right) &=\\
&\int_{\Gamma_h(t)} \left(\nabla_{\Gamma_h}\cdot\boldsymbol{V}_h^a - 2\mathcal{D}(\boldsymbol{V}_h^a)\right)\nabla_{\Gamma_h(t)}\Phi_h(\cdot,t)\cdot\nabla_{\Gamma_h(t)}\Psi_h(\cdot,t)\\
\tilde{b}_h\left(\Phi_h(\cdot,t),\Psi_h(\cdot,t);\boldsymbol{W}_h(\cdot,t);\boldsymbol{V}_h^a(\cdot,t)\right) &=\\
&\int_{\Gamma_h(t)} \nabla_{\Gamma_h}\cdot\boldsymbol{V}_h^a\left(\Phi\boldsymbol{W}_h\cdot\nabla_{\Gamma_h}\Psi\right) + \int_{\Gamma_h(t)} \Phi\boldsymbol{W}_h\cdot\mathcal{B}(\boldsymbol{V}_h^a,\boldsymbol{\nu}_h)\nabla_{\Gamma_h}\Psi,
\end{aligned}
$$

with the deformation tensors $\mathcal{B}(\cdot,\cdot)$ and $\mathcal{D}(\cdot)$ as defined in Lemma A.1.

3.5. **Transport formula.** We recall some results proved in [2] and [3] that state (time continuous) transport formulas on the triangulated surfaces and define an adequate notion of discrete in time transport formulas and certain corollaries. The proofs of the transport formulas on the lifted surface (i.e., the smooth surface) follow from the formula given in Lemma A.1, the corresponding proofs on the triangulated surface Γ_h follow once we note that we may apply the same transport formula stated in Lemma A.1 (with the velocity of Γ_h replacing the velocity of $\Gamma(t)$) element by element.

We note the transport formula are shown for a triangulated surface with a material velocity that is the interpolant of a velocity that has both normal and tangential components. Hence the formula may be applied directly to the present setting where the velocity of the triangulated surface $\boldsymbol{V}_h^a$ is the interpolant of the velocity $\boldsymbol{v}_a$.

3.6. **Lemma** (Triangulated surface transport formula)**.** *Let $\Gamma_h(t)$ be an evolving admissible triangulated surface with material velocity $\boldsymbol{V}_h^a$. Then for $\Phi_h, \Psi_h, \boldsymbol{W}_h \in \mathcal{S}_h(t)\times\mathcal{S}_h(t)\times(\mathcal{S}_h(t))^{m+1}$,*

$$
\frac{\mathrm{d}}{\mathrm{d}t}m_h\left(\Phi_h,\Psi_h\right) = m_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a}\Phi_h,\Psi_h\right) + m_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a}\Psi_h,\Phi_h\right) + g_h\left(\Phi_h,\Psi_h;\boldsymbol{V}_h^a\right) \tag{3.21}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}a_h\left(\Phi_h,\Psi_h\right) = a_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a}\Phi_h,\Psi_h\right) + a_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a}\Psi_h,\Phi_h\right) + \tilde{a}_h\left(\Phi_h,\Psi_h;\boldsymbol{V}_h^a\right) \tag{3.22}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}b_h\left(\Phi_h,\Psi_h;\boldsymbol{W}_h\right) &= b_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a}\Phi_h,\Psi_h;\boldsymbol{W}_h\right) + b_h\left(\Phi_h,\partial^\bullet_{h,\boldsymbol{V}_h^a}\Psi_h;\boldsymbol{W}_h\right)\\
&\quad + b_h\left(\Phi_h,\Psi_h;\partial^\bullet_{\boldsymbol{V}_h^a}\boldsymbol{W}_h\right) + \tilde{b}_h\left(\Phi_h,\Psi_h;\boldsymbol{W}_h;\boldsymbol{V}_h^a\right).
\end{aligned} \tag{3.23}
$$

Let $\Gamma(t)$ be an evolving surface made up of curved elements $k(t)$ whose edges move with velocity $\boldsymbol{v}_h^a$. Then for $\varphi, \psi, \boldsymbol{w} \in \mathrm{H}^1(\mathcal{G}_T) \times \mathrm{H}^1(\mathcal{G}_T) \times (\mathrm{C}^1(\mathcal{G}_T))^{m+1}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} m(\varphi,\psi) = m\left(\partial^\bullet_{h,\boldsymbol{v}_h^a}\varphi,\psi\right) + m\left(\varphi,\partial^\bullet_{h,\boldsymbol{v}_h^a}\psi\right) + g(\varphi,\psi;\boldsymbol{v}_h^a) \tag{3.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} a(\varphi,\psi) = a\left(\partial^\bullet_{h,\boldsymbol{v}_h^a}\varphi,\psi\right) + a\left(\varphi,\partial^\bullet_{h,\boldsymbol{v}_h^a}\psi\right) + \tilde{a}(\varphi,\psi;\boldsymbol{v}_h^a) \tag{3.25}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} b(\varphi,\psi;\boldsymbol{w}) = {} & b\left(\partial^\bullet_{h,\boldsymbol{v}_h^a}\varphi,\psi;\boldsymbol{w}\right) + b\left(\varphi,\partial^\bullet_{h,\boldsymbol{v}_h^a}\psi;\boldsymbol{w}\right) \\ & + b\left(\varphi,\psi;\partial^\bullet_{h,\boldsymbol{v}_h^a}\boldsymbol{w}\right) + \tilde{b}(\varphi,\psi;\boldsymbol{w};\boldsymbol{v}_h^a).\end{aligned} \tag{3.26}$$

We find it convenient to introduce the following notation for $W_h \in \mathcal{S}_h(t)$ and $w_h \in \mathrm{H}^1(\Gamma(t)), t \in [t^{n-1}, t^{n+1}]$ and for a given $\Phi_h^{n+1} \in \mathcal{S}_h^{n+1}$ and corresponding lift $\varphi_h^{n+1} \in \mathcal{S}_h^{n+1,l}$

$$\begin{aligned}\mathcal{L}_{2,h}\left(W_h,\Phi^{n+1}\right) = {} & \frac{3}{2\tau}\left(m_h\left(W_h(\cdot,t^{n+1}),\underline{\Phi}_h^{n+1}(\cdot,t^{n+1})\right) - m_h\left(W_h(\cdot,t^n),\underline{\Phi}_h^{n+1}(\cdot,t^n)\right)\right) \\ & - \frac{1}{2\tau}\left(m_h\left(W_h(\cdot,t^n),\underline{\Phi}_h^{n+1}(\cdot,t^n)\right) - m_h\left(W_h(\cdot,t^{n-1}),\underline{\Phi}_h^{n+1}(\cdot,t^{n-1})\right)\right),\end{aligned} \tag{3.27}$$

$$\begin{aligned}\mathcal{L}_2\left(w_h,\varphi^{n+1}\right) = {} & \frac{3}{2\tau}\left(m\left(w_h(\cdot,t^{n+1}),\underline{\varphi}_h^{n+1}(\cdot,t^{n+1})\right) - m\left(w_h(\cdot,t^n),\underline{\varphi}_h^{n+1}(\cdot,t^n)\right)\right) \\ & - \frac{1}{2\tau}\left(m\left(w_h(\cdot,t^n),\underline{\varphi}_h^{n+1}(\cdot,t^n)\right) - m\left(w_h(\cdot,t^{n-1}),\underline{\varphi}_h^{n+1}(\cdot,t^{n-1})\right)\right).\end{aligned} \tag{3.28}$$

The following Lemma defines an adequate notion of discrete in time transport and follows easily from the transport formula (3.21).

3.7. **Lemma** (Discrete in time transport formula). *For $W_h \in \mathcal{S}_h(t)$ and $w_h \in \mathrm{H}^1(\Gamma(t)), t \in [t^n, t^{n+1}]$ and for a given $\Phi_h^{n+1} \in \mathcal{S}_h^{n+1}$ and corresponding lift $\varphi_h^{n+1} \in \mathcal{S}_h^{n+1,l}$*

$$\begin{aligned}&\mathcal{L}_{2,h}\left(W_h,\Phi_h^{n+1}\right) = \\ &\frac{3}{2\tau}\int_{t^n}^{t^{n+1}} \frac{\mathrm{d}}{\mathrm{d}t} m_h\left(W_h(\cdot,t),\underline{\Phi}_h^{n+1}(\cdot,t)\right)\mathrm{d}t - \frac{1}{2\tau}\int_{t^{n-1}}^{t^n} \frac{\mathrm{d}}{\mathrm{d}t} m_h\left(W_h(\cdot,t),\underline{\Phi}_h^{n+1}(\cdot,t)\right)\mathrm{d}t \\ &= \frac{3}{2\tau}\int_{t^n}^{t^{n+1}} m_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a} W_h(\cdot,t),\underline{\Phi}_h^{n+1}(\cdot,t)\right) + g_h\left(W_h(\cdot,t),\underline{\Phi}_h^{n+1}(\cdot,t);\boldsymbol{V}_h^a(\cdot,t)\right)\mathrm{d}t \\ &- \frac{1}{2\tau}\int_{t^{n-1}}^{t^n} m_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a} W_h(\cdot,t),\underline{\Phi}_h^{n+1}(\cdot,t)\right) + g_h\left(W_h(\cdot,t),\underline{\Phi}_h^{n+1}(\cdot,t);\boldsymbol{V}_h^a(\cdot,t)\right)\mathrm{d}t\end{aligned} \tag{3.29}$$

$$\begin{aligned}&\mathcal{L}_2\left(w_h,\varphi_h^{n+1}\right) = \\ &\frac{3}{2\tau}\int_{t^n}^{t^{n+1}} \frac{\mathrm{d}}{\mathrm{d}t} m\left(w_h(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t)\right)\mathrm{d}t - \frac{1}{2\tau}\int_{t^{n-1}}^{t^n} \frac{\mathrm{d}}{\mathrm{d}t} m\left(w_h(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t)\right)\mathrm{d}t \\ &= \frac{3}{2\tau}\int_{t^n}^{t^{n+1}} m\left(\partial^\bullet_{h,\boldsymbol{v}_h^a} w_h(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t)\right) + g\left(w_h(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t);\boldsymbol{v}_h^a(\cdot,t)\right)\mathrm{d}t \\ &- \frac{1}{2\tau}\int_{t^{n-1}}^{t^n} m\left(\partial^\bullet_{h,\boldsymbol{v}_h^a} w_h(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t)\right) + g\left(w_h(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t);\boldsymbol{v}_h^a(\cdot,t)\right)\mathrm{d}t\end{aligned} \tag{3.30}$$

For $t \in [t^{n-1}, t^{n+1}]$ and $\tau \leq \tau_0$ the following bounds hold. The result was proved for $t \in [t^n, t^{n+1}]$ in [3]. The proof may be extended for $t \in [t^{n-1}, t^{n+1}]$ as $\partial^\bullet_{h,\boldsymbol{V}^a_h} \underline{\Phi}^{n+1}_h = 0$ and $\partial^\bullet_{h,\boldsymbol{v}^a_h} \underline{\varphi}^{n+1}_h = 0$,

$$\left| m\left(\varphi_h^{n+1}, \varphi_h^{n+1}\right) - m\left(\underline{\varphi}_h^{n+1}(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t)\right)\right| \leq c\tau m\left(\varphi_h^{n+1}, \varphi_h^{n+1}\right) \tag{3.31}$$

$$\left| a\left(\varphi_h^{n+1}, \varphi_h^{n+1}\right) - a\left(\underline{\varphi}_h^{n+1}(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t)\right)\right| \leq c\tau a\left(\varphi_h^{n+1}, \varphi_h^{n+1}\right). \tag{3.32}$$

If $\partial^\bullet_{h,\boldsymbol{V}^a_h}\Phi_h = 0$ and $\partial^\bullet_{h,\boldsymbol{V}^a_h}\Psi_h = 0$ then

$$\begin{aligned} &\left| m_h\left(\Phi_h(\cdot,t^{k+1}), \Psi_h(\cdot,t^{k+1})\right) - m_h\left(\Phi_h(\cdot,t^k), \Psi_h(\cdot,t^k)\right)\right| \\ &\qquad \leq c\int_{t^k}^{t^{k+1}} m_h\left(\Phi_h(\cdot,t), \Phi_h(\cdot,t)\right)^{1/2} m_h\left(\Psi_h(\cdot,t), \Psi_h(\cdot,t)\right)^{1/2} \end{aligned} \tag{3.33}$$

$$\begin{aligned} &\left| a_h\left(\Phi_h(\cdot,t^{k+1}), \Phi_h(\cdot,t^{k+1})\right) - a_h\left(\Phi_h(\cdot,t^k), \Phi_h(\cdot,t^k)\right)\right| \\ &\qquad \leq c\int_{t^k}^{t^{k+1}} a_h\left(\Phi_h(\cdot,t), \Phi_h(\cdot,t)\right) \end{aligned} \tag{3.34}$$

For $t \in [t^{n-1}, t^{n+1}]$ and $\tau \leq \tau_0$

$$\left\|\underline{\Phi}_h^{n+1}(\cdot,t)\right\|_{\mathrm{L}_2(\Gamma_h(t))} \leq c\left\|\Phi_h^{n+1}\right\|_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}, \quad \left\|\underline{\varphi}_h^{n+1}(\cdot,t)\right\|_{\mathrm{L}_2(\Gamma(t))} \leq c\left\|\varphi_h^{n+1}\right\|_{\mathrm{L}_2(\Gamma^{n+1})} \tag{3.35}$$

$$\begin{aligned} &\left\|\nabla_{\Gamma_h(t)}\underline{\Phi}_h^{n+1}(\cdot,t)\right\|_{\mathrm{L}_2(\Gamma_h(t))} \leq c\left\|\nabla_{\Gamma_h^{n+1}}\Phi_h^{n+1}\right\|_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}, \\ &\left\|\nabla_{\Gamma(t)}\underline{\varphi}_h^{n+1}(\cdot,t)\right\|_{\mathrm{L}_2(\Gamma(t))} \leq c\left\|\nabla_{\Gamma^{n+1}}\varphi_h^{n+1}\right\|_{\mathrm{L}_2(\Gamma^{n+1})}. \end{aligned} \tag{3.36}$$

4. SEMIDISCRETE ALE-ESFEM

4.1. **Semidiscrete scheme.**

Given $U_h^0 \in \mathcal{S}_h(0)$ find $U_h(t) \in \mathcal{S}_h(t)$ such that for all $\Phi_h(t) \in \mathcal{S}_h(t)$ and $t \in (0,T]$

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h\left(U_h, \Phi_h\right) + a_h\left(U_h, \Phi_h\right) = m_h\left(U_h, \partial^\bullet_{h,\boldsymbol{V}^a_h}\Phi_h\right) - b_h\left(U_h, \Phi_h; \boldsymbol{T}^a_h\right), U_h(\cdot,0) = U_h^0. \tag{4.1}$$

By the transport property of the basis functions (3.8) we have the equivalent definition

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h\left(U_h, \chi_j\right) + a_h\left(U_h, \chi_j\right) = -b_h\left(U_h, \chi_j; \boldsymbol{T}^a_h\right), \quad U_h(\cdot,0) = U_h^0, \text{ for } j = 1,\ldots,J. \tag{4.2}$$

Thus a matrix vector formulation of the scheme is given $\boldsymbol{\alpha}(0)$ find a coefficient vector $\boldsymbol{\alpha}(t), t \in (0,T]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{M}(t)\boldsymbol{\alpha}(t)\right) + \left(\boldsymbol{S}(t) + \boldsymbol{B}(t)\right)\boldsymbol{\alpha}(t) = 0, \tag{4.3}$$

where $\boldsymbol{M}^n = \boldsymbol{M}(t^n), \boldsymbol{S}^n = \boldsymbol{S}(t^n)$ and $\boldsymbol{B}^n = \boldsymbol{B}(t^n)$ are time dependent mass, stiffness and nonsymmetric matrices with coefficients given by

$$\begin{aligned} &M(t)_{ij} = \int_{\Gamma_h(t)} \chi_i(\cdot,t)\chi_j(\cdot,t), \quad A(t)_{ij} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)}\chi_i(\cdot,t)\nabla_{\Gamma_h(t)}\chi_j(\cdot,t), \\ &B(t)_{ij} = \int_{\Gamma_h(t)} \chi_i(\cdot,t)\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}(\boldsymbol{X}_j(t),t)\chi_j(\cdot,t)\cdot\nabla_{\Gamma_h(t)}\chi_j(\cdot,t). \end{aligned} \tag{4.4}$$

4.2. **Lemma** (Stability of the semidiscrete scheme)**.** *The finite element solution U_h to (4.1) satisfies the following bounds*

$$\|U_h\|^2_{\mathrm{L}_2(\Gamma_h(t))} + \int_0^T \left\|\nabla_{\Gamma_h(s)} U_h\right\|^2_{\mathrm{L}_2(\Gamma_h(s))} \mathrm{d}s \le c \|U_h\|^2_{\mathrm{L}_2(\Gamma_h(0))}, \tag{4.5}$$

$$\|u_h\|^2_{\mathrm{L}_2(\Gamma(t))} + \int_0^T \left\|\nabla_{\Gamma(s)} u_h\right\|^2_{\mathrm{L}_2(\Gamma(s))} \mathrm{d}s \le c \|u_h\|^2_{\mathrm{L}_2(\Gamma(0))}, \tag{4.6}$$

$$\int_0^T \left\|\partial^\bullet_{h,\boldsymbol{V}_h^a} U_h\right\|^2_{\mathrm{L}_2(\Gamma_h(s))} \mathrm{d}s + \left\|\nabla_{\Gamma_h(t)} U_h\right\|^2_{\mathrm{L}_2(\Gamma_h(t))} \le c \|U_h\|^2_{\mathrm{H}^1(\Gamma_h(0))}, \tag{4.7}$$

$$\int_0^T \left\|\partial^\bullet_{h,\boldsymbol{v}_h^a} u_h\right\|^2_{\mathrm{L}_2(\Gamma(s))} \mathrm{d}s + \left\|\nabla_{\Gamma(t)} u_h\right\|^2_{\mathrm{L}_2(\Gamma(t))} \le c \|u_h\|^2_{\mathrm{H}^1(\Gamma(0))}. \tag{4.8}$$

Proof . We start with (4.5), testing with U_h in (4.1) and applying the transport formula (3.21) as in [1] yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} m_h(U_h, U_h) + a_h(U_h, U_h) &= -b_h(U_h, U_h; \boldsymbol{T}_h^a) - \frac{1}{2} g_h(U_h, U_h; \boldsymbol{V}_h^a) \\ &= \int_{\Gamma_h(t)} -\frac{1}{2}\boldsymbol{T}_h^a \cdot \nabla_{\Gamma_h(t)}(U_h^2) - \frac{1}{2}U_h^2 \nabla_{\Gamma_h(t)} \boldsymbol{V}_h^a.\end{aligned}$$

Noting that the boundary of Γ_h is either empty or that the boundary conditions are zero-flux an integration by parts gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U_h\|^2_{\mathrm{L}_2(\Gamma_h(t))} + \left\|\nabla_{\Gamma_h(t)} U_h\right\|^2_{\mathrm{L}_2(\Gamma_h(t))} &= \int_{\Gamma_h(t)} \frac{1}{2} U_h^2 \nabla_{\Gamma_h(t)} \cdot (\boldsymbol{T}_h^a - \boldsymbol{V}_h^a) \\ &\le c \|U\|^2_{\mathrm{L}_2(\Gamma_h(t))}.\end{aligned}$$

A Gronwall argument implies the desired result.

For the proof of (4.7) we apply the transport formula (3.21) to rewrite (4.1) as

$$m_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a} U_h, \Phi_h\right) + a_h(U_h, \Phi_h) = -g_h(U_h, \Phi_h; \boldsymbol{V}_h^a) - b_h(U_h, \Phi_h; \boldsymbol{T}_h^a),$$

testing with $\partial^\bullet_{h,\boldsymbol{V}_h^a} U_h$ gives

$$\left\|\partial^\bullet_{h,\boldsymbol{V}_h^a} U_h\right\|^2_{\mathrm{L}_2(\Gamma_h(t))} + a_h\left(U_h, \partial^\bullet_{h,\boldsymbol{V}_h^a} U_h\right) = -b_h\left(U_h, \partial^\bullet_{h,\boldsymbol{V}_h^a} U_h; \boldsymbol{T}_h^a\right) - g_h\left(U_h, \partial^\bullet_{h,\boldsymbol{V}_h^a} U_h; \boldsymbol{V}_h^a\right). \tag{4.9}$$

From the transport formula (3.22) we have

$$a_h\left(U_h, \partial^\bullet_{h,\boldsymbol{V}_h^a} U_h\right) = \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t} a_h(U_h, U_h) - \tilde{a}_h(U_h, U_h; \boldsymbol{V}_h^a)\right). \tag{4.10}$$

Using (4.10) in (4.9) we have

$$\begin{aligned}&\left\|\partial^\bullet_{h,\boldsymbol{V}_h^a} U_h\right\|^2_{\mathrm{L}_2(\Gamma_h(t))} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Gamma_h(t) U\|^2_{\mathrm{L}_2(\Gamma_h(t))} = \\ &\quad \frac{1}{2}\tilde{a}_h(U_h, U_h; \boldsymbol{V}_h^a) - b_h\left(U_h, \partial^\bullet_{h,\boldsymbol{V}_h^a} U_h; \boldsymbol{T}_h^a\right) - g_h\left(U_h, \partial^\bullet_{h,\boldsymbol{V}_h^a} U_h; \boldsymbol{V}_h^a\right).\end{aligned} \tag{4.11}$$

The Cauchy-Schwarz inequality together with the smoothness of the velocity $\boldsymbol{v}_a$ (and hence $\boldsymbol{V}_h^a$), yields the following estimates

$$\tilde{a}_h(U_h, U_h; \boldsymbol{V}_h^a) \le c \left\|\nabla_{\Gamma_h(t)} U_h\right\|^2_{\mathrm{L}_2(\Gamma_h(t))}, \tag{4.12}$$

$$g_h\left(U_h, \partial^\bullet_{h,\boldsymbol{V}_h^a} U_h; \boldsymbol{V}_h^a\right) \le c \|U_h\|_{\mathrm{L}_2(\Gamma_h(t))} \left\|\partial^\bullet_{h,\boldsymbol{V}_h^a} U_h\right\|_{\mathrm{L}_2(\Gamma_h(t))} \tag{4.13}$$

and

$$
\begin{aligned}
(4.14)\qquad b_h\left(U_h, \partial^\bullet_{h,\boldsymbol{V}^a_h} U_h; \boldsymbol{T}^a_h\right) &= \int_{\Gamma_h(t)} U_h \boldsymbol{T}^a_h \cdot \nabla_{\Gamma_h(t)} \partial^\bullet_{h,\boldsymbol{V}^a_h} U_h \\
&= \int_{\Gamma_h(t)} -\nabla_{\Gamma_h(t)} \cdot \left(U_h \boldsymbol{T}^a_h\right) \partial^\bullet_{h,\boldsymbol{V}^a_h} U_h \\
&\leq c\left(\left\|\nabla_{\Gamma_h(t)} U_h\right\|_{\mathrm{L}_2(\Gamma_h(t))} + \left\|U_h\right\|_{\mathrm{L}_2(\Gamma_h(t))}\right)\left\|\partial^\bullet_{h,\boldsymbol{V}^a_h} U_h\right\|_{\mathrm{L}_2(\Gamma_h(t))}.
\end{aligned}
$$

Applying estimates (4.12)-(4.14) in (4.11) gives

$$
\begin{aligned}
&\left\|\partial^\bullet_{h,\boldsymbol{V}^a_h} U_h\right\|^2_{\mathrm{L}_2(\Gamma_h(t))} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\Gamma_h(t) U\right\|^2_{\mathrm{L}_2(\Gamma_h(t))} \leq \\
&\qquad c\left\|\partial^\bullet_{h,\boldsymbol{V}^a_h} U_h\right\|_{\mathrm{L}_2(\Gamma_h(t))}\left(\left\|U_h\right\|_{\mathrm{L}_2(\Gamma_h(t))} + \left\|\nabla_{\Gamma_h(t)} U_h\right\|_{\mathrm{L}_2(\Gamma_h(t))}\right) + c\left\|\nabla_{\Gamma_h(t)} U_h\right\|^2_{\mathrm{L}_2(\Gamma_h(t))}
\end{aligned}
$$

Young's inequality, the estimate (4.5) and a Gronwall argument completes the proof of (4.7).

Due to the equivalence of the L_2 norm and the H^1 seminorm on Γ_h and $\Gamma(t)$ (c.f., [1]), the estimates (4.5) and (4.7) imply the estimates (4.6) and (4.8) respectively. □

4.3. **Theorem** (Error bound for the semidiscrete scheme)**.** *Let u be a sufficiently smooth solution of (1.1) and let the geometry be sufficiently regular. Furthermore let $u_h(t), t \in [0,T]$ denote the lift of the solution of the semidiscrete scheme (4.1) and assume that the approximation of the initial data satisfies*

$$
(4.15)\qquad \left\|u(\cdot,0) - u_h(\cdot,0)\right\|_{\mathrm{L}_2(\Gamma(t))} \leq ch^2.
$$

Then for $0 < h \leq h_0$ with h_0 dependent on the data of the problem, the following error bound holds

$$
\begin{aligned}
(4.16)\qquad \sup_{t\in(0,T)}\left\|u(\cdot,t) - u_h(\cdot,t)\right\|^2_{\mathrm{L}_2(\Gamma(t))} + h^2\int_0^T\left\|\nabla_\Gamma\left(u(\cdot,t) - u_h(\cdot,t)\right)\right\|^2_{\mathrm{L}_2(\Gamma(t))} \\
\leq ch^4\left(\left\|u\right\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a} u\right\|^2_{\mathrm{H}^2(\Gamma(t))}\right).
\end{aligned}
$$

We shall prove some preliminary Lemmas before proving the Theorem.

4.4. **Error decomposition.** It is convenient in the analysis to decompose the error as follows

$$
(4.17)\qquad u - u_h = \rho + \theta, \quad \rho := u - \mathrm{R}^h\, u, \quad \theta = \mathrm{R}^h\, u - u_h \in \mathcal{S}^l_h,
$$

with R^h the Ritz projection defined in (C.1).

4.5. **Lemma** (Semidiscrete error relation)**.** *We have the following error relation between the semidiscrete solution and the Ritz projection. For $\varphi_h \in \mathcal{S}^l_h$*

$$
(4.18)\qquad \frac{\mathrm{d}}{\mathrm{d}t} m\left(\theta, \varphi_h\right) + a\left(\theta, \varphi_h\right) - m\left(\theta, \partial^\bullet_{h,\boldsymbol{v}^a_h}\varphi_h\right) + b\left(\theta, \varphi_h; \boldsymbol{t}^a_h\right) = F_2(\varphi_h) - F_1(\varphi_h),
$$

where

$$
\begin{aligned}
(4.19)\qquad F_1(\varphi_h) &= m\left(\partial^\bullet_{h,\boldsymbol{v}^a_h} u_h, \varphi_h\right) - m_h\left(\partial^\bullet_{h,\boldsymbol{V}^a_h} U_h, \Phi_h\right) + a\left(u_h, \varphi_h\right) - a_h\left(U_h, \Phi_h\right) \\
&\quad - b_h\left(U_h, \Phi_h; \boldsymbol{T}^a_h\right) + b\left(u_h, \varphi_h; \boldsymbol{t}^a_h\right) + g\left(u_h, \varphi_h; \boldsymbol{v}^a_h\right) - g_h\left(U_h, \Phi_h; \boldsymbol{V}^a_h\right),
\end{aligned}
$$

$$
\begin{aligned}
(4.20)\qquad F_2(\varphi_h) &= m\left(-\partial^\bullet_{h,\boldsymbol{v}^a_h}\rho, \varphi_h\right) - g\left(\rho, \varphi_h; \boldsymbol{v}^a_h\right) - b\left(\rho, \varphi_h; \boldsymbol{t}^a_h\right) \\
&\quad + m\left(u, \partial^\bullet_{\boldsymbol{v}_a}\varphi_h - \partial^\bullet_{h,\boldsymbol{v}^a_h}\varphi_h\right) - b\left(u, \varphi_h; \boldsymbol{a}_{\boldsymbol{\mathcal{T}}} - \boldsymbol{t}^a_h\right).
\end{aligned}
$$

Proof . From the definition of the semidiscrete scheme (4.1) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m\left(u_h,\varphi_h\right)+a\left(u_h,\varphi_h\right)-m\left(u_h,\partial^\bullet_{h,\boldsymbol{v}_h^a}\varphi_h\right)+b\left(u_h,\varphi_h;\boldsymbol{t}_h^a\right)=&\\ \frac{\mathrm{d}}{\mathrm{d}t}m\left(u_h,\varphi_h\right)+a\left(u_h,\varphi_h\right)-m\left(u_h,\partial^\bullet_{h,\boldsymbol{v}_h^a}\varphi_h\right)+b\left(u_h,\varphi_h;\boldsymbol{t}_h^a\right)&\\ -\frac{\mathrm{d}}{\mathrm{d}t}m_h\left(U_h,\Phi_h\right)-a_h\left(U_h,\Phi_h\right)+m_h\left(U_h,\partial^\bullet_{h,\boldsymbol{V}_h^a}\Phi_h\right)-b_h\left(U_h,\Phi_h;\boldsymbol{T}_h^a\right)&\\ =F_1(\varphi_h),&\end{aligned}\tag{4.21}$$

where we have used the transport formulas (3.21) and (3.24) for the last step. Using the variational formulation of the continuous equation (1.5) we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}m\left(\mathrm{R}^h u,\varphi_h\right)+a\left(\mathrm{R}^h u,\varphi_h\right)-m\left(\mathrm{R}^h u,\partial^\bullet_{h,\boldsymbol{v}_h^a}\varphi_h\right)+b\left(\mathrm{R}^h u,\varphi_h;\boldsymbol{t}_h^a\right)\\ &=\frac{\mathrm{d}}{\mathrm{d}t}m\left(\mathrm{R}^h u,\varphi_h\right)+a\left(\mathrm{R}^h u,\varphi_h\right)-m\left(\mathrm{R}^h u,\partial^\bullet_{h,\boldsymbol{v}_h^a}\varphi_h\right)+b\left(\mathrm{R}^h u,\varphi_h;\boldsymbol{t}_h^a\right)\\ &\quad-\frac{\mathrm{d}}{\mathrm{d}t}m\left(u,\varphi_h\right)-a\left(u,\varphi_h\right)+m\left(u,\partial^\bullet_{\boldsymbol{v}_a}\varphi_h\right)-b\left(u,\varphi_h;\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}\right)\\ &=\frac{\mathrm{d}}{\mathrm{d}t}m\left(-\rho,\varphi_h\right)+m\left(\rho,\partial^\bullet_{h,\boldsymbol{v}_h^a}\varphi_h\right)+m\left(u,\partial^\bullet_{\boldsymbol{v}_a}\varphi_h-\partial^\bullet_{h,\boldsymbol{v}_h^a}\right)\\ &\quad-b\left(\rho,\varphi_h;\boldsymbol{t}_h^a\right)-b\left(u,\varphi_h;\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}-\boldsymbol{t}_h^a\right)\\ &=F_2(\varphi_h),\end{aligned}\tag{4.22}$$

where we have used (C.1) in the second step and the transport theorem (3.24) in the final step. Subtracting (4.21) from (4.22) yields the desired error relation. □

We estimate the two terms on the right hand side of (4.18) as follows. From Lemma B.3 we have

$$\begin{aligned}|F_1(\varphi_h)|\le&ch^2\Big(\left\|\partial^\bullet_{h,\boldsymbol{v}_h^a}u_h\right\|_{\mathrm{L}_2(\Gamma(t))}\left\|\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))}+\left\|\nabla_{\Gamma(t)}u_h\right\|_{\mathrm{L}_2(\Gamma(t))}\left\|\nabla_{\Gamma(t)}\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))}\\ &+\left\|u_h\right\|_{\mathrm{L}_2(\Gamma(t))}\left\|\nabla_{\Gamma(t)}\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))}+\left\|u_h\right\|_{\mathrm{H}^1(\Gamma(t))}\left\|\varphi_h\right\|_{\mathrm{H}^1(\Gamma(t))}\Big).\end{aligned}\tag{4.23}$$

We apply Young's inequality to conclude that with $\epsilon>0$ a positive constant of our choice

$$|F_1(\varphi_h)|\le c(\epsilon)h^4\Big(\left\|\partial^\bullet_{h,\boldsymbol{v}_h^a}u_h\right\|^2_{\mathrm{L}_2(\Gamma(t))}+\left\|u_h\right\|^2_{\mathrm{H}^1(\Gamma(t))}\Big)+c(\epsilon)\left\|\varphi_h\right\|^2_{\mathrm{L}_2(\Gamma(t))}+\epsilon\left\|\nabla_{\Gamma(t)}\varphi_h\right\|^2_{\mathrm{L}_2(\Gamma(t))}\tag{4.24}$$

For the term F_2 on the right hand side of (4.18), we have

$$\begin{aligned}|F_2(\varphi_h)|\le&\left|m\left(-\partial^\bullet_{h,\boldsymbol{v}_h^a}\rho,\varphi_h\right)\right|+\left|g\left(\rho,\varphi_h;\boldsymbol{v}_h^a\right)\right|+\left|b\left(\rho,\varphi_h;\boldsymbol{t}_h^a\right)\right|\\ &+\left|m\left(u,\partial^\bullet_{\boldsymbol{v}_a}\varphi_h-\partial^\bullet_{h,\boldsymbol{v}_h^a}\varphi_h\right)\right|+\left|b\left(u,\varphi_h;\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}-\boldsymbol{t}_h^a\right)\right|\\ :=&\left|I\right|+\left|II\right|+\left|III\right|+\left|IV\right|+\left|V\right|.\end{aligned}\tag{4.25}$$

Using (C.3) we have

$$|I|\le\left\|\partial^\bullet_{h,\boldsymbol{v}_h^a}\rho\right\|_{\mathrm{L}_2(\Gamma(t))}\left\|\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))}\le ch^2\left(\left\|u\right\|_{\mathrm{H}^2(\Gamma(t))}+\left\|\partial^\bullet_{\boldsymbol{v}_a}u\right\|_{\mathrm{H}^2(\Gamma(t))}\right)\left\|\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))}.\tag{4.26}$$

We estimate the second and third terms with (C.2) as follows

$$|II|\le c\left\|\rho\right\|_{\mathrm{L}_2(\Gamma(t))}\left\|\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))}\le ch^2\left\|u\right\|_{\mathrm{H}^2(\Gamma(t))}\left\|\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))},\tag{4.27}$$

$$|III|\le c\left\|\rho\right\|_{\mathrm{L}_2(\Gamma(t))}\left\|\nabla_{\Gamma(t)}\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))}\le ch^2\left\|u\right\|_{\mathrm{H}^2(\Gamma(t))}\left\|\nabla_{\Gamma(t)}\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))}.\tag{4.28}$$

For the next term we use (B.5) to conclude

$$|IV| \leq \|u\|_{\mathrm{L}_2(\Gamma(t))} \left\|\partial^\bullet_{\boldsymbol{v}_a}\varphi_h - \partial^\bullet_{h,\boldsymbol{v}_h^a}\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))} \leq ch^2 \|u\|_{\mathrm{L}_2(\Gamma(t))} \|\varphi_h\|_{\mathrm{H}^1(\Gamma(t))}. \tag{4.29}$$

Finally for the last term we apply (B.4) which yields

$$|V| \leq ch^2 \|u\|_{\mathrm{L}_2(\Gamma(t))} \left\|\nabla_{\Gamma(t)}\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))}. \tag{4.30}$$

Combining the estimates (4.26)-(4.30) we have

$$\begin{aligned}|F_2(\varphi_h)| \leq & ch^2\Bigg(\left(\|u\|_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a}u\right\|_{\mathrm{H}^2(\Gamma(t))}\right)\|\varphi_h\|_{\mathrm{L}_2(\Gamma(t))} + \\ & \left(\|u\|_{\mathrm{L}_2(\Gamma(t))} + \|u\|_{\mathrm{H}^2(\Gamma(t))}\right)\left\|\nabla_{\Gamma(t)}\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))} + \|u\|_{\mathrm{L}_2(\Gamma(t))}\|\varphi_h\|_{\mathrm{H}^1(\Gamma(t))}\Bigg).\end{aligned} \tag{4.31}$$

We apply Young's inequality to conclude that with $\epsilon > 0$ a positive constant of our choice

$$|F_2(\varphi_h)| \leq c(\epsilon)h^4\left(\|u\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a}u\right\|^2_{\mathrm{H}^2(\Gamma(t))}\right) + c(\epsilon)\|\varphi_h\|^2_{\mathrm{L}_2(\Gamma(t))} + \epsilon\left\|\nabla_{\Gamma(t)}\varphi_h\right\|^2_{\mathrm{L}_2(\Gamma(t))}. \tag{4.32}$$

Proof of Theorem 4.3 We test with θ in the error relation (4.18) which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}m(\theta,\theta) + a(\theta,\theta) - m\left(\theta, \partial^\bullet_{h,\boldsymbol{v}_h^a}\theta\right) + b\left(\theta,\theta;\boldsymbol{t}_h^a\right) = F_2(\theta) - F_1(\theta). \tag{4.33}$$

Applying the transport formula (3.24) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}m(\theta,\theta) + a(\theta,\theta) = F_2(\theta) - F_1(\theta) - g\left(\theta,\theta;\boldsymbol{v}_h^a\right) - b\left(\theta,\theta;\boldsymbol{t}_h^a\right). \tag{4.34}$$

Integrating by parts the last term gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}m(\theta,\theta) + a(\theta,\theta) = F_2(\theta) - F_1(\theta) - g\left(\theta,\theta;\boldsymbol{v}_h^a + \frac{1}{2}\boldsymbol{t}_h^a\right). \tag{4.35}$$

Application of the estimates (4.24) and (4.32) gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2_{\mathrm{L}_2(\Gamma(t))} + (1-\epsilon)\left\|\nabla_{\Gamma(t)}\theta\right\|^2_{\mathrm{L}_2(\Gamma(t))} \leq c(\epsilon)\|\theta\|^2_{\mathrm{L}_2(\Gamma(t))} \\ &\quad + c(\epsilon)h^4\left(\left\|\partial^\bullet_{h,\boldsymbol{v}_h^a}u_h\right\|^2_{\mathrm{L}_2(\Gamma(t))} + \|u_h\|^2_{\mathrm{H}^1(\Gamma(t))} + \|u\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a}u\right\|^2_{\mathrm{H}^2(\Gamma(t))}\right).\end{aligned} \tag{4.36}$$

A Gronwall argument, the stability estimates in Lemma 4.2, the error decomposition (4.17) and the estimates on the error in the Ritz projection (C.2) complete the proof. □

5. Fully discrete ALE-ESFEM

We consider a second order time discretisation of the semidiscrete scheme (4.1) based on a (second order backward differentiation formula) BDF2 time discretisation defined as follows;

5.1. **Fully discrete BDF2 ALE-ESFEM scheme.** Given $U_h^0 \in \mathcal{S}_h^0$ and $U_h^1 \in \mathcal{S}_h^1$ find $U_h^n \in \mathcal{S}_h^n, n = \{2,\dots,N\}$ such that for all $\Phi_h^{n+1} \in \mathcal{S}_h^{n+1}$ and for $n \in \{1,\dots,N-1\}$

$$\mathcal{L}_{2,h}\left(U_h, \Phi_h^{n+1}\right) + a_h\left(U_h^{n+1}, \Phi_h^{n+1}\right) = -b_h\left(U_h^{n+1}, \Phi_h^{n+1}; (\boldsymbol{T}_h^a)^{n+1}\right) \tag{5.1}$$

For the basis functions we note that by definition for $\alpha = -1, 0, 1$,

$$\underline{\chi}_j^{n+1}(\cdot, t^{n+\alpha}) = \chi_j^{n+\alpha} \in \mathcal{S}_h^{n+\alpha}. \tag{5.2}$$

Therefore the matrix vector formulation of the scheme (5.1) is for $n = \{1, \ldots, N-1\}$ given $\boldsymbol{\alpha}^n, \boldsymbol{\alpha}^{n-1}$ find a coefficient vector $\boldsymbol{\alpha}^{n+1}$

$$\left(\frac{3}{2}\boldsymbol{M}^{n+1} + \tau\left(\boldsymbol{S}^{n+1} + \boldsymbol{B}^{n+1}\right)\right)\boldsymbol{\alpha}^{n+1} = 2\boldsymbol{M}^n\boldsymbol{\alpha}^n - \frac{1}{2}\boldsymbol{M}^{n-1}\boldsymbol{\alpha}^{n-1}, \tag{5.3}$$

where $\boldsymbol{M}^n = \boldsymbol{M}(t^n), \boldsymbol{S}^n = \boldsymbol{S}(t^n)$ and $\boldsymbol{B}^n = \boldsymbol{B}(t^n)$ are time dependent mass, stiffness and nonsymmetric matrices (see (4.4)).

5.2. **Proposition** (Solvability of the fully discrete scheme). *For $\tau < \tau_0$, where τ_0 depends on the data of the problem and the arbitrary tangential velocity $\boldsymbol{a}_{\mathcal{T}}$, and for each $n \in \{2, \ldots, N\}$, the finite element solution U_h^n to the scheme (5.1) exists and is unique.*

Proof . For the scheme (5.1) integrating by parts, we have for $\Phi_h^n \in \mathcal{S}_h^n$

$$\begin{aligned}\frac{3}{2}m_h\left(\Phi_h^n, \Phi_h^n\right) &+ \tau\left(a_h\left(\Phi_h^n, \Phi_h^n\right) + b_h\left(\Phi_h^n, \Phi_h^n; (\boldsymbol{T}_h^a)^n\right)\right) \\ &= \frac{3}{2}m_h\left(\Phi_h^n, \Phi_h^n\right) + \tau\left(a_h\left(\Phi_h^n, \Phi_h^n\right) - \frac{1}{2}g_h\left(\Phi_h^n, \Phi_h^n; (\boldsymbol{T}_h^a)^n\right)\right) \\ &\geq (\frac{3}{2} - c\tau)m_h\left(\Phi_h^n, \Phi_h^n\right) + \tau a_h\left(\Phi_h^n, \Phi_h^n\right),\end{aligned} \tag{5.4}$$

hence for $\tau \leq \tau_0$, the system matrix $\boldsymbol{A}^n = \left(\frac{3}{2}\boldsymbol{M}^n + \tau\left(\boldsymbol{S}^n + \boldsymbol{B}^n\right)\right), n = 1, \ldots, N$ is positive definite. □

We now prove the fully discrete analogues to the stability bounds of Lemma 4.2. We make use of the following result from [4] that provides basic estimates. There is a constant μ (independent of the discretisation parameters τ, h and the length of the time interval T) such that for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^J$, for $\tau \leq \tau_0$, for $k, j = -1, 0, 1, j \geq k$ and for $n \in 1, \ldots, N-1$ we have

$$\left(\boldsymbol{M}^{n+j} - \boldsymbol{M}^{n+k}\right)\boldsymbol{\alpha}\cdot\boldsymbol{\beta} \leq 2\mu(j-k)\tau\left(\boldsymbol{M}^{n+k}\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\right)^{\frac{1}{2}}\left(\boldsymbol{M}^{n+k}\boldsymbol{\beta}\cdot\boldsymbol{\beta}\right)^{\frac{1}{2}}. \tag{5.5}$$

5.3. **Lemma** (Stability of the fully discrete scheme (5.1)). *Assume the starting value for the scheme satisfies the bound*

$$\left\|U_h^1\right\|^2_{L_2(\Gamma_h^1)} \leq c\left\|U_h^0\right\|^2_{L_2(\Gamma_h^0)}, \tag{5.6}$$

then the fully discrete solution $U_h^n, n = 2, \ldots, N$ of the BDF2 scheme (5.1) satisfies the following bounds for $\tau \leq \tau_0$, where τ_0 depends on the data of the problem and the arbitrary tangential velocity $\boldsymbol{a}_{\mathcal{T}}$,

$$\left\|U_h^n\right\|^2_{L_2(\Gamma_h^n)} + \tau\sum_{i=2}^n\left\|\nabla_{\Gamma_h^i}U_h^i\right\|^2_{L_2(\Gamma_h^i)} \leq c\left\|U_h^0\right\|^2_{L_2(\Gamma_h^0)}, \tag{5.7}$$

$$\left\|u_h^n\right\|^2_{L_2(\Gamma^n)} + \tau\sum_{i=2}^n\left\|\nabla_{\Gamma^i}u_h^i\right\|^2_{L_2(\Gamma^i)} \leq c\left\|u_h^0\right\|^2_{L_2(\Gamma^0)}. \tag{5.8}$$

Furthermore if, along with (5.6), we assume the starting values satisfy the bound

$$\tau\left\|\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot, t^1 - 0)\right\|^2_{L_2(\Gamma_h^2)} + \left\|\nabla_{\Gamma_h^1}U_h^1\right\|^2_{L_2(\Gamma_h^1)} \leq c\left(\left\|U_h^0\right\|^2_{L_2(\Gamma_h^0)} + \left\|\nabla_{\Gamma_h^0}U_h^0\right\|^2_{L_2(\Gamma_h^0)}\right), \tag{5.9}$$

then for $n \in 2, \ldots, N$, we have the stability bounds

$$\tau\sum_{i=1}^{n-1}\left\|\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot, t^{i+1} - 0)\right\|^2_{L_2(\Gamma_h^i)} + \left\|\nabla_{\Gamma_h^n}U_h^n\right\|^2_{L_2(\Gamma_h^n)} \leq c\left\|U_h^0\right\|^2_{H^1(\Gamma_h^0)}, \tag{5.10}$$

$$\tau\sum_{i=1}^{n-1}\left\|\partial^\bullet_{h,\boldsymbol{v}_h^a}u_h^L(\cdot, t^{i+1} - 0)\right\|^2_{L_2(\Gamma^i)} + \left\|\nabla_{\Gamma^n}u_h^n\right\|^2_{L_2(\Gamma^n)} \leq c\left\|u_h^0\right\|^2_{H^1(\Gamma^0)}. \tag{5.11}$$

Proof . We begin with the proof of (5.7). We work with the matrix vector form of the scheme (5.3) and we multiply by a vector $\boldsymbol{U}^{n+1}$ which gives

$$\frac{3}{2\tau}\boldsymbol{M}^{n+1}\boldsymbol{U}^{n+1}\cdot\boldsymbol{U}^{n+1}-\frac{2}{\tau}\boldsymbol{M}^{n}\boldsymbol{U}^{n}\cdot\boldsymbol{U}^{n+1} + \frac{1}{2\tau}\boldsymbol{M}^{n-1}\boldsymbol{U}^{n-1}\cdot\boldsymbol{U}^{n+1}+\left(\boldsymbol{A}^{n+1}+\boldsymbol{B}^{n+1}\right)\boldsymbol{U}^{n+1}\cdot\boldsymbol{U}^{n+1}=0. \tag{5.12}$$

We first note that a calculation yields for $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\kappa}\in\mathbb{R}^J$

$$\left(\frac{3}{2}\boldsymbol{\alpha}-2\boldsymbol{\beta}+\frac{1}{2}\boldsymbol{\kappa}\right)\cdot\boldsymbol{\alpha}=\frac{1}{4}\left(|\boldsymbol{\alpha}|^2-|\boldsymbol{\beta}|^2+|2\boldsymbol{\alpha}-\boldsymbol{\beta}|^2-|2\boldsymbol{\beta}-\boldsymbol{\kappa}|^2\right)+\frac{1}{4}\left|\boldsymbol{\alpha}-2\boldsymbol{\beta}+\boldsymbol{\kappa}\right|^2. \tag{5.13}$$

Using this result we see that

$$\begin{aligned}
&\frac{3}{2}\boldsymbol{M}^{n+1}\boldsymbol{U}^{n+1}\cdot\boldsymbol{U}^{n+1}-2\boldsymbol{M}^{n}\boldsymbol{U}^{n}\cdot\boldsymbol{U}^{n+1}+\frac{1}{2}\boldsymbol{M}^{n-1}\boldsymbol{U}^{n-1}\cdot\boldsymbol{U}^{n+1}\\
&=\frac{3}{2}\left(\boldsymbol{M}^{n+1}-\boldsymbol{M}^{n}\right)\boldsymbol{U}^{n+1}\cdot\boldsymbol{U}^{n+1}+\frac{1}{2}\left(\boldsymbol{M}^{n-1}-\boldsymbol{M}^{n}\right)\boldsymbol{U}^{n-1}\cdot\boldsymbol{U}^{n+1}\\
&\quad+\frac{1}{4}\Bigg(\boldsymbol{M}^{n}\boldsymbol{U}^{n+1}\cdot\boldsymbol{U}^{n+1}-\boldsymbol{M}^{n}\boldsymbol{U}^{n}\cdot\boldsymbol{U}^{n}\\
&\quad+\boldsymbol{M}^{n}\left(2\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)\cdot\left(2\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)-\boldsymbol{M}^{n}\left(2\boldsymbol{U}^{n}-\boldsymbol{U}^{n-1}\right)\cdot\left(2\boldsymbol{U}^{n}-\boldsymbol{U}^{n-1}\right)\\
&\quad+\boldsymbol{M}^{n}\left(\boldsymbol{U}^{n+1}-2\boldsymbol{U}^{n}+\boldsymbol{U}^{n-1}\right)\cdot\left(\boldsymbol{U}^{n+1}-2\boldsymbol{U}^{n}+\boldsymbol{U}^{n-1}\right)\Bigg)\\
&=\frac{1}{4}\boldsymbol{M}^{n+1}\boldsymbol{U}^{n+1}\cdot\boldsymbol{U}^{n+1}-\frac{1}{4}\boldsymbol{M}^{n}\boldsymbol{U}^{n}\cdot\boldsymbol{U}^{n}\\
&\quad+\frac{1}{4}\boldsymbol{M}^{n}\left(2\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)\cdot\left(2\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)\\
&\quad-\frac{1}{4}\boldsymbol{M}^{n-1}\left(2\boldsymbol{U}^{n}-\boldsymbol{U}^{n-1}\right)\cdot\left(2\boldsymbol{U}^{n}-\boldsymbol{U}^{n-1}\right)\\
&\quad+\frac{1}{4}\boldsymbol{M}^{n}\left(\boldsymbol{U}^{n+1}-2\boldsymbol{U}^{n}+\boldsymbol{U}^{n-1}\right)\cdot\left(\boldsymbol{U}^{n+1}-2\boldsymbol{U}^{n}+\boldsymbol{U}^{n-1}\right)\\
&\quad+\frac{5}{4}\left(\boldsymbol{M}^{n+1}-\boldsymbol{M}^{n}\right)\boldsymbol{U}^{n+1}\cdot\boldsymbol{U}^{n+1}+\frac{1}{2}\left(\boldsymbol{M}^{n-1}-\boldsymbol{M}^{n}\right)\boldsymbol{U}^{n-1}\cdot\boldsymbol{U}^{n+1}\\
&\quad+\frac{1}{4}\left(\boldsymbol{M}^{n-1}-\boldsymbol{M}^{n}\right)\left(2\boldsymbol{U}^{n}-\boldsymbol{U}^{n-1}\right)\cdot\left(2\boldsymbol{U}^{n}-\boldsymbol{U}^{n-1}\right).
\end{aligned} \tag{5.14}$$

The last three terms on the right hand side may be estimated as follows. Using (3.31)

$$\begin{aligned}
\frac{5}{4}\left(\boldsymbol{M}^{n+1}-\boldsymbol{M}^{n}\right)\boldsymbol{U}^{n+1}\cdot\boldsymbol{U}^{n+1}&=\frac{5}{4}\left(m_h\left(U_h^{n+1},U_h^{n+1}\right)-m_h\left(\underline{U}_h^{n+1}(\cdot,t^n),\underline{U}_h^{n+1}(\cdot,t^n)\right)\right)\\
&\geq -c\tau\left\|U_h^{n+1}\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}.
\end{aligned} \tag{5.15}$$

Using (5.5), Young's inequality and (3.35) we have

$$\begin{aligned}
\frac{1}{2}\left(\boldsymbol{M}^{n-1}-\boldsymbol{M}^{n}\right)\boldsymbol{U}^{n-1}\cdot\boldsymbol{U}^{n+1}&\geq-\frac{\mu}{2}\tau\left(m_h\left(\underline{U}_h^{n+1}(\cdot,t^{n-1}),\underline{U}_h^{n+1}(\cdot,t^{n-1})\right)+\left\|U_h^{n-1}\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n-1}\right)}\right)\\
&\geq-c\tau\left(\left\|U_h^{n+1}\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}+\left\|U_h^{n-1}\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n-1}\right)}\right).
\end{aligned} \tag{5.16}$$

For the third term we use (5.5) to conclude

$$\begin{aligned}\frac{1}{4}\left(\boldsymbol{M}^{n-1}-\boldsymbol{M}^{n}\right)\left(2\boldsymbol{U}^{n}-\boldsymbol{U}^{n-1}\right)\cdot\left(2\boldsymbol{U}^{n}-\boldsymbol{U}^{n-1}\right)\geq\\ -c\tau m_h\left(2\underline{U}_h^n(\cdot,t^{n-1})-U_h^{n-1}(\cdot,t^{n-1}),2\underline{U}_h^n(\cdot,t^{n-1})-U_h^{n-1}(\cdot,t^{n-1})\right).\end{aligned}\tag{5.17}$$

Applying (5.14)—(5.17) in (5.12) and reverting to the bilinear forms, we arrive at

$$\begin{aligned}&\frac{1}{4}\partial_\tau\Big(m_h\left(U_h^n,U_h^n\right)+m_h\left(2\underline{U}_h^n(\cdot,t^{n-1})-U_h^{n-1}(\cdot,t^{n-1}),2\underline{U}_h^n(\cdot,t^{n-1})-U_h^{n-1}(\cdot,t^{n-1})\right)\Big)\\ &+\left\|\nabla_{\Gamma_h^{n+1}}U_h^{n+1}\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}\leq c\Big(\left\|U_h^{n+1}\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}+\left\|U_h^n\right\|^2_{\mathrm{L}_2\left(\Gamma_h^n\right)}\\ &+m_h\left(2\underline{U}_h^n(\cdot,t^{n-1})-U_h^{n-1}(\cdot,t^{n-1}),2\underline{U}_h^n(\cdot,t^{n-1})-U_h^{n-1}(\cdot,t^{n-1})\right)\Big),\end{aligned}\tag{5.18}$$

where we have integrated by parts the non-symmetric term. Summing over n and multiplying by τ gives

$$\begin{aligned}&\frac{1}{4}\Big(\left\|U_h^k\right\|^2_{\mathrm{L}_2\left(\Gamma_h^k\right)}+m_h\Big(2\underline{U}_h^k(\cdot,t^{k-1})-U_h^{k-1}(\cdot,t^{k-1}),2\underline{U}_h^k(\cdot,t^{k-1})-U_h^{k-1}(\cdot,t^{k-1})\Big)\Big)\\ &+\tau\sum_{i=2}^{k}\left\|\nabla_{\Gamma_h^i}U_h^i\right\|^2_{\mathrm{L}_2\left(\Gamma_h^i\right)}\leq c\tau\sum_{i=0}^{k}\left\|U_h^i\right\|^2_{\mathrm{L}_2\left(\Gamma_h^i\right)}\\ &+c\tau\sum_{i=1}^{k}m_h\left(2\underline{U}_h^i(\cdot,t^{i-1})-U_h^{i-1}(\cdot,t^{i-1}),2\underline{U}_h^i(\cdot,t^{i-1})-U_h^{i-1}(\cdot,t^{i-1})\right)\Big)\\ &+\frac{1}{4}\Big(\left\|U_h^1\right\|^2_{\mathrm{L}_2\left(\Gamma_h^1\right)}+m_h\left(2\underline{U}_h^1(\cdot,t^0)-U_h^0(\cdot,t^0),2\underline{U}_h^1(\cdot,t^0)-U_h^0(\cdot,t^0)\right)\Big).\end{aligned}\tag{5.19}$$

With the assumptions on the starting values, a Gronwall argument completes the proof. The estimate (5.8) follows by the usual norm equivalence.

For the bound (5.10) we first state the following basic identity given in [18], suppose we have vectors $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\kappa}\in\mathbb{R}^J$, we set $\boldsymbol{z}=\boldsymbol{\alpha}-\boldsymbol{\beta},\boldsymbol{y}=\boldsymbol{\beta}-\boldsymbol{\kappa}$

$$\frac{3}{2}\boldsymbol{\alpha}\cdot\boldsymbol{z}-2\boldsymbol{\beta}\cdot\boldsymbol{z}+\frac{1}{2}\boldsymbol{\kappa}\cdot\boldsymbol{z}=|\boldsymbol{z}|^2+\frac{1}{4}\Big(|\boldsymbol{z}|^2-|\boldsymbol{y}|^2+|\boldsymbol{z}-\boldsymbol{y}|^2\Big).\tag{5.20}$$

We work with the matrix vector form of the scheme (5.3), multiplying with $\boldsymbol{U}^{n+1}-\boldsymbol{U}^n$ and using (5.20) we have

$$\begin{aligned}&\frac{1}{\tau}\Bigg(\boldsymbol{M}^n\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^n\right)\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^n\right)+\frac{1}{4}\Big(\boldsymbol{M}^n\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^n\right)\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^n\right)\\ &-\boldsymbol{M}^n\left(\boldsymbol{U}^n-\boldsymbol{U}^{n-1}\right)\cdot\left(\boldsymbol{U}^n-\boldsymbol{U}^{n-1}\right)+\boldsymbol{M}^n\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n-1}\right)\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n-1}\right)\Big)\Bigg)\\ &+\left(\boldsymbol{S}^{n+1}+\boldsymbol{B}^{n+1}\right)\boldsymbol{U}^{n+1}\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^n\right)+\frac{1}{2\tau}\left(\boldsymbol{M}^{n-1}-\boldsymbol{M}^n\right)\boldsymbol{U}^{n-1}\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^n\right)\\ &+\frac{3}{2\tau}\left(\boldsymbol{M}^{n+1}-\boldsymbol{M}^n\right)\boldsymbol{U}^{n+1}\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^n\right)=0.\end{aligned}\tag{5.21}$$

Dropping a positive term and rearranging gives

$$
\begin{aligned}
(5.22)\quad & \boldsymbol{M}^{n+1}\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)+\frac{\tau}{4}\partial_\tau\left(\boldsymbol{M}^{n}\left(\boldsymbol{U}^{n}-\boldsymbol{U}^{n-1}\right)\cdot\left(\boldsymbol{U}^{n}-\boldsymbol{U}^{n-1}\right)\right)\\
&+\frac{\tau}{2}\left(\boldsymbol{S}^{n+1}\boldsymbol{U}^{n+1}\cdot\boldsymbol{U}^{n+1}-\boldsymbol{S}^{n}\boldsymbol{U}^{n}\cdot\boldsymbol{U}^{n}\right)\\
&\geq\\
&-\frac{\tau}{2}\boldsymbol{S}^{n+1}(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n})\cdot(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n})\\
&+\frac{\tau}{2}\left(\boldsymbol{S}^{n+1}-\boldsymbol{S}^{n}\right)\boldsymbol{U}^{n}\cdot\boldsymbol{U}^{n}-\tau\boldsymbol{B}^{n+1}\boldsymbol{U}^{n+1}\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)\\
&+\frac{1}{2}\left(\boldsymbol{M}^{n}-\boldsymbol{M}^{n-1}\right)\boldsymbol{U}^{n-1}\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)-\frac{3}{2}\left(\boldsymbol{M}^{n+1}-\boldsymbol{M}^{n}\right)\boldsymbol{U}^{n+1}\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)\\
&+\frac{5}{4}\left(\boldsymbol{M}^{n+1}-\boldsymbol{M}^{n}\right)\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right).\\
&:=I+II+III+IV+V+VI.
\end{aligned}
$$

For the first two terms on the right hand side of (5.22) we proceed as in [3] using (3.32) and get the following bound,

$$
\begin{aligned}
(5.23)\quad I+II=&-\frac{\tau^3}{2}a_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot,t^{n+1}-0),\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot,t^{n+1}-0)\right)\\
&+\frac{\tau}{2}\left(a_h\left(\underline{U}_h^n(\cdot,t^{n+1}),\underline{U}_h^n(\cdot,t^{n+1})\right)-a_h\left(\underline{U}_h^n(\cdot,t^{n}),\underline{U}_h^n(\cdot,t^{n})\right)\right)\\
&\leq c\tau^2\left\|\nabla_{\Gamma_h^{n+1}}U_h^{n+1}\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}.
\end{aligned}
$$

For the third term on the right hand side of (5.22) integrating by parts, we have

$$
\begin{aligned}
(5.24)\quad III\leq&\left|\tau^2 b_h\left(U_h^{n+1},\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot,t^{n+1}-0);(\boldsymbol{T}_h^a)^{n+1}\right)\right|\\
=&\tau^2\left|-\int_{\Gamma_h^{n+1}}\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot,t^{n+1}-0)\nabla_{\Gamma_h^{n+1}}\cdot\left((\boldsymbol{T}_h^a)^{n+1}U_h^{n+1}\right)\right|\\
\leq&c\tau^2\left\|\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot,t^{n+1}-0)\right\|_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}\left(\left\|U_h^{n+1}\right\|_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}+\left\|\nabla_{\Gamma_h^{n+1}}U_h^{n+1}\right\|_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}\right)\\
\leq&\epsilon\tau^2\left\|\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot,t^{n+1}-0)\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}+c(\epsilon)\tau^2\left\|U_h^{n+1}\right\|^2_{\mathrm{H}^1(\Gamma_h^{n+1})}.
\end{aligned}
$$

where ϵ is a positive constant of our choice. For the fourth term using (5.5) we have

$$
(5.25)\quad |IV|\leq\mu\tau\left(\boldsymbol{M}^{n-1}\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)\right)^{1/2}\left\|U_h^{n-1}\right\|_{\mathrm{L}_2\left(\Gamma_h^{n-1}\right)}.
$$

Applying Young's inequality and (3.35) gives, with ϵ a positive constant of our choice

$$
(5.26)\quad |IV|\leq c\tau^2\left(c(\epsilon)\left\|U_h^{n-1}\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n-1}\right)}+\epsilon\left\|\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot,t^{n+1}-0)\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}\right).
$$

For the fifth term we use (3.31) and (3.33) to give for all $\epsilon>0$,

$$
\begin{aligned}
(5.27)\quad |V|=&\frac{3\tau}{2}\left(m_h\left(U_h^{n+1},\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot,t^{n+1}-0)\right)-m_h\left(\underline{U}_h^{n+1}(\cdot,t^{n}),\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot,t^{n}+0)\right)\right)\\
\leq&c(\epsilon)\tau^2\left\|U_h^{n+1}\right\|_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}+\epsilon\tau^2\left\|\partial^\bullet_{h,\boldsymbol{V}_h^a}U_h^L(\cdot,t^{n+1}-0)\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}.
\end{aligned}
$$

For the sixth term we apply (3.31) to obtain

$$|VI| \leq c\tau \boldsymbol{M}^{n+1}\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right)\cdot\left(\boldsymbol{U}^{n+1}-\boldsymbol{U}^{n}\right) = c\tau^3 \left\|\partial^\bullet_{h,\boldsymbol{V}_h^a} U_h^L(\cdot,t^{n+1}-0)\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{n+1}\right)}. \tag{5.28}$$

Writing (5.22) in terms of the bilinear forms, applying the estimates (5.23), (5.24) and (5.26)—(5.28) and summing gives,

$$\begin{aligned}\sum_{i=2}^{n}\tau^2\left\|\partial^\bullet_{h,\boldsymbol{V}_h^a} U_h^L(\cdot,t^{i}-0)\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{i}\right)} + \tau\left\|\nabla_{\Gamma_h^n} U_h^n\right\|_{\mathrm{L}_2\left(\Gamma_h^{n}\right)} &\leq c\tau^2\left\|\partial^\bullet_{h,\boldsymbol{V}_h^a} U_h^L(\cdot,t^{1}-0)\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{1}\right)} \\ &+ c\tau\left\|\nabla_{\Gamma_h^1} U_h^1\right\|_{\mathrm{L}_2\left(\Gamma_h^{1}\right)} + c\tau^2\sum_{i=0}^{n}\left\|U_h^i\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{i}\right)} + c\tau^2\sum_{i=2}^{n}\left\|\nabla_{\Gamma_h^i} U_h^i\right\|^2_{\mathrm{L}_2\left(\Gamma_h^{i}\right)}.\end{aligned} \tag{5.29}$$

Dividing by τ, applying the stability bound (5.7) and the assumptions on the starting data (5.6) and (5.9) completes the proof of (5.10). As usual the equivalence of norms yields (5.11). ☐

5.4. **Theorem** (Error bound for the fully discrete scheme (5.1))**.** *Let u be a sufficiently smooth solution of (1.1) and let the geometry be sufficiently regular. Furthermore let $u_h^i, (i=0,\dots,N)$ denote the lift of the solution of the BDF2 fully discrete scheme (5.1) and assume that the approximation of the initial data and the starting values satisfy*

$$\|u(\cdot,0)-u_h(\cdot,0)\|_{\mathrm{L}_2(\Gamma(t))} \leq ch^2, \tag{5.30}$$

and

$$\left\|u(\cdot,t^1)-u_h(\cdot,t^1)\right\|_{\mathrm{L}_2(\Gamma(t))} \leq c(h^2+\tau^2). \tag{5.31}$$

Furthermore assume the starting values satisfy the stability assumptions (5.6) and (5.9). Then for $0 < h \leq h_0, 0 < \tau \leq \tau_0$, with h_0 dependent on the data of the problem and τ_0 dependent on the data of the problem and the arbitrary tangential velocity $\boldsymbol{a_\mathcal{T}}$, the following error bound holds. For $n \in \{2,\dots,N\}$ the solution of the fully discrete BDF2 scheme satisfies

(5.32)

$$\begin{aligned}&\|u(\cdot,t^n)-u_h^n\|^2_{\mathrm{L}_2(\Gamma^n)} + c_1h^2\tau\sum_{i=2}^{n}\left\|\nabla_\Gamma\left(u(\cdot,t^i)-u_h^i\right)\right\|^2_{\mathrm{L}_2(\Gamma^i)} \\ &\leq c\left(h^4+\tau^4\right)\left(\sup_{s\in[0,T]}\|u\|^2_{\mathrm{H}^2(\Gamma(s))} + \int_0^T \|u\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a} u\right\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a}(\partial^\bullet_{\boldsymbol{v}_a} u)\right\|^2_{\mathrm{H}^1(\Gamma(t))}\,\mathrm{d}t\right).\end{aligned}$$

We follow a similar strategy to that employed in the semidiscrete case to prove the theorem. We decompose the error as in §4.4 setting

$$u(\cdot,t^n)-u_h^n = \rho^n+\theta^n, \quad \rho^n=\rho(\cdot,t^n)=u(\cdot,t^n)-\mathrm{R}^h\,u(\cdot,t^n), \quad \theta^n=\mathrm{R}^h\,u(\cdot,t^n)-u_h^n \in \mathcal{S}_h^l, \tag{5.33}$$

with R^h the Ritz projection defined in (C.1) and u_h^n the lift of the solution to the fully discrete scheme at time t^n.

From the scheme (5.1) on the interval $[t^{n-1},t^{n+1}]$ we have

$$\begin{aligned}\mathcal{L}_2\left(u_h,\varphi_h^{n+1}\right) + a\left(u_h^{n+1},\varphi_h^{n+1}\right) + b\left(u_h^{n+1},\varphi_h^{n+1};(\boldsymbol{t}_h^a)^{n+1}\right) &\\ =\mathcal{L}_2\left(u_h,\varphi_h^{n+1}\right) - \mathcal{L}_{2,h}\left(U_h,\Phi_h^{n+1}\right) + a\left(u_h^{n+1},\varphi_h^{n+1}\right) - a_h\left(U_h^{n+1},\Phi_h^{n+1}\right)&\\ + b\left(u_h^{n+1},\varphi_h^{n+1};(\boldsymbol{t}_h^a)^{n+1}\right) - b_h\left(U_h^{n+1},\Phi_h^{n+1};(\boldsymbol{T}_h^a)^{n+1}\right)&\\ :=H_1(\varphi_h^{n+1}).&\end{aligned} \tag{5.34}$$

From the definition of the Ritz projection (C.1) we have

$$\begin{aligned}(5.35)\qquad &\mathcal{L}_2\left(\mathrm{R}^h\, u, \varphi_h^{n+1}\right) + a\left(\mathrm{R}^h\, u^{n+1}, \varphi_h^{n+1}\right) + b\left(\mathrm{R}^h\, u^{n+1}, \varphi_h^{n+1}; (\boldsymbol{t}_h^a)^{n+1}\right)\\ &= -\mathcal{L}_2\left(\rho, \varphi_h^{n+1}\right) + \mathcal{L}_2\left(u, \varphi_h^{n+1}\right) + a\left(u^{n+1}, \varphi_h^{n+1}\right) + b\left(\mathrm{R}^h\, u^{n+1}, \varphi_h^{n+1}; (\boldsymbol{t}_h^a)^{n+1}\right)\\ &:= H_2(\varphi_h^{n+1}).\end{aligned}$$

Taking the difference of (5.35) and (5.34) we arrive at the error relation between the fully discrete solution and the Ritz projection, for $\varphi_h^{n+1} = (\Phi^{n+1})^l \in \mathcal{S}_h^{n+1,l}$

$$(5.36)\qquad \mathcal{L}_2\left(\theta, \varphi_h^{n+1}\right) + a\left(\theta^{n+1}, \varphi_h^{n+1}\right) + b\left(\theta^{n+1}, \varphi_h^{n+1}; (\boldsymbol{t}_h^a)^{n+1}\right) = H_2(\varphi_h^{n+1}) - H_1(\varphi_h^{n+1}).$$

5.5. **Lemma.** *For H_1 defined in (5.34) and for all $\epsilon > 0$, we have the estimate*

$$\begin{aligned}(5.37)\qquad \left|H_1(\varphi_h^{n+1})\right| \leq &\frac{c(\epsilon)}{\tau}h^4\int_{t^{n-1}}^{t^{n+1}} \left\|u_h^L\right\|^2_{\mathrm{H}^1(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_h^a} u_h^L\right\|^2_{\mathrm{L}_2(\Gamma(t))}\,\mathrm{d}t\\ &+ c(\epsilon)h^4\left\|u_h^{n+1}\right\|^2_{\mathrm{H}^1(\Gamma^{n+1})} + c\left\|\varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})} + \epsilon\left\|\nabla_{\Gamma^{n+1}}\varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}\end{aligned}$$

Proof . From the definition of H_1 (5.34) we have

$$\begin{aligned}(5.38)\qquad \left|H_1(\varphi_h^{n+1})\right| \leq &\left|\mathcal{L}_2\left(u_h, \varphi_h^{n+1}\right) - \mathcal{L}_{2,h}\left(U_h, \Phi_h^{n+1}\right)\right| + \left|a\left(u_h^{n+1}, \varphi_h^{n+1}\right) - a_h\left(U_h^{n+1}, \Phi_h^{n+1}\right)\right|\\ &+ \left|b\left(u_h^{n+1}, \varphi_h^{n+1}; (\boldsymbol{t}_h^a)^{n+1}\right) - b_h\left(U_h^{n+1}, \Phi_h^{n+1}; (\boldsymbol{T}^a{}_h)^{n+1}\right)\right|\\ &:= I + II + III.\end{aligned}$$

For the first term, we follow [3, Proof of Lemma 4.3], using the transport formulas (3.29) and (3.30) together with (B.9) and (B.11) we have

$$\begin{aligned}(5.39)\qquad I =&\leq \frac{c}{\tau}\Bigg|\int_{t^{n-1}}^{t^{n+1}} m_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a} U_h(\cdot,t), \underline{\Phi}_h^{n+1}(\cdot,t)\right) + g_h\left(U_h(\cdot,t), \underline{\Phi}_h^{n+1}(\cdot,t); \boldsymbol{V}^a{}_h(\cdot,t)\right)\\ &\quad - m\left(\partial^\bullet_{h,\boldsymbol{v}_h^a} u_h(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t)\right) + g\left(u_h(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t); \boldsymbol{v}_h^a(\cdot,t)\right)\mathrm{d}t\Bigg|.\\ &\leq \frac{ch^2}{\tau}\int_{t^{n-1}}^{t^{n+1}}\left(\left\|\partial^\bullet_{h,\boldsymbol{v}_h^a} u_h^L\right\|_{\mathrm{L}_2(\Gamma(t))}\left\|\underline{\varphi}_h^{n+1}\right\|_{\mathrm{L}_2(\Gamma(t))} + \left\|u_h\right\|_{\mathrm{H}^1(\Gamma(t))}\left\|\underline{\varphi}_h^{n+1}\right\|_{\mathrm{H}^1(\Gamma(t))}\right)\mathrm{d}t\\ &\leq \epsilon\left\|\nabla_{\Gamma^{n+1}}\varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})} + c\left\|\varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}\\ &\quad + \frac{c(\epsilon)}{\tau}h^4\int_{t^{n-1}}^{t^{n+1}}\left\|\partial^\bullet_{h,\boldsymbol{v}_h^a} u_h^L\right\|^2_{\mathrm{L}_2(\Gamma(t))} + \left\|u_h^L\right\|^2_{\mathrm{H}^1(\Gamma(t))}\mathrm{d}t\end{aligned}$$

where ϵ is a positive constant of our choice. Using (B.10) we conclude that for all $\epsilon > 0$

$$\begin{aligned}(5.40)\qquad II &\leq ch^2\left\|\nabla_{\Gamma^{n+1}}\varphi_h^{n+1}\right\|_{\mathrm{L}_2(\Gamma^{n+1})}\left\|\nabla_{\Gamma^{n+1}}u_h^{n+1}\right\|_{\mathrm{L}_2(\Gamma^{n+1})}\\ &\leq c(\epsilon)h^4\left\|\nabla_{\Gamma^{n+1}}u_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})} + \epsilon\left\|\nabla_{\Gamma^{n+1}}\varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}\end{aligned}$$

Using (B.12) we have for all $\epsilon > 0$

$$\begin{aligned}(5.41)\qquad III &\leq ch^2\left\|u_h^{n+1}\right\|_{\mathrm{L}_2(\Gamma^{n+1})}\left\|\nabla_{\Gamma^{n+1}}\varphi_h^{n+1}\right\|_{\mathrm{L}_2(\Gamma^{n+1})}\\ &\leq c(\epsilon)h^4\left\|u_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})} + \epsilon\left\|\nabla_{\Gamma^{n+1}}\varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}.\end{aligned}$$

Applying the estimates (5.39)—(5.41) in (5.38) completes the proof of the Lemma. □

5.6. **Lemma.** *For H_2 defined in (5.35) and for all $\epsilon > 0$, we have the estimate*

$$
\begin{aligned}
\left|H_2(\varphi_h^{n+1})\right| \leq & \frac{c}{\tau} h^4 \int_{t^{n-1}}^{t^{n+1}} \|u\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^{\bullet}_{\boldsymbol{v}_a} u\right\|^2_{\mathrm{H}^2(\Gamma(t))} \mathrm{d}t \\
& + c\tau^3 \int_{t^{n-1}}^{t^{n+1}} \|u\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^{\bullet}_{\boldsymbol{v}_a} u\right\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^{\bullet}_{\boldsymbol{v}_a}(\partial^{\bullet}_{\boldsymbol{v}_a} u)\right\|^2_{\mathrm{H}^1(\Gamma(t))} \mathrm{d}t \\
& + ch^4 \|u\|^2_{\mathrm{H}^2(\Gamma^{n+1})} + c\left\|\varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})} + \epsilon \left\|\nabla_{\Gamma^{n+1}} \varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}
\end{aligned} \tag{5.42}
$$

Proof . We set

$$
\sigma(t) = \begin{cases} \frac{3}{2\tau} & t \in [t^n, t^{n+1}] \\ -\frac{1}{2\tau} & t \in [t^{n-1}, t^n]. \end{cases} \tag{5.43}
$$

We start by noting that using the transport formula (3.30),

(5.44)

$$
\begin{aligned}
\left|\mathcal{L}_2\left(\rho, \varphi_h^{n+1}\right)\right| &= \left| \int_{t^{n-1}}^{t^{n+1}} \sigma(t) \left(m\left(\partial^{\bullet}_{h,\boldsymbol{v}_h^a} \rho(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t)\right) + g\left(\rho(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t); \boldsymbol{v}_h^a\right) \mathrm{d}t \right) \right| \\
&\leq \frac{c}{\tau} \int_{t^{n-1}}^{t^{n+1}} \left(\left\|\partial^{\bullet}_{h,\boldsymbol{v}_h^a} \rho(\cdot,t)\right\|_{\mathrm{L}_2(\Gamma(t))} + \|\rho(\cdot,t)\|_{\mathrm{L}_2(\Gamma(t))} \right) \left\|\underline{\varphi}_h^{n+1}(\cdot,t)\right\|_{\mathrm{L}_2(\Gamma(t))} \mathrm{d}t.
\end{aligned}
$$

Young's inequality, (3.35), (C.2) and (C.3), yield the estimate

(5.45)

$$
\left|\mathcal{L}_2\left(\rho, \varphi_h^{n+1}\right)\right| \leq \frac{ch^4}{\tau} \int_{t^{n-1}}^{t^{n+1}} \left(\left\|\partial^{\bullet}_{\boldsymbol{v}_a} u(\cdot,t)\right\|^2_{\mathrm{H}^2(\Gamma(t))} + \|u(\cdot,t)\|^2_{\mathrm{H}^2(\Gamma(t))} \mathrm{d}t \right) + \left\|{\varphi_h}^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}.
$$

Integrating in time the variational form (2.5) over the interval $[t^n, t^{n+1}]$ with $\varphi = \underline{\varphi}_h^{n+1}$ we have

$$
\begin{aligned}
m\left(u^{n+1}, \underline{\varphi}_h^{n+1}(\cdot, t^{n+1})\right) - m\left(u^n, \underline{\varphi}_h^{n+1}(\cdot, t^n)\right) + \int_{t^n}^{t^{n+1}} a\left(u(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t)\right) \mathrm{d}t \\
= \int_{t^n}^{t^{n+1}} - b\left(u(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t); \boldsymbol{a}_{\boldsymbol{\mathcal{T}}}\right) + m\left(u(\cdot,t), \partial^{\bullet}_{\boldsymbol{v}_a} \underline{\varphi}_h^{n+1}(\cdot,t)\right) \mathrm{d}t.
\end{aligned} \tag{5.46}
$$

Similarly integrating in time the variational form (2.5) over the interval $[t^{n-1}, t^{n+1}]$ with $\varphi = \underline{\varphi}_h^{n+1}$ we have

$$
\begin{aligned}
m\left(u^{n+1}, \underline{\varphi}_h^{n+1}(\cdot, t^{n+1})\right) - m\left(u^{n-1}, \underline{\varphi}_h^{n+1}(\cdot, t^{n-1})\right) + \int_{t^{n-1}}^{t^{n+1}} a\left(u(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t)\right) \mathrm{d}t \\
= \int_{t^{n-1}}^{t^{n+1}} - b\left(u(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t); \boldsymbol{a}_{\boldsymbol{\mathcal{T}}}\right) + m\left(u(\cdot,t), \partial^{\bullet}_{\boldsymbol{v}_a} \underline{\varphi}_h^{n+1}(\cdot,t)\right) \mathrm{d}t.
\end{aligned} \tag{5.47}
$$

From the definition (3.28), we observe that

$$
\begin{aligned}
\mathcal{L}_2\left(u,\varphi^{n+1}\right) =&\frac{2}{\tau}\left(m\left(u^{n+1},\underline{\varphi}_h^{n+1}(\cdot,t^{n+1})\right)-m\left(u^{n},\underline{\varphi}_h^{n+1}(\cdot,t^{n})\right)\right)\\
&-\frac{1}{2\tau}\left(m\left(u^{n+1},\underline{\varphi}_h^{n+1}(\cdot,t^{n+1})\right)-m\left(u^{n-1},\underline{\varphi}_h^{n+1}(\cdot,t^{n-1})\right)\right)\\
=&\int_{t^{n-1}}^{t^{n+1}}\sigma(t)\Bigg(m\left(u(\cdot,t),\partial^\bullet_{\boldsymbol{v}_a}\underline{\varphi}_h^{n+1}(\cdot,t)\right)-a\left(u(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t)\right)\\
&-b\left(u(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t);\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}(\cdot,t)\right)\Bigg)\mathrm{d}t,
\end{aligned}
\tag{5.48}
$$

with σ as defined in (5.43). Thus we have

$$
\begin{aligned}
&\mathcal{L}_2\left(u,\varphi_h^{n+1}\right)+a\left(u^{n+1},\varphi_h^{n+1}\right)+b\left(\mathrm{R}^h\,u^{n+1},\varphi_h^{n+1};(\boldsymbol{t}_h^a)^{n+1}\right)=\\
&\left(b\left(\mathrm{R}^h\,u^{n+1},\varphi_h^{n+1};(\boldsymbol{t}_h^a)^{n+1}\right)-b\left(u^{n+1},\varphi_h^{n+1};\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}^{n+1}\right)\right)\\
&+\left(b\left(u^{n+1},\varphi_h^{n+1};\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}^{n+1}\right)-\int_{t^{n-1}}^{t^{n+1}}\sigma(t)b\left(u(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t);\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}(\cdot,t)\right)\mathrm{d}t\right)\\
&+\left(a\left(u^{n+1},\varphi_h^{n+1}\right)-\int_{t^{n-1}}^{t^{n+1}}\sigma(t)a\left(u(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t)\right)\mathrm{d}t\right)\\
&+\int_{t^{n-1}}^{t^{n+1}}\sigma(t)m\left(u(\cdot,t),\partial^\bullet_{\boldsymbol{v}_a}\underline{\varphi}_h^{n+1}(\cdot,t)\right)\\
&:=I+II+III+IV
\end{aligned}
\tag{5.49}
$$

The first term on the right of (5.49) is estimated as follows, we have

$$
|I|\leq\left|-b\left(\rho^{n+1},\varphi_h^{n+1};(\boldsymbol{t}_h^a)^{n+1}\right)\right|+\left|b\left(u^{n+1},\varphi_h^{n+1};(\boldsymbol{t}_h^a)^{n+1}-\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}^{n+1}\right)\right|
\tag{5.50}
$$

For the first term on the right hand side of (5.50) we use (C.2) to see that for all $\epsilon>0$

$$
\left|-b\left(\rho^{n+1},\varphi_h^{n+1};(\boldsymbol{t}_h^a)^{n+1}\right)\right|\mathrm{d}t\leq c(\epsilon)h^4\left\|u\right\|^2_{\mathrm{H}^2(\Gamma^{n+1})}+\epsilon\left\|\nabla_{\Gamma^{n+1}}\varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}.
\tag{5.51}
$$

For the next term on the right hand side of (5.50) we apply (B.4) and observe that for all $\epsilon>0$

$$
\left|b\left(u^{n+1},\varphi_h^{n+1};(\boldsymbol{t}_h^a)^{n+1}-\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}^{n+1}\right)\right|\mathrm{d}t\leq c(\epsilon)h^4\left\|u\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}+\epsilon\left\|\nabla_{\Gamma^{n+1}}\varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}.
\tag{5.52}
$$

Thus we have

$$
|I|\leq c(\epsilon)h^4\left\|u\right\|^2_{\mathrm{H}^2(\Gamma^{n+1})}+\epsilon\left\|\nabla_{\Gamma^{n+1}}\varphi_h^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})},
\tag{5.53}
$$

for all $\epsilon>0$. For the second term on the right of (5.49) we have

$$
\begin{aligned}
|II|\leq&\frac{1}{\tau}\Bigg(\int_{t^{n}}^{t^{n+1}}(t^{n+1}-t)(t^{n+1}-3t-4t^n)\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}b\left(u(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t);\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}(\cdot,t)\right)\right|\mathrm{d}t\\
&+\int_{t^{n-1}}^{t^{n}}(t-t^{n-1})^2\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}b\left(u(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t);\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}(\cdot,t)\right)\right|\mathrm{d}t\Bigg).
\end{aligned}
\tag{5.54}
$$

The estimate (A.6) and the fact that $\partial^\bullet_{h,\boldsymbol{v}_h^a}\underline{\varphi}_h^{n+1} = 0$ yield

$$(5.55)\qquad |II| \le \tau \int_{t^{n-1}}^{t^{n+1}} \Big(\|u\|_{\mathrm{L}_2(\Gamma(t))} + \left\|\partial^\bullet_{h,\boldsymbol{v}_h^a} u\right\|_{\mathrm{L}_2(\Gamma(t))} + \left\|\partial^\bullet_{h,\boldsymbol{v}_h^a}(\partial^\bullet_{h,\boldsymbol{v}_h^a} u)\right\|_{\mathrm{L}_2(\Gamma(t))} \Big) \left\|\nabla_{\Gamma(t)}\underline{\varphi}_h^{n+1}\right\|_{\mathrm{L}_2(\Gamma(t))} \mathrm{d}t.$$

Young's inequality and (3.36) give for all $\epsilon > 0$,

$$(5.56)\qquad |II| \le c(\epsilon)\tau^3 \int_{t^{n-1}}^{t^{n+1}} \left(\|u\|^2_{\mathrm{L}_2(\Gamma(t))} + \left\|\partial^\bullet_{h,\boldsymbol{v}_h^a} u\right\|^2_{\mathrm{L}_2(\Gamma(t))} + \left\|\partial^\bullet_{h,\boldsymbol{v}_h^a}(\partial^\bullet_{h,\boldsymbol{v}_h^a} u)\right\|^2_{\mathrm{L}_2(\Gamma(t))} \right) \mathrm{d}t + \epsilon \left\|\nabla_{\Gamma^{n+1}}{\varphi_h}^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}.$$

The third term on the right of (5.49) is estimated in the same way using (A.5) and (3.36) to give for all $\epsilon > 0$,

$$(5.57)\qquad |III| \le c(\epsilon)\tau^3 \int_{t^{n-1}}^{t^{n+1}} \Big(\left\|\nabla_{\Gamma(t)}\partial^\bullet_{h,\boldsymbol{v}_h^a}(\partial^\bullet_{h,\boldsymbol{v}_h^a} u)\right\|^2_{\mathrm{L}_2(\Gamma(t))} + \left\|\nabla_{\Gamma(t)}\partial^\bullet_{h,\boldsymbol{v}_h^a} u\right\|^2_{\mathrm{L}_2(\Gamma(t))} + \left\|\nabla_{\Gamma(t)} u\right\|^2_{\mathrm{L}_2(\Gamma_h(t))} \Big) \mathrm{d}t + \epsilon \left\|\nabla_{\Gamma^{n+1}}{\varphi_h}^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}.$$

The fourth term on the right of (5.49) may be estimated using (B.5) together with the fact that $\partial^\bullet_{h,\boldsymbol{v}_h^a}\underline{\varphi}_h^{n+1} = 0$ which gives for all $\epsilon > 0$,

$$(5.58)\qquad |IV| \le \frac{c(\epsilon)}{\tau} h^4 \int_{t^n}^{t^{n+1}} \|u\|^2_{\mathrm{L}_2(\Gamma(t))} \mathrm{d}t + \epsilon \left\|\nabla_{\Gamma^{n+1}}{\varphi_h}^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})}.$$

The estimates (5.45), (5.53), (5.56), (5.57) and (5.58) together with the estimates (B.7) and (B.8) completes the proof of the Lemma. ☐

We may now finally complete the proof of Theorem 5.4.

Proof of Theorem 5.4 With the error decomposition of (5.33) and the estimates on the Ritz projection error C.2 it remains to bound θ. With the same argument as used in the proof of Lemma 5.3, i.e., (5.12)—(5.17) we have

$$(5.59)\qquad \frac{1}{4}\partial_\tau\Big(m(\theta^n,\theta^n) + m\left(2\underline{\theta}^n(\cdot,t^{n-1}) - \theta^{n-1}(\cdot,t^{n-1}), 2\underline{\theta}^n(\cdot,t^{n-1}) - \theta^{n-1}(\cdot,t^{n-1})\right)\Big) + \left\|\nabla_{\Gamma^{n+1}}\theta^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})} \le \left|H_1(\theta^{n+1})\right| + \left|H_2(\theta^{n+1})\right| + c\Big(\left\|\theta^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})} + \|\theta^n\|^2_{\mathrm{L}_2(\Gamma^n)} + m\left(2\underline{\theta}^n(\cdot,t^{n-1}) - \theta^{n-1}(\cdot,t^{n-1}), 2\underline{\theta}^n(\cdot,t^{n-1}) - \theta^{n-1}(\cdot,t^{n-1})\right)\Big).$$

Inserting the bounds from Lemmas 5.5 and 5.6 we obtain

$$(5.60)\qquad \frac{1}{4}\partial_\tau\Big(m(\theta^n,\theta^n) + m\left(2\underline{\theta}^n(\cdot,t^{n-1}) - \theta^{n-1}(\cdot,t^{n-1}), 2\underline{\theta}^n(\cdot,t^{n-1}) - \theta^{n-1}(\cdot,t^{n-1})\right)\Big) + \left\|\nabla_{\Gamma^{n+1}}\theta^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})} \le \epsilon\left\|\nabla_{\Gamma^{n+1}}\theta^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})} + c\Big(\left\|\theta^{n+1}\right\|^2_{\mathrm{L}_2(\Gamma^{n+1})} + \|\theta^n\|^2_{\mathrm{L}_2(\Gamma^n)} + m\left(2\underline{\theta}^n(\cdot,t^{n-1}) - \theta^{n-1}(\cdot,t^{n-1}), 2\underline{\theta}^n(\cdot,t^{n-1}) - \theta^{n-1}(\cdot,t^{n-1})\right)\Big) + \frac{c}{\tau}h^4 \int_{t^{n-1}}^{t^{n+1}} \|u\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a} u\right\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|u_h^L\right\|^2_{\mathrm{H}^1(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_h^a} u_h^L\right\|^2_{\mathrm{L}_2(\Gamma(t))} \mathrm{d}t + c\tau^3 \int_{t^{n-1}}^{t^{n+1}} \|u\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a} u\right\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a}(\partial^\bullet_{\boldsymbol{v}_a} u)\right\|^2_{\mathrm{H}^1(\Gamma(t))} \mathrm{d}t + ch^4\|u\|^2_{\mathrm{H}^2(\Gamma^{n+1})} + ch^4\left\|u_h^{n+1}\right\|^2_{\mathrm{H}^1(\Gamma^{n+1})}$$

Summing over time, multiplying by τ and choosing $\epsilon > 0$ suitably (note the dependence of the constants on ϵ has been suppressed) yields (where we have dropped a positive term), for $n \in \{2, \ldots, N\}$

$$\begin{aligned}(5.61)\quad \|\theta^n\|^2_{\mathrm{L}_2(\Gamma^n)} + c_1\tau\sum_{k=2}^{n}\left\|\nabla_{\Gamma^k}\theta^k\right\|^2_{\mathrm{L}_2(\Gamma^k)} &\le \left\|\theta^1\right\|^2_{\mathrm{L}_2(\Gamma^1)} + c\tau\sum_{i=1}^{n}\left\|\theta^i\right\|^2_{\mathrm{L}_2(\Gamma^i)}\\ &+ m\left(2\underline{\theta}^1(\cdot,t^0)-\theta^0(\cdot,t^0), 2\underline{\theta}^1(\cdot,t^0)-\theta^0(\cdot,t^0)\right)\\ &+ ch^4\int_0^{t^n}\|u\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a}u\right\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|u_h^L\right\|^2_{\mathrm{H}^1(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_h^a}u_h^L\right\|^2_{\mathrm{L}_2(\Gamma(t))}\mathrm{d}t\\ &+ c\tau^4\int_0^{t^n}\|u\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a}u\right\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a}(\partial^\bullet_{\boldsymbol{v}_a}u)\right\|^2_{\mathrm{H}^1(\Gamma(t))}\mathrm{d}t\\ &+ c\tau h^4\sum_{i=2}^{n}\left(\|u\|^2_{\mathrm{H}^2(\Gamma^i)} + ch^4\left\|u_h^i\right\|^2_{\mathrm{H}^1(\Gamma^i)}\right).\end{aligned}$$

A Gronwall argument together with the stability bounds of Lemmas 4.2 5.3 and the assumptions on the approximation of the initial data and starting values complete the proof. □

5.7. **Fully discrete BDF1 ALE-ESFEM scheme.** We could also have considered an implicit Euler time discretisation of the semidiscrete scheme (4.1) as follows. Given $U_h^0 \in \mathcal{S}_h^0$ find $U_h^n \in \mathcal{S}_h^n, n = \{1, \ldots, N\}$ such that for all $\Phi_h^n \in \mathcal{S}_h^n, \Phi_h^{n+1} \in \mathcal{S}_h^{n+1}$ and for $n \in \{0, \ldots, N-1\}$

$$(5.62)\quad \partial_\tau m_h\left(U_h^n, \Phi_h^n\right) + a_h\left(U_h^{n+1}, \Phi_h^{n+1}\right) = m_h\left(U_h^n, \partial^\bullet_{h,\boldsymbol{V}_h^a}\Phi_h^n\right) - b_h\left(U_h^{n+1}, \Phi_h^{n+1}; (\boldsymbol{T}_h^a)^{n+1}\right).$$

Using the ideas in the analysis presented above it is a relatively straight forward extension of [3] to show the following error bound.

5.8. **Corollary** (Error bound for an implicit Euler time discretisation)**.** *Let u be a sufficiently smooth solution of (1.1) and let the geometry be sufficiently regular. Furthermore let $u_h^i, (i = 0, \ldots, N)$ denote the lift of the solution of the implicit Euler fully discrete scheme (5.62) and assume that the approximation of the initial data satisfies*

$$(5.63)\qquad \|u(\cdot,0) - u_h(\cdot,0)\|_{\mathrm{L}_2(\Gamma(t))} \le ch^2.$$

Then for $0 < h \le h_0, 0 < \tau \le \tau_0$ (with h_0 dependent on the data of the problem and τ_0 dependent on the data of the problem and the arbitrary tangential velocity $\boldsymbol{a}_{\mathcal{T}}$) the following error bound holds. For $n \in \{0, \ldots, N\}$

$$\begin{aligned}(5.64)\qquad \|u(\cdot,t^n) - u_h^n\|^2_{\mathrm{L}_2(\Gamma^n)} + c_1h^2\tau\sum_{i=1}^{n}\left\|\nabla_\Gamma\left(u(\cdot,t^i) - u_h^i\right)\right\|^2_{\mathrm{L}_2(\Gamma^i)}&\\ \le c\left(h^4+\tau^2\right)\left(\|u\|^2_{\mathrm{H}^2(\Gamma(t))} + \left\|\partial^\bullet_{\boldsymbol{v}_a}u\right\|^2_{\mathrm{H}^2(\Gamma(t))}\right).&\end{aligned}$$

5.9. **Remark** (Surfaces with curved boundary)**.** Our assumption that $\Gamma(t)$ is smooth necessitates that the boundary of $\Gamma(t)$ (if nonempty) is curved. For the natural boundary conditions we consider it is possible to define a conforming piecewise linear finite element space on a triangulation with curved boundary elements such that the lift of the triangulated surface is the smooth surface i.e., $(\Gamma_h(t))^l = \Gamma(t)$, see for example [19]. This is the setup we assume in the analysis and hence we do not need to consider the error due to boundary approximation. However we note that this theoretical formulation necessitates integration over curved simplices which in practice must be evaluated by numerical quadrature, we neglect the analysis of this variational crime in this work.

5.10. **Remark** (Application to ALE schemes for PDEs in bulk domains)**.** The analysis holds for flat hyper surfaces with boundary. Hence it is valid for the case that $\Gamma_h(t)$ is a flat three dimensional hypersurface in $\mathbb{R}^4$ with smooth boundary, this corresponds to a moving domain in three dimensions. Specifically the analysis we present is valid for a flat three dimensional surface with zero normal velocity but nonzero tangential and conormal velocity. In this case as the domain is flat the geometric errors we estimate in the subsequent sections are zero. Therefore as a consequence of our analysis we get an error estimate for an ALE scheme for a linear parabolic equation on an evolving three dimensional bulk domain in which all of the analysis is all carried out on the physical domain.

6. Numerical experiments

We report on numerical simulations that support our theoretical results and illustrate that, for certain material velocities, the arbitrary tangential velocity may be chosen such that the meshes generated during the evolution are more suitable for computation than in the Lagrangian case. We also report on an experiment in which we investigate numerically the long time behaviour of solutions to (1.1) with different initial data when the evolution of the surface is a periodic function of time. The code for the simulations made use of the finite element library ALBERTA [20] and for the visualisation we used PARAVIEW [21].

6.1. **Example** (Benchmarking experiments)**.** We define the level set function

$$d(\boldsymbol{x},t) = \frac{x_1^2}{a(t)} + x_2^2 + x_3^2 - 1, \tag{6.1}$$

and consider the surface

$$\Gamma(t) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \,|\, d(\boldsymbol{x},t) = 0, x_3 \geq 0 \right\}. \tag{6.2}$$

The surface is the surface of a hemiellipsoid with time dependent axis. We set $a(t) = 1 + 0.25\sin(t)$ and we assume that the material velocity of the surface $\boldsymbol{v}$ has zero tangential component. Therefore the material velocity of the surface is given by [1]

$$\boldsymbol{v} = \frac{-\partial_t d}{|\nabla d|}\frac{\nabla d}{|\nabla d|}. \tag{6.3}$$

We insert a suitable right hand side in (1.1) such that the exact solution is $u(\boldsymbol{x},t) = \sin(t)x_1x_2$ and consider a time interval $[0,2]$.

To investigate the performance of the proposed BDF2-ALE ESFEM scheme we report on two numerical experiments. First we consider the Lagrangian scheme i.e., $\boldsymbol{a}_{\mathcal{T}} = \mathbf{0}$. Secondly we consider an evolution in which the arbitrary tangential velocity is nonzero. The velocity is defined as follows;

$$v_1^a(\boldsymbol{x},t) = \frac{0.25\cos(t)}{2(1+0.25\sin(t))^{1/2}}x_0, \quad v_2^a(\boldsymbol{x},t) = v_3^a(\boldsymbol{x},t) = 0, \quad \boldsymbol{x}_0 \in \Gamma(0). \tag{6.4}$$

The arbitrary tangential velocity is then determined by $\boldsymbol{a}_{\mathcal{T}} = \boldsymbol{v}^a - \boldsymbol{v}$ where $\boldsymbol{v}^a$ and $\boldsymbol{v}$ are defined by (6.4) and (6.3) respectively. We note that $\boldsymbol{v}^a \cdot \boldsymbol{\mu} = 0$ as the conormal to the boundary of $\Gamma(t)$ is given by $(0,0,-1)^T$.

6.2. **Definition.** Experimental order of convergence (EOC) For a series of triangulations $\left\{\mathcal{T}_i\right\}_{i=0,\dots,N}$ we denote by $\{e_i\}_{i=0,\dots,N}$ the error and by h_i the mesh size of $\mathcal{T}_i$. The EOC is given by

$$EOC(e_{i,i+1}, h_{i,i+1}) = \ln(e_{i+1}/e_i)/\ln(h_{i+1}/h_i). \tag{6.5}$$

In Tables 1 and 2 we report on the mesh size at the final time together with the errors and EOCs in the norms appearing in Theorem 5.4 for the two numerical simulations considered in Example 6.1. The EOCs were computed using the mesh size at the final time and the timestep was coupled to the initial mesh size. The starting values for the scheme were taken to be the interpolant of the exact solution. We observe that the EOCs support the error bounds of Theorem 5.4 and that for this example the errors with the Lagrangian and ALE schemes are similar in magnitude.

h	$\sup_{n\in[2,\ldots,N]} \Vert u(\cdot,t^n) - u_h^n \Vert_{\mathrm{L}_2(\Gamma^n)}$	*EOC*	$\sum_{i=2}^N \left(\tau \left\Vert \nabla_{\Gamma^i} \left(u(\cdot,t^i) - u_h^i\right)\right\Vert^2_{\mathrm{L}_2(\Gamma^n)}\right)^{1/2}$	*EOC*
0.88146	0.07772	-	0.63634	-
0.47668	0.02087	2.13842	0.36133	0.92064
0.24445	0.00546	2.00845	0.18755	0.98184
0.12307	0.00140	1.97958	0.09480	0.99420
0.06165	0.00036	1.96828	0.04754	0.99823

TABLE 1. Errors and EOC in the $\mathrm{L}_\infty\left(0,T;\mathrm{L}_2\right)$ norm and the $\mathrm{L}_2\left(0,T;\mathrm{H}^1\right)$ norm for Example 6.1 with the Lagrangian scheme ($\boldsymbol{a}_{\mathcal{T}} = \mathbf{0}$).

h	$\sup_{n\in[2,\ldots,N]} \Vert u(\cdot,t^n) - u_h^n \Vert_{\mathrm{L}_2(\Gamma^n)}$	*EOC*	$\sum_{i=2}^N \left(\tau \left\Vert \nabla_{\Gamma^i} \left(u(\cdot,t^i) - u_h^i\right)\right\Vert^2_{\mathrm{L}_2(\Gamma^n)}\right)^{1/2}$	*EOC*
0.85679	0.07876	-	0.63090	-
0.44695	0.02134	2.00683	0.35151	0.89884
0.22693	0.00560	1.97379	0.18173	0.97332
0.11415	0.00143	1.98248	0.09177	0.99437
0.05722	0.00036	1.98228	0.04601	0.99973

TABLE 2. Errors and EOC in the $\mathrm{L}_\infty\left(0,T;\mathrm{L}_2\right)$ norm and the $\mathrm{L}_2\left(0,T;\mathrm{H}^1\right)$ norm for Example 6.1 with the velocity defined by (6.4) which includes a nonzero arbitrary tangential component $\boldsymbol{a}_{\mathcal{T}}$.

6.3. **Example** (Comparison of the Lagrangian and ALE schemes)**.** We define the level set function

$$d(\boldsymbol{x},t) = \frac{x_1^2}{a(t)^2} + G(x_2^2) + G\left(\frac{x_3^2}{L(t)^2}\right) - 1, \tag{6.6}$$

where $G(s) = 31.25s(s-0.36)(s-0.95)$ and consider the surface

$$\Gamma(t) = \left\{\boldsymbol{x}\in\mathbb{R}^3 \,|d(\boldsymbol{x},t) = 0\right\}. \tag{6.7}$$

To compare the Lagrangian and the ALE numerical schemes we first consider a numerical scheme where the nodes are moved with the material velocity, which we assume is the normal velocity. For this Lagrangian scheme we approximate the nodal velocity by solving the ODE (6.3) at each node numerically with d as in (6.9). Secondly we consider an evolution of the form proposed in [10] where the arbitrary tangential velocity is nonzero. The evolution is defined as follows; for each node $(X_j(t), Y_j(t), Z_j(t))^T := \boldsymbol{X}_j, j = 1,\ldots,J$, given nodes $\boldsymbol{X}_j(0), j=1,\ldots,J$ on $\Gamma(0)$, we set

$$X_j(t) = X_j(0)\frac{a(t)}{a(0)},\ Y_j(t) = Y_j(0) \text{ and } Z_j(t) = Z_j(0)\frac{L(t)}{L(0)}, \quad t\in[0,T]. \tag{6.8}$$

Thus $d(\boldsymbol{X}^j(t),t) = 0, j=1,\ldots,J, t\in[0,T]$.

We set $a(t) = 0.1 + 0.01\sin(2\pi t)$ and $L(t) = 1 + 0.3\sin(4\pi t)$. We insert a suitable right hand side in (1.1) such that the exact solution is $u(\boldsymbol{x},t) = \cos(\pi t)x_1x_2x_3$ and consider a time interval $[0,1]$.

We used CGAL [22] to generate an initial triangulation $\Gamma_h(0)$ of $\Gamma(0)$. The mesh had 15991 vertices (the righthand mesh at $t = 1$ in Figure 1 is identical to the initial mesh). We used the same initial triangulation for both schemes. We considered a time interval corresponding to a single period of the evolution, i.e., $[0,1]$ and selected a timestep of 10^{-3} and used a BDF2 time discretisation, i.e., the scheme (5.1). The starting values for the scheme were taken to be the interpolant of the exact solution.

Figure 1 shows snapshots of the meshes obtained with the two different velocities. We clearly observe that moving the vertices of the mesh with the velocity with a nonzero tangential component generates meshes that appear much more suitable for computation than the meshes obtained when the vertices are moved with the material velocity. Figure 2 shows the interpolant of the error, i.e., the Figure shows plots of the function $e_h(\cdot,t) \in \mathcal{S}_h(t)$ with nodal values given by $e_h(\boldsymbol{X}_j, t^n) = |(U_h(\boldsymbol{X}_j))^n - u(\boldsymbol{X}_j, t^n)|, j = 1,\ldots,J$. We observe that the ALE scheme has a significantly smaller error than the Lagrangian scheme.

6.4. **Example** (Simulation on a surface with changing conormal)**.** We define the level set function

$$d(\boldsymbol{x},t) = x_1^2 + x_2^2 + a(t)^2\frac{x_3^2}{b(t)^2} - a(t)^2, \tag{6.9}$$

We set $a(t) = 1 + 0.7\sin^2(\pi t)$ and $b(t) = 1 - 0.7\sin^2(\pi t)$ and define the initial surface as follows

$$\Gamma(0) = \left\{\boldsymbol{x} \in \mathbb{R}^3 \left| d(\boldsymbol{x},0) = 0, x_1^2 + x_2^2 \le 0.25a(0)^2 \right.\right\}. \tag{6.10}$$

We assume the material velocity of the surface is given by

$$\boldsymbol{v} = \frac{-\partial_t d}{|\nabla d|}\frac{\nabla d}{|\nabla d|}, \tag{6.11}$$

with d as in (6.9). Therefore the surface is a portion of an ellipsoid with time dependent axes, with the initial surface a portion of the surface of the unit sphere. We compare the Lagrangian scheme where the vertices are moved with the material velocity with an ALE scheme where the arbitrary tangential velocity $\boldsymbol{a}_{\mathcal{T}}$ is defined as follows. For each node $\boldsymbol{X}_j(t), j = 1,\ldots,J$, given nodes $\boldsymbol{X}_j(0), j = 1,\ldots,J$ on $\Gamma(0)$, we define $\tilde{\boldsymbol{v}}(\boldsymbol{X}_j(t),t)$ as

$$\tilde{v}_1(\boldsymbol{X}_j(t),t) = g\left(\boldsymbol{X}_j(0)\right)\frac{\dot{a}(t)}{a(0)}X_j(0),\ \tilde{v}_2(\boldsymbol{X}_j(t),t) = g\left(\boldsymbol{X}_j(0)\right)\frac{\dot{a}(t)}{a(0)}Y_j(0) \tag{6.12}$$

$$\text{and } \tilde{v}_3(\boldsymbol{X}_j(t),t) = g\left(\boldsymbol{X}_j(0)\right)\frac{\dot{b}(t)}{b(0)}Z_j(0), \quad t \in [0,T],$$

with $g\left(\boldsymbol{X}_j(0)\right) = (a(0)^2 - 4(X_j(0)^2 + Y_j(0)^2))^4$. We set

$$\boldsymbol{v}^a(\boldsymbol{X}_j(t),t) = \boldsymbol{v}(\boldsymbol{X}_j(t),t) + \boldsymbol{a}_{\mathcal{T}}(\boldsymbol{X}_j(t),t), \tag{6.13}$$

where $\boldsymbol{v}(\boldsymbol{X}_j(t),t)$ is the material velocity of the surface and the arbitrary tangential velocity is given by;

$$\boldsymbol{a}_{\mathcal{T}}(\boldsymbol{X}_j(t),t) = \left(\tilde{\boldsymbol{v}}(\boldsymbol{X}_j(t),t) - \tilde{\boldsymbol{v}}(\boldsymbol{X}_j(t),t)\cdot\boldsymbol{\nu}(\boldsymbol{X}_j(t),t)\boldsymbol{\nu}(\boldsymbol{X}_j(t),t)\right), \tag{6.14}$$

where $\boldsymbol{\nu}$ denotes the normal to $\Gamma(t)$. We note that $\boldsymbol{a}_{\mathcal{T}} = \boldsymbol{0}$ on the boundary of the surface so (1.4) holds. We consider the following equation on the surface (6.10);

$$\partial^\bullet_{\boldsymbol{v}} u + u\nabla_{\Gamma(t)}\cdot\boldsymbol{v} - 0.01\Delta_{\Gamma(t)}u = \sin(5\pi x_3^2) \quad \text{on } \Gamma(t), t \in [0,1], \tag{6.15}$$

with boundary conditions of the form (1.2). We take the initial data $u(\boldsymbol{x},0) = 0$. We used a mesh with 33025 vertices and selected a timestep of 10^{-4}. We employed the BDF1 (implicit Euler) scheme (5.62) to compute the discrete solutions. Figure 3 shows snapshots of the discrete solution for the two different velocities. We observe that at $t = 0.4, 0.5$ and 0.6 the mesh generated by the Lagrangian scheme does not appear to resolve the surface adequately while the mesh generated by the ALE scheme where the arbitrary tangential velocity is given by an approximation of the velocity (6.14) appears to generate a mesh more suitable for computation. Moreover, at $t = 0.5$ the discrete solutions obtained with Lagrangian and ALE schemes exhibit qualitative differences in the region where the mesh generated by the Lagrangian scheme is coarse.

6.5. **Example** (Long time simulations on a surface with periodic evolution)**.** We consider a surface

$$\Gamma(t) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \frac{x_1^2}{a(t)^2} + \frac{x_2^2}{b(t)^2} + \frac{x_3^2}{c(t)^2} - 1 = 0 \right\}, \tag{6.16}$$

with $a(t) = 1 - 0.1\sin(\pi t), b(t) = 1 - 0.2\sin(\pi t)$ and $c(t) = 1 + 0.1\sin(\pi t)$. The surface is therefore an ellipsoid with time dependent axes and the initial surface at $t = 0$ is the surface of the unit sphere. We assume the material velocity of the surface is the normal velocity. We consider (1.1) posed on the surface with four different initial conditions

$$u_1(\boldsymbol{x}, 0) = 1 \quad \boldsymbol{x} \in \Gamma(0), \tag{6.17}$$

$$u_2(\boldsymbol{x}, 0) = 1 + \sin(2\pi x_1) \quad \boldsymbol{x} \in \Gamma(0), \tag{6.18}$$

$$u_3(\boldsymbol{x}, 0) = 1 + 4\sin(8\pi x_1) + 3\cos(6\pi x_2) + 2\sin(8\pi x_3) \quad \boldsymbol{x} \in \Gamma(0), \tag{6.19}$$

$$u_4(\boldsymbol{x}, 0) = 1 + 8\sin(16\pi x_1) + 7\cos(14\pi x_2) + 6\sin(24\pi x_3) \quad \boldsymbol{x} \in \Gamma(0). \tag{6.20}$$

We used the Lagrangian BDF1 scheme (5.62) to simulate the equation on a triangulation of the sphere with 16386 vertices and selected a timestep of 10^{-4}. We approximated the initial data for the numerical method as follows

$$U_{h,1}(\boldsymbol{x}, 0) = 1 \quad \boldsymbol{x} \in \Gamma_h(0), \tag{6.21}$$

$$U_{h,2}(\boldsymbol{x}, 0) = \tilde{I}_h u_2(\boldsymbol{x}, 0) + \int_{\Gamma_h(0)} \left(1 - \tilde{I}_h u_2(\cdot, 0)\right) \quad \boldsymbol{x} \in \Gamma_h(0), \tag{6.22}$$

$$U_{h,3}(\boldsymbol{x}, 0) = \tilde{I}_h u_3(\boldsymbol{x}, 0) + \int_{\Gamma_h(0)} \left(1 - \tilde{I}_h u_3(\cdot, 0)\right) \quad \boldsymbol{x} \in \Gamma_h(0), \tag{6.23}$$

$$U_{h,4}(\boldsymbol{x}, 0) = \tilde{I}_h u_4(\boldsymbol{x}, 0) + \int_{\Gamma_h(0)} \left(1 - \tilde{I}_h u_4(\cdot, 0)\right) \quad \boldsymbol{x} \in \Gamma_h(0), \tag{6.24}$$

where $\tilde{I}_h : \mathrm{C}(\Gamma(0)) \to \mathcal{S}_h(0)$ denotes the linear Lagrange interpolation operator. The approximations of the initial conditions for the numerical scheme were chosen such that the initial approximations have the same total mass. We note that the approximations of the the initial conditions satisfy (5.30).

Figure 4 shows plots of the initial conditions (6.22), (6.23) and (6.24) on the discrete surface. Figure 5 shows snapshots of the discrete solution for the case of constant initial data (6.21) we observe that the numerical solution appears to converge rapidly to a periodic function. We wish to investigate numerically the effect of the initial data on this periodic solution, to this end we compute the numerical solution with the initial conditions (6.22), (6.23) and (6.24) and compare these numerical solutions to that obtained with constant initial data. Figure 6 shows the $\mathrm{L}_2(\Gamma_h(t))$ norm of the difference between the numerical solutions with the non-constant initial data and the numerical solution with the constant initial data versus time. It appears that the numerical solutions converge to the same periodic solution for all four different initial conditions.

APPENDIX A. TRANSPORT FORMULA

The following transport formula play a fundamental role in the formulation and analysis of the numerical method.

A.1. **Lemma** (Transport formula)**.** *Let $\mathcal{M}(t)$ be a smoothly evolving surface with material velocity $\boldsymbol{v}$, let f and g be sufficiently smooth functions and $\boldsymbol{w}$ a sufficiently smooth vector field such that all the* *following quantities exist. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}(t)} f = \int_{\mathcal{M}(t)} \partial^\bullet_{\boldsymbol{v}} f + f\nabla_\Gamma\cdot\boldsymbol{v}, \tag{A.1}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}(t)} f\boldsymbol{w}\cdot\nabla_\Gamma g = &\int_{\mathcal{M}(t)} \left(\partial^\bullet_{\boldsymbol{v}} f\boldsymbol{w}\cdot\nabla_\Gamma g + f\partial^\bullet_{\boldsymbol{v}}\boldsymbol{w}\cdot\nabla_\Gamma g + f\boldsymbol{w}\cdot\nabla_\Gamma\partial^\bullet_{\boldsymbol{v}} g\right)\\ &+\int_{\mathcal{M}(t)}\nabla_\Gamma\cdot\boldsymbol{v}\,(f\boldsymbol{w}\cdot\nabla_\Gamma g) + \int_{\mathcal{M}(t)} f\boldsymbol{w}\cdot\mathcal{B}(\boldsymbol{v},\boldsymbol{\nu})\nabla_\Gamma g\end{aligned} \tag{A.2}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}(t)}\nabla_\Gamma f\cdot\nabla_\Gamma g = &\int_{\mathcal{M}(t)}\left(\nabla_\Gamma\partial^\bullet_{\boldsymbol{v}} f\cdot\nabla_\Gamma g + \nabla_\Gamma\partial^\bullet_{\boldsymbol{v}} g\cdot\nabla_\Gamma f\right)\\ &+\int_{\mathcal{M}(t)}\left(\nabla_\Gamma\cdot\boldsymbol{v} - 2\mathcal{D}(\boldsymbol{v})\right)\nabla_\Gamma f\cdot\nabla_\Gamma g,\end{aligned} \tag{A.3}$$

with the deformation tensors defined by

$$\mathcal{B}(\boldsymbol{v},\boldsymbol{\nu})_{ij} = (\nabla_\Gamma)_i v_j - \sum_{l=1}^{m+1}\nu_l\nu_i(\nabla_\Gamma)_j v_l \quad \textit{and}\; \mathcal{D}(\boldsymbol{v})_{ij} = \frac{1}{2}\left((\nabla_\Gamma)_i v_j + (\nabla_\Gamma)_j v_i\right),$$

respectively.

Proof . Proofs of (A.1) and (A.3) are given in [1]. The proof of (A.2) is as follows, (for further details see the proof of (A.3) in [1, Appendix]) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}(t)} f\boldsymbol{w}\cdot\nabla_\Gamma g &= \int_{\mathcal{M}(t)}\partial^\bullet_{\boldsymbol{v}}\left(f\boldsymbol{w}\cdot\nabla_\Gamma g\right) + \nabla_\Gamma\cdot\boldsymbol{v}\,(f\boldsymbol{w}\cdot\nabla_\Gamma g)\\ &= \int_{\mathcal{M}(t)}\partial^\bullet_{\boldsymbol{v}} f\,(\boldsymbol{w}\cdot\nabla_\Gamma g) + f\,(\partial^\bullet_{\boldsymbol{v}}\boldsymbol{w})\cdot\nabla_\Gamma g + f\boldsymbol{w}\cdot(\partial^\bullet_{\boldsymbol{v}}\nabla_\Gamma g) + \nabla_\Gamma\cdot\boldsymbol{v}\,(f\boldsymbol{w}\cdot\nabla_\Gamma g),\end{aligned} \tag{A.4}$$

the following result from [7]

$$\partial^\bullet_{\boldsymbol{v}}\nabla_{\Gamma(t)} g = \nabla_{\Gamma(t)}\partial^\bullet_{\boldsymbol{v}} f - \mathcal{B}(\boldsymbol{v},\boldsymbol{\nu})\nabla_{\Gamma(t)} g,$$

completes the proof. □

For the analysis of the second order scheme we note that repeated application of the transport formula together with the smoothness of the velocity yields the following bounds, see [4] for a similar discussion. Let $\mathcal{M}(t)$ be a smoothly evolving surface with material velocity $\boldsymbol{v}$, let f and g be sufficiently smooth functions and $\boldsymbol{w}$ a sufficiently smooth vector and further assume $\partial^\bullet_{\boldsymbol{v}} g = 0$ then

$$\begin{aligned}\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_\Gamma\nabla_\Gamma f\cdot\nabla_\Gamma g\mathrm{d}t\right| \le &\left|\int_\Gamma\nabla_\Gamma\partial^\bullet_{\boldsymbol{v}}(\partial^\bullet_{\boldsymbol{v}} f)\cdot\nabla_\Gamma g\mathrm{d}t\right|\\ &+ c\left(\left|\int_\Gamma\nabla_\Gamma\partial^\bullet_{\boldsymbol{v}} f\cdot\nabla_\Gamma g\mathrm{d}t\right| + \left|\int_\Gamma\nabla_\Gamma f\cdot\nabla_\Gamma g\mathrm{d}t\right|\right),\end{aligned} \tag{A.5}$$

$$\begin{aligned}\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_\Gamma f\boldsymbol{w}\cdot\nabla_\Gamma g\mathrm{d}t\right| \le &\left|\int_\Gamma\partial^\bullet_{\boldsymbol{v}}(\partial^\bullet_{\boldsymbol{v}} f)\boldsymbol{w}\cdot\nabla_\Gamma g\mathrm{d}t\right|\\ &+ c\left(\left|\int_\Gamma\partial^\bullet_{\boldsymbol{v}} f\boldsymbol{w}\cdot\nabla_\Gamma g\mathrm{d}t\right| + \left|\int_\Gamma f\boldsymbol{w}\cdot\nabla_\Gamma g\mathrm{d}t\right|\right).\end{aligned} \tag{A.6}$$

APPENDIX B. APPROXIMATION RESULTS

The following approximation Lemma was shown in [23]. For a function $\eta\in C^0(\Gamma(t))$ we denote by $I_h\eta\in\mathcal{S}_h^l$ the lift of the linear Lagrange interpolant of $\tilde{I}_h\eta\in\mathcal{S}_h$, i.e., $I_h\eta = (\tilde{I}_h\eta)^l$.

B.1. **Lemma** (Interpolation bounds). *For an $\eta \in \mathrm{H}^2\,\Gamma(t)$ there exists a unique $I_h\eta \in \mathcal{S}_h^l(t)$ such that*

$$\|\eta - I_h\eta\|_{\mathrm{L}_2(\Gamma(t))} + h \left\|\nabla_{\Gamma(t)}(\eta - I_h\eta)\right\|_{\mathrm{L}_2(\Gamma(t))} \le ch^2 \|\eta\|_{\mathrm{H}^2(\Gamma(t))}\,. \tag{B.1}$$

The following results provide estimates for the difference between the continuous velocity (here we mean the velocity that includes the arbitrary tangential motion and *not* the material velocity) and the discrete velocity of the smooth surface together with an estimate on the material derivative.

B.2. **Lemma.** *Velocity and material derivative estimates*

$$|\boldsymbol{v}_a - \boldsymbol{v}_h^a| + h \left|\nabla_{\Gamma(t)}\left(\boldsymbol{v}_a - \boldsymbol{v}_h^a\right)\right| \le ch^2 \quad \text{on } \Gamma \tag{B.2}$$

$$\left|\partial^\bullet_{h,\boldsymbol{v}_h^a}\left(\boldsymbol{v}_a - \boldsymbol{v}_h^a\right)\right| + h\left|\nabla_{\Gamma(t)}\partial^\bullet_{h,\boldsymbol{v}_h^a}\left(\boldsymbol{v}_a - \boldsymbol{v}_h^a\right)\right| \le ch^2 \quad \text{on } \Gamma \tag{B.3}$$

$$\|\boldsymbol{a}_{\boldsymbol{\mathcal{T}}} - \boldsymbol{t}_h^a\|_{\mathrm{L}_2(\Gamma(t))} + h\left\|\nabla_{\Gamma(t)}\left(\boldsymbol{a}_{\boldsymbol{\mathcal{T}}} - \boldsymbol{t}_h^a\right)\right\|_{\mathrm{L}_2(\Gamma(t))} \le ch^2 \|\boldsymbol{a}_{\boldsymbol{\mathcal{T}}}\|_{\mathrm{H}^2(\Gamma(t))}\,. \tag{B.4}$$

$$\left\|\partial^\bullet_{\boldsymbol{v}_a} z - \partial^\bullet_{h,\boldsymbol{v}_h^a} z\right\|_{\mathrm{L}_2(\Gamma(t))} \le ch^2 \|z\|_{\mathrm{H}^1(\Gamma(t))} \tag{B.5}$$

$$\left\|\nabla_{\Gamma(t)}\left(\partial^\bullet_{\boldsymbol{v}_a} z - \partial^\bullet_{h,\boldsymbol{v}_h^a} z\right)\right\|_{\mathrm{L}_2(\Gamma(t))} \le ch \|z\|_{\mathrm{H}^2(\Gamma(t))} \tag{B.6}$$

$$\left\|\partial^\bullet_{\boldsymbol{v}_a}\partial^\bullet_{\boldsymbol{v}_a} z - \partial^\bullet_{\boldsymbol{v}_h^a}\partial^\bullet_{h,\boldsymbol{v}_h^a} z\right\|_{\mathrm{L}_2(\Gamma(t))} \le ch^2 \|\partial^\bullet_{\boldsymbol{v}} z\|_{\mathrm{H}^1(\Gamma(t))} \tag{B.7}$$

$$\left\|\nabla_{\Gamma(t)}\left(\partial^\bullet_{\boldsymbol{v}_a}\partial^\bullet_{\boldsymbol{v}_a} z - \partial^\bullet_{\boldsymbol{v}_h^a}\partial^\bullet_{h,\boldsymbol{v}_h^a} z\right)\right\|_{\mathrm{L}_2(\Gamma(t))} \le ch \|\partial^\bullet_{\boldsymbol{v}} z\|_{\mathrm{H}^2(\Gamma(t))}\,. \tag{B.8}$$

Proof . The estimate (B.2) is shown in [2, Lemma 5.6], (B.3) is shown in [8, Lemma 7.3], (B.4) follows from Lemma B.1 and the fact that $\boldsymbol{T}_h^a$ is the interpolant of the arbitrary tangential velocity and $\boldsymbol{t}_h^a$ is its lift. Estimates (B.5) and (B.6) are shown in [2, Cor. 5.7]. The estimates (B.7) and (B.8) follow easily from (B.3), (B.5) and (B.6). □

We now state some results on the error due to the approximation of the surface

B.3. **Lemma** (Geometric perturbation errors). *For any $(\Psi_h(\cdot,t), \Phi_h(\cdot,t)) \in \mathcal{S}_h(t)\times\mathcal{S}_h(t)$ with corresponding lifts $(\psi_h(\cdot,t), \varphi_h(\cdot,t)) \in \mathcal{S}_h^l(t)\times\mathcal{S}_h^l(t)$, the following bounds hold:*

$$|m(\psi_h,\varphi_h) - m_h(\Psi_h,\Phi_h)| \le ch^2 \|\psi_h\|_{\mathrm{L}_2(\Gamma(t))}\|\varphi_h\|_{\mathrm{L}_2(\Gamma(t))} \tag{B.9}$$

$$|a(\psi_h,\varphi_h) - a_h(\Psi_h,\Phi_h)| \le ch^2 \left\|\nabla_{\Gamma(t)}\psi_h\right\|_{\mathrm{L}_2(\Gamma(t))}\left\|\nabla_{\Gamma(t)}\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))} \tag{B.10}$$

$$|g(\psi_h,\varphi_h;\boldsymbol{v}_h^a) - g_h(\Psi_h,\Phi_h;\boldsymbol{V}_h^a)| \le ch^2 \|\psi_h\|_{\mathrm{H}^1(\Gamma(t))}\|\varphi_h\|_{\mathrm{H}^1(\Gamma(t))} \tag{B.11}$$

$$|b(\psi_h,\varphi_h;\boldsymbol{t}_h^a) - b_h(\Psi_h,\Phi_h;\boldsymbol{T}_h^a)| \le ch^2 \|\psi_h\|_{\mathrm{L}_2(\Gamma(t))}\left\|\nabla_{\Gamma(t)}\varphi_h\right\|_{\mathrm{L}_2(\Gamma(t))}, \tag{B.12}$$

$$\left|m\left(\partial^\bullet_{h,\boldsymbol{v}_h^a}\psi_h,\varphi_h\right) - m_h\left(\partial^\bullet_{h,\boldsymbol{V}_h^a}\Psi_h,\Phi_h\right)\right| \le ch^2\left\|\partial^\bullet_{h,\boldsymbol{v}_h^a}\psi_h\right\|_{\mathrm{L}_2(\Gamma(t))}\|\varphi_h\|_{\mathrm{L}_2(\Gamma(t))} \tag{B.13}$$

with $\boldsymbol{V}_h^a, \boldsymbol{T}_h^a, \boldsymbol{v}_h^a$ and $\boldsymbol{t}_h^a$ as defined in §3.

Proof . A proof of (B.9), (B.10) and (B.11) is given in [2, Lemma 5.5]. The proof of (B.13) is in [2, Lemma 5.8]. We now prove (B.12). We start by introducing some notation. We denote by δ_h the quotient between the discrete and smooth surface measures which satisfies [1, Lemma 5.1]

$$\sup_{t\in(0,T)}\sup_{\Gamma_h(t)}|1-\delta_h| \le ch^2 \tag{B.14}$$

We introduce $\boldsymbol{P}, \boldsymbol{P}_h$ the projections onto the tangent planes of $\Gamma(t)$ and Γ_h respectively. We denote by $\boldsymbol{\mathcal{H}}$ the Weingarten map ($\mathcal{H}_{ij} = \partial_{x_j}\nu_i$).

$$|b(\psi_h,\varphi_h;\boldsymbol{t}_h^a) - b_h(\Psi_h,\Phi_h;\boldsymbol{T}_h^a)| = \left|\int_{\Gamma(t)}\psi_h\boldsymbol{t}_h^a\cdot\nabla_{\Gamma(t)}\varphi_h - \int_{\Gamma_h(t)}\Psi_h\boldsymbol{T}_h^a\cdot\nabla_{\Gamma_h(t)}\Phi_h\right| \tag{B.15}$$

From [1] we have

$$\nabla_{\Gamma_h}\eta = \boldsymbol{B}_h \nabla_\Gamma \eta^l, \tag{B.16}$$

where $\boldsymbol{B}_h = \boldsymbol{P}_h(\boldsymbol{I} - d\boldsymbol{\mathcal{H}})$. We have with $\boldsymbol{p}, \boldsymbol{x}$ as in (3.1),

$$\begin{aligned}\boldsymbol{T}_h^a(\boldsymbol{x},\cdot)\cdot\nabla_{\Gamma_h}\Phi_h(\boldsymbol{x},\cdot) &= \boldsymbol{P}_h\boldsymbol{T}_h^a(\boldsymbol{x},\cdot)\cdot\nabla_{\Gamma_h}\Phi_h(\boldsymbol{x},\cdot)\\ &= \boldsymbol{P}_h\boldsymbol{t}_h^a(\boldsymbol{p},\cdot)\cdot\boldsymbol{P}_h(\boldsymbol{I}-d\boldsymbol{\mathcal{H}})\boldsymbol{P}\nabla_\Gamma\varphi_h(\boldsymbol{p},\cdot)\\ &= \boldsymbol{P}_h\boldsymbol{t}_h^a(\boldsymbol{p},\cdot)\cdot\boldsymbol{P}_h\boldsymbol{P}(\boldsymbol{I}-d\boldsymbol{\mathcal{H}})\nabla_\Gamma\varphi_h(\boldsymbol{p},\cdot)\\ &= (\boldsymbol{I}-d\boldsymbol{\mathcal{H}})\boldsymbol{P}\boldsymbol{P}_h\boldsymbol{t}_h^a(\boldsymbol{p},\cdot)\cdot\nabla_\Gamma\varphi_h(\boldsymbol{p},\cdot)\\ &= \boldsymbol{\mathcal{Q}}_h\boldsymbol{t}_h^a(\boldsymbol{p},\cdot)\cdot\nabla_\Gamma\varphi_h(\boldsymbol{p},\cdot)\end{aligned} \tag{B.17}$$

where the last equality defines $\boldsymbol{\mathcal{Q}}_h$. We denote the lifted version by $\boldsymbol{\mathcal{Q}}_h^l$ Thus we may write (B.15) as

$$\begin{aligned}\left|b\left(\psi_h,\varphi_h;\boldsymbol{t}_h^a\right)-b_h\left(\Psi_h,\Phi_h;\boldsymbol{T}_h^a\right)\right| &= \left|\int_{\Gamma(t)}\psi_h\boldsymbol{t}_h^a\cdot\nabla_{\Gamma(t)}\varphi_h-\int_{\Gamma(t)}\frac{1}{\delta_h^l}\psi_h\boldsymbol{\mathcal{Q}}_h^l\boldsymbol{t}_h^a\cdot\nabla_{\Gamma(t)}\varphi_h\right|\\ &\le\left|\int_{\Gamma(t)}\psi_h(\boldsymbol{I}-\boldsymbol{\mathcal{Q}}_h^l)\boldsymbol{P}\boldsymbol{t}_h^a\cdot\nabla_{\Gamma(t)}\varphi_h\right|+ch^2,\end{aligned} \tag{B.18}$$

where we have used (B.14). We now apply the following result from [1, Lem 5.1]

$$\sup_{t\in(0,T)}\sup_{\Gamma_h(t)}\left|\left(\boldsymbol{I}-\boldsymbol{\mathcal{Q}}_h\right)\boldsymbol{P}\right|\le ch^2, \tag{B.19}$$

which yields the desired bound. □

APPENDIX C. RITZ PROJECTION ESTIMATES

It is proves helpful in the analysis to introduce the Ritz projection $\mathrm{R}^h : \mathrm{H}^1(\Gamma) \to \mathcal{S}_h^l$ defined as follows: for $z \in \mathrm{H}^1(\Gamma)$ with $\int_\Gamma z = 0$,

$$a\left(\mathrm{R}^h z, \varphi_h\right) = a\left(z, \varphi_h\right) \quad \forall \varphi_h \in \mathcal{S}_h^l. \tag{C.1}$$

C.1. **Lemma** (Ritz projection estimates)**.** *We recall the following estimates proved in [2] that hold for the mesh-size h sufficiently small*

$$\left\|z-\mathrm{R}^h z\right\|_{\mathrm{L}_2(\Gamma)}+h\left\|\nabla_\Gamma\left(z-\mathrm{R}^h z\right)\right\|_{\mathrm{L}_2(\Gamma)}\le ch^2\left\|z\right\|_{\mathrm{H}^2(\Gamma)}. \tag{C.2}$$

$$\begin{aligned}\left\|\partial^\bullet_{h,\boldsymbol{v}_h^a}\left(z-\mathrm{R}^h z\right)\right\|_{\mathrm{L}_2(\Gamma)}+h\left\|\nabla_\Gamma\left(\partial^\bullet_{h,\boldsymbol{v}_h^a}z-\partial^\bullet_{h,\boldsymbol{v}_h^a}\left(\mathrm{R}^h z\right)\right)\right\|_{\mathrm{L}_2(\Gamma)}\le\\ ch^2\left(\left\|z\right\|_{\mathrm{H}^2(\Gamma)}+\left\|\partial^\bullet_{\boldsymbol{v}_a}z\right\|_{\mathrm{H}^2(\Gamma)}\right).\end{aligned} \tag{C.3}$$

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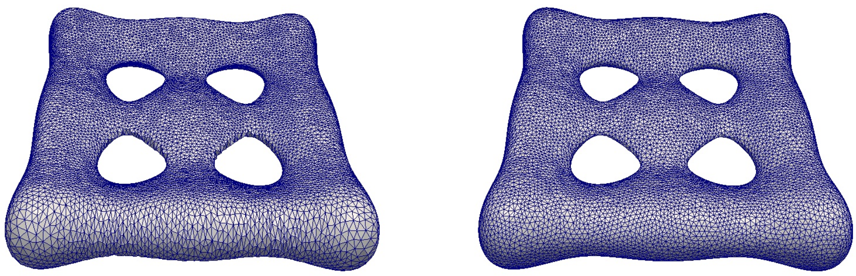
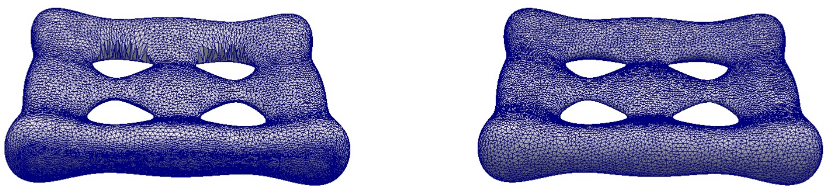


(a) $t = 0.2$



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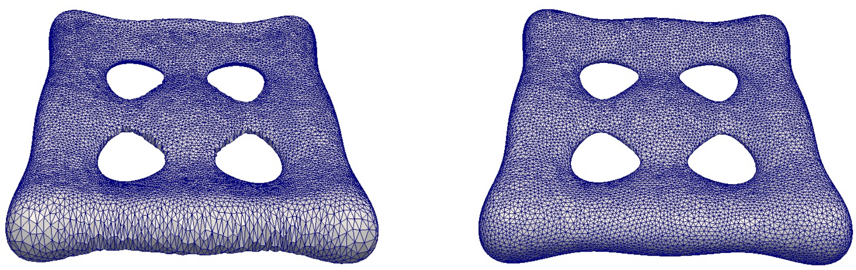
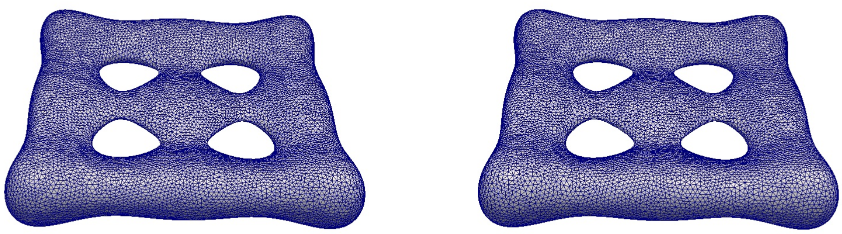
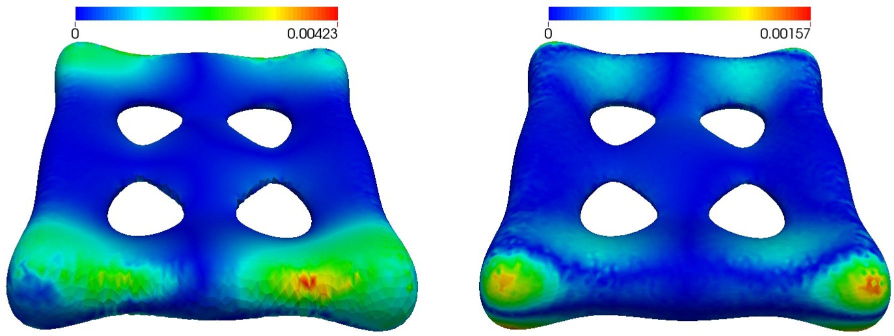


(c) $t = 0.7$

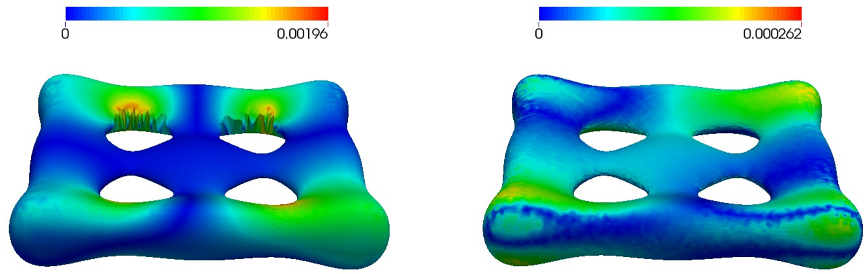


(d) $t = 1$

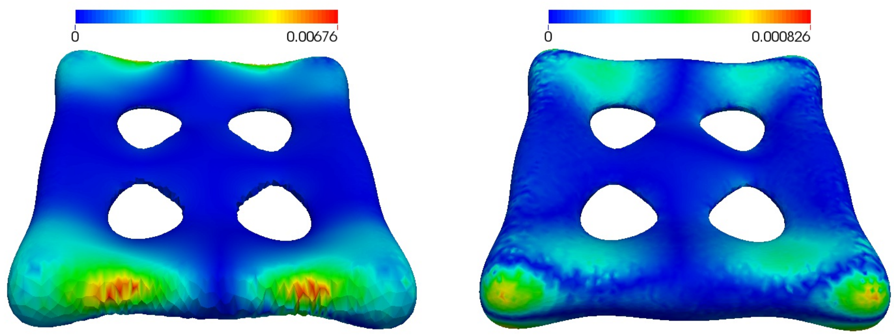
FIGURE 1. Meshes obtained for Example 6.3 with an approximation of the Lagrangian (zero tangential) velocity (lefthand column) and with the ALE velocity (6.8) (righthand column).



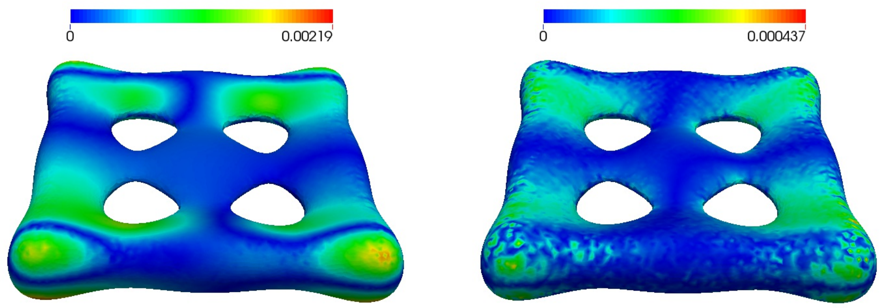
(a) $t = 0.2$



(b) $t = 0.4$

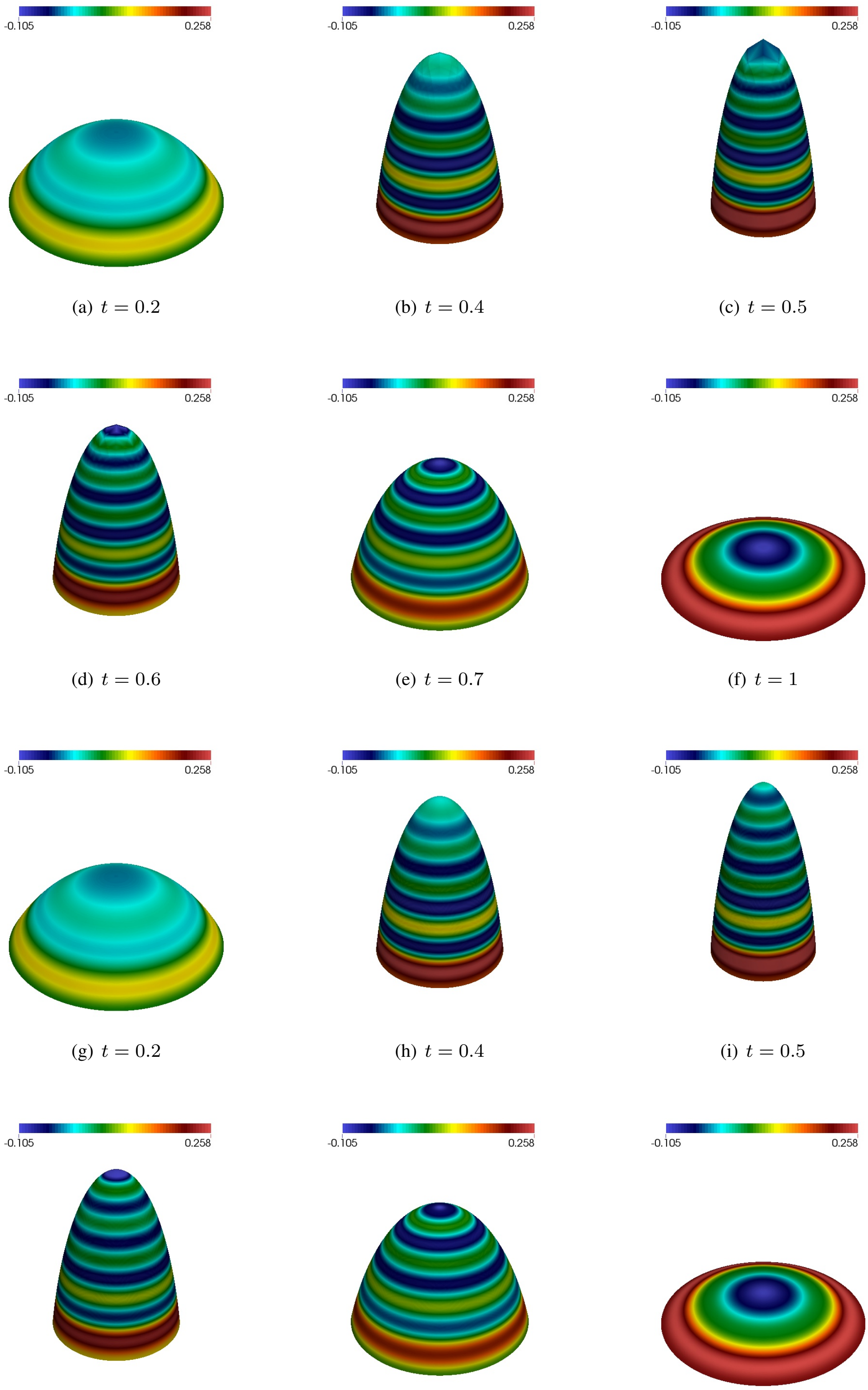


(c) $t = 0.7$



(d) $t = 1$

FIGURE 2. Snapshots of the interpolant of the error using the two different schemes for Example 6.3, the left hand column corresponds to the Lagrangian scheme and the righthand column corresponds to the ALE scheme.



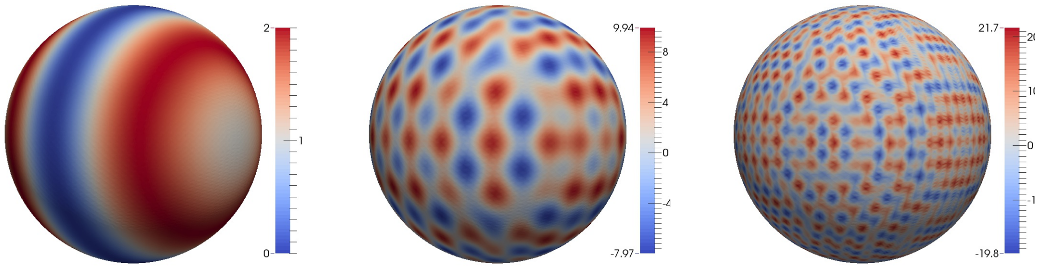
(a) $t = 0.2$ (b) $t = 0.4$ (c) $t = 0.5$

(d) $t = 0.6$ (e) $t = 0.7$ (f) $t = 1$

(g) $t = 0.2$ (h) $t = 0.4$ (i) $t = 0.5$

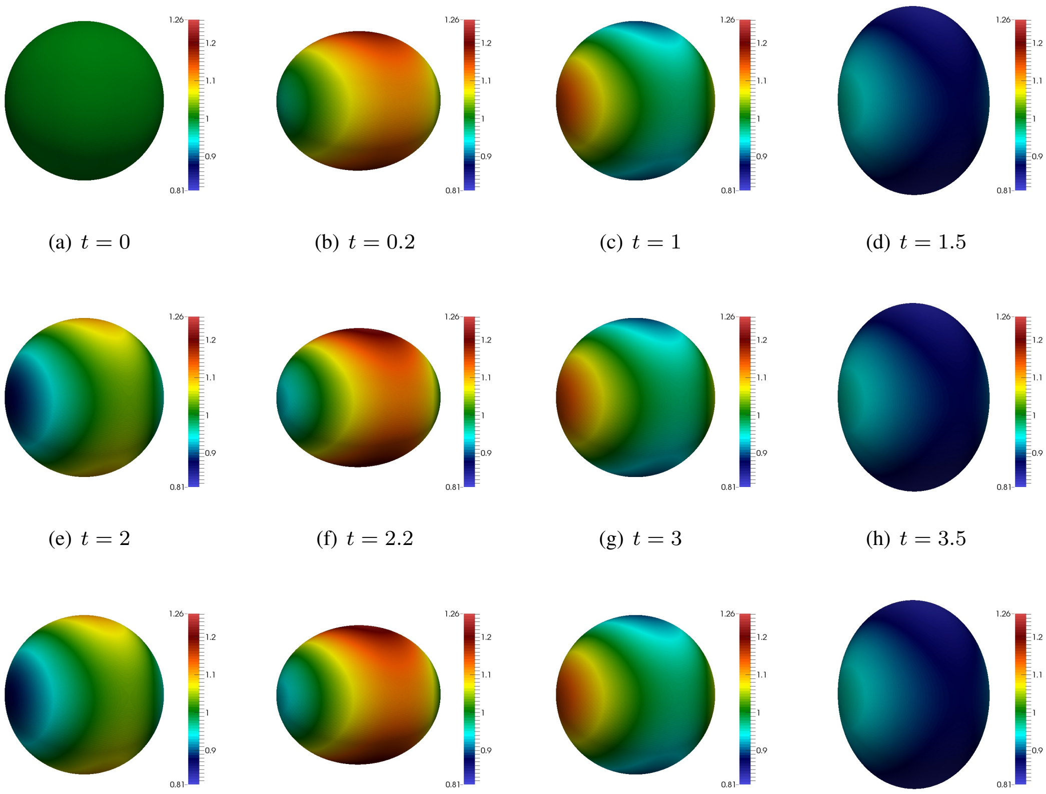
(j) $t = 0.6$ (k) $t = 0.7$ (l) $t = 1$

FIGURE 3. Snapshots of the discrete solution using the two different schemes for Example 6.4, the top two rows correspond to the Lagrangian scheme and the bottom two rows correspond to the ALE scheme.



(a) $U_{h,2}(\cdot,0)$ (b) $U_{h,3}(\cdot,0)$ (c) $U_{h,4}(\cdot,0)$

FIGURE 4. Initial conditions (6.22), (6.23) and (6.24) for the simulations of Example 6.5.



(a) $t = 0$ (b) $t = 0.2$ (c) $t = 1$ (d) $t = 1.5$

(e) $t = 2$ (f) $t = 2.2$ (g) $t = 3$ (h) $t = 3.5$

(i) $t = 4$ (j) $t = 4.2$ (k) $t = 5$ (l) $t = 5.5$

FIGURE 5. Snapshots of the numerical solution of Example 6.5 with constant initial data (6.21).

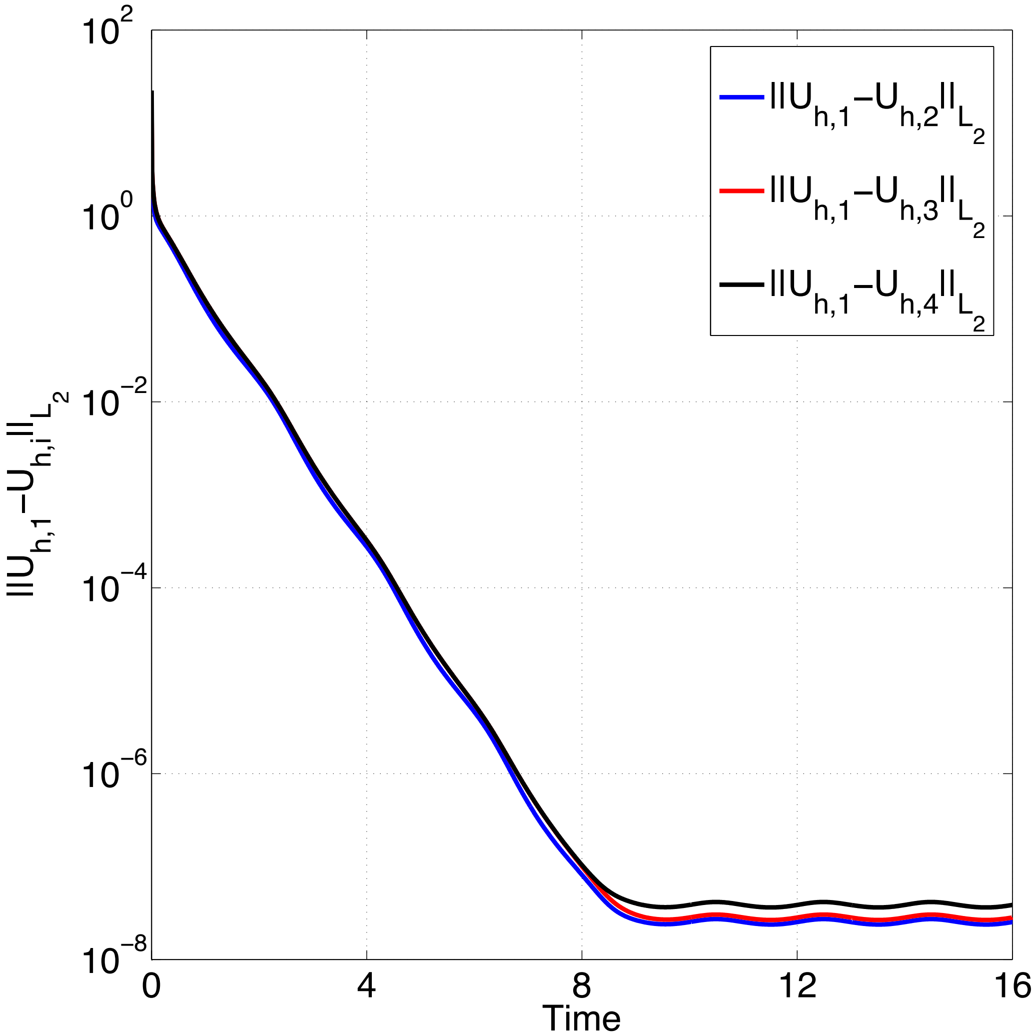


FIGURE 6. The $\mathrm{L}_2(\Gamma_h(t))$ norm of the difference between numerical solution with constant initial data (6.21) and the numerical solutions corresponding to the initial conditions (6.22), (6.23) and (6.24) (blue, red and black lines respectively).